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# Hybrid optimal control problems for partial differential equations

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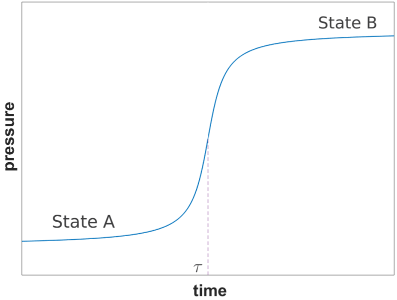
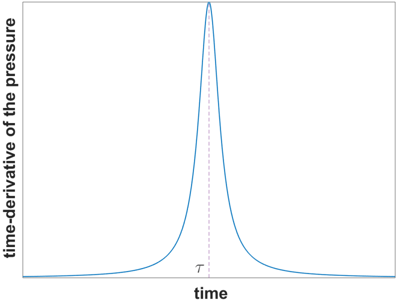
# Motivation

- In cardiac electrophysiology, we want to act on a pressure, in order to improve the efficiency of **defibrillation** .
- Qualitative behavior of the pressure:

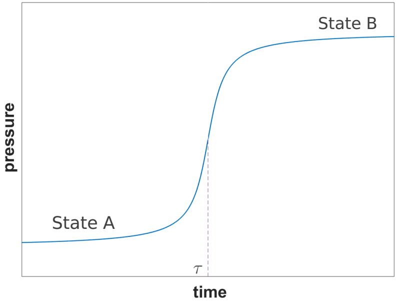


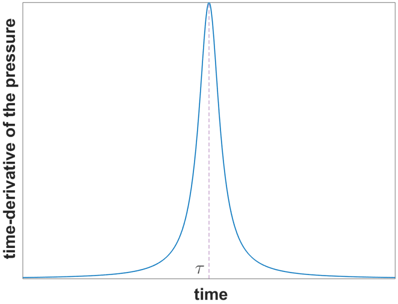




- We want to maximize the time-derivative of this pressure, which can be expressed in terms of the **displacement** inside of the cardiac tissue (hyperelastic material).

# Goal - optimal time for a maximum

- Qualitative behavior of a pressure:




- We don’t know when this time $\tau$ has to occur, it’s not important.

# Goal - optimal time for a maximum

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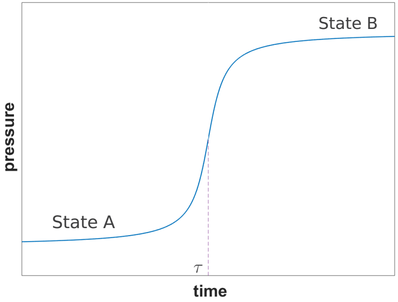


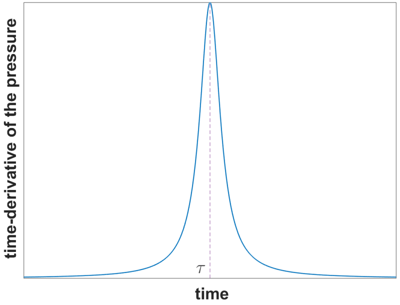


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- We could impose it whenever we want.

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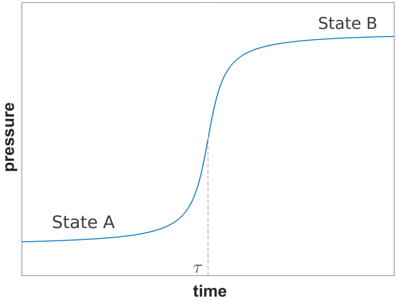


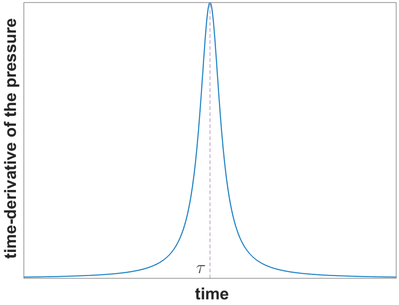


- We don't know when this time $\tau$ has to occur, it's not important.
- We could impose it whenever we want.
- Let's choose it **optimal**. $\rightarrow$ **"MaxMax"** problem

# Optimal control formulation

$$\begin{cases} \displaystyle\max_{\tau \in (0,T),\, u \in \mathrm{L}^2(0,T;U)} \int_0^T \ell(y,u)\mathrm{d}t + \Phi_1(y(\tau)) + \Phi_2(y(T)) \\ \text{subject to: } \dot{y} = f(y,u), \quad y(0) = y_0. \end{cases} \qquad (\mathcal{P})$$

Example:

$$\begin{cases} \dot{y} = f(y,u) & \text{in } \Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

with $\ell(y,u) = -\frac{\alpha}{2}\|u\|_U^2$ and $\quad f(y,u) = \Delta y + N(y) + Bu$.

⚠ The solution $t \mapsto y(t)$ is not necessarily differentiable at time $\tau$.

# Transformation of the problem

- Change of (time) variable:
  Define $\pi : (0, T) \to (0, T)$ sufficiently smooth, such that

$$\pi(0) = 0, \quad \pi(T/2) = \tau , \quad \pi(T) = T.$$

- Change of variables:

$$\tilde{y}(s) = y(\pi(s)), \qquad \tilde{u}(s) = u(\pi(s)).$$

- **Goals**:
  $\rightarrow$ Uncouple the state $y$ and the parameter $\tau$;
  $\rightarrow$ Circumvent the lack of differentiability;
  $\rightarrow$ Derive optimality conditions.

# Transformation of the problem

- The transformed system:

$$\begin{cases} \dot{\tilde{y}} = \dot{\pi} f(\tilde{y}, \tilde{u}) & \text{in } \Omega \times (0, T), \\ \tilde{y}(0) = y_0 & \text{in } \Omega. \end{cases}$$

- The transformed optimal control problem:

$$\begin{cases} \displaystyle \max_{\tau \in (0,T),\, \tilde{u} \in \mathrm{L}^2(0,T;U)} \int_0^T \dot{\pi} \ell(\tilde{y}, \tilde{u}) \mathrm{d}s + \Phi_1(\tilde{y}(T/2)) + \Phi_2(\tilde{y}(T)) \\ \text{subject to: } \dot{\tilde{y}} = \dot{\pi} f(\tilde{y}, \tilde{u}), \quad \tilde{y}(0) = y_0. \end{cases} \qquad (\tilde{\mathcal{P}})$$

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- Functional framework, see:
  M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich.
  **Optimization with PDE constraints**, volume 23 of Mathematical Modelling: Theory and Applications. Springer, New York, 2009.

# Plan

# First-order optimality conditions

The cost functional: $J(\tilde{u},\tau) = \int_0^T \dot{\pi}\ell(\tilde{y},\tilde{u})\mathrm{d}s + \Phi_1(\tilde{y}(T/2)) + \Phi_2(\tilde{y}(T))$,
where $\tilde{y}$ is the state of the (modified) system, with $\tilde{u}$ and $y_0$ as data.

The Hamiltonian: $H(\tilde{y},\tilde{u},\tilde{p}) = \ell(\tilde{y},\tilde{u}) + \langle \tilde{p}; f(\tilde{y},\tilde{u})\rangle_{Z';Z}$.
**First order derivatives**:

$$J_u(\tilde{u},\tau) = \dot{\pi}(\cdot,\tau)H_u(\tilde{y},\tilde{u},\tilde{p}), \qquad J_\tau(\tilde{u},\tau) = \int_0^T \dot{\pi}_\tau H(\tilde{y},\tilde{u},\tilde{p})\mathrm{d}s,$$

where

$$\begin{cases} -\dot{\tilde{p}} = \dot{\pi}(\cdot,\tau)H_y(\tilde{y},\tilde{u},\tilde{p}) \quad \text{on } (0,T/2)\cup(T/2,T), \\ \tilde{p}(T) = D\Phi_2(\bar{y}(T)), \\ \tilde{p}((T/2)^+) - \tilde{p}((T/2)^-) + D\Phi_1(\bar{y}(T/2)) = 0. \end{cases}$$

**If the pair $(\tilde{u},\tau)$ is optimal, then $DJ(\tilde{u},\tau) = 0$.**

# Second-order optimality conditions

The second-order derivative of $J$ is given by

$$D^2 J(\tilde{u}, \tau) \quad = \quad \begin{pmatrix} \mathbf{S}_u^*(\tilde{u}, \tau) & \mathrm{I} & 0 \\ \mathbf{S}_\tau^*(\tilde{u}, \tau) & 0 & 1 \end{pmatrix} D^2 L(\tilde{\mathbf{y}}, \tilde{u}, \tau, \tilde{p}) \begin{pmatrix} \mathbf{S}_u(\tilde{u}, \tau) & \mathbf{S}_\tau(\tilde{u}, \tau) \\ \mathrm{I} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\mathbf{S}$ is the control-to-state mapping $(\tilde{u}, \tau) \mapsto (\tilde{y}(T/2), \tilde{y}(T), \tilde{y})$, and

$$L\big((a_1, a_2, y), u, \tau, p\big) = \Phi_1(a_1) + \Phi_2(a_2) + \int_0^2 \Big(\dot{\pi} H(y, u, p) - \langle p, \dot{y} \rangle_{Z';Z}\Big) \mathrm{d}s$$
$$-\langle p(0), y(0) - y_0 \rangle_X + \langle p(T), y(T) - a_2 \rangle_X - \langle [p]_{T/2}, y(T/2) - a_1 \rangle_X.$$

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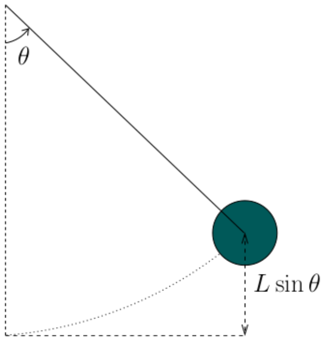
- If $(\bar{u}, \bar{\tau})$ is solution, then $t \mapsto H(\bar{y}, \bar{u}, \bar{p})$ is constant. In particular, it has **no jump** at time $t = \tau$.

- Moreover, for all $(v, \theta) \in \mathrm{L}^2(0, T; U) \times \mathbb{R}$, we have: $D^2 J(\bar{u}, \bar{\tau}).(v, \theta)^2 \leq 0$.

# Plan

# Let us start with ODEs

The simple damped pendulum:

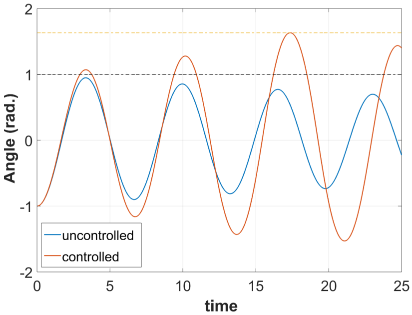


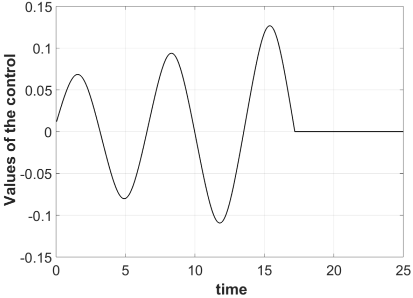

With $(y_1, y_2) = (\theta, \dot{\theta})$:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -\lambda y_2 - \mu \sin y_1 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

- We want to maximize $\Phi_1(y) := y_1$, at optimal time $\tau$ (to be determined).
- No terminal cost: $\Phi_2 \equiv 0$'.

# The simple damped pendulum




Optimal time at which the maximum is reached: $\tau \approx 17.22$.

- The control compensates the damping, and creates a swing effect.
- The control is sparse in time.

# A coupled system

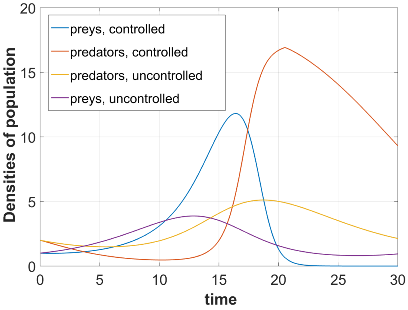
Consider the Lotka-Volterra system with control $u = (u_1, u_2)$.
$y_1$: density of preys. $y_2$: density of predators.

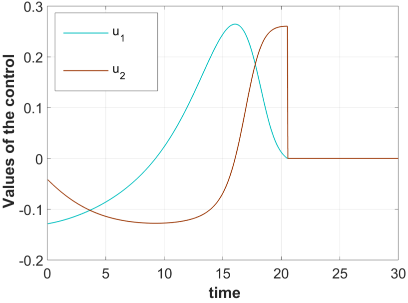
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} (y_1(a - by_2) + u_1 y_1)(1 - c_1 y_1) \\ (y_2(qy_1 - r) + u_2 y_2)(1 - c_2 y_2) \end{pmatrix}.$$

- We maximize at time $\tau$ the density of predators: $\Phi_1(y) = y_2$.
- First without terminal cost: $\Phi_2 \equiv 0$.
- Next with terminal cost (for saving the preys):

$$\Phi_2(y) = -\beta \log\left(\left|\frac{y_1}{y_{\mathrm{des}}}\right|\right)^2,$$

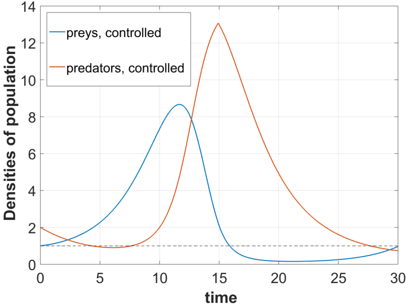
# The Lotka-Volterra system: Without terminal cost

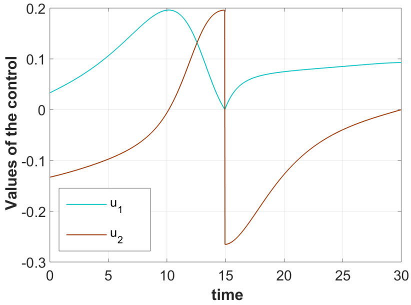



- At the beginning the control starts with killing preys and predators.
- Next the control introduces preys, that feed the predators.
- The maximum of predators is reached at $\tau \approx 20.57$.
- After that, two many predators leads to global extinction...

# The Lotka-Volterra system: With terminal cost

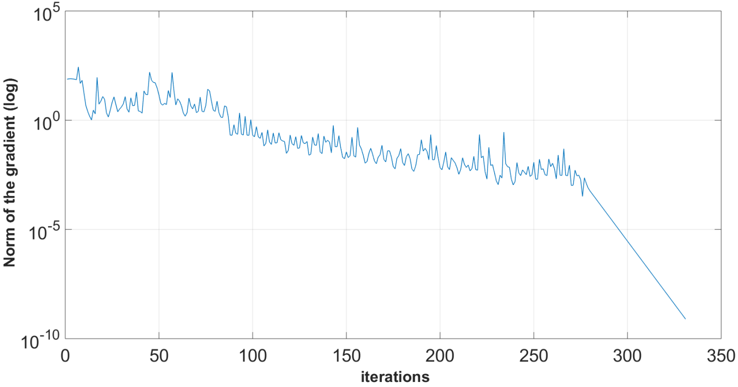



- Once the maximum is reached ($\tau \equiv 15$), the control kills predators, so that the preys can survive.
- We observe a jump for the control at $t = \tau$.
- And lack of differentiability of the state at $t = \tau$.

# Convergence

Barzilai-Borwein gradient steps $\rightarrow$ Newton method



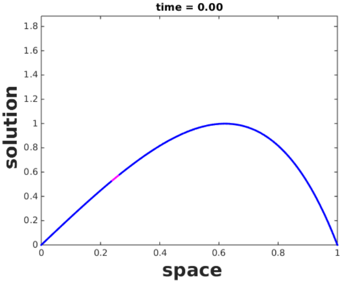
# Example: Burgers equation

The 1D Burgers equation with homogeneous Dirichlet conditions:

$$\begin{cases} \dot{y} + yy_x = \nu y_{xx} + \mathbb{1}_\omega u, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = y(1,t) = 0, & t \in (0,T), \\ y(x,0) = 10.(1 - e^{-(1-x)})(e^{-(1-x)} - e^{-1}), & x \in (0,1). \end{cases}$$

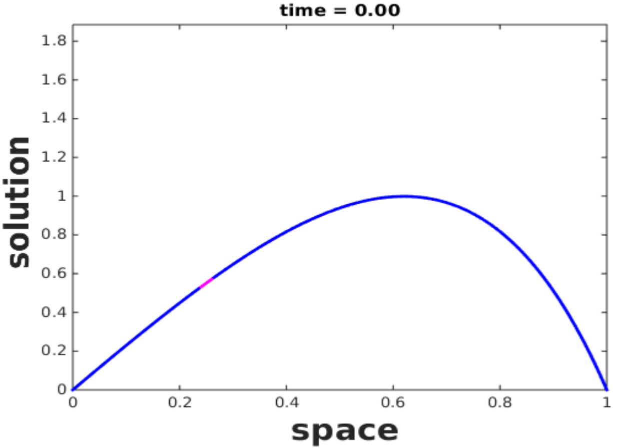
with: $\nu = 10^{-4}$, $\omega = [0.00; 0.25]$.

- P1 finite elements
- Crank-Nicolson for time-stepping (state equation),
- implicit in time for the adjoint state.



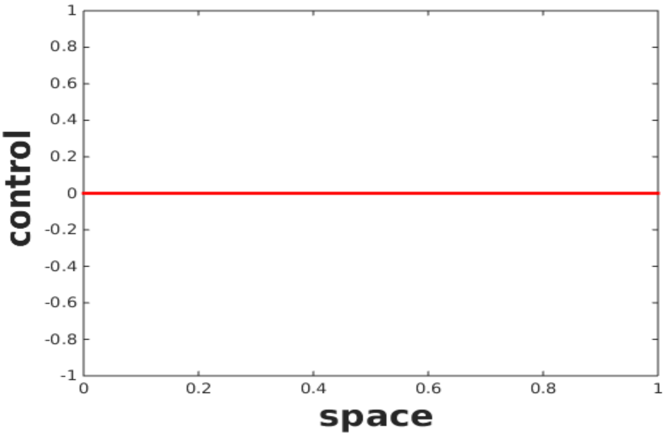
# Results - Burgers equation - The state

We maximize at $t = \tau \simeq 4.78$ the functional $\|y\|^2_{[0.25;0.27]}$.



# Results - Burgers equation - The control

The control is distributed on $\omega = [0.00; 0.25]$.



# Qualitative comments and remarks

- The control is **sparse** in time:
  before and after its main period of activation.

- Moreover, a **delay** is observed, between the end of the activation of the control and the optimal time $\tau$.
  $\rightarrow$ This corresponds to the **time of propagation** of the information/transport of the mass.

- "Local" aspect, non-convex problem, *many* local maximum.
  $\rightarrow$ Importance of initialization

# Plan

# The model

- The system:

$$
\left\{
\begin{aligned}
\ddot{u} - \kappa\Delta\dot{u} - \mathrm{div}((\mathrm{I}+\nabla u)\Sigma(u)) &= f \quad \text{in } \Omega\times(0,T)\\
\kappa\frac{\partial\dot{u}}{\partial n} + (\mathrm{I}+\nabla u)\Sigma(u)n + p\,\mathrm{cof}(\mathrm{I}+\nabla u)n &= g \quad \text{on } \Gamma_N\times(0,T)\\
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \det(\mathrm{I}+\nabla u)\mathrm{d}\Omega = \int_{\Gamma_N} \dot{u}\cdot\mathrm{cof}(\mathrm{I}+\nabla u)n\mathrm{d}\Gamma_N &= 0 \quad \text{in } (0,T)\\
u(0) = u_0, \quad \dot{u}(0) &= u_1 \quad \text{in } \Omega\\
+\text{compatibility conditions}&
\end{aligned}
\right.
$$

- Energy estimate: With $\frac{\partial\mathcal{W}}{\partial E}(E) = \check{\Sigma}(E)$ and $\Sigma(u) = \check{\Sigma}(E(u))$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\dot{u}\|^2_{[\mathrm{L}^2(\Omega)]^d} + \int_\Omega \mathcal{W}(E(u))\mathrm{d}\Omega\right) + \kappa\|\nabla\dot{u}\|^2_{[\mathrm{L}^2(\Omega)]^{d\times d}} = \int_\Omega f\cdot\dot{u}\,\mathrm{d}\Omega + \int_{\Gamma_N} g\cdot\dot{u}\,\mathrm{d}\Gamma_N
$$

# Without pressure, without damping

Some comments for the pure hyperelastic model:

$$\begin{cases} \ddot{u} - \mathrm{div}((\mathrm{I} + \nabla u)\Sigma(u)) = f & \text{in } \Omega \times (0,T) \\ (\mathrm{I} + \nabla u)\Sigma(u)n = g & \text{on } \Gamma_D \times (0,T) \\ u(0) = u_0, \quad \dot{u}(0) = u_1 & \text{in } \Omega \end{cases}$$

- The question of global existence is a **tough** problem.
- Honorable mentions: Sideris 1996 & 2000, Agemi 2000.
- Recently: Ciarlet & mardare 2016, nonlinear Korn's inequality ($p > d$):

$$\|u\|_{[\mathrm{W}^{2,p}(\Omega)]^d} \leq C \|E(u)\|_{[\mathrm{W}^{1,p}(\Omega)]^{d\times d}}$$

# In the parabolic world

In the context of the (strong) $\mathrm{L}^p$-maximal regularity:

- The displacement solutions are considered such that

$$\begin{array}{rcl} \dot{u} & \in & \mathrm{L}^p(0,T;\mathbf{W}^{2,p}(\Omega)) \cap \mathrm{W}^{1,p}(0,T;\mathbf{L}^p(\Omega)) =: \dot{\mathcal{U}}_{p,T}(\Omega), \\ u & \in & \mathrm{W}^{1,p}(0,T;\mathbf{W}^{2,p}(\Omega)) \cap \mathrm{W}^{2,p}(0,T;\mathbf{L}^p(\Omega)) := \mathcal{U}_{p,T}(\Omega). \end{array}$$

- Right-hand-sides:

$$\begin{array}{rcl} f & \in & \mathrm{L}^p(0,T;\mathbf{L}^p(\Omega)) =: \mathcal{F}_{p,T}(\Omega), \\ g & \in & \mathrm{W}^{1/(2p'),p}(0,T;\mathbf{L}^p(\Gamma_N)) \cap \mathrm{L}^p(0,T;\mathbf{W}^{1/p',p}(\Gamma_N)) =: \mathcal{G}_{p,T}(\Gamma_N). \end{array}$$

- Trace spaces:

$$(u_0,u_1) \in \left(\mathbf{W}^{2,p}(\Omega) \cap \mathbf{W}^{1,p}_{0,D}(\Omega)\right) \times \left(\mathbf{W}^{2/p',p}(\Omega) \cap \mathbf{W}^{1/p',p}_{0,D}(\Omega)\right).$$

See Prüss 2002, and more (& Hieber), about the $\mathrm{L}^p$-maximal regularity.

# Theoretical results

- The linearized system
- Local existence of solutions $u \in \mathcal{U}_{p,T}(\Omega)$.
- Necessary optimality conditions derived as previously (rigorously, to be implemented).
- The pressure is an expression of the displacement:

$$p \approx \frac{-1}{|\Gamma_N|} \int_{\Gamma_N} (\mathrm{I} + \nabla u)\Sigma(u)n \cdot n \, \mathrm{d}\Gamma_N$$

- Example of maximization problem:

$$\max_{\xi \in \mathcal{X}_{p,T}(\omega), \tau \in (0,T)} \quad \frac{p(\tau+\varepsilon) - p(\tau)}{\varepsilon} \quad - \quad \frac{\alpha}{2} \|\xi\|^2_{\mathrm{L}^p(\omega)}$$

# Illustrations

With the Saint-Venant–Kirchhoff model:

$$\mathcal{W}(E) = \mu \, \mathrm{trace}(E^2) + \frac{\lambda}{2} \mathrm{trace}(E)^2$$

$u \mapsto E(u) = \frac{1}{2}\left((\mathrm{I} + \nabla u)^T(\mathrm{I} + \nabla u) - \mathrm{I}\right)$: Green–Saint-Venant strain tensor.

$$\check{\Sigma}(E) = 2\mu E + \lambda(\mathrm{trace}(E))\mathrm{I}$$
$$(\mathrm{I} + \nabla u)\Sigma(u) = (\mathrm{I} + \nabla u)\check{\Sigma}(E(u))$$

# Plan

1. **Motivation/Introduction**
2. **On the optimality conditions**
3. **Numerical experiments**
4. **A pressure coupled with a damped hyperelastic model**
5. **Hybrid optimal control with a *space* parameter**

**Objective:**

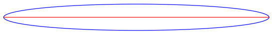
Maximizing a terminal cost, **somewhere** in a domain $\Omega$, a location we also optimize as a parameter.

**Analogy** with the "*Maxmax* in time":

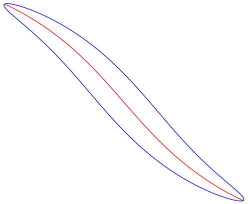
- For the "Maxmax" problem in time, the parameter $\tau$ transforms the shape of the interval $(0, T)$ into $(0, \tau) \cup (\tau, T)$.
- The parameter is geometric: a point in 1D, a submanifold of codimension 1 in higher dimension. We denote it by $\Gamma$.
- The topology of the space domain is then transformed by this geometric object $\Gamma$.
- Without control, this is *Shape Optimization*.
  $\rightarrow$ similar techniques, but calculations not as simple as before.

# Analogy with the *Maxmax in time* problem

- Define a change of variables $X[\eta]$ parameterized by $\eta$ such that:



Reference configuration: $\Gamma_0$



Deformed configuration: $\Gamma = X(\Gamma_0)$

- For example:

$$\dot{y} = f(y,u) = \operatorname{div} F(y) + Bu \;\Rightarrow\; \dot{\tilde{y}} = \frac{1}{\det \nabla X}\left(\operatorname{div}((\mathrm{cof} \nabla X) F(\tilde{y})) + B\tilde{u}\right)$$

with: $\tilde{y} := y \circ X, \quad \tilde{u} := (\det \nabla X)(u \circ X), \quad \tilde{p} := (\det \nabla X)(p \circ X)$.

- Analogy: $\quad \dot{\pi} f(\tilde{y}, \tilde{u}) \;\leftrightarrow\; \dfrac{f(\tilde{y}, \tilde{u})}{\det \nabla X}$

# First-order optimality conditions

- Choose: $J(\tilde{u},\eta) = \Phi\left(\tilde{y}_{|\Gamma_0}(\cdot,T)\right) - \frac{\alpha}{2}\int_0^T\int_\omega |\tilde{u}|^2_{\mathbb{R}^l}\mathrm{d}y\mathrm{d}t.$

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- Then: $J_\eta(\tilde{u},\eta) = -\int_0^T\int_\Omega (\mathrm{div}\, w)\mathcal{H}(y,u,p)\,\mathrm{d}x\,\mathrm{d}t,$

  where

$$\begin{cases} \mathcal{H}(y,u,p) = -\frac{\alpha}{2}|u|^2_{\mathbb{R}^l} + p\cdot \mathrm{div}(F(y)) + p\cdot Bu, \\ w = \frac{\partial X[\eta]}{\partial \eta}\circ X[\eta]^{-1}. \end{cases}$$

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- Furthermore: $J_\eta(\tilde{u},\eta) = 0 \quad \Leftrightarrow \quad \mathcal{H}(y,u,p)(t) = C(t)$ a.e. in $\Omega$.
- Adjoint system:

$$\begin{cases} -\dot{p} - F'(u)^*.\nabla p = 0 & \text{in } \Omega\times(0,T), \\ p(\cdot,T) = \delta_\Gamma * \nabla\Phi(u(\cdot,T)) & \text{in } \Omega, \end{cases}$$

## Example: The Shallow-Water Equations

$\rightarrow$ Conservation law system, no viscosity, coupled.

H: Height of the water flow, $\mathbf{v}$: velocity field (horizontal).

$$
\left\{
\begin{aligned}
\frac{\partial H}{\partial t} + \mathrm{div}(H\mathbf{v}) &= 0 && \text{in } \Omega \times (0,T), \\
\frac{\partial}{\partial t}(H\mathbf{v}) + \mathrm{div}(H\mathbf{v}\otimes \mathbf{v}) + \nabla\left(\frac{g}{2}H^2\right) &= \mathbb{1}_\omega u && \text{in } \Omega \times (0,T). \\
\mathbf{v} &= 0 && \text{on } \partial\Omega \times (0,T), \\
(H, H\mathbf{v})_{t=0} &= (H_0, 0) && \text{in } \Omega.
\end{aligned}
\right.
$$

The optimal control problem:

$$
\max_{u\in\mathscr{C},\Gamma\in\mathscr{G}} -\frac{\alpha}{2}\int_0^T\int_\omega |u|^2_{\mathbb{R}^2}\mathrm{d}\omega\mathrm{d}t + \int_\Gamma H(x,T)^2\mathrm{d}\Gamma(x),
$$

where Γ is a point (1D) or a curve (2D), whose the location is to be optimized.

# Numerical simulation in 1D

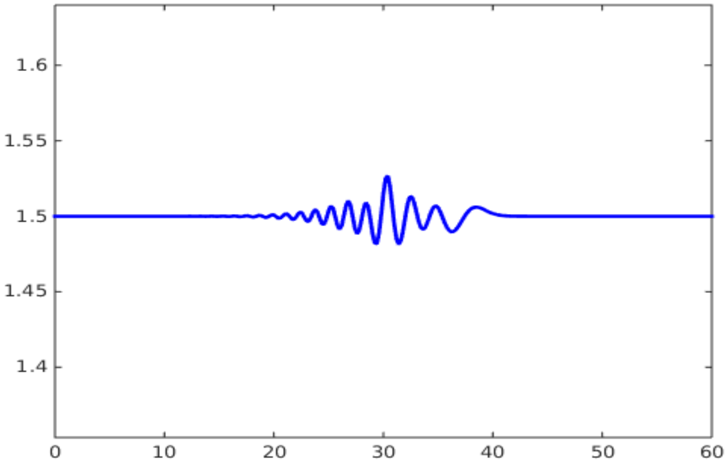


**Figure:** Maxmax problem for the 1D Shallow-Water equations.

# Some words on the implementation

- Finite differences on a uniform grid, Lax-Wendroff for the state equation.

- Scheme for the adjoint equation:
  Implicit in time, centered in space (1st-order).

- When $d = 2$, $\eta \mapsto \gamma \mapsto \Gamma = X[\eta](\Gamma_0)$ parameterizes the curve: $\gamma(s) = \displaystyle\sum_{k=1}^{4} \eta_k \cos\left(\frac{2\pi k}{f} s\right)$
- Sensitivity w.r.t $\mathrm{cof}(\nabla X[\eta])$ needs to be derived.
- The geometric parameter (a curve when $d = 2$) is taken into account with a *fictitious domain approach* (à la papa).

- Barzilai-Borwein (L-BFGS style) for the gradient rule.
- Everything works surprisingly well, residual converging below $1.e^{-12}$.

# Numerical simulation in 2D

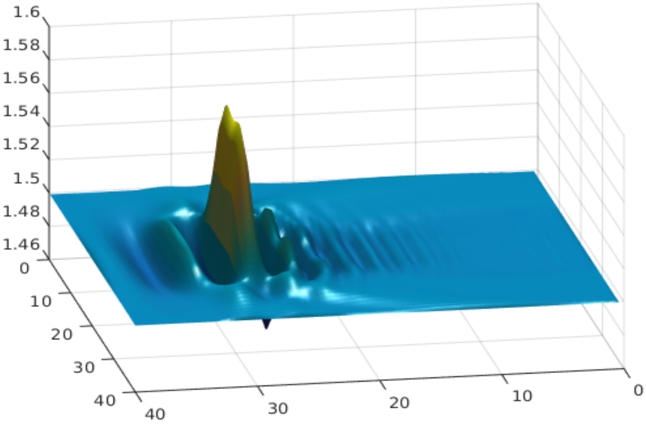


**Figure:** Maxmax problem for the 2D Shallow-Water equations.

# References

- S.C., K. Kunisch, L. Pfeiffer, *Optimal control for a class of infinite dimensional systems involving an $\mathrm{L}^\infty$-term in the Cost Functional*, ZAMM 2018.

- S.C., K. Kunisch, L. Pfeiffer, *Hybrid optimal control problems for a class of semilinear parabolic equations*, DCDS-S 2018.

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- S.C., *A hybrid optimal control problem for maximizing variations of a pressure coupled with a damped hyperelastic model*, in preparation.

**THANK YOU FOR YOUR ATTENTION!**

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