Spacetime finite element methods for control problems subject to the wave equation

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Based on a joint work with
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Control problem

Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$ be connected, bounded, open, with smooth $\partial\Omega$. Let χ to be a cutoff function (that will be made more precise later).

Problem. For a fixed initial state (u_0, u_1), find such a control function ϕ that the solution u of

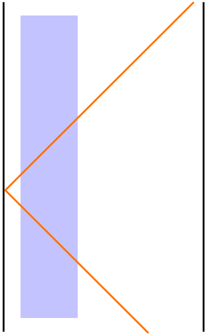
$$\begin{cases} \partial_t^2 u - \Delta u = \chi\phi, & \text{in } (0,T)\times\Omega, \\ u|_{x\in\partial\Omega} = 0, & \\ u|_{t=0} = u_0, \ \partial_t u|_{t=0} = u_1, & \end{cases}$$

satisfies $u|_{t=T} = 0$ and $\partial_t u|_{t=T} = 0$.

The problem has a solution under the geometric control condition (GCC) [Bardos-Lebeau-Rauch'92].

Geometric control condition

A cylinder $(a, b) \times \omega \subset (0, T) \times \Omega$ satisfies GCC if every (generalized) light ray intersects it.



GCC implies the observability estimate

$$\|\phi_0\|_{L^2(\Omega)} + \|\phi_1\|_{H^{-1}(\Omega)} \lesssim \|\phi\|_{L^2((a,b)\times\omega)}$$

for the solution ϕ of

$$\begin{cases} \partial_t^2 \phi - \Delta\phi = 0, & \text{in } (0, T) \times \Omega, \\ \phi|_{x\in\partial\Omega} = 0, \\ \phi|_{t=0} = \phi_0,\ \partial_t \phi|_{t=0} = \phi_1. \end{cases}$$

Control problem (precise formulation)

Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$ be connected, bounded, open, with smooth $\partial\Omega$.

Consider a cutoff function $\chi(t, x) = \chi_0(t)\chi_1^2(x)$ satisfying:
χ_j smooth, $0 \leq \chi_j \leq 1$, $\chi_0 = 0$ near $t = 0, T$, and
(A) $\chi = 1$ on open $(a, b) \times \omega \subset (0, T) \times \Omega$ satisfying GCC.

Problem. For a fixed initial state $(u_0, u_1) \in C_0^\infty(\Omega)^2$, find such a control function ϕ that the solution u of

$$\begin{cases} \partial_t^2 u - \Delta u = \chi\phi, & \text{in } (0, T) \times \Omega \\ u|_{x\in\partial\Omega} = 0, & \\ u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1, & \end{cases}$$

satisfies $u|_{t=T} = 0$ and $\partial_t u|_{t=T} = 0$.

Numerical analysis of the control problem

Problem. For $(u_0, u_1) \in C_0^\infty(\Omega)^2$, find ϕ such that the solution u of

$$\begin{cases} \partial_t^2 u - \Delta u = \chi\phi, & \text{in } (0,T)\times\Omega \\ u|_{x\in\partial\Omega} = 0, \\ u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1, \end{cases}$$

satisfies $u|_{t=T} = 0$ and $\partial_t u|_{t=T} = 0$.

GCC implies existence of a solution $\phi \in C^\infty((0,T)\times\Omega)$.

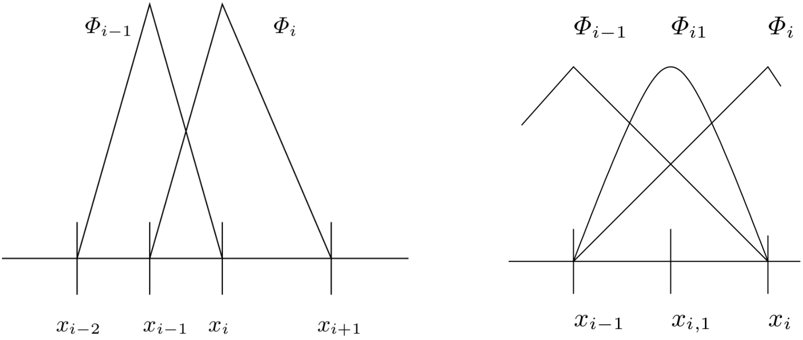
The solution is not unique, but for a certain (minimal) solution ϕ, we show how to compute a finite element approximation ϕ_h satisfying

$$\|\chi(\phi_h - \phi)\|_{L^2((0,T)\times\Omega)} \lesssim h^p$$

with $h > 0$ the mesh size and p the polynomial order of the basis functions [E.B-Feizmohammadi-Münch-Oksanen].

Finite element bases in 1d

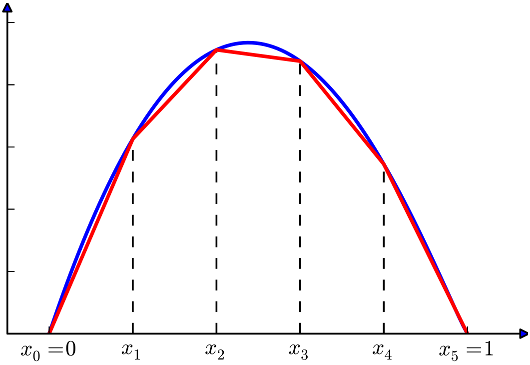
When the unit interval $[0, 1]$ is discretized by $0 = x_0 < x_1 < \cdots < x_N = 1$, the mesh size is $h = \max_{i=1,\ldots,N} |x_i - x_{i-1}|$.



The basis functions of order p whose support intersects (x_{i-1}, x_i).
Left. $p = 1$. *Right.* $p = 2$.

Finite element approximation

For $u \in C_0^\infty(0,1)$ there is a finite element approximation u_h satisfying $\|u - u_h\|_{L^2(0,1)} \lesssim h^{p+1}$. Our proven convergence rate h^p can be suboptimal of at most one degree.



Finite element approximation with polynomial order $p = 1$.

On previous literature

Naive discretizations of the control problem fail to converge due to spurious high frequency modes, see e.g. [Glowinski-Lions'95].

There are two traditional approaches:

1. Control theory on discrete level, with filtering of high frequencies
    - Numerical schemes on uniform meshes [Infante-Zuazua'99], ...
    - Continuum theory implies discrete theory with inexplicit control time [Ervedoza'09, Miller'12]
2. Discretize an iterative method formulated in the continuum
    - Based on Russell's stabilization implies control principle [Cîndea-Micu-Tucsnak'11]
    - Based on the Hilbert Uniqueness Method [Ervedoza-Zuazua'13]

Approach 1 is not aligned with the geometric control condition, while stopping criteria for the iteration in approach 2 are hard to design.

On previous literature: direct methods

Discretization of a direct (i.e. non-iterative) method in the continuum [Cîndea-Münch'15]

- Convergence of numerical experiments
- Convergence analysis conditional to uniform boundedness of certain discrete inf-sup constants
- Spacetime finite element method (FEM)

We discretize and stabilize (i.e. regularize) a direct method

- Proven convergence
- Spacetime FEM
- Earlier work [E.B.-Feizmohammadi-Oksanen'20] uses piecewise affine finite elements in space and finite differences in time

Duality between inverse and control problems

Consider the map $A(\phi_0, \phi_1) = \phi|_{(0,T)\times\omega}$ where ϕ is the solution of

$$\begin{cases} \partial_t^2 \phi - \Delta \phi = 0, & \text{in } (0,T)\times\Omega, \\ \phi|_{x\in\partial\Omega} = 0, \\ \phi|_{t=0} = \phi_0,\ \partial_t \phi|_{t=0} = \phi_1. \end{cases}$$

The transpose is $A^* f = (u|_{t=0}, \partial_t u|_{t=0})$ where u is the solution of

$$\begin{cases} \partial_t^2 u - \Delta u = f, & \text{in } (0,T)\times\Omega, \\ u|_{x\in\partial\Omega} = 0, \\ u|_{t=T} = 0,\ \partial_t u|_{t=T} = 0 \end{cases}$$

and f is supported on $(0,T)\times\omega$. The observability estimate

$$\|\phi_0\|_{L^2(\Omega)} + \|\phi_1\|_{H^{-1}(\Omega)} \lesssim \|\phi\|_{L^2((0,T)\times\omega)}$$

implies that A^* is surjective from $L^2((0,T)\times\omega)$ to $H_0^1(\Omega)\times L^2(\Omega)$.

Stabilized FEMs for unique continuation

Problem. Find (ϕ_0, ϕ_1) given $\phi|_{(0,T)\times\omega}$ for the solution ϕ of

$$\begin{cases} \partial_t^2 \phi - \Delta\phi = 0, & \text{in } (0,T)\times\Omega, \\ \phi|_{x\in\partial\Omega} = 0, & \\ \phi|_{t=0} = \phi_0,\ \partial_t\phi|_{t=0} = \phi_1. & \end{cases}$$

We can view the above inverse initial source problem as a unique continuation problem: given ϕ in $(0,T)\times\omega$ find ϕ in $(0,T)\times\Omega$. Stabilized FEMs have been designed e.g. for

- elliptic Cauchy problem [E.B.'14]
- stable and unstable unique continuation for the heat equation [E.B.–Ish-Horowicz–Oksanen'18]
- stable unique continuation for the wave equation [E.B.–Feizmohammadi–Münch–Oksanen]

Smoothness of minimal control

Theorem [Ervedoza–Zuazua'10]. Suppose GCC and $(u_0, u_1) \in C_0^\infty(\Omega)^2$. Then there is a solution $\phi \in C^\infty((0,T)\times\Omega)$ to the control problem s.t.

$$\begin{cases} \partial_t^2\phi - \Delta\phi = 0, & \text{in } (0,T)\times\Omega, \\ \phi|_{x\in\partial\Omega} = 0, & \\ \phi|_{t=T} = \phi_0,\ \partial_t u|_{t=T} = \phi_1, & \end{cases}$$

where (ϕ_0, ϕ_1) is the unique minimizer over $L^2(\Omega)\times H^{-1}(\Omega)$ of

$$J(\phi_0,\phi_1) = \frac{1}{2}\int_0^T\int_\Omega \chi(t,x)|\phi(t,x)|^2 dxdt + \langle u_0, \partial_t\phi|_{t=0}\rangle_{H_0^1(\Omega)\times H^{-1}(\Omega)} - (u_1, \phi|_{t=0})_{L^2(\Omega)}.$$

Direct formulation of the control problem

By [Ervedoza–Zuazua'10] there is $(u, \phi) \in C^\infty((0,T) \times \Omega)^2$ solving[1]

$$\begin{cases} \Box u = \chi\phi, \\ u|_{x\in\partial\Omega} = 0, \\ u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1, \\ u|_{t=T} = 0,\ \partial_t u|_{t=0} = 0, \end{cases} \qquad \begin{cases} \Box\phi = 0, \\ \phi|_{x\in\partial\Omega} = 0. \end{cases}$$

Lemma. The solution to the above system is unique.

Proof. Let $(u_{(j)}, \phi_{(j)})$, $j = 1, 2$, be solutions and write $u = u_{(1)} - u_{(2)}$ and $\phi = \phi_{(1)} - \phi_{(2)}$. Then (u, ϕ) satisfies the system with $u_0 = u_1 = 0$, and

$$(\chi\phi, \phi)_{L^2((0,T)\times\Omega)} = (\Box u, \phi)_{L^2((0,T)\times\Omega)} = (u, \Box\phi)_{L^2((0,T)\times\Omega)} = 0.$$

The observability estimate implies $\phi = 0$, and $u = 0$ follows. □

[1] Here $\Box = \partial_t^2 - \Delta$

Weak formulation of the control problem

Let g be the Minkowski metric on $\mathbb{R}^{1+n}$ and write $M = (0, T) \times \Omega$. Set

$$
\begin{aligned}
a(u, v) &= \int_M g(du, dv)dx - (u, \partial_\nu v)_{L^2(\partial M)} - (\partial_\nu u, v)_{L^2((0,T)\times\partial\Omega)}, \\
L(v) &= (u_1, v|_{t=0})_{L^2(\Omega)} - (u_0, \partial_t v|_{t=0})_{L^2(\Omega)},
\end{aligned}
$$

and $c(\phi, v) = (\chi\phi, v)_{L^2((0,T)\times\Omega)}$. If smooth (u, ϕ) solves

$$
\begin{cases}
\Box u = \chi\phi, \\
u|_{x\in\partial\Omega} = 0, \\
u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1, \\
u|_{t=T} = 0,\ \partial_t u|_{t=0} = 0,
\end{cases}
\qquad
\begin{cases}
\Box\phi = 0, \\
\phi|_{x\in\partial\Omega} = 0.
\end{cases}
$$

then for all smooth enough ψ, v

$$
a(u, \psi) = c(\phi, \psi) + L(\psi), \quad a(v, \phi) = 0.
$$

Stabilized FEM for the contol problem

Let $\mathcal{T}_h$, $h > 0$, be a quasi-uniform family of triangulations of[2] $(0, T) \times \Omega$, parametrized by the mesh size $h > 0$. Let $\mathbb{P}_p(K)$ be the space of polynomials of degree $\leq p$ on a set $K \subset \mathbb{R}^{1+n}$ and define

$$V_h^p = \{u \in C(M) : u|_K \in \mathbb{P}_p(K) \text{ for all } K \in \mathcal{T}_h\}.$$

We write $U_0 = (u_0, u_1)$ for the data and define the "energy"

$$E(U_0) = h^{-1} \|u_0\|^2_{L^2(\Omega)} + h \|u_1\|^2_{L^2(\Omega)}.$$

Our finite element method has the form: find the critical point of the Lagrangian $\mathcal{L}(u, \phi) : V_h^p \times V_h^q \to \mathbb{R}$,

$$\mathcal{L}(u, \phi) = \frac{1}{2} c(\phi, \phi) - \frac{1}{2} E(U|_{t=0} - U_0) - \frac{1}{2} Q(u, \phi) - a(u, \phi) + L(\phi),$$

where $U = (u, \partial_t u)$ and Q is a quadratic form giving the stabilization.

[2] Triangles adjacent to $(0, T) \times \partial\Omega$ have curved faces so that $\bigcup_{K \in \mathcal{T}_h} K = (0, T) \times \Omega$.

Stabilization

We write $\mathcal{F}_h$ for the set of internal faces of the triangulation $\mathcal{T}_h$, and $[\![\cdot]\!]$ for the jump over $F \in \mathcal{F}_h$. The stabilization is given by

$$Q(u,\phi) = \sum_{K \in \mathcal{T}_h} h^2 \|\Box u - \chi\phi\|^2_{L^2(K)} - \sum_{K \in \mathcal{T}_h} h^2 \|\Box\phi\|^2_{L^2(K)}$$
$$+ s(u,u) - s(\phi,\phi) + E(U|_{t=T}),$$
$$s(u,u) = \sum_{F \in \mathcal{F}_h} h \|[\![\partial_\nu u]\!]\|^2_{L^2(F)} + h^{-1} \|u\|^2_{L^2((0,T)\times\partial\Omega)} \,.$$

Observe that $Q(u,\phi) = 0$ for a smooth solution (u,ϕ) to

$$\begin{cases} \Box u = \chi\phi, \\ u|_{x\in\partial\Omega} = 0, \\ u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1, \\ u|_{t=T} = 0,\ \partial_t u|_{t=0} = 0, \end{cases} \qquad \begin{cases} \Box\phi = 0, \\ \phi|_{x\in\partial\Omega} = 0. \end{cases}$$

Numerical analysis results

- Solutions with limited regularity (partial result):
  Assume that $u_0 = 0$ and $u_1 \in L^2(\Omega)$ then
  $$(u_h, \phi_h) \rightharpoonup (u, \phi) \text{ in } [L^2(M)]^2.$$
- Smooth solutions:
  Error estimates reflecting the continuum stability, the smoothness of the solution and the approximation properties of the finite element space.

Error estimate

Theorem [E.B.-Feizmohammadi-Münch-Oksanen]. Suppose that the GCC holds. Let $p, q \geq 1$ and let $(u, \phi) \in H^{p+1}(M) \times H^{q+1}(M)$ be the unique solution to

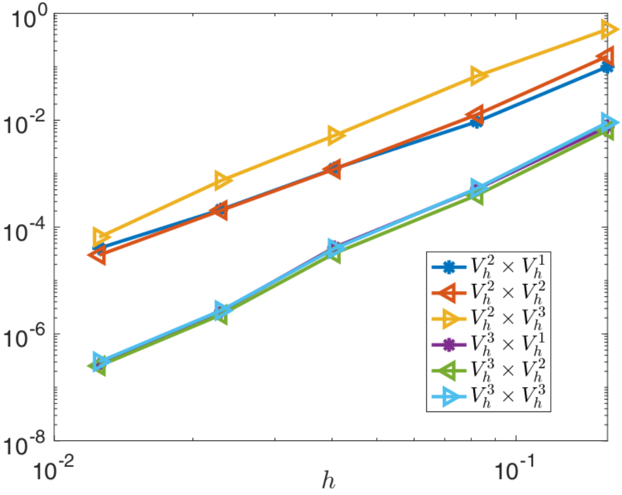
$$\begin{cases} \Box u = \chi\phi, \\ u|_{x\in\partial\Omega} = 0, \\ u|_{t=0} = u_0, \ \partial_t u|_{t=0} = u_1, \\ u|_{t=T} = 0, \ \partial_t u|_{t=0} = 0, \end{cases} \qquad \begin{cases} \Box\phi = 0, \\ \phi|_{x\in\partial\Omega} = 0. \end{cases}$$

Then the Lagrangian $\mathcal{L}$ has a unique critical point $(u_h, \phi_h) \in V_h^p \times V_h^q$ and

$$\|\chi(\phi - \phi_h)\|_{L^2(M)} \lesssim h^p \|u\|_{H^{p+1}(M)} + h^q \|\phi\|_{H^{q+1}(M)}.$$

The critical point can be computed by a solving a finite dimensional linear system.

Computational experiments on the **inverse** source problem



Convergence rate in a $1+1$d test case is roughly the optimal one h^{p+1} for $p = 2$ (orange) and $p = 3$ (light blue).

Brief history of stabilized finite element methods

- Unless very fine meshes are used, typical finite element methods can be unstable for convection dominated convection–diffusion equations,

$$\underbrace{\epsilon\Delta u}_{\text{diffusion}} + \underbrace{b \cdot \nabla u}_{\text{convection}} = 0.$$

- To remedy this, [Douglas–Dupont'76] introduced regularization using the jumps of the normal derivatives

$$\sum_{F \in \mathcal{F}_h} \|[\![\partial_\nu u]\!]\|^2_{L^2(F)}. \tag{J}$$

- Galerkin-least squares methods was introduced and analysed by [Hughes–Brooks'79] and [Johnson–Nävert–Pitkäranta'84].
- Space-time FEM for wave equations in [Hughes–Hulbert'88].
- [Burman–Hansbo'04] analyzed (J), leading to the present work.

Linear system for the critical points of the Lagrangian

Take $p = q = 1$ for simplicity, and consider the simplified Lagrangian

$$\begin{aligned}\mathcal{L}(u,\phi) = \frac{1}{2}c(\phi,\phi) - a(u,\phi) + L(\phi) \\ - \frac{1}{2}E(U|_{t=0} - U_0) - \frac{1}{2}\left(s(u,u) - s(\phi,\phi) + E(U|_{t=T})\right)\end{aligned}$$

.

The equation $d\mathcal{L}(u,\phi) = 0$ for the critical points of $\mathcal{L}$ on $V_h^1 \times V_h^1$ reads

$$\begin{aligned} a(v,\phi) + s(u,v) + \sum_{\tau=0,T} e(U|_{t=\tau}, V|_{t=\tau}) = e(U_0, V|_{t=0}) \quad & \forall v \in V_h^1, \\ a(u,\psi) - s(\phi,\psi) - c(\phi,\psi) = L(\psi) \quad & \forall \psi \in V_h^1, \end{aligned}$$

where $V = (v, \partial_t v)$ and e is the bilinear form associated to the quadratic form E, that is, $e(U,U) = E(U)$.

Linear system for the critical points of the Lagrangian

Take $p = q = 1$ for simplicity, and consider the simplified Lagrangian

$$\begin{aligned}\mathcal{L}(u,\phi) = \frac{1}{2}c(\phi,\phi) - a(u,\phi) + L(\phi) \\ - \frac{1}{2}E(U|_{t=0} - U_0) - \frac{1}{2}\left(s(u,u) - s(\phi,\phi) + E(U|_{t=T})\right)\end{aligned}$$

.

The equation $d\mathcal{L}(u,\phi) = 0$ can also be written equivalently as

$$A[(u,\phi),(v,\psi)] = L(\psi) + e(U_0, V|_{t=0}) \quad \text{for all } (v,\psi) \in V_h^1 \times V_h^1,$$

where the bilinear form A is given by

$$\begin{aligned}A[(u,\phi),(v,\psi)] = -s(\phi,\psi) - c(\phi,\psi) + s(u,v) + \sum_{\tau=0,T} e(U|_{t=\tau}, V|_{t=\tau}) \\ + a(v,\phi) + a(u,\psi).\end{aligned}$$

Existence of a unique critical point

$$A[(u,\phi),(v,\psi)] = L(\psi) + e(U_0, V|_{t=0}) \quad \text{for all } (v,\psi) \in V_h^1 \times V_h^1, \quad (1)$$

defines a square system of linear equations. Hence existence is equivalent to uniqueness. Suppose that $(u,\phi) \in V_h^1 \times V_h^1$ solves (1) with $U_0 = 0$. (In this case also $L = 0$). It remains to show that $(u,\phi) = 0$. Recall

$$A[(u,\phi),(v,\psi)] = s(u,v) - s(\phi,\psi) - c(\phi,\psi) + \sum_{\tau=0,T} e(U|_{t=\tau}, V|_{t=\tau}) \\ + a(v,\phi) + a(u,\psi).$$

Now $A[(u,\phi),(u,-\phi)] = 0$ implies

$$|||(u,\phi)|||^2 := s(u,u) + s(\phi,\phi) + c(\phi,\phi) + \sum_{\tau=0,T} E(U|_{t=\tau}) = 0.$$

From $c(\phi,\phi) = s(\phi,\phi) = 0$ we get $\phi = 0$ for $x \in \omega \cup \partial\Omega$ and $[\![\partial_\nu \phi]\!] = 0$ for all $F \in \mathcal{F}_h$. Hence $\phi = 0$. Also $u = 0$ from $E(U|_{t=0}) = s(u,u) = 0$.

Briefly on convergence

Let (u_h, ϕ_h) be the solution to

$$A[(u_h, \phi_h), (v, \psi)] = e(U_0, V|_{t=0}) + L(\psi) \quad \text{for all } (v, \psi) \in V_h^1 \times V_h^1,$$

and let smooth (u, ϕ) solve the weak formulation of the control problem

$$a(u, \psi) = c(\phi, \psi) + L(\psi), \quad a(v, \phi) = 0 \quad \text{for all smooth enough } (v, \psi).$$

Our error estimate will follow immediately from

$$|||(u - u_h, \phi - \phi_h)||| \lesssim h \left(\|u\|_{H^2(M)} + \|\phi\|_{H^2(M)} \right).$$

Sketch of proof. As the stabilization vanishes when applied to (u, ϕ),

$$A[(u, \phi), (v, \psi)] = e(U_0, V|_{t=0}) + L(\psi) \quad \text{for all } (v, \psi) \in V_h^1 \times V_h^1.$$

That is, the FEM is consistent. This implies the Galerkin orthogonality

$$A[(u - u_h, \phi - \phi_h), (v, \psi)] = 0 \quad \text{for all } (v, \psi) \in V_h^1 \times V_h^1.$$

Sketch of proof continues

Let $\pi_h : H^2((0,T)\times\Omega) \to V_h^1$ be h^2-accurate interpolant preserving vanishing boundary value (e.g. the Scott–Zhang interpolant). We write

$$e = u - u_h, \quad e_h = \pi_h u - u_h, \quad \eta = \phi - \phi_h, \quad \eta_h = \pi_h \phi - \phi_h.$$

The relation between A and $|||\cdot|||$, and the Galerkin orthogonality, imply

$$|||(e,\eta)|||^2 = A[(e,\eta),(e,-\eta)] = A[(e,\eta),(e-e_h,\eta_h-\eta)].$$

We write $e_\pi = u - \pi_h u = e - e_h$ and $\eta_\pi = \phi - \pi_h\phi = \eta - \eta_h$, and use:

(1) Continuity for a certain[3] norm $\|\cdot\|_*$

$$A[X,Y] \lesssim |||X|||\, \|Y\|_*\,.$$

(2) Smallness of the interpolation error

$$\|(e_\pi, -\eta_\pi)\|_* \lesssim h\left(\|u\|_{H^2(M)} + \|\phi\|_{H^2(M)}\right).$$

[3] Specifically $\|(u,\phi)\|_* = |||(u,\phi)||| + \|u\|_{**} + \|q\|_{**}$ where $\|u\|_{**}^2 = h^{-2}\|u\|_{L^2(M)}^2 + h^{-1}\|u\|_{L^2(\partial M)}^2 + h\|\partial_\nu u\|_{L^2(\partial M)}^2 + h^{-1}\sum_{F\in\mathcal{F}_h}\|u\|_{L^2(F)}^2$.

Conclusion

Theorem [E.B.-Feizmohammadi-Münch-Oksanen]. Suppose that the GCC holds. Let $(u, \phi) \in H^2(M) \times H^2(M)$ be the unique solution to

$$\begin{cases} \Box u = \chi\phi, \\ u|_{x\in\partial\Omega} = 0, \\ u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1, \\ u|_{t=T} = 0,\ \partial_t u|_{t=0} = 0, \end{cases} \qquad \begin{cases} \Box \phi = 0, \\ \phi|_{x\in\partial\Omega} = 0. \end{cases}$$

Then the Lagrangian $\mathcal{L}$ has a unique critical point $(u_h, \phi_h) \in V_h^1 \times V_h^1$ and

$$\|\chi(\phi - \phi_h)\|_{L^2(M)} \lesssim h \left(\|u\|_{H^2(M)} + h \|\phi\|_{H^2(M)} \right).$$

Reference:

- For details on the low regularity case and boundary control see:

  Erik Burman, Ali Feizmohammadi, Arnaud Munch, Lauri Oksanen, *Spacetime finite element methods for control problems subject to the wave equation*, arXiv:2109.07890, 2021.