On the reachable set of perturbed heat equations

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Outline

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Very (very) brief introduction to control theory:
Dynamical system $y' = f(y, u)$.

- y is the state;
- f describes the dynamics;
- u is the control.

Control theory

Describe the possible actions of the control on the state.

Examples:

- Park a car;
- Swim;
- $\dots$,

Tout est dans le contrôle. - M. Platini.

Control of linear systems:

$$y' = Ay + Bu, \quad t \geqslant 0, \qquad y(0) = y_0.$$

- A is a linear operator, generating a C^0 semi-group $(e^{tA})_{t\geqslant 0}$ on an Hilbert space H.
- $y \in C^0([0,T]; H)$ is the state.
- B is the control operator, $\in \mathcal{L}(U; H)$.
- $u \in L^2(0,T; U)$ is the control.

Objective

Describe the reachable set $\mathscr{R}(T, y_0)$ defined by

$$\mathscr{R}(T, y_0) = \{\, y(T),\ u \in L^2(0,T;U)\}.$$

Theorem (Finite dimension) [Kalman Ho Larendra '63]

Let $n \in \mathbb{N}$, $A \in \mathbb{R}^{n\times n}$, $H = \mathbb{R}^n$. Then for all $T > 0$,

$$\mathscr{R}(T, y_0) = e^{TA} y_0 + \mathrm{Ran}(B|AB|A^2B| \cdots A^{n-1}B).$$

In particular, denoting $\mathscr{R} = \mathrm{Ran}(B|AB|A^2B| \cdots A^{n-1}B)$:

- If $\mathscr{R} = \mathbb{R}^n$, the system $y' = \tilde{A}y + \tilde{B}v$ is controllable for any operators $(\tilde{A}, \tilde{B})$ close enough to (A, B).
- If $\mathscr{R} \neq \mathbb{R}^n$, then
  - $A_{\mathscr{R}} = A|_{\mathscr{R}} \in \mathcal{L}(\mathscr{R})$, $B \in \mathcal{L}(U, \mathscr{R})$;
  - the system $z' = A_{\mathscr{R}} z + Bv$ is exactly controllable on $\mathscr{R}$.

Much more delicate in infinite dimensional settings:

- There are vector spaces which are not closed;
- Cayley Hamilton's theorem does not apply.

Question

What happens for infinite dimensional systems ?

$\rightsquigarrow$ A typical example: the heat equation

$$\begin{cases} \partial_t y - \partial_{xx} y = 0, & \text{in } (0, T) \times (-L, L), \\ y(t, -L) = u_-(t), & \text{on } (0, T), \\ y(t, L) = u_+(t), & \text{on } (0, T), \\ y(0, x) = y_0(x), & \text{in } (-L, L). \end{cases}$$

Outline

Setting

Control of linear systems:

$$y' = Ay + Bu, \quad t \geqslant 0, \qquad y(0) = y_0.$$

- A is a linear operator, generating a C^0 semi-group $(e^{tA})_{t\geqslant 0}$ on an Hilbert space H.
- $y \in C^0([0,T]; H)$ is the state.
- B is the control operator, $\in \mathcal{L}(U; H)$. *(or admissible)*
- $u \in L^2(0,T; U)$ is the control.

Standing assumptions:

System $y' = Ay + Bu$ is null-controllable in any time $T > 0$: $\forall y_0 \in H$, $\exists u \in L^2(0,T;U)$ such that $y(T) = 0$.

$\rightsquigarrow$ Satisfied for heat type equations.

[Fursikov-Imanuvilov '96, Lebeau Robbiano '95].

The reachable set

Definition

The reachable set $\mathscr{R}(T, y_0)$ is defined by

$$\mathscr{R}(T, y_0) = \{ y(T), \ u \in L^2(0, T; U)\}.$$

Theorem

The reachable set $\mathscr{R}(T, y_0)$ is independent of $T > 0$ and $y_0 \in H$, now simply denoted $\mathscr{R}$.

- Null-controllable $\Rightarrow \mathscr{R}(T, y_0) = \mathscr{R}(T, 0)$.
- For $T_1 < T_2$, $\mathscr{R}(T_1, 0) \subset \mathscr{R}(T_2, 0)$.
- Null-controllable $\Rightarrow$ Exactly controllable to trajectories.
  $\Rightarrow$ For $T_1 < T_2$, we also have $\mathscr{R}(T_1, 0) \supset \mathscr{R}(T_2, 0)$.

Proposition

$\mathscr{R}$ is a Hilbert space when endowed with the norm

$$\|z\|_{\mathscr{R}(T)} = \inf \{\|u\|_{L^2(0,T;U)}, \\ \text{s.t. } z = y(T), \text{ with } y' = Ay + Bu, \; y(0) = 0.\}$$

- For $T_1 < T_2$, $\forall z \in \mathscr{R}$, $\|z\|_{\mathscr{R}(T_2)} \leqslant \|z\|_{\mathscr{R}(T_1)}$
- For $T_1 < T_2$, $\exists C = C(T_1, T_2)$, $\|z\|_{\mathscr{R}(T_1)} \leqslant C\|z\|_{\mathscr{R}(T_2)}$.

$\rightsquigarrow$ All these norms are equivalent.

Main result

Theorem [S.E., K. Le Balc'h, M. Tucsnak 2021]

For $\tau > 0$, we set

$$\mathbb{T}_t = e^{tA}|_{\mathscr{R}(\tau)}, \qquad (t \geqslant 0).$$

Then the family $\mathbb{T} = \left(\mathbb{T}_t|_{\mathscr{R}(\tau)}\right)_{t\geqslant 0}$

- does not depend on the choice of $\tau > 0$,
- forms a C^0 semigroup on $\mathscr{R}(\tau)$,
- has generator $\tilde{A}$ defined by $\mathscr{D}(\tilde{A}) = \mathscr{D}(A) \cap \mathscr{R}(\tau)$ and $\tilde{A}z = Az$ for $z \in \mathscr{D}(\tilde{A})$.

Finally, the system $z' = \tilde{A}z + Bu$ is exactly controllable in $\mathscr{R}(\tau)$ in any time $T > 0$.

Main ingredient of the proof

Lemma

$\exists c_\tau > 0$ s.t. $\forall t \in [0, \tau]$, $z \in \mathscr{R}(\tau)$,

$$\|\mathbb{T}_t z\|_{\mathscr{R}(\tau)} \leqslant c_\tau \|z\|_{\mathscr{R}(\tau)}.$$

Proof. For $t \in [0, \tau]$, $z \in \mathscr{R}(\tau)$

$$\|\mathbb{T}_t z\|_{\mathscr{R}(\tau)} \leqslant C_\tau \|\mathbb{T}_t z\|_{\mathscr{R}(2\tau)} \leqslant C_\tau \|z\|_{\mathscr{R}(2\tau - t)} \leqslant C_\tau \|z\|_{\mathscr{R}(\tau)}.$$

Abstract Applications

$\rightsquigarrow$ By standard perturbation arguments for the exactly controllable system $z' = \tilde{A}z + Bu$ (on $\mathscr{R}(\tau)$).

Theorem [S.E., K. Le Balc'h, M. Tucsnak 2021]

For all $\tau > 0$, there exists $\varepsilon_\tau > 0$ such that if $P \in \mathcal{L}(H) \cap \mathcal{L}(\mathscr{R}(\tau))$ with

$$\|P\|_{\mathcal{L}(\mathscr{R}(\tau))} \leqslant \varepsilon_\tau,$$

then the reachable set $\mathscr{R}^P(\tau)$ of the system

$$y' = Ay + Py + Bu, \quad t \geqslant 0, \qquad y(0) = 0,$$

coincides with $\mathscr{R}(\tau)$.

Abstract Applications (2)

Proposition [S.E., K. Le Balc'h, M. Tucsnak 2021]

For $T > 0$, $\exists C > 0$ and $\exists$ a continuous linear map

$$\mathcal{L} : \mathscr{R}(\tau) \times L^1([0,T];\mathscr{R}(\tau)) \to L^2([0,T];U)$$

such that $\forall \eta \in \mathscr{R}(\tau)$ and $f \in L^1([0,T];\mathscr{R}(\tau))$ the solution of

$$z'(t) = \tilde{A}z(t) + Bu(t) + f(t), \quad (t \in [0,T]), \qquad z(0) = 0,$$

associated to the control $u = \mathcal{L}(\eta, f)$, satisfies
$z \in C^0([0,T];\mathscr{R}(\tau))$, together with $z(T) = \eta$, and

$$\|z\|_{C^0([0,T];\mathscr{R}(\tau))} + \|u\|_{L^2([0,T];U)} \leqslant C \left(\|\eta\|_{\mathscr{R}(\tau)} + \|f\|_{L^1([0,T];\mathscr{R}(\tau))} \right).$$

Abtsract Applications (3)

Corollary [S.E., K. Le Balc'h, M. Tucsnak 2021]

Suppose that $f : C^0([0,T];\mathscr{R}(\tau)) \to L^1([0,T];\mathscr{R}(\tau))$ satisfies $f(0) = 0$ and, for all $z_1, z_2 \in C^0([0,T];\mathscr{R}(\tau))$ we have

$$\|f(z_1) - f(z_2)\|_{L^1([0,T];\mathscr{R}(\tau))} \leqslant C\|(z_1,z_2)\|_{(C^0([0,T];\mathscr{R}(\tau))^2}\|z_1 - z_2\|_{C^0([0,T];\mathscr{R}(\tau))}.$$

Then $\exists \delta > 0$, $\forall \eta \in \mathscr{R}(\tau)$ satisfying $\|\eta\|_{\mathscr{R}(\tau)} \leqslant \delta$, $\exists$ a control function $u \in L^2([0,T];U)$ and a controlled trajectory $z \in C^0([0,T];\mathscr{R}(\tau))$ satisfying

$$z'(t) = Az(t) + Bu(t) + f(z)(t), \quad (t \in [0,T]), \qquad z(0) = 0,$$

and $z(T) = \eta$.

Outline

The 1-d heat equation

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) - \dfrac{\partial^2 z}{\partial x^2}(t,x) = 0 & (t \geqslant 0,\ x \in (0,\pi)), \\ \dfrac{\partial z}{\partial x}(t,0) = u_0(t), \quad \dfrac{\partial z}{\partial x}(t,\pi) = u_\pi(t) & (t \geqslant 0), \\ z(0,x) = 0 & (x \in (0,\pi)), \end{cases}$$

$$A = \frac{\partial^2}{\partial x^2} \text{ on } H = L^2(0,\pi),$$

$$\mathscr{D}(A) = \left\{ z \in H^2(0,\pi), \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(\pi) = 0 \right\}.$$

$$B \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix} = -u_0 \delta_0 + u_\pi \delta_\pi.$$

Null-controllable in any time $T > 0$. [Fattorini Russell 1971]

Known result

Theorem [Hartmann-Orsoni 2021]

The reachable space of the above 1d heat equation is independent of the time horizon $\tau > 0$ and, for all $\tau > 0$,

$$\mathscr{R}(\tau) = A^{1,2}(S),$$

where

$$S = \{s = x + iy \in \mathbb{C} \mid |y| < x \text{ and } |y| < \pi - x\}.$$

and $A^{1,2}(S) = \{f \in \mathrm{Hol}\,(S) \cap W^{1,2}(S)\}$.

Exact characterization, following several attempts: [Fattorini Russell '71], [Martin Rosier Rouchon '16], [Dardé Ervedoza '18], [Hartmann Kellay Tucsnak '20], [Kellay Normand Tucsnak '19], [Orsoni '19],

First consequence

Theorem

The heat equation

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) - \dfrac{\partial^2 z}{\partial x^2}(t,x) = 0 & (t \geqslant 0,\ x \in (0,\pi)), \\ \dfrac{\partial z}{\partial x}(t,0) = \dfrac{\partial z}{\partial x}(t,\pi) = 0 & (t \geqslant 0), \\ z(0,x) = z_0 & (x \in (0,\pi)), \end{cases}$$

is well-posed in $A^{1,2}(S)$.

- Difficult to prove by hand !!

Application 1: Small potentials

Theorem [S.E., K. Le Balc'h, M. Tucsnak 2021]

There exists $\varepsilon > 0$, such that if $p \in \mathrm{Hol}\,(S) \cap W^{1,\infty}(S)$ with $\|p\|_{W^{1,\infty}(S)} \leqslant \varepsilon$, the reachable set for the equation

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) - \dfrac{\partial^2 z}{\partial x^2}(t,x) + p(x)z(t,x) = 0 & (t \geqslant 0,\ x \in (0,\pi)), \\ \dfrac{\partial z}{\partial x}(t,0) = u_0(t), \quad \dfrac{\partial z}{\partial x}(t,\pi) = u_\pi(t) & (t \geqslant 0), \\ z(0,x) = 0 & (x \in (0,\pi)), \end{cases}$$

is independent of the time horizon and coincides with $A^{1,2}(S)$.

Proof. For $z \in A^{1,2}(S)$, $\|pz\|_{A^{1,2}(S)} \leqslant C\|p\|_{W^{1,\infty}(S)}\|z\|_{A^{1,2}(S)}$.

Application 2: Non-local quadratic terms potentials

For $z \in C^0([0, T]; L^2[0, \pi])$, we define

$$f(z)(t,x) = \left(\int_0^\pi z(t,y)\,\mathrm{d}y \right) z(t,x),$$

Theorem [S.E., K. Le Balc'h, M. Tucsnak 2021]

Let $T > 0$ Then $\exists \delta > 0$ such that $\forall \eta \in A^{1,2}(S)$ satisfying $\|\eta\|_{W^{1,2}(S)} \leqslant \delta$, there exist control functions $u_0, u_\pi \in L^2[0, \tau]$ such that the solution z of

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) - \dfrac{\partial^2 z}{\partial x^2}(t,x) = f(z)(t,x) & (t \geqslant 0,\ x \in (0,\pi)), \\ \dfrac{\partial z}{\partial x}(t,0) = u_0(t), \quad \dfrac{\partial z}{\partial x}(t,\pi) = u_\pi(t) & (t \geqslant 0), \\ z(0,x) = 0 & (x \in (0,\pi)), \end{cases}$$

satisfies the terminal condition $z(T, \cdot) = \eta$.

Application 3: Semi-linear equations

$$f(z)(t,x) = \sum_{k=2}^{\infty} a_k(t,x)(z(t,x))^k, \quad (t \in [0,T],\ x \in [0,\pi]).$$

Difficulty: $A^{1,2}(S)$ is not an algebra.

Theorem [Kellay, Normand, Tucsnak 2020]

For $\tau > 0$, let $H_L^1(0,\tau)$ be the set of all functions $v \in H^1(0,\tau)$ satisfying $v(0) = 0$.
Then, for every $\tau > 0$ the set $\mathscr{R}_1(\tau)$ of states which can be reached with controls in $H_L^1((0,\tau);\mathbb{C}^2)$ is

$$A^{3,2}(S) = \mathrm{Hol}\,(S) \cap W^{3,2}(S).$$

Rk: $A^{3,2}(S)$ is an algebra.

$$f(z)(t,x) = \sum_{k=2}^{\infty} a_k(t,x)(z(t,x))^k, \quad (t \in [0,T],\ x \in [0,\pi]).$$

Theorem [S.E., K. Le Balc'h, M. Tucsnak 2021]

Let $T > 0$, f as above s.t. $f_k(t,x) \in L^1([0,T]; A^{3,2}(S))$ and

$$\exists \rho > 0, \quad \sum_{k=2}^{\infty} k \|f_k\|_{L^1([0,T];A^{3,2}(S))} \rho^k < \infty.$$

Then $\exists \delta > 0$ such that $\forall \eta \in A^{3,2}(S)$, satisfying $\|\eta\|_{A^{3,2}(S)} \leqslant \delta$, $\exists$control functions $u_0,\ u_\pi \in L^2[0,\tau]$ such that the solution z of

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) - \dfrac{\partial^2 z}{\partial x^2}(t,x) = f(t,x,z) & (t \geqslant 0,\ x \in (0,\pi)), \\ \dfrac{\partial z}{\partial x}(t,0) = u_0(t), \ \ \dfrac{\partial z}{\partial x}(t,\pi) = u_\pi(t) & (t \geqslant 0), \\ z(0,x) = 0 & (x \in (0,\pi)), \end{cases}$$

satisfies $z(T,\cdot) = \eta$.

To be compared with [Laurent Rosier 2021]:

- Allows first order terms without any smallness condition;
- Handles analytic functions in z and $\partial_x z$, but no dependence in time;
- Requires stronger analyticity conditions on the coefficients in z;
- Shows that the states which are holomorphic on a ball $B_{\mathbb{C}}(\pi/2, R)$ for some $R > \widehat{R} = (2\pi)e^{(2e)^{-1}}$ are reachable.

Outline

Further comments and open problems

- One can also develop perturbative arguments based on compactness results
  - Requires *unique continuation properties*.
  - Well-adapted to deal with *non-local in space operators*.
    $\rightsquigarrow$ Allows to recover [Fernandez-Cara-Lu-Zuazua-2016].
- An interesting question is the following one:

> If $(e^{tA})_{t \geqslant 0}$ is an analytic semigroup on H which is null-controllable in any positive time, is its restriction to its reachable space an analytic semigroup?

- Our approach is a strong motivation to better describe reachable sets for parabolic models.

[Strohmaier Waters 2021, Hartmann-Orsoni 2021]

Thanks for your attention!

Based on the work:
Reachability results for perturbed heat equations,
S.E., Kévin Le Balc'h, and Marius Tucsnak, in preparation