A game theory approach for the groundwater pollution control

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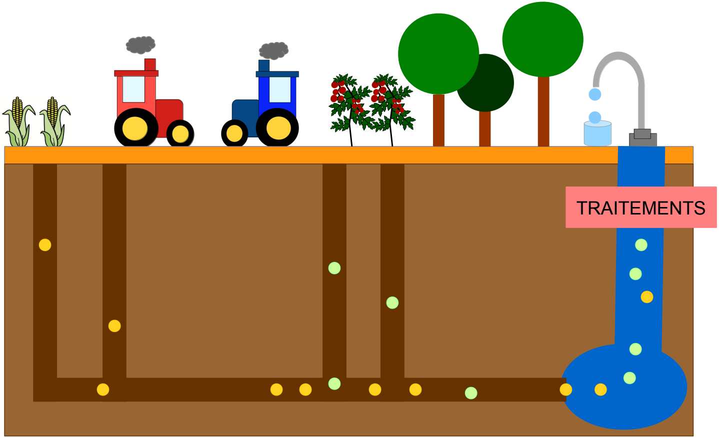

TRAITEMENTS

- Derives at 66% from agricultural activities.
- Nitrates concentration in the fertilizer $>$ part of nitrates consumed by the plant.
- Nitrates Directive (91/676/CEE) : regulation drinking water : nitrates $\leq$ 50mg/L.

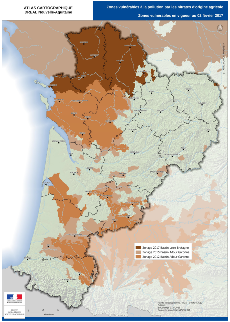


FIGURE – Vulnerable zones in 2017

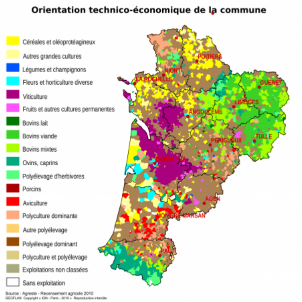


FIGURE – Agricultural crops in Nouvelle Aquitaine in 2010

- Description of the model
  - Hydrogeological modelling
  - Economic modelling
- Nash equilibrium
  - Existence
  - Characterization of a Nash equilibrium by the adjoint problem
  - Uniqueness
- Numerical results
  - Algorithm
  - With cooperation
  - Without cooperation

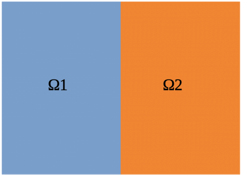
Optimal control problem constrained by a PDE's system.

Four variables...

- fertilizer load $p_i(x, t)$ spread over time by player i on the ground surface
- concentration $c(x, t)$ of the pollutant in the underground
- Darcy velocity of the solution in the underground $v(x, t)$.

... strongly coupled

- because of conservation principles
- because of the coupling between the constraints and the optimization process



$\Omega \subset \mathbb{R}^N$, $N \geq 1$, bounded domain

Areas of the players : $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$

$\Omega_1 \cap \Omega_2 = \emptyset$.

The mass conservation is modeled by :

$$(1)\begin{cases} R\psi\partial_t c + v\cdot\nabla c - \mathrm{div}(\psi S(v)\nabla c) = -r(c) + \chi_{\Omega_1} p_1 + \chi_{\Omega_2} p_2 - gc \\ \mathrm{div}(v) = g \\ v = -\kappa\nabla\phi \\ +CI + CB \end{cases}$$

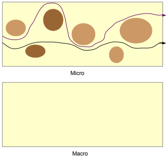
- **porosity** ψ ; - **mobility** of the fluid κ ; - **hydraulic head** ϕ ;

- **source term** describing groundwater recharge or discharge g ;

- nonlinear term modelling the **chemical reactions** $r(c)$ with classical reaction models for $c \in [0,1]$:
  - Freundlich type : $r(c) = k_1 c^m$, $(k_1, m) \in \mathbb{R}_+^2$,
  - Langmuir type : $r(c) = \frac{k_2 c}{1+k_3 c}$, $(k_2, k_3) \in \mathbb{R}_+^2$.

  Assume that r is a continuous function s.t for some $r_+ \in \mathbb{R}_+$: $|r(x)x| \leq r_+ |x|^2$, $\forall x \in \mathbb{R}_+$.

- **dispersion tensor** $S(v) = S_m Id + S_p(v)$ defined by $S_p(v) = |v|\,(\alpha_L \epsilon(v) + \alpha_T(Id - \epsilon(v)))$, with $\epsilon(v)_{i,j} = \frac{v_i v_j}{|v|^2}$ and $S(v)\xi\cdot\xi \geq (S_m + \alpha_T |v|)\,|\xi|^2$, $\forall \xi \in \mathbb{R}^N$.



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Lemma 1 **E. Augeraud, C. Choquet, E. C., M. Diédhiou**
Let (p_1, p_2) given in $L^2(\Omega_1 \times (O, T)) \times L^2(\Omega_2 \times (0, T))$. Under reasonable assumptions on the data, there exists a unique solution (c, ϕ) of (1) associated to (p_1, p_2) in the sense of weak formulation definition of (1).

Objective player 1 :

$$J_1(p_1, p_2, c) = \int_0^T \left(\int_\Omega \left(f_1(x, p_1(x,t))\chi_{\Omega_1}(x) - D_1(x, c(x,t; p_1, p_2)) \right) dx \right) e^{-\rho t}\, dt - \nu e^{-\rho T} \int_\Omega D_1(x, c(x, T; p_1, p_2))\, dx$$

Objective player 2 :

$$J_2(p_1, p_2, c) = \int_0^T \left(\int_\Omega \left(f_2(x, p_2(x,t))\chi_{\Omega_2}(x) - D_2(x, c(x,t; p_1, p_2)) \right) dx \right) e^{-\rho t}\, dt - \nu e^{-\rho T} \int_\Omega D_2(x, c(x, T; p_1, p_2))\, dx$$

- benefits of player i : f_i, depending on the fertilizer load : inscreasing, strictly concave, on $\mathbb{R}^+$ inf-continuous
- decontamination cost D_i, depending on the concentration : increasing, strictly convexe, hemicontinuous

Objective player 1 :

$$J_1(p_1, p_2, c) = \int_0^T \left(\int_\Omega \left(f_1(x, p_1(x,t))\chi_{\Omega_1}(x) - D_1(x, c(x,t;p_1,p_2)) \right) dx \right) e^{-\rho t}\, dt$$
$$- \nu e^{-\rho T} \int_\Omega D_1(x, c(x,T;p_1,p_2))\, dx$$

Objective player 2 :

$$J_2(p_1, p_2, c) = \int_0^T \left(\int_\Omega \left(f_2(x, p_2(x,t))\chi_{\Omega_2}(x) - D_2(x, c(x,t;p_1,p_2)) \right) dx \right) e^{-\rho t}\, dt$$
$$- \nu e^{-\rho T} \int_\Omega D_2(x, c(x,T;p_1,p_2))\, dx$$

Optimal control problem :

Find $\max_{p_1 \in E_1} J_1(p_1, p_2, c)$ where $p_2 \in E_2$ et $\max_{p_2 \in E_2} J_2(p_1, p_2, c)$ where $p_1 \in E_1$ subject to the state equation (1) for c and v.

$E_i \subset \{q \in L^2(0,T; L^2(\Omega)), 0 \leq q \leq \bar{p}, \text{ a.e. in} \Omega \times (0,T)\}$

Definition 1: Players reaction functions
For a.e. $(x,t) \in \Omega_T$, *the reaction functions* $p_i^*(t,x;p_{-i})$, $i = 1,2$, *are defined by*

$$p_1^*(x,t;p_2) = argmax_{q_1 \in E_1} J_1(q_1;p_2) \quad \forall p_2 \in E_2 \tag{5.1}$$
$$p_2^*(x,t;p_1) = argmax_{q_2 \in E_2} J_2(q_2;p_1) \quad \forall p_1 \in E_1 \tag{5.2}$$

Lemma 2 [E. Augeraud, C. Choquet, E.C., M. M. Diédhiou]
For a given $p_i \in E_i$, $i = 1, 2$, there exists a unique pair of functions $(p_{-i}^*, c^*(x,t;p_{-i}^*(x,t;p_i),p_i))$ such that $p_{-i}^*(x,t;p_i) = argmax_{q_{-i} \in E_{-i}} J_{-i}(q_{-i},p_i)$ and $(c(x,t;p_{-i}^*(x,t;p_i),p_i),\phi)$ is the weak solution of (1) associated with the load (p_{-i}^*,p_i).

Definition 2: Nash equilibrium (NE)
The quadruplet $(p_1^\flat, p_2^\flat, c^\flat, \phi)$ *is a Nash equilibrium iff :*

$$J_1(p_1^\flat; p_2^\flat,) \geq J_1(p_1; p_2^\flat,) \quad \forall p_1 \in E_1,$$

$$J_2(p_2^\flat; p_1^\flat,) \geq J_2(p_2; p_1^\flat,) \quad \forall p_2 \in E_2,$$

where $c^\flat = c^\flat(\cdot; p_1^\flat, p_2^\flat)$ *is the solution given by* ***Lemma 1*** *of the problem (1).*

Theorem 1: Existence of a Nash Equilibrium [E. Augeraud, C. Choquet, E. C., M. M. Diédhiou]
There exists a Nash equilibrium in the sense of Definition 2.

Proof :

Let $p_1^* = argmax_{q_1 \in E_1} J_1(q_1, p_2)$ and $p_2^* = argmax_{q_2 \in E_2} J_2(q_2, p_1)$ uniquely defined by Lemma 2.

Set $c_1^*(x,t) = c_1(x,t;p_1^*,p_2)$ and $c_2^*(x,t) = c_2(x,t;p_1,p_2^*)$.

The aim is to prove that

$$\mathcal{C} : (p_1, p_2) \in E_1 \times E_2 \mapsto (p_1^*, p_2^*) \in E_1 \times E_2$$

admits a fixed point.

Setp 1 : $\mathcal{C}$ is continuous for the weak topology of $L^2(\Omega_T)$

Setp 2 : $\mathcal{C}$ is a compact application in $L^2(\Omega_T) \times L^2(\Omega_T)$

Setp 3 : Schauder fixed point theorem

Setp 1 : $\mathcal{C}$ is continuous for the weak topology of $L^2(\Omega_T)$

Consider $(p_1^n, p_2^n)_{n\geq 0} \in E_1 \times E_2$ such that

$$p_1^n \rightharpoonup p_1 \text{ and } p_2^n \rightharpoonup p_2 \text{ weakly in } L^2(\Omega_T)$$

associated with $(p_1^{*,n}, p_2^{*,n}) := \mathcal{C}(p_1^n, p_2^n)$, $c_1^{*,n}(x,t) = c(x,t;p_1^{*,n}, p_2^n)$, $c_2^{*,n}(x,t) = c(x,t;p_1^n, p_2^{*,n})$ uniquely defined by

$$p_i^{*,n} = argmax_{q_i \in E_i} J_i(q_i; p_{-i}^n) \tag{5.3}$$

$$R\psi\partial_t c_i^{*,n} - \text{div}(\psi S(v)\nabla c_i^{*,n}) + v\cdot\nabla c_i^{*,n} = -r(c_i^{*,n}) + p_i^{*,n}\chi_{\Omega_i} + p_{-i}^n\chi_{\Omega_{-i}} - gc_i^{*,n} + \gamma \tag{5.4}$$

$$+IC + BC$$

We want to prove that $p_i^{*,n} \rightharpoonup p_i^*$ weakly in $L^2(\Omega_T)$, where

$$p_i^* = argmax_{q_i \in E_i} J_i(q_i; p_{-i}) \tag{5.5}$$

is associated to $c_i^* = c_i(\cdot; p_i^*, p_{-i})$ solution in Ω_T of

$$R\psi\partial_t c_i^* - \text{div}(\psi S(v)\nabla c_i^*) + v\cdot\nabla c_i^* = -r(c_i^*) + p_i^*\chi_{\Omega_i} + p_{-i}\chi_{\Omega_{-i}} - gc_i^* + \gamma \tag{5.6}$$

Setp 1 : $\mathcal{C}$ is continuous for the weak topology of $L^2(\Omega_T)$

Due to the definition of E_i and according to Cauchy-Schwarz and Young inequalities, and Gronwall Lemma we get

$$\left\|p_i^{*,n}\right\|_{L^2(\Omega_T)} \leq C\,,\ \ i = 1, 2. \tag{5.7}$$

$$\left\|c_i^{*,n}\right\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq C, \quad i = 1, 2. \tag{5.8}$$

We deduce the existence of limit functions $p_i^0 \in L^2(\Omega_T)$ and $c_i^0 \in L^2(0, T; H^1(\Omega))$ and of a subsequence, s.t :

$$p_i^{*,n} \rightharpoonup p_i^0 \text{ weakly in } L^2(\Omega_T),\ i = 1, 2,$$
$$c_i^{*,n} \to c_i^0 \text{ in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T,\ i = 1, 2,$$
$$c_i^{*,n} \rightharpoonup c_i^0 \text{ weakly in } L^2(0, T; H^1(\Omega)),\ i = 1, 2,$$

Thus, we can **pass to the limit in the weak formulation of** (5.4) **(state equation)**. According to the uniqueness result in Lemma 1,

$$c_1^0(\cdot) = c(\cdot; p_1^0, p_2) \text{ and } c_2^0(\cdot) = c(\cdot; p_1, p_2^0). \tag{5.9}$$

Setp 1 : $\mathcal{C}$ is continuous for the weak topology of $L^2(\Omega_T)$

It remains to prove that $p_i^0 = p_i^*$, by using the limit behavior of the control problem.

Difficulty : no compactness result for $p_i^{*,n}$, appearing in nonlinear f_i
Idea : use convexity arguments

We know that $J_i(p_i^0; p_{-i}) \underbrace{\leq}_{\text{def}} J_i(p_i^*; p_{-i})$.

We prove that

$$\boxed{\underbrace{J_i(p_i^*; p_{-i}) \leq}_{?} J_i(p_i^0; p_{-i}) \underbrace{\leq}_{\text{def}} J_i(p_i^*; p_{-i})}$$

Because of the concavity of f_i, and since

$$p_i^{*,n} \rightharpoonup p_i^0 \text{ weakly in } L^2(\Omega_T),\ i = 1, 2,$$
$$c_i^{*,n} \to c_i^0 \text{ in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T,\ i = 1, 2,$$

according to a **lower semi-continuity argument** : $J_i(p_i^0; p_{-i}) \geq \overline{\lim}_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n)$.
By definition of the optimum $p_i^{*,n}$, $J_i(p_i^{*,n}; p_{-i}^n) \geq J_i(q_i; p_{-i}^n) \quad \forall q_i \in E_i$, thus

$$J_i(p_i^0; p_{-i}) \geq \overline{\lim}_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n) \geq \overline{\lim}_{n\to\infty} J_i(q_i; p_{-i}^n)$$

Setp 1 : $\mathcal{C}$ is continuous for the weak topology of $L^2(\Omega_T)$

$$\boxed{\underbrace{J_i(p_i^*; p_{-i}) \leq}_{?} J_i(p_i^0; p_{-i}) \underbrace{\leq}_{\text{def}} J_i(p_i^*; p_{-i})}$$

$J_i(p_i^0; p_{-i})$ upper bound of $J_i(q_i; p_{-i})$ and $c(\cdot; q_i, p_{-i}^n) \to c(\cdot; q_i, p_{-i})$ in $L^2(\Omega_T)$, a.e. in Ω_T. Therefore :

$$\begin{aligned}
\lim_{n\to\infty} J_i(q_i; p_{-i}^n) &= \lim_{n\to\infty} \Big(\int_0^T \Big(\int_{\Omega_i} f_i(x, q_i)\,dx \;-\; \int_\Omega D_i(x, c(t, x; q_i, p_{-i}^n))\,dx \Big) e^{-\rho t}\,dt \\
&\quad - \nu e^{-\rho T} \int_\Omega D_i(x, c(T, x; q_i, p_{-i}^n))\,dx \Big) \\
&= \int_0^T \Big(\int_{\Omega_1} f_i(x, q_i)\,dx \;-\; \int_\Omega D_i(x, c(t, x; q_i, p_{-i}))\,dx \Big) e^{-\rho t}\,dt \\
&\quad - \nu e^{-\rho T} \int_\Omega D_i(x, c(T, x; q_i, p_{-i}))\,dx = J_i(q_i, p_{-i})
\end{aligned}$$

Thus $\quad J_i(p_i^0; p_{-i}) \geq J_i(q_i; p_{-i})$ for any $q_i \in E_i$.

For $q_i = p_i^*$, we get $\quad J_i(p_i^0; p_{-i}) \geq J_i(p_i^*; p_{-i}) \quad$ and $\quad J_i(p_i^0; p_{-i}) = J_i(p_i^*, p_{-i})$.
According to Lemma 1,

$$p_i^0 = p_i^*$$

and the sequence $p_i^{*,n} \rightharpoonup p_i^*$ in $L^2(\Omega_T)$ for $i = 1, 2$:
$\mathcal{C}$ is continuous for the weak topology of $L^2(\Omega_T) \times L^2(\Omega_T)$.

Setp 2 : $\mathcal{C}$ is a compact application in $L^2(\Omega_T) \times L^2(\Omega_T)$

We now know that $p_i^* = p_i^0$, and $p_i^{*,n} \rightharpoonup p_i^*$ in $L^2(\Omega_T)$ thus

$$J_i(p_i^*; p_{-i}) \geq \overline{\lim}_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n)$$

By definition, $J_i(p_i^{*,n}; p_{-i}^n) \geq J_i(p_i^*; p_{-i}^n)$. Thus

$$\begin{aligned}\underline{\lim}_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n) \geq \underline{\lim}_{n\to\infty} J_i(p_i^*; p_{-i}^n) = &\int_0^T \int_{\Omega_i} f_i(x, p_i^*) e^{-\rho t}\, dx\, dt \\ &- \underline{\lim}_{n\to\infty} \Big(\int_0^T \int_\Omega D_i(x; c(t, x; p_i^*, p_{-i}^n)) e^{-\rho t}\, dx\, dt \\ &- \nu e^{-\rho T} \int_\Omega D_i(x; c(T, x; p_i^*, p_{-i}^n))\, dx \Big)\end{aligned}$$

where $c(t, x; p_i^*, p_{-i}^n)$ satisfies

$$\begin{aligned}&R\psi \partial_t c(t, x; p_i^*, p_{-i}^n) - \operatorname{div}(\psi S(v) \nabla c(t, x; p_i^*, p_{-i}^n)) + v \cdot \nabla c(t, x; p_i^*, p_{-i}^n) = \\ &\quad -r(c(t, x; p_i^*, p_{-i}^n)) + p_i^* \chi_{\Omega_i} + p_{-i}^n \chi_{\Omega_{-i}} - gc(t, x; p_i^*, p_{-i}^n) + \gamma \text{ in } \Omega_T, \\ &+IC + IB\end{aligned}$$

Setp 2 : $\mathcal{C}$ is a compact application in $L^2(\Omega_T) \times L^2(\Omega_T)$

There exists a subsequence of $c(\cdot; p_i^*, p_{-i}^n)$ which strongly converges in $L^2(\Omega_T)$, a.e. in Ω_T and weakly in $L^2(0,T;H^1(\Omega))$ to the unique solution of :

$$R\psi\partial_t c - \mathrm{div}(\psi S(v)\nabla c) + v\cdot\nabla c = -r(c) + p_i^*\chi_{\Omega_i} + p_{-i}\chi_{\Omega_{-i}} - gc + \gamma \text{ in } \Omega_T,$$
$$+IC + IB$$

that is

$$c(\cdot; p_i^*, p_{-i}^n) \to c(\cdot; p_i^*, p_{-i}) \text{ in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T.$$

Thanks to the hemicontinuity of D_i,

$$\begin{aligned}\underline{\lim}_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n) \geq \underline{\lim}_{n\to\infty} J_i(p_i^*; p_{-i}^n) &= \int_0^T \int_{\Omega_i} f_i(x, p_i^*) e^{-\rho t}\, dx\, dt \\ &- \underline{\lim}_{n\to\infty} \Big(\int_0^T \int_\Omega D_i(x; c(t,x;p_i^*, p_{-i}^n)) e^{-\rho t}\, dx\, dt \\ &- \nu e^{-\rho T} \int_\Omega D_i(x; c(T,x;p_i^*, p_{-i}^n))\, dx \Big) \\ &= J_i(p_i^*; p_{-i})\end{aligned}$$

As $J_i(p_i^*; p_{-i}) \geq \overline{\lim}_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n) \geq \underline{\lim}_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n) \geq J_i(p_i^*; p_{-i})$, we get
$J_i(p_i^*; p_{-i}) = \lim_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n)$

Setp 2 : $\mathcal{C}$ is a compact application in $L^2(\Omega_T) \times L^2(\Omega_T)$

Moreover, $c(\cdot; p_i^{*,n}, p_{-i}^n) \to c(\cdot; p_i^*, p_{-i})$ in $L^2(\Omega_T)$ and a.e. in Ω_T and, according to the hemicontinuity of D_i, as $n \to \infty$,

$$\int_{\Omega_T} D_i(x, c(t,x; p_i^{*,n}, p_{-i}^n))e^{-\rho t}\,dx\,dt \to \int_{\Omega_T} D_i(x, c(t,x; p_i^*, p_{-i}))e^{-\rho t}\,dx\,dt,$$
$$\int_{\Omega} D_i(x, c(T,x; p_i^{*,n}, p_{-i}^n))e^{-\rho T}\,dx \to \int_{\Omega} D_i(x, c(T,x; p_i^*, p_{-i}))e^{-\rho T}\,dx.$$

Thus from $J_i(p_i^*; p_{-i}) = \lim_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n)$ we get :

$$\lim_{n\to\infty} \int_0^T \int_{\Omega_i} f_i(x, p_i^{*,n})e^{-\rho t}\,dx\,dt = \int_0^T \int_{\Omega_i} f_i(x, p_i^*)e^{-\rho t}\,dx\,dt.$$

f_i being continuous and strictly concave, we get from Visintin [1]

$$p_i^{*,n} \to p_i^* \text{ in } L^2(\Omega_i \times (0,T)), \quad i = 1, 2.$$

$\mathcal{C}$ is a compact application in $L^2(\Omega_T) \times L^2(\Omega_T)$.

1. A. Visintin. Strong convergence results related to strict convexity, *Comm. in Partial Diff. Eq.*, 9(5), 1984.

Setp 3 : Schauder fixed point theorem

There exists $(p_1^\flat, p_2^\flat) \in E_1 \times E_2$ such that

$$\mathcal{C}(p_1^\flat, p_2^\flat) = (p_1^\flat, p_2^\flat).$$

By definition of $\mathcal{C}$, it satisfies

$$\begin{cases} p_1^\flat = argmax_{q_1 \in E_1} J_1(q_1; p_2^\flat) \text{ then } J_1(p_1^\flat; p_2^\flat) \geq J_1(q_1; p_2^\flat) \text{ for all } q_1 \in E_1, \\ p_2^\flat = argmax_{q_2 \in E_2} J_2(q_2; p_1^\flat) \text{ then } J_2(p_2^\flat; p_1^\flat) \geq J_2(q_2; p_1^\flat) \text{ for all } q_2 \in E_2. \end{cases}$$

Thus $(p_1^\flat, p_2^\flat)$ **is a Nash equilibrium.**

Aim : to prove the uniqueness of the NE

Difficulty : Nonlinearities both in the objectives functions and in the state equation.

Idea : turn to another formulation of NE using the **adjoint problem**.

Cost : additional assumptions on the objective functions.

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Lemma : [E. Augeraud, C. Choquet, E. C., M. M. Diédhiou]
Assume that $f_i : p \in [0, \bar{p}] \mapsto f_i(x, p)$ and $c \in \mathbb{R}_+ \mapsto D_i(x, c)$, $i = 1, 2$, are $\mathcal{C}^1$ functions for almost every $x \in \Omega$.
Let $(p_1^\flat, p_2^\flat)$ be a NE defined by Definition 2. Let $c^\flat(t, x) = c(t, x; p_1^\flat, p_2^\flat)$.
There exists $(\mu_1^\flat, \mu_2^\flat) \in (L^2(0, T; H^1(\Omega)))^2$ such that for $i = 1, 2$:

$$\begin{cases} \frac{\partial f_i}{\partial p}(x, p_i^\flat(t, x)) = \mu_i^\flat(t, x)\chi_{\Omega_i}(x) \text{ in } \Omega_T, \\ R\psi\partial_t\mu_i^\flat + v \cdot \nabla\mu_i^\flat + \mathrm{div}(\psi S(v)\nabla\mu_i^\flat) = r'(c^\flat)\mu_i^\flat + R\mu_i^\flat\psi\rho - \frac{\partial D_i}{\partial c}(x, c^\flat) \text{ in } \Omega_T, \\ R\psi\mu_i^\flat(T, x) = \nu\frac{\partial D_i}{\partial c}(x, c^\flat(T, x)) \text{ in } \Omega \\ +BC \end{cases}$$

Definition 3: Adjoint problem

Problem $\mathcal{P}_{\rm adj}$ *consists in finding* $(c^\flat, \mu_1^\flat, \mu_2^\flat)$ *satisfying*

$$\begin{cases} R\psi\partial_t c^\flat + v\cdot\nabla c^\flat - \mathrm{div}(\psi S(v)\nabla c^\flat) = -r(c^\flat) - gc^\flat + \chi_{\Omega_1}F_1(\chi_{\Omega_1}\mathcal{C}_T\mu_1^\flat) + \chi_{\Omega_2}F_2(\chi_{\Omega_2}\mathcal{C}_T\mu_2^\flat), \\ R\psi\partial_t\mu_i^\flat - \mathcal{C}_T v\cdot\nabla\mu_i^\flat - \mathrm{div}(\psi S(\mathcal{C}_T v)\nabla\mu_i^\flat) + r'(\mathcal{C}_T c^\flat)\mu_i^\flat + R\psi\rho\mu_i^\flat - \frac{\partial D_i}{\partial c}(\cdot, \mathcal{C}_T c^\flat) = 0 \text{ in } \Omega_T, i = 1, 2, \\ R\psi\mu_i^\flat{}_{|t=0} = \nu\frac{\partial D_i}{\partial c}(\cdot, c^\flat{}_{|t=T}) \quad i = 1, 2, \quad +BC \end{cases}$$

with $\mathcal{C}_T u(t) = u(T-t)$ *for* $t \in [0, T]$, *and* $\frac{\partial f_i}{\partial p}(x, F_i(y)) = y$.

Proposition 1 : [E. Augeraud, C. Choquet, E. C., M. M. Diédhiou]

There exists a weak solution $(c^\flat, \mu_1^\flat, \mu_2^\flat)$ **of Problem** $\mathcal{P}_{\rm adj}$ belonging to $(\mathcal{C}([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)))^3$.

Moreover, if $c \in \mathbb{R}_+ \mapsto D_i(x, c)$ is increasing a.e. in Ω_T, then $\mu_i^\flat \geq 0$ a.e. in Ω_T, $i = 1, 2$.

Definition 3: Adjoint problem

Problem $\mathcal{P}_{\text{adj}}$ consists in finding $(c^\flat, \mu_1^\flat, \mu_2^\flat)$ satisfying

$$\begin{cases} R\psi\partial_t c^\flat + v\cdot\nabla c^\flat - \operatorname{div}(\psi S(v)\nabla c^\flat) = -r(c^\flat) - gc^\flat + \chi_{\Omega_1}F_1(\chi_{\Omega_1}\mathcal{C}_T\mu_1^\flat) + \chi_{\Omega_2}F_2(\chi_{\Omega_2}\mathcal{C}_T\mu_2^\flat), \\ R\psi\partial_t\mu_i^\flat - \mathcal{C}_T v\cdot\nabla\mu_i^\flat - \operatorname{div}(\psi S(\mathcal{C}_T v)\nabla\mu_i^\flat) + r'(\mathcal{C}_T c^\flat)\mu_i^\flat + R\psi\rho\mu_i^\flat - \frac{\partial D_i}{\partial c}(\cdot, \mathcal{C}_T c^\flat) = 0 \text{ in } \Omega_T, i = 1, 2, \\ R\psi\mu_i^\flat{}_{|t=0} = \nu\frac{\partial D_i}{\partial c}(\cdot, c^\flat{}_{|t=T}) \quad i = 1, 2, \quad +BC \end{cases}$$

with $\mathcal{C}_T u(t) = u(T-t)$ for $t \in [0, T]$, and $\frac{\partial f_i}{\partial p}(x, F_i(y)) = y$.

Proposition 1 : [E. Augeraud, C. Choquet, E. C., M. M. Diédhiou]
There exists a weak solution $(c^\flat, \mu_1^\flat, \mu_2^\flat)$ of Problem $\mathcal{P}_{\text{adj}}$ belonging to $(\mathcal{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)))^3$.
Moreover, if $c \in \mathbb{R}_+ \mapsto D_i(x, c)$ is increasing a.e. in Ω_T, then $\mu_i^\flat \geq 0$ a.e. in Ω_T, $i = 1, 2$.

Theorem 2: Uniqueness of the Nash Equilibrium [E. Augeraud, C. Choquet, E. C., M. M. Diédhiou]

Assume that $\mu_i^\flat$ given in Proposition 1 belongs to $L^\infty(\Omega_T)$ for $i = 1, 2$.
Assume that the functions r', F_i and $\partial_c D_i$, $i = 1, 2$, are Lipschitz continuous.
*Then the **solution of Problem $\mathcal{P}_{\text{adj}}$ is unique.***

Proof :
Let (c, μ_1, μ_2) and $(\underline{c}, \underline{\mu}_1, \underline{\mu}_2)$ two solutions of Problem $\mathcal{P}_{\mathrm{adj}}$. Thus the difference solves

$$
\begin{aligned}
&R\psi\partial_t(c-\underline{c}) + v\cdot\nabla(c-\underline{c}) - \mathrm{div}(\psi S(v)\nabla(c-\underline{c})) = -(r(c)-r(\underline{c})) - g(c-\underline{c})\\
&\qquad\qquad + \sum_{i=1,2}\chi_{\Omega_i}\left(F_i(\mathcal{C}_T\mu_i) - F_i(\mathcal{C}_T\underline{\mu}_i)\right) \text{ in } \Omega_T,\\
&R\psi\partial_t(\mu_i-\underline{\mu}_i) - \mathcal{C}_T v\cdot\nabla(\mu_i-\underline{\mu}_i) - \mathrm{div}(\psi S(\mathcal{C}_T v)\nabla(\mu_i-\underline{\mu}_i)) + r'(\mathcal{C}_T c)(\mu_i-\underline{\mu}_i)\\
&\qquad + (r'(\mathcal{C}_T c) - r'(\mathcal{C}_T\underline{c}))\underline{\mu}_i + R\psi\rho(\mu_i-\underline{\mu}_i) - (\partial_c D_i(x,\mathcal{C}_T c) - \partial_c D_i(x,\mathcal{C}_T\underline{c})) = 0 \text{ in } \Omega_T,\\
&(\mu_i-\underline{\mu}_i)_{|t=0} = \nu\partial_c D_i(x, c_{|t=T}) - \nu\partial_c D_i(x,\underline{c}_{|t=T}) \text{ in } \Omega,\\
&+BC.
\end{aligned}
$$

Let $T_0 \in (0,T)$ and $\tau \in (0,T_0)$. [... ...]. We prove that $c = \underline{c}$ and $\mu_i = \underline{\mu}_i$, $i = 1, 2$, a.e. in $\Omega\times(0,T_0)$, if

$$
\begin{aligned}
T_0 < T_+ := \frac{R\psi_-}{2}\min\Big\{&\Big(\frac{\|v\|_\infty^2}{2\psi_- S_m} + r_+ + 1 + \|\mu_i\|_\infty r'_+ + (1+R\psi_+\nu)(\partial_c D_i)_+\Big)^{-1};\\
&\Big(\frac{F_{i,+}^2}{2} + r_+ + \frac{\|\mu_i\|_\infty r'_+}{2} + \frac{(\partial_c D_i)_+}{2} + \frac{\|v\|_\infty^2}{2\psi_- S_m} + \|g\|_\infty\Big)^{-1}\Big\}
\end{aligned}
$$

We can extend this uniqueness result in the small, and reiterate until covering $[0,T]$.
The global uniqueness result of Theorem 2 is proved.

Numerical simulations

Let (p_1, p_2) given in $E_1 \times E_2$ and find (p_1^*, p_2^*)

Step 1 : to find ϕ and v by solving

$$\mathrm{div}(v) = g, \quad v = -\kappa \nabla \phi,$$

Step 2 : to find c and v_c by solving

$$R\psi \partial_t c + \mathrm{div}(v_c) = -r(c) + \mathcal{X}_{\Omega_1} p_1 + \mathcal{X}_{\Omega_2} p_2, \quad v_c = -S(v)\psi \nabla c + cv,$$

Step 3 : to find μ_1, μ_2 and v_{μ_1}, v_{μ_2} by solving

$$\begin{aligned} &R\psi \partial_t \mu_1 + \mathrm{div}(v_{\mu_1}) = -\big(r'(\mathcal{T}_T c) + R\psi\rho + \mathcal{T}_T \theta\big)\mu_1 + \omega_1 \partial_c D(x, \mathcal{T}_T c), \\ &v_{\mu_1} = -S(v)\psi \nabla \mu_1 - \mu_1 \mathcal{T}_T v, \\ &c_{|t=0} = c_0, \quad R\psi \mu_{1|t=0} = \nu \omega_1 \partial_c D(\cdot, c_{|t=T}). \end{aligned}$$

$$\begin{aligned} &R\psi \partial_t \mu_2 + \mathrm{div}(v_{\mu_2}) = -\big(r'(\mathcal{T}_T c) + R\psi\rho + \mathcal{T}_T \theta\big)\mu_2 + \omega_2 \partial_c D(x, \mathcal{T}_T c), \\ &v_{\mu_2} = -S(v)\psi \nabla \mu_2 - \mu_2 \mathcal{T}_T v, \\ &c_{|t=0} = c_0, \quad R\psi \mu_{2|t=0} = \nu \omega_2 \partial_c D(\cdot, c_{|t=T}). \end{aligned}$$

Step 4 : to compute $p_1^* = (f_1')^{-1}(\mu_1 \chi_{\Omega_1}), \quad p_2^* = (f_2')^{-1}(\mu_2 \chi_{\Omega_2})$

Step 5 : stop criterion $\|(p_1 + p_2) - (p_1^* + p_2^*)\| < \epsilon$

Benefit function from [Godart et al., 2002] : $f_i(p_i) = \alpha_i(K_1 - K_2 e^{-K_3 \times p_i})$

Damages function : $D_i(c) = \omega_i D(c) = \omega_i \beta c^2 \chi_{well}$ where χ_{well} is the characteristic function of the well

Reaction function : $r(c) = \gamma c^2$

Data

$\alpha_1 = \alpha_2 = 1$ ($f_1 = f_2$), $\beta = 100$, $\gamma = 0.001$

$K_1 = 11.7888$, $K_2 = 8.6\times10^{-3}$, $K_3 = 5.0465\times10^{1}$,

$\omega_1 = \omega_2 = 0.5$ **(symmetric repartition of the cost)**,

$S_p = 0$, $S_m = 0.01$, $R = 1$, $\nu = 1$,

$\kappa = 39.04$, $\psi = 0.3$, $\rho = 0.05$,

$\Omega = [0, 1] \times [0, 1]$,

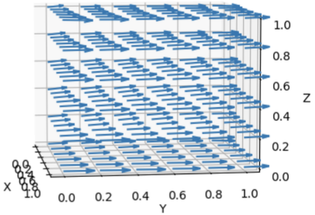
$p^0 = 0.005$, $c^0 = 0.005$,

$\Delta t = 0.01$, $T = 1$,

$N = 88$, $dx = dy = 1/N$,

$\varepsilon = 5 \times 10^{-6}$,

$v_g = v_d = 0.1$,



Darcy velocity

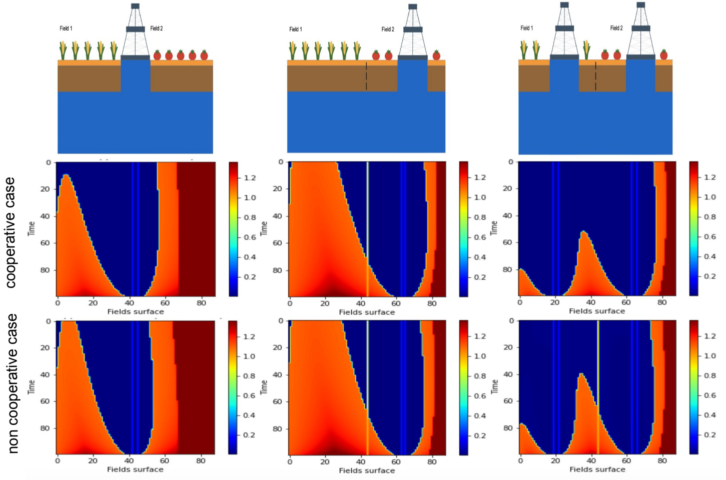
Non cooperative case : maximizing $J_1(p_1, p_2, c(\cdot, ; p_1, p_2))$ and maximizing $J_2(p_2, p_1, c(\cdot, ; p_1, p_2))$ where (c, μ_1, μ_2) satisfies

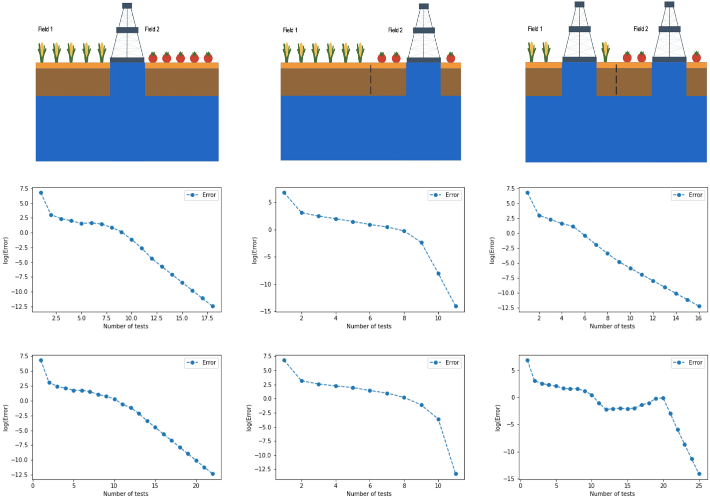
$$\begin{cases} R\psi\partial_t c + v\cdot\nabla c - \mathrm{div}(\psi S(v)\nabla c) = -r(c) - gc + \chi_{\Omega_1} p_1 + \chi_{\Omega_2} p_2, \\ R\psi\partial_t \mu_i - \mathcal{C}_T v\cdot\nabla\mu_i - \mathrm{div}(\psi S(\mathcal{C}_T v)\nabla\mu_i) + r'(\mathcal{C}_T c)\mu_i + R\psi\rho\mu_i - \frac{\partial D_i}{\partial c}(\cdot, \mathcal{C}_T c^\flat) = 0 \text{ in } \Omega_T, \quad i = 1, 2, \\ R\psi\mu_i^\flat{}_{|t=0} = \nu\frac{\partial D_i}{\partial c}(\cdot, c^\flat{}_{|t=T}) \quad i = 1, 2, \quad +BC \end{cases}$$

Cooperative case : classical optimal control problem, maximizing
$J(p_1, p_2, c(\cdot, ; p_1, p_2)) = \int_0^T \left(\int_\Omega \sum_{i=1}^2 f_i(x, p_i(t, x))\chi_{\Omega_i}(x) - D(x, c(t, x; p_1, p_2))\, dx\right) e^{-\rho t}\, dt.$
where (c, μ) satisfies

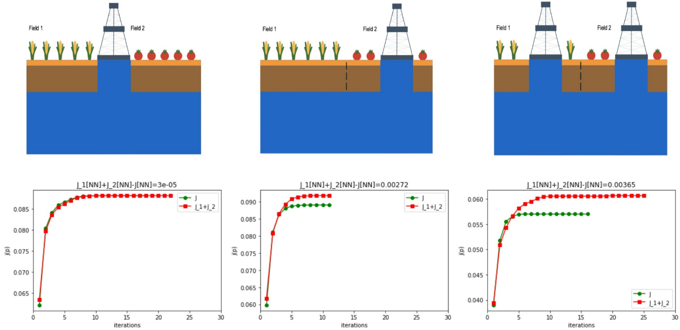
$$\begin{cases} R\psi\partial_t c + v\cdot\nabla c - \mathrm{div}(\psi S(v)\nabla c) = -r(c) - gc + p + \gamma \text{ in } \Omega_T, \\ R\psi\partial_t \mu - \mathcal{C}_T v\cdot\nabla\mu - \mathrm{div}(\psi S(\mathcal{C}_T v)\nabla\mu) + r'(\mathcal{C}_T c)\mu + R\psi\rho\mu - \frac{\partial D}{\partial c}(\cdot, \mathcal{C}_T c) = 0 \text{ in } \Omega_T, \\ R\psi\mu_{|t=0} = \nu\frac{\partial D}{\partial c}(\cdot, c_{|t=T}) \quad + BC \end{cases}$$

with $p = \sum_i \chi_{S_i}(f_i')^{-1}(\chi_{S_i}\mu)$.


Field 1
Field 2
cooperative case
non cooperative case
Time
Fields surface

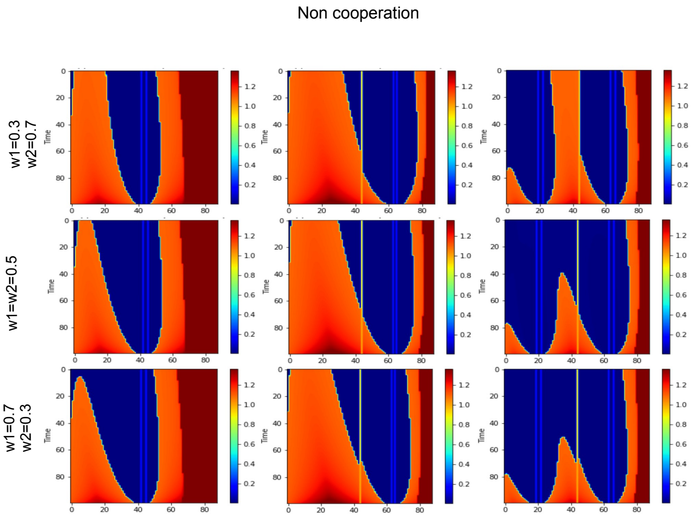


Field 1
Field 2
Field 1
Field 2
Field 1
Field 2
Error
log(Error)
Number of tests



Optimal control (cooperation case) *versus* game theory (non cooperative case)

J in green *versus* $J_1 + J_2$ in red


Non cooperation
w1=0.3
w2=0.7
w1=w2=0.5
w1=0.7
w2=0.3
Time

Thank you for your attention

1. E. Augeraud-Véron, C. Choquet and **É. C.**
Optimal control for a groundwater pollution ruled by a convection-diffusion-reaction problem.
Journal of Optimization, Theory and Applications, 173(3), 941-966, 2017.

2. E. Augeraud-Véron, C. Choquet, and **É. C.**
Existence, uniqueness and asymptotic analysis of optimal control problems for a model of groundwater pollution.
ESAIM : Control, Optimization and Calculus of Variations, 25, Article number 53, 2019.

3. C. Godart, P. A. Jayet, B. Niang, L. Bamiere, S. De Cara, E. Debove, E. Baranger.
Rapport Final, APR GICC 2002, Convention de recherche MEDD num 02-00019, UMR Economie Publique INRAE-INAPG, 2002.

4. E. Augeraud-Véron, C. Choquet, **É. C.** and M. M. Diédhiou.
A game theory approach for the groundwater pollution control (**under revision**).