On the reachable set of perturbed heat equations

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Outline

1. Introduction
2. Small Time Null-Controllable Linear Systems
3. Applications to the heat equation
4. Further comments
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Very (very) brief introduction to control theory:
Dynamical system $y' = f(y, u)$.

- $y$ is the state;
- $f$ describes the dynamics;
- $u$ is the control.

**Control theory**

Describe the possible actions of the control on the state.

Examples:
- Park a car;
- Swim;
- . . . ,

*Tout est dans le contrôle.* - M. Platini.
Control of linear systems:

\[ y' = Ay + Bu, \quad t \geq 0, \quad y(0) = y_0. \]

- \( A \) is a linear operator, generating a \( C^0 \) semi-group \( (e^{tA})_{t \geq 0} \) on an Hilbert space \( H \).
- \( y \in C^0([0, T]; H) \) is the state.
- \( B \) is the control operator, \( \in \mathcal{L}(U; H) \).
- \( u \in L^2(0, T; U) \) is the control.

**Objective**

Describe the reachable set \( \mathcal{R}(T, y_0) \) defined by

\[ \mathcal{R}(T, y_0) = \{ y(T), u \in L^2(0, T; U) \}. \]
Theorem (Finite dimension) [Kalman Ho Larendra ’63]

Let \( n \in \mathbb{N}, A \in \mathbb{R}^{n \times n}, H = \mathbb{R}^n \). Then for all \( T > 0 \),

\[
\mathcal{R}(T, y_0) = e^{TA}y_0 + \text{Ran}(B|AB|A^2B|\cdots|A^{n-1}B).
\]

In particular, denoting \( \mathcal{R} = \text{Ran}(B|AB|A^2B|\cdots|A^{n-1}B) \):

- If \( \mathcal{R} = \mathbb{R}^n \), the system \( y' = \tilde{A}y + \tilde{B}v \) is controllable for any operators \((\tilde{A}, \tilde{B})\) close enough to \((A, B)\).

- If \( \mathcal{R} \neq \mathbb{R}^n \), then
  - \( A_\mathcal{R} = A|_\mathcal{R} \in \mathcal{L}(\mathcal{R}), B \in \mathcal{L}(U, \mathcal{R}) \);
  - the system \( z' = A_\mathcal{R}z + Bv \) is exactly controllable on \( \mathcal{R} \).
Much more delicate in **infinite dimensional settings**:

- There are vector spaces which are not closed;
- Cayley Hamilton’s theorem does not apply.

**Question**

What happens for **infinite dimensional** systems?

⇝ A typical example: **the heat equation**

\[
\begin{align*}
\partial_t y - \partial_{xx} y &= 0, & \text{in } (0, T) \times (-L, L), \\
y(t, -L) &= u_-(t), & \text{on } (0, T), \\
y(t, L) &= u_+(t), & \text{on } (0, T), \\
y(0, x) &= y_0(x), & \text{in } (-L, L).
\end{align*}
\]
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Control of linear systems:

\[ y' = Ay + Bu, \quad t \geq 0, \quad y(0) = y_0. \]

- \( A \) is a linear operator, generating a \( C^0 \) semi-group \( (e^{tA})_{t \geq 0} \) on an Hilbert space \( H \).
- \( y \in C^0([0, T]; H) \) is the state.
- \( B \) is the control operator, \( B \in \mathcal{L}(U; H) \) (or admissible).
- \( u \in L^2(0, T; U) \) is the control.

**Standing assumptions:**

System \( y' = Ay + Bu \) is null-controllable in any time \( T > 0 \): 
\[ \forall y_0 \in H, \exists u \in L^2(0, T; U) \text{ such that } y(T) = 0. \]

\[ \Rightarrow \] Satisfied for heat type equations.

[Fursikov-Imanuvilov ’96, Lebeau Robbiano ’95].
The reachable set

**Definition**

The reachable set $\mathcal{R}(T, y_0)$ is defined by

$$\mathcal{R}(T, y_0) = \{ y(T), \ u \in L^2(0, T; U) \}.$$

**Theorem**

The reachable set $\mathcal{R}(T, y_0)$ is independent of $T > 0$ and $y_0 \in H$, now simply denoted $\mathcal{R}$.

- Null-controllable $\Rightarrow \mathcal{R}(T, y_0) = \mathcal{R}(T, 0)$.
- For $T_1 < T_2$, $\mathcal{R}(T_1, 0) \subset \mathcal{R}(T_2, 0)$.
- Null-controllable $\Rightarrow$ Exactly controllable to trajectories. $\Rightarrow$ For $T_1 < T_2$, we also have $\mathcal{R}(T_1, 0) \supset \mathcal{R}(T_2, 0)$.
Proposition

$\mathcal{R}$ is a Hilbert space when endowed with the norm

$$\|Z\|_{\mathcal{R}(T)} = \inf \left\{ \|U\|_{L^2(0,T;U)}, \right.$$  

s.t. $z = y(T)$, with $y' = Ay + Bu$, $y(0) = 0$.}

- For $T_1 < T_2$, $\forall z \in \mathcal{R}$, $\|Z\|_{\mathcal{R}(T_2)} \leq \|Z\|_{\mathcal{R}(T_1)}$
- For $T_1 < T_2$, $\exists C = C(T_1, T_2)$, $\|Z\|_{\mathcal{R}(T_1)} \leq C \|Z\|_{\mathcal{R}(T_2)}$.

$\Rightarrow$ All these norms are equivalent.
Theorem [S.E., K. Le Balc’h, M. Tucsnak 2021]

For $\tau > 0$, we set

$$\mathcal{T}_t = e^{tA}|_{\mathcal{R}(\tau)}, \quad (t \geq 0).$$

Then the family $\mathcal{T} = \left( \mathcal{T}_t|_{\mathcal{R}(\tau)} \right)_{t \geq 0}$

- does not depend on the choice of $\tau > 0$,
- forms a $C^0$ semigroup on $\mathcal{R}(\tau)$,
- has generator $\tilde{A}$ defined by $\mathcal{D}(\tilde{A}) = \mathcal{D}(A) \cap \mathcal{R}(\tau)$ and $\tilde{A}z = Az$ for $z \in \mathcal{D}(\tilde{A})$.

Finally, the system $z' = \tilde{A}z + Bu$ is exactly controllable in $\mathcal{R}(\tau)$ in any time $T > 0$. 
Lemma

\[ \exists c_\tau > 0 \text{ s.t. } \forall t \in [0, \tau], \ z \in \mathcal{R}(\tau), \]

\[ \| \mathbb{T}_t z \|_{\mathcal{R}(\tau)} \leq c_\tau \| z \|_{\mathcal{R}(\tau)}. \]

Proof. For \( t \in [0, \tau], \ z \in \mathcal{R}(\tau) \)

\[ \| \mathbb{T}_t z \|_{\mathcal{R}(\tau)} \leq C_\tau \| \mathbb{T}_t z \|_{\mathcal{R}(2\tau)} \leq C_\tau \| z \|_{\mathcal{R}(2\tau-t)} \leq C_\tau \| z \|_{\mathcal{R}(\tau)}. \]
By standard perturbation arguments for the exactly controllable system $z' = \tilde{A}z + Bu$ (on $\mathcal{R}(\tau)$).

**Theorem** [S.E., K. Le Balc’h, M. Tucsnak 2021]

For all $\tau > 0$, there exists $\varepsilon_\tau > 0$ such that if

$$P \in \mathcal{L}(H) \cap \mathcal{L}(\mathcal{R}(\tau))$$

with

$$\|P\|_{\mathcal{L}(\mathcal{R}(\tau))} \leq \varepsilon_\tau,$$

then the reachable set $\mathcal{R}^P(\tau)$ of the system

$$y' = Ay + Py + Bu, \quad t \geq 0, \quad y(0) = 0,$$

coinsides with $\mathcal{R}(\tau)$. 

For $T > 0$, $\exists C > 0$ and $\exists$ a continuous linear map

$$\mathcal{L} : \mathcal{R}(\tau) \times L^1([0, T]; \mathcal{R}(\tau)) \rightarrow L^2([0, T]; U)$$

such that $\forall \eta \in \mathcal{R}(\tau)$ and $f \in L^1([0, T]; \mathcal{R}(\tau))$ the solution of

$$z'(t) = \tilde{A}z(t) + Bu(t) + f(t), \quad (t \in [0, T]), \quad z(0) = 0,$$

associated to the control $u = \mathcal{L}(\eta, f)$, satisfies $z \in C^0([0, T]; \mathcal{R}(\tau))$, together with $z(T) = \eta$, and

$$\|z\|_{C^0([0, T]; \mathcal{R}(\tau))} + \|u\|_{L^2([0, T]; U)} \leq C \left( \|\eta\|_{\mathcal{R}(\tau)} + \|f\|_{L^1([0, T]; \mathcal{R}(\tau))} \right).$$
Corollary

Suppose that $f : C^0([0, T]; \mathbb{R}(\tau)) \rightarrow L^1([0, T]; \mathbb{R}(\tau))$ satisfies $f(0) = 0$ and, for all $z_1, z_2 \in C^0([0, T]; \mathbb{R}(\tau))$ we have

$$
\|f(z_1) - f(z_2)\|_{L^1([0, T]; \mathbb{R}(\tau))} \\
\leq C\|(z_1, z_2)\|_{(C^0([0, T]; \mathbb{R}(\tau))^2} \|z_1 - z_2\|_{C^0([0, T]; \mathbb{R}(\tau))}.
$$

Then $\exists \delta > 0, \forall \eta \in \mathbb{R}(\tau)$ satisfying $\|\eta\|_{\mathbb{R}(\tau)} \leq \delta$, $\exists$ a control function $u \in L^2([0, T]; U)$ and a controlled trajectory $z \in C^0([0, T]; \mathbb{R}(\tau))$ satisfying

$$z'(t) = A z(t) + B u(t) + f(z)(t), \quad (t \in [0, T]), \quad z(0) = 0,$$

and $z(T) = \eta$. 

[S.E., K. Le Balc’h, M. Tucsnak 2021]
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The 1-d heat equation

\[
\begin{aligned}
\frac{\partial z}{\partial t}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) &= 0 \quad (t \geq 0, \ x \in (0, \pi)), \\
\frac{\partial z}{\partial x}(t, 0) &= u_0(t), \ \frac{\partial z}{\partial x}(t, \pi) = u_\pi(t) \quad (t \geq 0), \\
z(0, x) &= 0 \quad (x \in (0, \pi)),
\end{aligned}
\]

\[A = \frac{\partial^2}{\partial x^2} \text{ on } H = L^2(0, \pi),\]

\[\mathcal{D}(A) = \left\{ z \in H^2(0, \pi), \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(\pi) = 0 \right\}.\]

\[B \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix} = -u_0 \delta_0 + u_\pi \delta_\pi.\]

Null-controllable in any time \( T > 0.\) \[\text{[Fattorini Russell 1971]}\]
Theorem [Hartmann-Orsoni 2021]

The reachable space of the above 1d heat equation is independent of the time horizon \( \tau > 0 \) and, for all \( \tau > 0 \),

\[
\mathcal{R}(\tau) = A^{1,2}(S),
\]

where

\[
S = \{ s = x + iy \in \mathbb{C} \mid |y| < x \text{ and } |y| < \pi - x \}.
\]

and \( A^{1,2}(S) = \{ f \in \text{Hol}(S) \cap W^{1,2}(S) \} \).

Exact characterization, following several attempts: [Fattorini Russell ’71], [Martin Rosier Rouchon ’16], [Dardé Ervedoza ’18], [Hartmann Kellay Tucsnak ’20], [Kellay Normand Tucsnak ’19], [Orsoni ’19],
The heat equation

\[
\begin{align*}
\frac{\partial z}{\partial t}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) &= 0 \quad (t \geq 0, \ x \in (0, \pi)), \\
\frac{\partial z}{\partial x}(t, 0) &= \frac{\partial z}{\partial x}(t, \pi) = 0 \quad (t \geq 0), \\
z(0, x) &= z_0 \quad (x \in (0, \pi)),
\end{align*}
\]

is well-posed in $A^{1,2}(S)$.

- Difficult to prove by hand !!
Application 1: Small potentials

There exists \( \varepsilon > 0 \), such that if \( p \in \text{Hol} (S) \cap W^{1,\infty}(S) \) with \( \|p\|_{W^{1,\infty}(S)} \leq \varepsilon \), the reachable set for the equation

\[
\begin{aligned}
\frac{\partial z}{\partial t} (t, x) - \frac{\partial^2 z}{\partial x^2} (t, x) + p(x)z(t, x) &= 0 \quad (t \geq 0, \ x \in (0, \pi)), \\
\frac{\partial z}{\partial x} (t, 0) &= u_0(t), \quad \frac{\partial z}{\partial x} (t, \pi) = u_\pi(t) \quad (t \geq 0), \\
z(0, x) &= 0
\end{aligned}
\]

is independent of the time horizon and coincides with \( A^{1,2}(S) \).

Proof. For \( z \in A^{1,2}(S) \), \( \|pz\|_{A^{1,2}(S)} \leq C\|p\|_{W^{1,\infty}(S)}\|z\|_{A^{1,2}(S)} \).
Application 2: Non-local quadratic terms potentials

For \( z \in C^0([0, T]; L^2[0, \pi]) \), we define

\[
f(z)(t, x) = \left( \int_0^\pi z(t, y) \, dy \right) z(t, x),
\]

Theorem \([S.E., K. Le Balc'h, M. Tucsnak 2021]\)

Let \( T > 0 \) Then \( \exists \delta > 0 \) such that \( \forall \eta \in A^{1,2}(S) \) satisfying

\[
\| \eta \|_{W^{1,2}(S)} \leq \delta,
\]

there exist control functions \( u_0, u_\pi \in L^2[0, \tau] \) such that the solution \( z \) of

\[
\begin{cases}
\frac{\partial z}{\partial t}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) = f(z)(t, x) & (t \geq 0, \ x \in (0, \pi)), \\
\frac{\partial z}{\partial x}(t, 0) = u_0(t), \quad \frac{\partial z}{\partial x}(t, \pi) = u_\pi(t) & (t \geq 0), \\
z(0, x) = 0 & (x \in (0, \pi)),
\end{cases}
\]

satisfies the terminal condition \( z(T, \cdot) = \eta \).
Application 3: Semi-linear equations

\[ f(z)(t, x) = \sum_{k=2}^{\infty} a_k(t, x)(z(t, x))^k, \quad (t \in [0, T], x \in [0, \pi]). \]

Difficulty: \( A^{1,2}(S) \) is not an algebra.

**Theorem** [Kellay, Normand, Tucsnak 2020]

For \( \tau > 0 \), let \( H^1_L(0, \tau) \) be the set of all functions \( v \in H^1(0, \tau) \) satisfying \( v(0) = 0 \).

Then, for every \( \tau > 0 \) the set \( R_1(\tau) \) of states which can be reached with controls in \( H^1_L((0, \tau); \mathbb{C}^2) \) is

\[ A^{3,2}(S) = \text{Hol}(S) \cap W^{3,2}(S). \]

Rk: \( A^{3,2}(S) \) is an algebra.
\[ f(z)(t, x) = \sum_{k=2}^{\infty} a_k(t, x)(z(t, x))^k, \quad (t \in [0, T], x \in [0, \pi]). \]

**Theorem**

[S.E., K. Le Balch'h, M. Tucsnak 2021]

Let \( T > 0 \), \( f \) as above s.t. \( f_k(t, x) \in L^1([0, T]; A^{3,2}(S)) \) and

\[ \exists \rho > 0, \quad \sum_{k=2}^{\infty} k \| f_k \|_{L^1([0, T]; A^{3,2}(S))} \rho^k < \infty. \]

Then \( \exists \delta > 0 \) such that \( \forall \eta \in A^{3,2}(S), \) satisfying \( \| \eta \|_{A^{3,2}(S)} \leq \delta, \)

\( \exists \) control functions \( u_0, u_\pi \in L^2[0, \tau] \) such that the solution \( z \) of

\[
\begin{aligned}
\frac{\partial z}{\partial t}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) &= f(t, x, z) \quad (t \geq 0, \ x \in (0, \pi)), \\
\frac{\partial z}{\partial x}(t, 0) &= u_0(t), \quad \frac{\partial z}{\partial x}(t, \pi) = u_\pi(t) \quad (t \geq 0), \\
z(0, x) &= 0 \quad (x \in (0, \pi)),
\end{aligned}
\]

satisfies \( z(T, \cdot) = \eta. \)
To be compared with [Laurent Rosier 2021]:

- Allows **first order terms** without any smallness condition;
- Handles analytic functions in $z$ and $\partial_x z$, but no dependence in time;
- Requires **stronger analyticity conditions** on the coefficients in $z$;
- Shows that the states which are **holomorphic on a ball** $B_C(\pi/2, R)$ for some $R > \hat{R} = (2\pi)e^{(2e)^{-1}}$ are reachable.
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Further comments and open problems

- One can also develop perturbative arguments based on compactness results
  - Requires unique continuation properties.
  - Well-adapted to deal with non-local in space operators.
    \[ \rightsquigarrow \text{Allows to recover [Fernandez-Cara-Lu-Zuazua-2016].} \]
- An interesting question is the following one:
  
  If \( (e^{tA})_{t \geq 0} \) is an analytic semigroup on \( H \) which is null-controllable in any positive time, is its restriction to its reachable space an analytic semigroup?

- Our approach is a strong motivation to better describe reachable sets for parabolic models.

[Strohmaier Waters 2021, Hartmann-Orsoni 2021]
Thanks for your attention!

Based on the work:

*Reachability results for perturbed heat equations, S.E., Kévin Le Balc’h, and Marius Tucsnak, in preparation*