Theoretical and numerical hierarchical control of some PDEs

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Outline

1. Background
   - The Stackelberg-Nash strategy
   - The main result
2. Numerical analysis and results
   - Computation of Nash equilibria
   - Computation of Pareto equilibria
   - Numerical solution of the Stackelberg-Nash null control problem
3. Additional results and comments

CONTROL PROBLEMS

What is usual: act to get good (or the best) results for

$$\begin{cases} E(y) = F(v) \\ + \dots \end{cases}$$

What is easier? Solving? Controlling?

Two classical approaches:

- Optimal control
- Controllability

OPTIMAL CONTROL

A general optimal control problem

$$\begin{array}{l} \text{Minimize } \; J(v) \\ \text{Subject to } \; v \in \mathcal{V}_{ad}, \;\; y \in \mathcal{Y}_{ad}, \; (v, y) \; \text{ satisfies} \end{array}$$

$$E(y) = F(v) \quad + \; \dots \qquad (S)$$

Main questions: $\exists$, uniqueness/multiplicity, characterization, computation, ...

We could also consider similar bi-objective optimal control:

$$\begin{array}{l} \text{"Minimize" } \; J_1(v), J_2(v) \\ \text{Subject to } \; v \in \mathcal{V}_{ad}, \dots \end{array}$$

CONTROLLABILITY

A null controllability problem

Find (v, y)
Such that $v \in \mathcal{V}_{ad}$, (v, y) satisfies (ES), $y(T) = 0$

with $y : [0, T] \mapsto H$,

$$E(y) \equiv y_t + A(y) = F(v) \quad + \ \dots \qquad (ES)$$

Again many interesting questions: $\exists$, uniqueness/multiplicity, characterization, computation, ...

A very rich subject for PDEs, see [Russell, J.-L. Lions, Coron, Zuazua, ...]

Question: How can we adopt both viewpoints together?

Example: Optimal-control / controllability problem
A simplified model for the autonomous car driving problem

The system:

$$x_t = f(x, u), \quad x(0) = x_0$$

Constraints:

$$\text{dist.}\,(x(t), Z(t)) \geq \varepsilon \quad \forall t$$
$$u \in \mathcal{U}_{ad} \quad (|u(t)| \leq C)$$

u determines direction and speed

Goals (prescribed x_T and $\hat{x}$):

- $x(T) = x_T$ (or $|x(T) - x_T| \leq \varepsilon \ldots$)
- Minimize $\sup_t |x(t) - \hat{x}(t)|$

[Sontag, Sussman-Tang, . . .]

Figure: The ICARE Project, INRIA, France. Autonomous car driving. Malis-Morin-Rives-Samson, 2004

The car in the street

Figure: Nissan ID. Autonomous car driving. 2015–2020

What was announced in 2014:

- Nissan ID 1.0 (2015), highways and traffic jams (no lane change) - OK
- ID 2.0 (2018), overtaking and lane change
- ID 3.0 (2020), complete autonomous driving in town

http://reports.nissan-global.com/EN/?p=17295

Another way to connect optimal control and controllability:
HIERARCHICAL CONTROL (Stackelberg-Nash, Stackelberg-Pareto, ...)
The main ideas in the context of Navier-Stokes:
Three controls: one leader, two followers

$$\begin{cases} y_t + (y\cdot\nabla)y - \Delta y + \nabla p = f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, & (x,t)\in\Omega\times(0,T) \\ \nabla\cdot y = 0, & (x,t)\in\Omega\times(0,T) \\ y = 0, & (x,t)\in\partial\Omega\times(0,T) \\ y(x,0) = y^0(x), & x\in\Omega \end{cases}$$

Disjoint domains $\mathcal{O}$, $\mathcal{O}_i$, $(i = 1, 2)$
Three objectives:

- "Simultaneously", $y \approx y_{i,d}$ in $\Omega\times(0,T)$, $i = 1, 2$, reasonable effort:

$$\text{Minimize } \alpha_i \iint_{\Omega\times(0,T)} |y - y_{i,d}|^2 + \mu \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2, \quad i = 1, 2$$

  Bi-objective optimal control - The task of the followers
  In practice, does an equilibrium $(v_1(f), v_2(f))$ exist for each f?
- Get $y(x, T) \equiv 0$
  Null controllability - The task of the leader
  Can we find f such that $y(T) = 0$?

$$\begin{cases} y_t+(y\cdot\nabla)y-\Delta y+\nabla p=f1_{\mathcal{O}}+v_1 1_{\mathcal{O}_1}+v_2 1_{\mathcal{O}_2}, & (x,t)\in\Omega\times(0,T) \\ \nabla\cdot y=0, & (x,t)\in\Omega\times(0,T) \\ y=0, & (x,t)\in\partial\Omega\times(0,T) \\ y(x,0)=y^0(x), & x\in\Omega \end{cases}$$

Many applications:

- Heating: Controlling temperatures
  Heat sources at different locations - Heat PDE (linear, semilinear, etc.)
- Tumor growth: Controlling tumor cell densities
  Radiotherapy strategies - Reaction-diffusion PDEs
  bilinear control
- Fluid mechanics: Controlling fluid velocity fields
  Several mechanical actions - Stokes, Navier-Stokes or similar
- Finances: Controlling the price of an option
  Agents at different stock prices, etc. - Backwards in time heat-like PDE
  Degenerate coefficients

Contributions: Lions, Díaz-Lions, Glowinski-Periaux-Ramos, Guillén, . . .
Optimal control + AC

A SIMPLIFIED PROBLEM FOR THE 1D HEAT PDE

Again three controls: one leader, two followers

$$(H) \quad \begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0, & t \in (0,T) \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases}$$

Different intervals $\mathcal{O}$, $\mathcal{O}_i$

Again three objectives:

- Simultaneously, $y \approx y_{i,d}$ in $\Omega \times (0,T)$, $i = 1,2$, reasonable effort:

$$\text{Minimize} \quad \alpha_i \iint_{\Omega \times (0,T)} |y - y_{i,d}|^2 + \mu \iint_{\mathcal{O}_i \times (0,T)} |v_i|^2, \quad i = 1,2$$

  Bi-objective optimal control - Followers' task

- Get $y(T) = 0$
  Null controllability - Leader's task

What can we do?

THE STACKELBERG-NASH STRATEGY

Step 1: f is fixed

$$J_i(v_1, v_2) := \alpha_i \iint_{\Omega\times(0,T)} |y - y_{i,d}|^2 + \mu \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2, \quad i = 1,2$$

Find a Nash equilibrium $(v_1(f), v_2(f))$ with $v_i(f) \in L^2(\mathcal{O}_i \times (0,T))$:

$$J_1(v_1(f), v_2(f)) \le J_1(v_1, v_2(f)) \quad \forall v_1 \in L^2(\mathcal{O}_1 \times (0,T))$$
$$J_2(v_1(f), v_2(f)) \le J_2(v_1(f), v_2) \quad \forall v_2 \in L^2(\mathcal{O}_2 \times (0,T))$$

Equivalent to an optimality system:

$$\begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} - \frac{1}{\mu}\phi_1 1_{\mathcal{O}_1} - \frac{1}{\mu}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i(y - y_{i,d}), \quad i = 1,2 \\ \phi_i(0,t) = \phi_i(1,t) = 0,\ y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x),\ \phi_i(x,T) = 0, \quad x \in (0,1) \end{cases}$$

$$v_i(f) = -\frac{1}{\mu}\phi_i|_{\mathcal{O}_i\times(0,T)}$$

$\exists (v_1(f), v_2(f))$? Uniqueness?

THE STACKELBERG-NASH STRATEGY

Step 2: Find f such that

$$(HSN)_1 \qquad \left\{ \begin{array}{l} y_t - y_{xx} = f 1_{\mathcal{O}} - \frac{1}{\mu}\phi_1 1_{\mathcal{O}_1} - \frac{1}{\mu}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}), \quad i = 1,2 \\ y|_{t=0} = y^0(x), \quad \phi_i|_{t=T} = 0, \quad \text{etc.} \end{array} \right.$$

$$(HSN)_2 \qquad y(x,T) = 0, \quad x \in (0,1)$$

with $\|f\|_{L^2(\mathcal{O}\times(0,T))} \le C\|y^0\|_{L^2}$

Equivalent to

$$\|\psi|_{t=0}\|^2 + \sum_{i=1}^{2} \iint_{\Omega\times(0,T)} \hat\rho^{-2}|\gamma^i|^2 \,dx\,dt \le C \iint_{\mathcal{O}\times(0,T)} |\psi|^2 \,dx\,dt$$

for all ψ^T, with

$$\left\{ \begin{array}{l} -\psi_t - \psi_{xx} = \sum_{i=1}^{2} \alpha_i \gamma^i, \quad \gamma^i_t - \gamma^i_{xx} = -\frac{1}{\mu}\psi 1_{\mathcal{O}_i} \\ \psi|_{t=T} = \psi^T(x), \quad \gamma^i|_{t=0} = 0, \quad \text{etc.} \end{array} \right.$$

True?

Theorem

Assume: large μ

$\exists\hat\rho$ such that, if $\iint_{\Omega\times(0,T)} \hat\rho^2|y_{i,d}|^2\,dx\,dt < +\infty,\ i = 1, 2,$ then:

$\forall y^0 \in L^2(\Omega)$ $\exists$ null controls $f \in L^2(\mathcal{O}\times(0,T))$ & Nash pairs $(v_1(f), v_2(f))$

Idea of the proof:

- Energy estimates for the optimality system for (y, ϕ_1, ϕ_2)
- Energy and Carleman estimates for the adjoint system for $(\psi, \gamma^1, \gamma^2)$

We do need: μ is large

FIRST, HOW CAN WE COMPUTE A NASH EQUILIBRIUM PAIR? (THE FOLLOWERS)

The goal: f is given. Solve the optimality system

$$\left\{ \begin{array}{l} y_t - \Delta y = f 1_{\mathcal{O}} - \frac{1}{\mu}\phi_1 1_{\mathcal{O}_1} - \frac{1}{\mu}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}), \quad i = 1,2 \\ y|_{t=0} = y^0(x), \quad \phi_i|_{t=T} = 0, \quad \text{etc.} \end{array} \right.$$

Then take $v_i = \frac{1}{\mu}\phi_i\big|_{\mathcal{O}_i \times (0,T)}$

For instance: ALG 1 - Fixed point
ALG 1: $(v_1, v_2) \to y \to (\phi_1, \phi_2) \to (v_1, v_2)$
Also: Gradient, Conjugate gradient, etc.

Standard approximations: P_ℓ-Lagrange FEM's, Implicit Euler schemes

A 2D numerical experiment with FreeFem++: http://www.freefem.org/

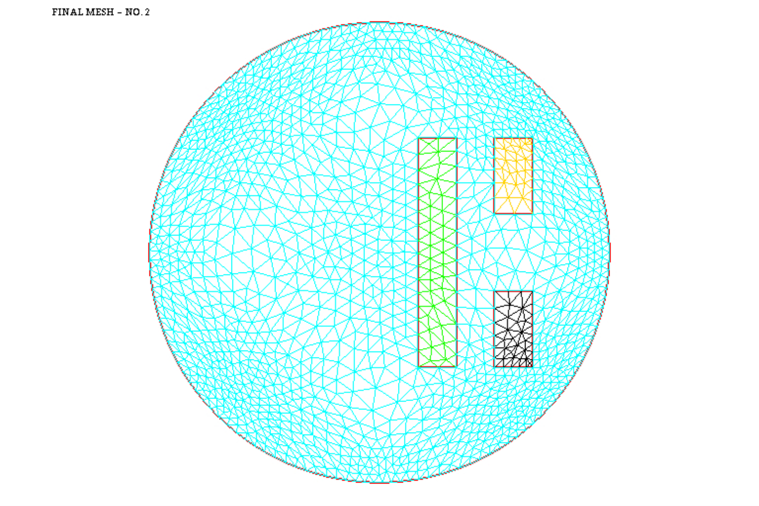


Figure: The final adapted mesh - Number of vertices: 1460 - Number of triangles: 2781

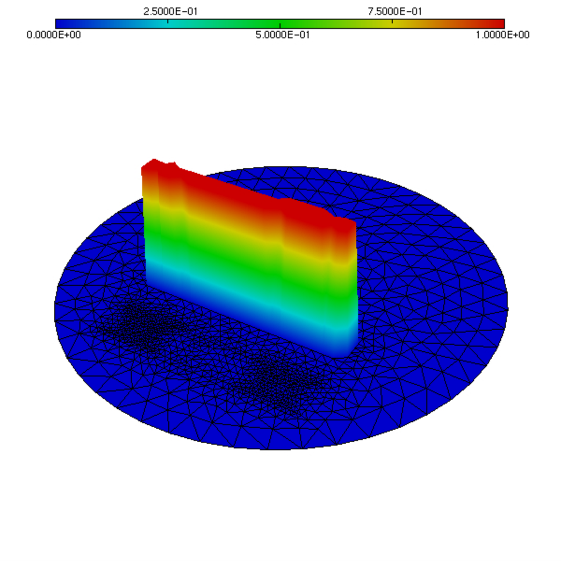


Figure: The (fixed) leader control f (constant in time)

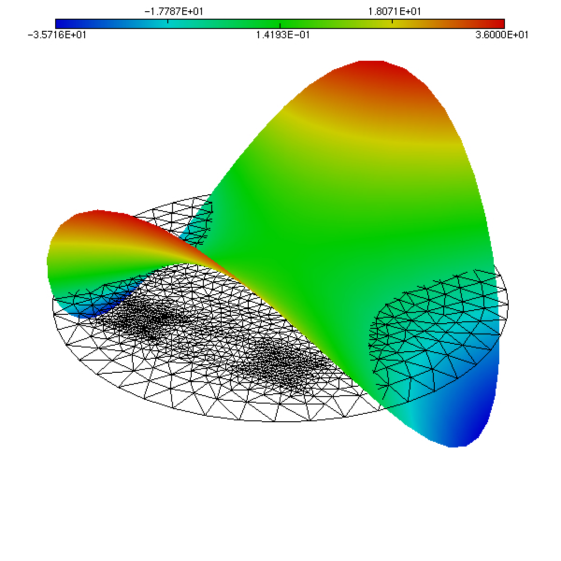


Figure: The target $y_{1,d}$ (constant in time)

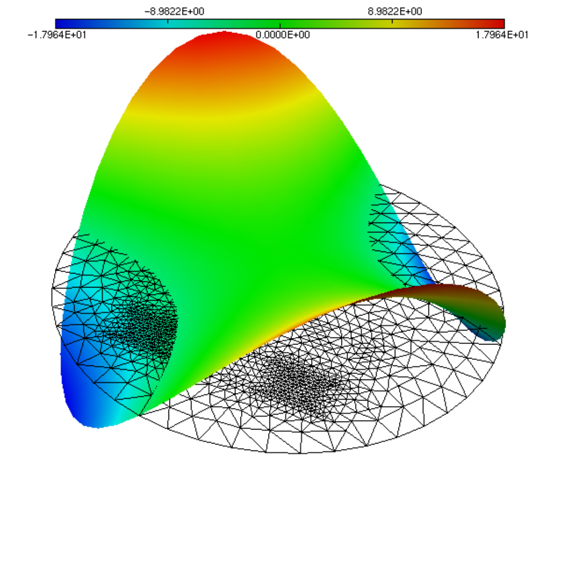


Figure: The target $y_{2,d}$ (constant in time)

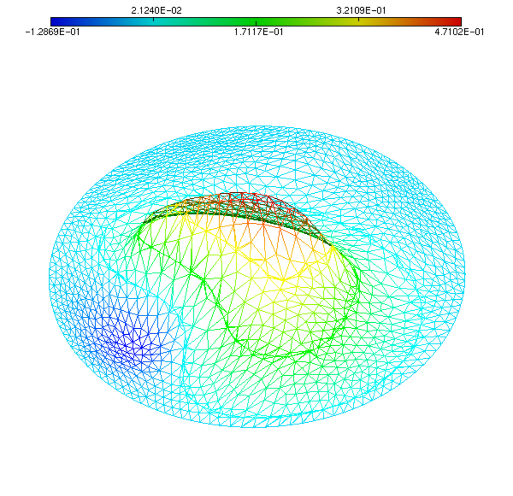


Figure: The state y at $t = T$ - Result for $y^0 = 0$, $\mu = 0.15$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$

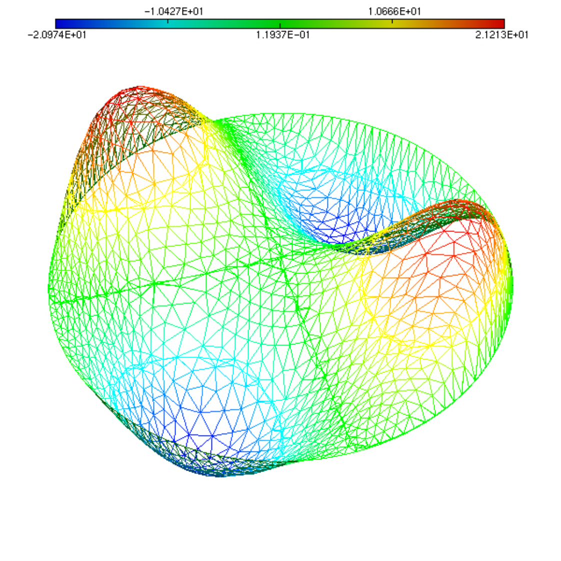


Figure: The adjoint state ϕ_1 at $t = 0$

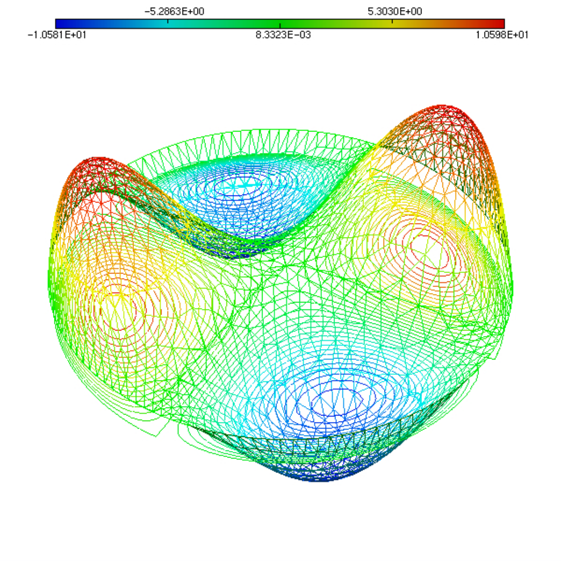


Figure: The adjoint state ϕ_2 at $t = 0$

Iterates versus μ:

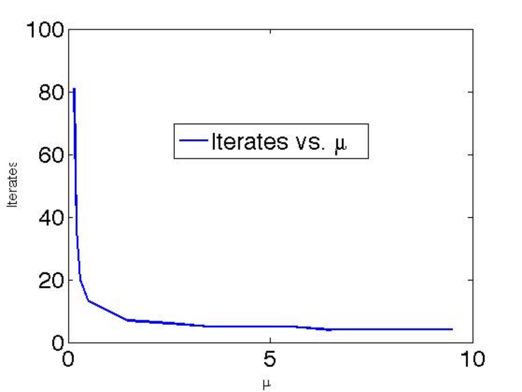


Figure: The number of iterates as a function of μ
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$

A similar semilinear problem: Compute a Nash equilibrium (v_1, v_2) for

$$\begin{cases} y_t - \Delta y + F(y) = f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2} \\ \text{etc.} \end{cases}$$

The task: solve the optimality system

$$\begin{cases} y_t - \Delta y + F(y) = f 1_{\mathcal{O}} - \frac{1}{\mu}\phi_1 1_{\mathcal{O}_1} - \frac{1}{\mu}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \Delta \phi_i + F'(y)\phi_i = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_{i,d}}, \quad i = 1,2 \\ y|_{t=0} = y^0(x), \quad \phi_i|_{t=T} = 0, \quad \text{etc.} \end{cases}$$

Then: $v_i = \frac{1}{\mu}\phi_i\big|_{\mathcal{O}_i \times (0,T)}$

For globally Lipschitz-continuous F: existence is ensured

For instance: ALG 2 and ALG 3 . . . - Fixed point strategies
ALG 2: $(v_1, v_2) \to \{y \to y\} \to (\phi_1, \phi_2) \to (v_1, v_2)$
ALG 3: $(v_1, v_2) \to y \to (\phi_1, \phi_2) \to (v_1, v_2)$

Numerical experiments with FreeFem - ALG 2 - Iterates versus μ

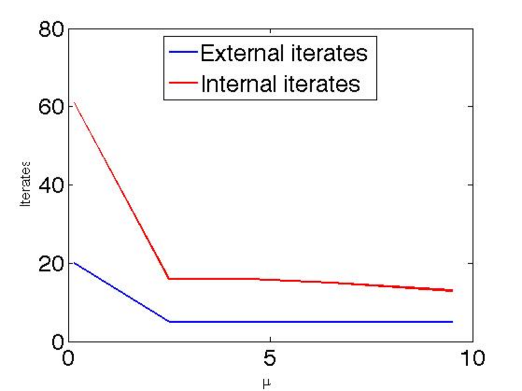


Figure: ALG 2 - The number of iterates as a function of μ - $y^0 = 0$ - $F(y) = y(1 + \sin y)$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$

Numerical experiments with FreeFem - ALG 3 - Iterates versus μ

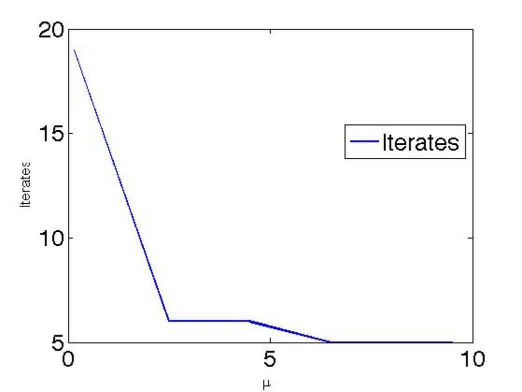


Figure: ALG 3 - The number of iterates as a function of μ - $y^0 = 0$ - $F(y) = y(1 + \sin y)$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$

Numerical experiments with FreeFem - ALG 3

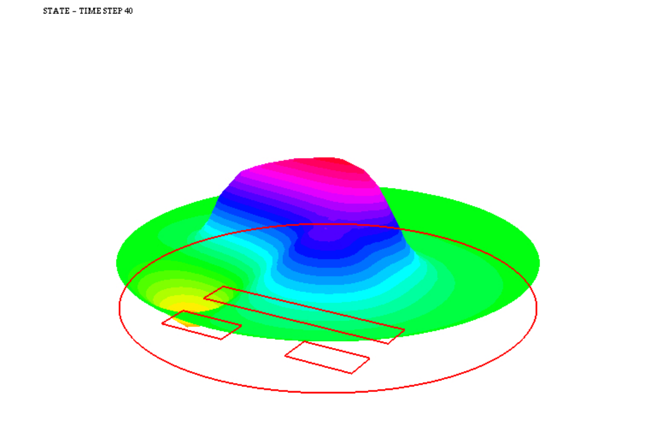


Figure: The state y at $t = T$ - Result for $y^0 = 0$ - $F(y) = y(1 + \sin y)$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$

Numerical experiments with FreeFem - ALG 3

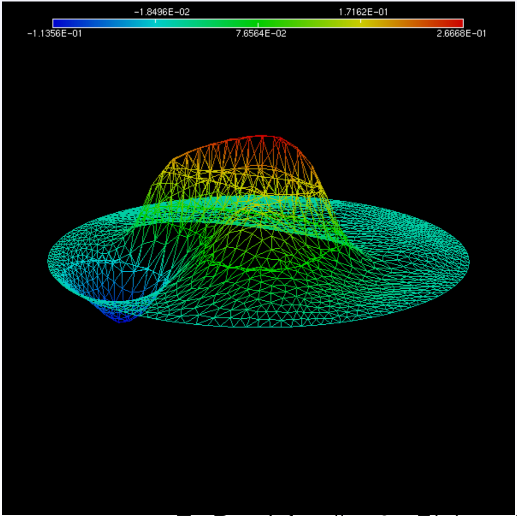


Figure: The state y at $t = T$ - Result for $y^0 = 0$ - $F(y) = y(1 + \sin y)$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$

Another semilinear system - ALG 3 - Iterates versus μ

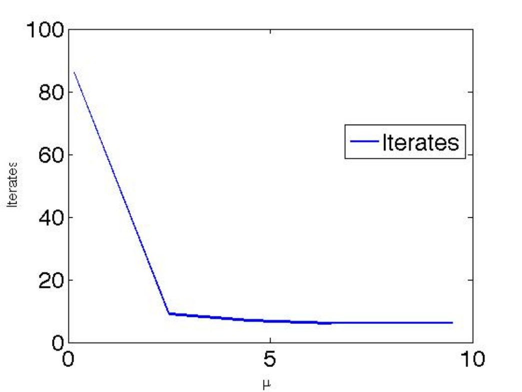


Figure: ALG 3 - The number of iterates as a function of μ - $y^0 = 0$
$F(y) = y\log(1+|y|)^a$, $1 < a < 2$, not sublinear!
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\|/\|v_{i,n+1}\| \leq 10^{-5}$

Another semilinear system - ALG 3

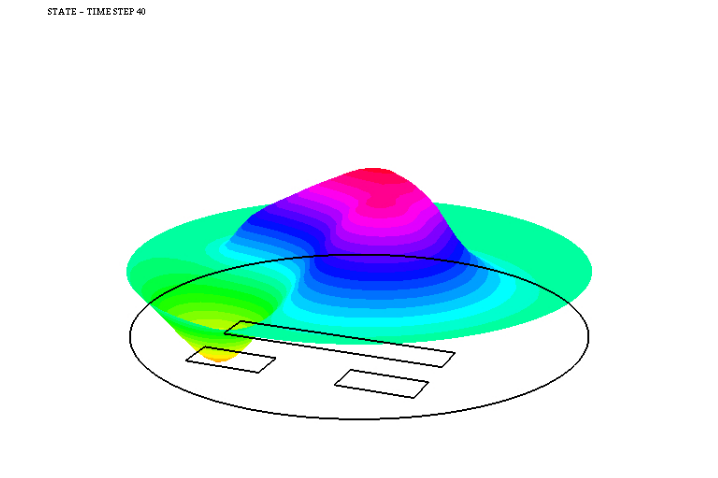
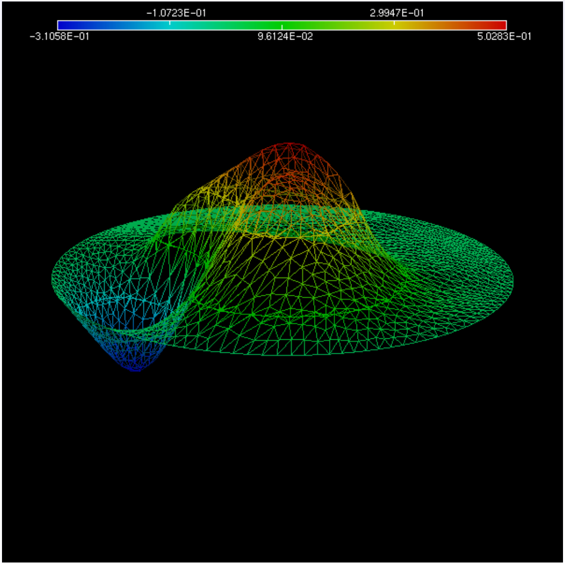


Figure: The state y at $t = T$ - Result for $y^0 = 0$, $\mu = 2.5$
$F(y) = y \log(1 + |y|)^a$, $1 < a < 2$, not sublinear!
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$

Another semilinear system - ALG 3



Another semilinear system - ALG 3

The results for $y^0 = 0$, $\mu = 2.5$
$F(y) = y\log(1 + |y|)^a$, $1 < a < 2$, not sublinear!
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$

FIXED-POINT ITERATE AND ERROR: 0 - 1383.98
FIXED-POINT ITERATE AND ERROR: 1 - 640.146
FIXED-POINT ITERATE AND ERROR: 2 - 39.2927
FIXED-POINT ITERATE AND ERROR: 3 - 3.13866
FIXED-POINT ITERATE AND ERROR: 4 - 0.283813
FIXED-POINT ITERATE AND ERROR: 5 - 0.0262522
FIXED-POINT ITERATE AND ERROR: 6 - 0.0024854
FIXED-POINT ITERATE AND ERROR: 7 - 0.000243612
FIXED-POINT ITERATE AND ERROR: 8 - 2.5955e-05
FIXED-POINT ITERATE AND ERROR: 9 - 2.86158e-06

FIXED-POINT RATE= 3.18113

ANOTHER HIERARCHIC STRATEGY: PARETO EQUILIBRIA

$(v_1(f), v_2(f))$ is a Pareto equilibrium if $\nexists (w_1, w_2)$ with

$$J_1(w_1, w_2) \le J_1(v_1(f), v_2(f)), \quad J_2(w_1, w_2) \le J_2(v_1(f), v_2(f))$$

and at least one strict inequality

The goal: f and $\lambda \in (0, 1)$ are given. Solve the optimality system

$$\begin{cases} y_t - \Delta y = f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 \phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}), \quad i = 1, 2 \\ y|_{t=0} = y^0(x), \quad \phi_i|_{t=T} = 0, \ \text{ etc.} \end{cases}$$

$$\begin{aligned} v_1 &= \frac{1}{\mu}(\phi_1 + \frac{1-\lambda}{\lambda}\phi_2)\Big|_{\mathcal{O}_1 \times (0,T)} \\ v_2 &= \frac{1}{\mu}(\frac{\lambda}{1-\lambda}\phi_1 + \phi_2)\Big|_{\mathcal{O}_2 \times (0,T)} \end{aligned}$$

Again: ALG 1bis - Fixed point
ALG 1bis: $(v_1, v_2) \to y \to (\phi_1, \phi_2) \to (v_1, v_2)$

Again: standard approximations

Numerical experiments with FreeFem - ALG 1bis - Iterates versus λ

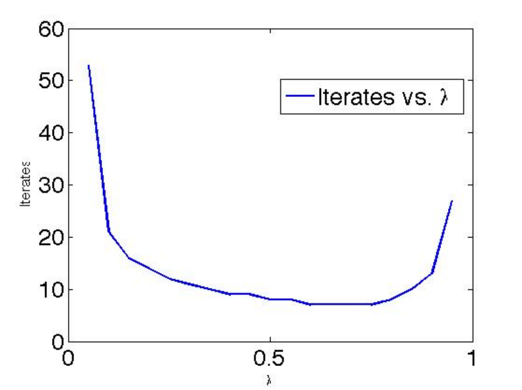


Figure: ALG 1bis - The number of iterates as a function of λ - $y^0 = 0$
Stopping test: $\sum_i \|v_{i,n+1} - v_{i,n}\| / \|v_{i,n+1}\| \leq 10^{-5}$

SOLVING NUMERICALLY THE STACKELBERG-NASH NC PROBLEM?
(COMPUTING THE LEADER AND THE ASSOCIATED FOLLOWERS)

The goal: Find f such that the solution to

$$(HN) \qquad \left\{ \begin{array}{l} y_t - \Delta y = f 1_{\mathcal{O}} - \frac{1}{\mu}\phi_1 1_{\mathcal{O}_1} - \frac{1}{\mu}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_{i,d}}, \quad i = 1,2 \\ y|_{t=0} = y^0(x), \quad \phi_i|_{t=T} = 0, \ \text{etc.} \end{array} \right.$$

satisfies

$$y(x, T) \equiv 0$$

The Fursikov-Imanuvilov approach:

$$\left\{ \begin{array}{l} \text{Minimize} \quad \displaystyle\iint \rho^2 |y|^2 + \iint_{\mathcal{O}\times(0,T)} \rho_0^2 |f|^2 \\ \text{Subject to } (HN) \end{array} \right.$$

ρ and ρ_0 are appropriate, blow up as $t \to T$

The advantage: y necessarily vanishes exactly at $t = T$ (and so does f)

The resulting task after applying Lagrange's principle:

Solve a 4th-order Lax-Milgram problem

$$\left\{ \begin{array}{l} a((\psi,\gamma_1,\gamma_2),(\psi',\gamma_1',\gamma_2')) = \langle \ell, (\psi',\gamma_1',\gamma_2')\rangle \\ \forall (\psi',\gamma_1',\gamma_2') \in W, \quad (\psi,\gamma_1,\gamma_2) \in W \end{array} \right.$$

$\exists!$ solution for appropriate ρ and ρ_0 (Carleman inequalities, large μ)

$$\begin{array}{l} \iint \left[\rho^{-2}\left\{(L^*\psi + \sum_i \alpha_i\gamma_i)(L^*\psi' + \sum_i \alpha_i\gamma_i') + \dots\right\} + \rho_0^{-2} 1_{\mathcal{O}}\psi\psi'\right] \\ \quad = \int_\Omega y^0(x)\psi'(x,0) \end{array}$$

$$\begin{array}{l} \iint [zz' + 1_{\mathcal{O}} mm'] + \iint (z' - \rho^{-1}(L^*(\rho_0 m') + \dots))\lambda \\ \quad = \int_\Omega y^0(x)\psi'(x,0) \end{array}$$

Reformulation: a 2nd-order mixed problem after integration by parts

$$\left\{ \begin{array}{l} \alpha((z,m),(z',m')) + \beta((z',m'),\lambda) = \langle \tilde{\ell}, (z',m')\rangle \\ \beta((z,m),\lambda') = 0 \\ \forall (z',m',\lambda') \in Z \times M \times \Lambda, \quad (z,m,\lambda) \in Z \times M \times \Lambda \end{array} \right.$$

$$\left\{\begin{array}{l}\alpha\left((z,m),(z',m')\right)+\beta\left((z',m'),\lambda\right)=\langle\tilde{\ell},(z',m')\rangle\\ \beta\left((z,m),\lambda'\right)=0\\ \forall(z',m',\lambda')\in Z\times M\times\Lambda,\quad (z,m,\lambda)\in Z\times M\times\Lambda\end{array}\right.$$

Approximation: mixed $P_1 - P_2$-Lagrange FEM's
Techniques already applied for NC of (nonlinear) heat, wave, Stokes, Navier-Stokes, ...
[EFC-Münch, Cindea-EFC-Münch, EFC-Münch-Souza, ...]

A 1D numerical experiment with FreeFem

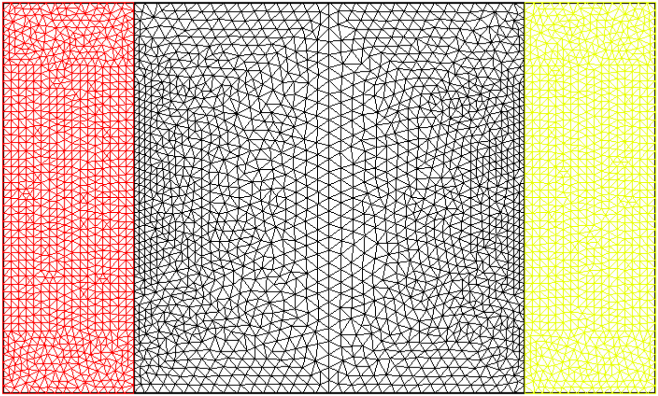


Figure: The domain and the mesh - $\Omega = (0,1)$, $\mathcal{O}_1 = (0, 0.2)$, $\mathcal{O}_2 = (0.8, 1)$ - $T = 0.5$ - Number of vertices (x_i, t_i): 3521 - Number of triangles: 6820

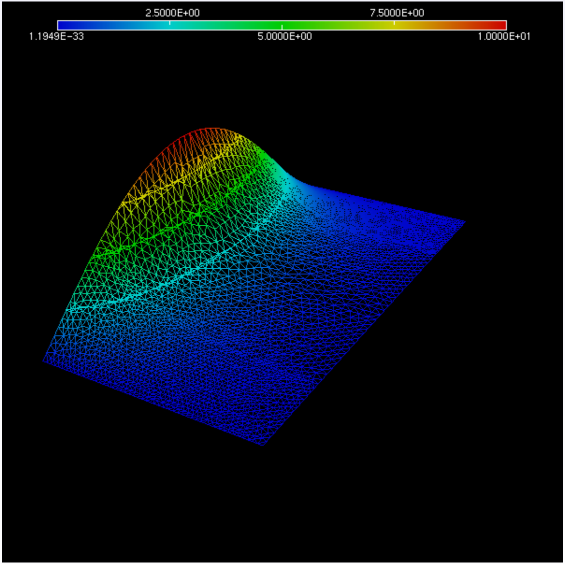


Figure: The state y - $y^0 \equiv 10 \sin x$ - $\mu = 1$ - $y_{1,d} = y_{2,d} = 0$

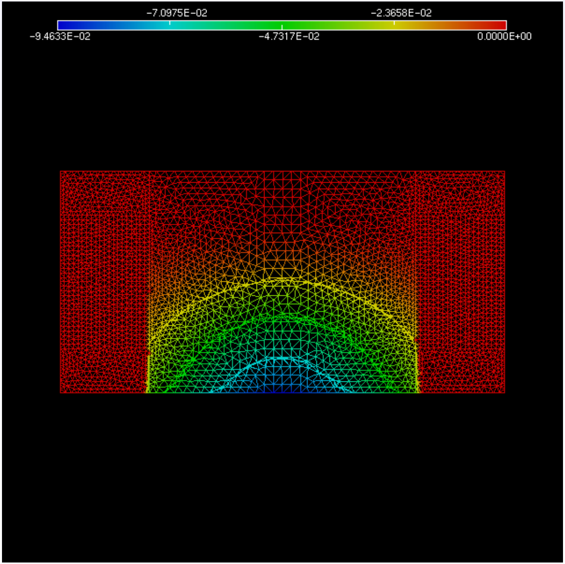


Figure: The leader f

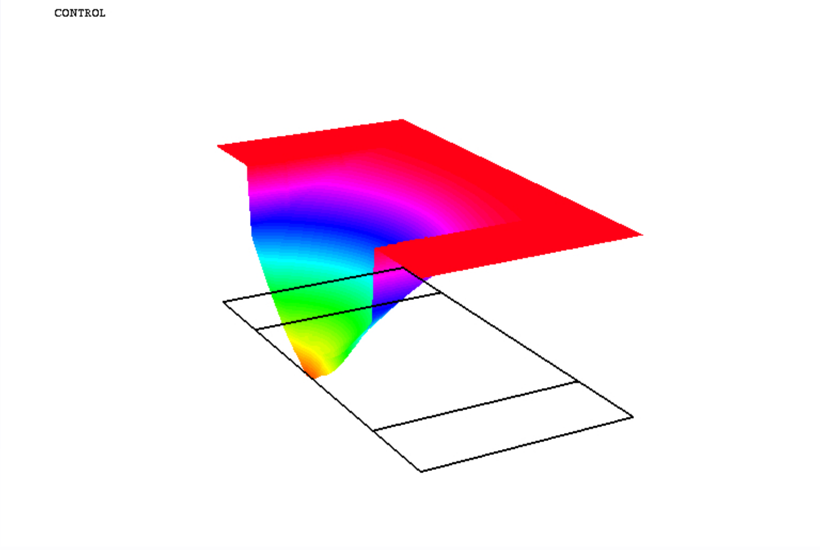


Figure: The leader f

EXTENSIONS

- Boundary followers, distributed leader: OK under similar conditions

$$\left\{\begin{array}{l} y_t - y_{xx} = f 1_{\mathcal{O}}, \quad (x,t) \in (0,1)\times(0,T) \\ y(0,t) = v_1(t), \ \ y(1,t) = v_2(t), \quad t \in (0,T) \\ y(x,0) = y^0(x), \quad x \in (0,1) \end{array}\right.$$

Costs: $\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 + \mu \int_0^T |v_i|^2\,dt, \ \ i = 1,2$

- Distributed followers, boundary leader: OK again

$$\left\{\begin{array}{l} y_t - y_{xx} = v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, \quad (x,t) \in (0,1)\times(0,T) \\ y(0,t) = f, \ \ y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \quad x \in (0,1) \end{array}\right.$$

- However: boundary followers + boundary leader is unknown!
  We would need: $\|\psi|_{t=0}\|^2 + \sum_{i=1}^2 \iint_Q \hat\rho^{-2}|\gamma^i|^2 \le C \int_0^T \rho_*^{-2}|\psi_x(0,t)|^2\,dt$ for

$$\left\{\begin{array}{l} -\psi_t - \psi_{xx} = \sum_{i=1}^2 \alpha_i \gamma^i 1_{\mathcal{O}_d}, \quad \gamma^i_t - \gamma^i_{xx} = -\frac{1}{\mu}\psi 1_{\mathcal{O}_i} \\ \psi|_{t=T} = \psi^T(x), \quad \gamma^i|_{t=0} = 0, \ \ \text{etc.} \end{array}\right.$$

EXTENSIONS (Cont.)

- More followers, coefficients, non-scalar parabolic systems, other functionals, boundary controls, higher dimensions, etc.
- Semilinear systems: OK for Lipschitz-continuous F

$$\begin{cases} y_t - y_{xx} = F(x,t;y) + f 1_{\mathcal{O}} + \sum_{i=1}^{m} v_i 1_{\mathcal{O}_i} \\ y(0,t) = y(1,t) = 0, \quad t \in (0,T), \quad \text{etc.} \end{cases}$$

- ECT: OK
- Local constraints: OK
  For instance, $v_i \in L^2(\mathcal{O}_i \times (0,T))$, $v_i(x,t) \in L_i$ (closed)

AN INTERESTING QUESTION:

All this holds for large μ - What about small μ?

Recall: $J_i(v_1, v_2) := \frac{\alpha_i}{2}\iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 + \frac{\mu}{2}\iint_{\mathcal{O}_i\times(0,T)} |v_i|^2, \quad i = 1, 2$

$$\left\{ \begin{array}{l} y_t - y_{xx} = f 1_{\mathcal{O}} - \frac{1}{\mu}(\phi_1 1_{\mathcal{O}_1} + \phi_2 1_{\mathcal{O}_2}) \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_{i,d}} \\ \text{etc.} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (\mathrm{Id.} - \frac{1}{\mu}\Lambda)(v_1, v_2) = (v_{1,0}, v_{2,0}) \\ v_i \in L^2(\mathcal{O}_i \times (0, T)) \end{array} \right.$$

for some compact, self-adjoint Λ

Fredholm's alternative + Hilbert-Schmidt

$\Rightarrow \exists \mu_1 > \mu_2 > \dots$ (independent of f), with $\mu_n \to 0^+$ such that $\dot{\exists}$ Nash equilibrium for all $\mu \neq \mu_n$ for all n

Do we have NC for these μ?

FINAL COMMENTS:

- Other hierarchical strategies? Stackelberg-Pareto controllability?

$$\lambda J_1'(v_1, v_2) + (1-\lambda) J_2'(v_1, v_2) = 0, \quad \lambda \in (0,1)$$

For each f, we get a family of equilibria $(v_\lambda^1(f), v_\lambda^2(f))$, with $\lambda \in (0,1)$

$$\begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} - \frac{1}{\mu}(\frac{1}{\lambda}\phi 1_{\mathcal{O}_1} + \frac{1}{1-\lambda}\phi 1_{\mathcal{O}_2}) \\ -\phi_t - \phi_{xx} = \alpha_1 \lambda (y - y_{1,d}) 1_{\mathcal{O}_{1,d}} + \alpha_2 (1-\lambda)(y - y_{2,d}) 1_{\mathcal{O}_{2,d}} \\ \dots \end{cases}$$

$$\begin{cases} -\psi_t - \psi_{xx} = \alpha_1 \lambda \gamma 1_{\mathcal{O}_{1,d}} + \alpha_2 (1-\lambda) \gamma 1_{\mathcal{O}_{2,d}} \\ \gamma_t^i - \gamma_{xx}^i = -\frac{1}{\mu}(\frac{1}{\lambda}\psi 1_{\mathcal{O}_1} + \frac{1}{1-\lambda}\psi 1_{\mathcal{O}_2}) \\ \dots \end{cases}$$

$\exists$ some kind of "common" null controls?
$\exists$ average null controls, i.e. f such that $(\int_0^1 y \, d\lambda)(T) = 0$?

- Navier-Stokes? OPEN, as well as he standard NC problem
  Work in progress: local results (for small y_0)
  [with Araruna, Guerrero and Santos]

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THANK YOU VERY MUCH . . .