Uniform observability for the 1D wave equation

Application to the optimization of the control's support

Journées Contrôle & Applications

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State system

Let $\Omega = (0,1)$, $T > 0$ and consider

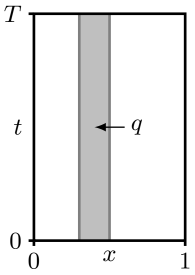
$$\begin{cases} \partial_{tt} y - \partial_{xx} y = u \mathbb{1}_q & \text{in } Q := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ (y, \partial_t y)(\cdot, 0) = (y^0, y^1) & \text{in } \Omega. \end{cases}$$

Let $\boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$ and $\boldsymbol{W} := L^2(\Omega) \times H^{-1}(\Omega)$.

Null controllability

For $q \subset Q$ open, the state system is said to be *null controllable* iff

$$\forall \boldsymbol{y}^0 \in \boldsymbol{V}, \quad \exists u \in L^2(q), \quad (y, \partial_t y)(\cdot, T; \boldsymbol{y}^0, u) = (0,0).$$



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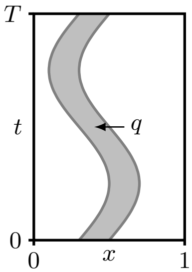

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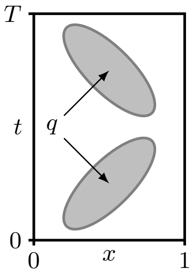

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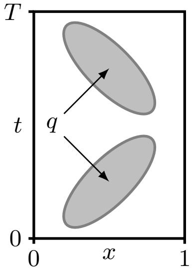
Let $\boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$ and $\boldsymbol{W} := L^2(\Omega) \times H^{-1}(\Omega)$.

Adjoint system

For $\varphi^0 \in \boldsymbol{W}$, consider

$$\begin{cases} \partial_{tt} \varphi - \partial_{xx} \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ (\varphi, \partial_t \varphi)(\cdot, 0) = (\varphi^0, \varphi^1) & \text{in } \Omega. \end{cases}$$



$$(\mathrm{Sta})\begin{cases}\partial_{tt}y-\partial_{xx}y=u\mathbb{1}_q & \text{in } Q\\ y=0 & \text{on } \Sigma\\ (y,\partial_t y)(\cdot,0)=(y^0,y^1) & \text{in } \Omega\end{cases} \qquad (\mathrm{Adj})\begin{cases}\partial_{tt}\varphi-\partial_{xx}\varphi=0 & \text{in } Q\\ \varphi=0 & \text{on } \Sigma\\ (\varphi,\partial_t\varphi)(\cdot,0)=(\varphi^0,\varphi^1) & \text{in } \Omega\end{cases}$$

Observability

For $q \subset Q$ open, the adjoint system is said to be *observable* if there exists $C_{\mathrm{obs}}(q) > 0$ such that

$$\|\boldsymbol{\varphi}^0\|_{\boldsymbol{W}}^2 \leq C_{\mathrm{obs}}(q)\,\|\varphi\|_{L^2(q)}^2, \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{W} := L^2(\Omega)\times H^{-1}(\Omega). \tag{$\mathrm{Obs}_{\boldsymbol{W}}$}$$

Controllability $\iff$ Observability

The state system is *null controllable* iff the adjoint system is *observable*.

An equivalent inequality

Inequality $(\mathrm{Obs}_{\boldsymbol{W}})$ is equivalent to the following inequality,

$$\|\boldsymbol{\varphi}^0\|_{\boldsymbol{V}}^2 \leq C_{\mathrm{obs}}(q)\,\|\partial_t\varphi\|_{L^2(q)}^2, \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V} := H_0^1(\Omega)\times L^2(\Omega). \tag{$\mathrm{Obs}_{\boldsymbol{V}}$}$$

Characteristic lines

For $x_0 \in \overline{\Omega}$, the characteristic lines starting from x_0 are

$$C_{x_0}^{\pm} := \left\{(x,t) \in \mathbb{R}^2; \quad x = |\mathfrak{m}(x_0 \pm t)|\right\},$$

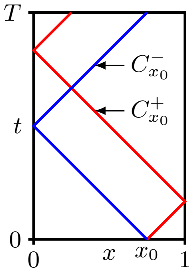
where $\mathfrak{m}(x) := x - 2k$ for $x \in (2k-1, 2k+1]$, $k \in \mathbb{Z}$.

Geometric Control Condition (GCC)

An open set $q \subset Q$ satisfies (GCC) if for all $x_0 \in \overline{\Omega}$, the characteristic lines $C_{x_0}^{\pm}$ meet q.

Observability $\iff$ (GCC)

- n-D cylindrical case : [Bardos *et al.* (92)]
- 1D non-cylindrical case : [Castro *et al.* (14)]
- n-D non-cylindrical case : [Le Rousseau *et al.* (17)]



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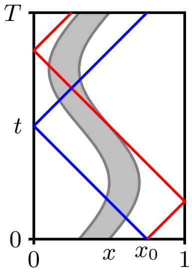


$$(\mathrm{Adj})\begin{cases}\partial_{tt}\varphi-\partial_{xx}\varphi=0 & \text{in } Q\\ \varphi=0 & \text{on } \Sigma\\ (\varphi,\partial_t\varphi)(\cdot,0)=(\varphi^0,\varphi^1) & \text{in } \Omega\end{cases}$$

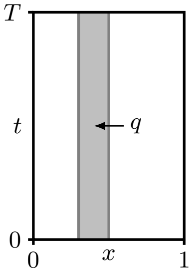
Uniform (w.r.t. q) observability inequality

Let $\mathcal{Q}_{\mathrm{ad}} \subset \{q \subset Q;\ \text{(GCC) holds for } q\}$. We want to find an observability constant that is uniform on $\mathcal{Q}_{\mathrm{ad}}$, i.e. find $\overline{C}_{\mathrm{obs}} > 0$ such that for all $q \in \mathcal{Q}_{\mathrm{ad}}$,

$$\|\boldsymbol{\varphi}^0\|_{\boldsymbol{V}}^2 \leq \overline{C}_{\mathrm{obs}}\, \|\partial_t\varphi\|_{L^2(q)}^2, \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V}.$$

In the sequel,

- first, we recall a result for the cylindrical case ;
- then, we present a new result for the non-cylindrical case.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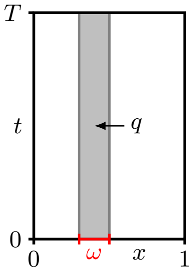


Admissible supports

Let $\delta > 0$ and consider

$$\mathcal{Q}_{\mathrm{ad}}^{\delta} := \left\{ q = \omega \times (0,T); \quad \omega \subset \Omega, \; |\omega| = \delta \right\}.$$

For $T \geq 2$, (GCC) holds for all $q \in \mathcal{Q}_{\mathrm{ad}}^{\delta}$.

Uniform observability [Periago (09)]

Let $\delta > 0$. For $T \geq 2$, we set $\overline{C}_{\mathrm{obs}} = \Big(\lfloor T/2 \rfloor \delta \big(1 - \mathrm{sinc}(\pi\delta)\big) \Big)^{-1}$, where $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$ if $x \neq 0$ and $\mathrm{sinc}(0) = 1$. Then, for all $q \in \mathcal{Q}_{\mathrm{ad}}^{\delta}$,

$$\|\boldsymbol{\varphi}^0\|_{\boldsymbol{V}}^2 \leq \overline{C}_{\mathrm{obs}} \, \|\partial_t \varphi\|_{L^2(q)}^2, \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V}.$$

Let $\delta > 0$, $T \geq 2$ and $q \in \mathcal{Q}_{\text{ad}}^{\delta}$.

For $\boldsymbol{\varphi}^0 \in \boldsymbol{V}$, we expand $\varphi^0(x) = \sum_{p\geq 1} a_p \sin(p\pi x)$ and $\varphi^1(x) = \sum_{p\geq 1} b_p \sin(p\pi x)$.

It follows that $\varphi(x,t) = \sum_{p\geq 1} \left(a_p \cos(p\pi t) + \frac{b_p}{p\pi} \sin(p\pi t) \right) \sin(p\pi x)$. Then,

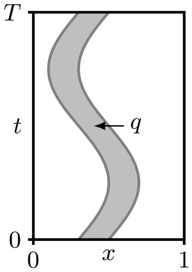
$$
\begin{aligned}
\|\partial_t \varphi\|^2_{L^2(q)} = \int_0^T \int_\omega |\partial_t \varphi|^2 \geq \int_0^{2\lfloor \frac{T}{2} \rfloor} \int_\omega |\partial_t \varphi|^2 = \lfloor T/2 \rfloor \int_0^2 \int_\omega |\partial_t \varphi|^2 \\
= \lfloor T/2 \rfloor \sum_{p\geq 1} \left((p\pi)^2 |a_p|^2 + |b_p|^2 \right) \int_\omega \sin^2(p\pi x)\, \mathrm{d}x.
\end{aligned}
$$

Lemma

For any $\omega \subset \Omega$ with $|\omega| = \delta$, $\quad \inf_{p\geq 1} \int_\omega \sin^2(p\pi x)\, \mathrm{d}x \geq \frac{\delta}{2}\big(1 - \mathrm{sinc}(\pi\delta)\big)$.

Using that $\|\boldsymbol{\varphi}^0\|^2_{\boldsymbol{V}} = \frac{1}{2} \sum_{p\geq 1} \left((p\pi)^2 |a_p|^2 + |b_p|^2 \right)$, we find

$$
\|\partial_t \varphi\|^2_{L^2(q)} \geq \lfloor T/2 \rfloor \delta \big(1 - \mathrm{sinc}(\pi\delta)\big) \|\boldsymbol{\varphi}^0\|^2_{\boldsymbol{V}}.
$$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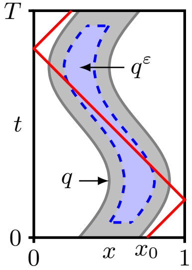
Admissible supports

Let $\varepsilon > 0$ and consider

$$\mathcal{Q}_{\text{ad}}^{\varepsilon} := \left\{ q \subset Q; \quad \text{(GCC) holds for } q^{\varepsilon} \right\},$$

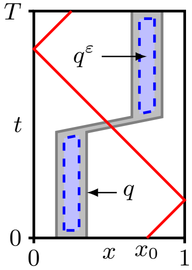
where q^{ε} is the ε-interior of q.

For all $q \in \mathcal{Q}_{\text{ad}}^{\varepsilon}$ and for any characteristic line $C_{x_0}^{\pm}$, the intersection $q \cap C_{x_0}^{\pm}$ has at least length ε.



Pathological case

We want to avoid the case where $q \cap C_{x_0}^{\pm}$ has arbitrarily small length, causing $C_{\text{obs}}(q)$ to be arbitrarily large.



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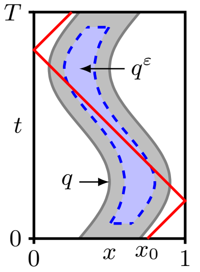

Uniform observability [B. *et al.* (21)]

Let $\varepsilon > 0$. There exists $\overline{C}_{\mathrm{obs}} > 0$ such that for all $q \in \mathcal{Q}_{\mathrm{ad}}^{\varepsilon}$,

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Reformulation

For $q \in \mathcal{Q}_{\mathrm{ad}}^{\varepsilon}$, we define the positive symmetric bilinear form

$$\mathfrak{F}(\boldsymbol{\varphi}^0, \overline{\boldsymbol{\varphi}}^0) := \iint_q \partial_t \varphi \, \partial_t \overline{\varphi}, \quad \forall \boldsymbol{\varphi}^0, \overline{\boldsymbol{\varphi}}^0 \in \boldsymbol{V}.$$

Then, the uniform observability property is equivalent to the following problem.

Find $\overline{\mathfrak{C}} > 0$ such that for all $q \in \mathcal{Q}_{\mathrm{ad}}^{\varepsilon}$,

$$\mathfrak{F}(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) \geq \overline{\mathfrak{C}} \, \|\boldsymbol{\varphi}^0\|_{\boldsymbol{V}}^2, \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V}.$$

Problem

$$\mathfrak{F}(\boldsymbol{\varphi}^0, \overline{\boldsymbol{\varphi}}^0) := \iint_q \partial_t \varphi \, \partial_t \overline{\varphi}, \quad \forall \boldsymbol{\varphi}^0, \overline{\boldsymbol{\varphi}}^0 \in \boldsymbol{V}$$

Find $\overline{\mathfrak{C}} > 0$ such that for all $q \in \mathcal{Q}_{\text{ad}}^{\varepsilon}$, $\quad \mathfrak{F}(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) \geq \overline{\mathfrak{C}} \|\boldsymbol{\varphi}^0\|_{\boldsymbol{V}}^2, \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V}$.

Sketch of the proof

- Let $q \in \mathcal{Q}_{\text{ad}}^{\varepsilon}$ and $\mathfrak{F}$ the associated form.
- Using a discretization of Ω, we define a new form $\mathfrak{F}_N$ such that

$$\mathfrak{F}(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) \geq \mathfrak{F}_N(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0), \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V}.$$

- We build an orthonormal basis of $\boldsymbol{V}$ that is orthogonal for $\mathfrak{F}_N$ after a certain rank.
- It reduces the problem to find $\overline{\mathfrak{C}} > 0$ independent of q such that

$$\mathfrak{F}_N(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) \geq \overline{\mathfrak{C}} \|\boldsymbol{\varphi}^0\|_{\boldsymbol{V}}^2, \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V}_N,$$

where $\boldsymbol{V}_N$ is a finite-dimensional subspace of $\boldsymbol{V}$.

- We conclude using that (GCC) holds for q^{ε}.

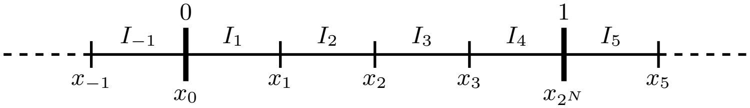
Goal

We want to define a new form $\mathfrak{F}_N$ such that $\mathfrak{F} \geq \mathfrak{F}_N$.

Discretization of Ω

Let $N \in \mathbb{N}$ such that $2^N > 1/\varepsilon$. We set $h = 1/2^N$ and $S_N = (x_k)_{0 \leq k \leq 2^N}$ the regular subdivision of $\overline{\Omega}$ in 2^N intervals, i.e. $x_k = kh$. We also set

$$I_k := \begin{cases} [x_{k-1}, x_k] & \text{if } k > 0, \\ [x_k, x_{k+1}] & \text{if } k < 0, \end{cases} \qquad \forall k \in \mathbb{Z}^*.$$

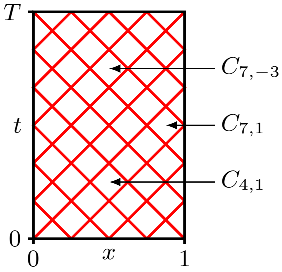


Decomposition of Q

For $i, j \in \mathbb{Z}^*$, we define the elementary square $C_{i,j}$ of indices (i,j) by

$$C_{i,j} := \Big\{(x,t) \in \mathbb{R}^2; \quad x+t \in I_i,\ x-t \in I_j\Big\}.$$

We also set $\mathcal{C}_N = \big\{C_{i,j};\ i,j \in \mathbb{Z}^*\big\}$ the set of elementary squares adapted to S_N.



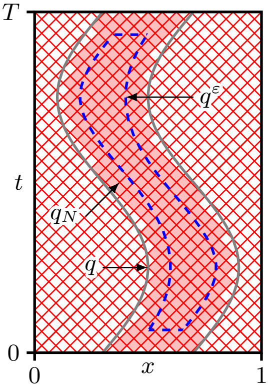
Decomposition of q

We set $\mathcal{C}_N(q) := \{C_{i,j} \in \mathcal{C}_N ;\ \mathring{C}_{i,j} \subset q\}$ and

$$q_N := \overbrace{\bigcup_{C_{i,j} \in \mathcal{C}_N(q)} C_{i,j}}^{\circ} .$$

Lemma (require $2^N > 1/\varepsilon$)

We have $q^\varepsilon \subset q_N \subset q$, so (GCC) holds for q_N.



New form

$$\mathfrak{F}(\boldsymbol{\varphi}^0, \overline{\boldsymbol{\varphi}}^0) := \iint_q \partial_t \varphi \, \partial_t \overline{\varphi}, \quad \forall \boldsymbol{\varphi}^0, \overline{\boldsymbol{\varphi}}^0 \in \boldsymbol{V}$$

We define the positive symmetric bilinear form

$$\mathfrak{F}_N(\boldsymbol{\varphi}^0, \overline{\boldsymbol{\varphi}}^0) := \iint_{q_N} \partial_t \varphi \, \partial_t \overline{\varphi}, \quad \forall \boldsymbol{\varphi}^0, \overline{\boldsymbol{\varphi}}^0 \in \boldsymbol{V}.$$

Since $q_N \subset q$, we have

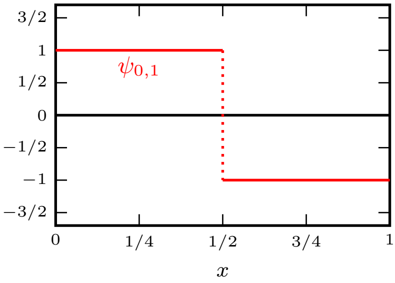
$$\mathfrak{F}(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) \geq \mathfrak{F}_N(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0), \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V}.$$

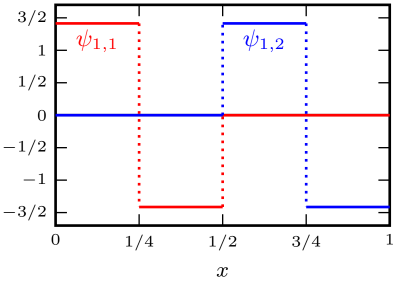
Goal

We want to build an ONB of $\boldsymbol{V}$ that is orthogonal for $\mathfrak{F}_N$ after a certain rank.

Haar wavelet basis [Haar (1910)]

Let the mother function $\psi = \mathbb{1}_{[0,1/2]} - \mathbb{1}_{[1/2,1]}$ and the scaling function $\psi_0 = \mathbb{1}_{[0,1]}$.
We set $\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k + 1)$ for $n \in \mathbb{N}$ and $1 \leq k \leq 2^n$.
Then, $\mathcal{B} := \left\{\psi_0,\ \psi_{n,k},\ n \in \mathbb{N},\ 1 \leq k \leq 2^n\right\}$ is an ONB of $L^2(\Omega)$.



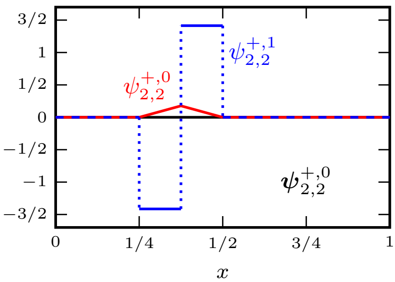


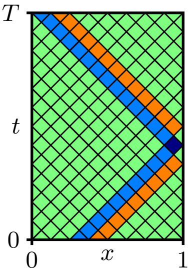
An orthonormal basis of V

We set $\psi_0^0(x) := 0$, $\psi_0^1(x) := \psi_0(x)$ and

$$\psi_{n,k}^{\pm,0}(x) := \frac{1}{\sqrt{2}} \int_0^x \psi_{n,k}, \quad \psi_{n,k}^{\pm,1}(x) := \mp \frac{1}{\sqrt{2}} \psi_{n,k}(x), \quad n \in \mathbb{N},\ 1 \leq k \leq 2^n.$$

Then, $\boldsymbol{\mathcal{B}} := \{\boldsymbol{\psi}_0^0,\ \boldsymbol{\psi}_{n,k}^{\pm,0},\ n \in \mathbb{N},\ 1 \leq k \leq 2^n\}$ is an ONB of $\boldsymbol{V}$.





$\partial_t \varphi$ for φ sol. of (Adj) asso. with $\boldsymbol{\psi}_{2,2}^{+,0}$.

Lemma (based on d'Alembert formula)

For $n \geq N$, $1 \leq k \leq 2^n$ and $s \in \{+,-\}$, we have

$$\mathfrak{F}_N(\boldsymbol{\psi}_{n,k}^{s,0}, \boldsymbol{\psi}^0) = 0, \quad \forall \boldsymbol{\psi}^0 \in \boldsymbol{\mathcal{B}},\ \boldsymbol{\psi}^0 \neq \boldsymbol{\psi}_{n,k}^{s,0}.$$

Finite-dimensional subspace $\boldsymbol{V}_N$ of $\boldsymbol{V}$

We set $\boldsymbol{\mathcal{B}}_N := \left\{\boldsymbol{\psi}_0^0,\ \boldsymbol{\psi}_{n,k}^{\pm,0},\ n < N,\ 1 \leq k \leq 2^n\right\}$ and $\boldsymbol{V}_N := \mathrm{Vect}(\boldsymbol{\mathcal{B}}_N)$.

New problem in finite dimension

For $\boldsymbol{\varphi}^0 \in \boldsymbol{V} = \boldsymbol{V}_N \oplus \widetilde{\boldsymbol{V}}$, we decompose $\boldsymbol{\varphi}^0 = \boldsymbol{\varphi}_N^0 + \widetilde{\boldsymbol{\varphi}}^0$ and we have

$$\mathfrak{F}_N(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) = \mathfrak{F}_N(\boldsymbol{\varphi}_N^0, \boldsymbol{\varphi}_N^0) + \mathfrak{F}_N(\widetilde{\boldsymbol{\varphi}}^0, \widetilde{\boldsymbol{\varphi}}^0).$$

We easily find $\overline{\mathfrak{C}} > 0$ independent of q (and $\widetilde{\boldsymbol{\varphi}}^0$) such that $\mathfrak{F}_N(\widetilde{\boldsymbol{\varphi}}^0, \widetilde{\boldsymbol{\varphi}}^0) \geq \overline{\mathfrak{C}} \|\widetilde{\boldsymbol{\varphi}}^0\|_{\boldsymbol{V}}^2$.
So we now need to find $\overline{\mathfrak{C}} > 0$ independent of q such that

$$\mathfrak{F}_N(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) \geq \overline{\mathfrak{C}} \|\boldsymbol{\varphi}^0\|_{\boldsymbol{V}}^2, \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V}_N.$$

Lemma (based on d'Alembert formula)

For $\boldsymbol{\varphi}^0 \in \boldsymbol{V}_N$, we expand $(\varphi^0)'(x) = \sum_{p=1}^{2^N} \alpha_p \mathbb{1}_{I_p}(x)$ and $\varphi^1(x) = \sum_{p=1}^{2^N} \beta_p \mathbb{1}_{I_p}(x)$.

For $1 \leq p \leq 2^N$, we set $\gamma_p^{\pm} = \alpha_p \pm \beta_p$. We then show that

$$\mathfrak{F}_N(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) = \frac{h^2}{8} \sum_{C_{i,j} \in \mathcal{C}_N(q)} \left(\gamma_{\mathfrak{p}_i}^{\mathfrak{s}_i} - \gamma_{\mathfrak{p}_j}^{\mathfrak{s}_j} \right)^2.$$

Conclusion

Since $q \in \mathcal{Q}_{\mathrm{ad}}^{\varepsilon}$ and $q^{\varepsilon} \subset q_N$, (GCC) holds for q_N.

If $\boldsymbol{\varphi}^0 \in \boldsymbol{V}_N$ is such that $\mathfrak{F}_N(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) = 0$, using that $q_N \cap C_{x_k}^{\pm} \neq \varnothing$, we show that $\gamma_p^{\pm} = 0$ for all p and we deduce $\boldsymbol{\varphi}^0 = 0$. Hence, $\mathfrak{F}_N$ is positive definite and there exists $\overline{\mathfrak{C}}_{q_N} > 0$ such that

$$\mathfrak{F}_N(\boldsymbol{\varphi}^0, \boldsymbol{\varphi}^0) \geq \overline{\mathfrak{C}}_{q_N} \|\boldsymbol{\varphi}^0\|_{\boldsymbol{V}}^2, \quad \forall \boldsymbol{\varphi}^0 \in \boldsymbol{V}_N.$$

Since the set of possible q_N is finite, we conclude by setting $\overline{\mathfrak{C}} := \min_{q_N} \overline{\mathfrak{C}}_{q_N} > 0$.

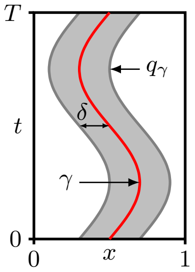
Admissible supports

Let $\delta > 0$ and $M > 0$. We consider supports of the form

$$q_\gamma := \Big\{(x,t) \in Q; \quad |x - \gamma(t)| < \delta\Big\}, \quad \forall \gamma \in \mathcal{G}_{\mathrm{ad}},$$

where $\mathcal{G}_{\mathrm{ad}} := \{\gamma \in W^{1,\infty}(0,T);\ \|\gamma'\|_{L^\infty} \leq M\}$.

For all $\gamma \in \mathcal{G}_{\mathrm{ad}}$, we have $q_\gamma \in \mathcal{Q}_{\mathrm{ad}}^{\varepsilon}$ for $\varepsilon = \frac{\delta}{4\sqrt{M^2+1}}$.



Increased control regularity [Ervedoza *et al.* (10)]

To gain a more regular control, in the state system, we substitute

$$\mathbb{1}_{q_\gamma}(x,t) = \mathbb{1}_{(-\delta,\delta)}(x - \gamma(t)) \qquad \text{by} \qquad \chi_\gamma(x,t) = \chi(x - \gamma(t)),$$

where $\chi \in C^2(\mathbb{R})$ and $\mathrm{Supp}(\chi) = [-\delta, \delta]$.

$$
\text{(Sta)} \begin{cases} \partial_{tt} y - \partial_{xx} y = u \chi_\gamma & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ (y, \partial_t y)(\cdot, 0) = (y^0, y^1) & \text{in } \Omega \end{cases}
$$

Optimization problem

For $\boldsymbol{y}^0 \in \boldsymbol{V}$ fixed, consider

$$
\min_{\gamma \in \mathcal{G}_{\mathrm{ad}}} J(\gamma), \quad \text{with} \quad J(\gamma) := \|u\|^2_{L^2_\chi(q_\gamma)} = \iint_{q_\gamma} \varphi^2 \chi_\gamma,
$$

and where $u = -\varphi_{\,|q_\gamma}$ is the control of minimal L^2-norm associated with $\boldsymbol{y}^0$ and q_γ.

Continuity of the support w.r.t. γ

Let $(\gamma_n)_{n\geq 0} \subset \mathcal{G}_{\rm ad}$ and $\gamma \in \mathcal{G}_{\rm ad}$. If $\gamma_n \to \gamma$ in $L^\infty(0,T)$, then $\chi_{\gamma_n} \to \chi_\gamma$ in $L^\infty(Q)$.

Continuity of J (use the uniform observability on $\mathcal{Q}^\varepsilon_{\rm ad}$)

The functional J is continuous on $\mathcal{G}_{\rm ad}$ for the $L^\infty(0,T)$ norm.

Existence of a minimum point for J

The functional J admit a minimum point on $\mathcal{G}_{\rm ad}$.

Note that this minimum point is a priori not unique.

Directional derivative of J

The directional derivative of J at γ in the direction $\overline{\gamma}$ can be written

$$\mathrm{d}J(\gamma;\overline{\gamma}) = \int_0^T \overline{\gamma} \int_\Omega \varphi^2 \chi'_\gamma, \quad \text{with } \chi'_\gamma(x,t) = \chi'(x-\gamma(t)).$$

"Numerical" optimization problem

Let $\eta > 0$. For $\boldsymbol{y}^0 \in \boldsymbol{V}$ fixed, consider

$$\min_{\gamma \in W^{1,\infty}} J_\eta(\gamma), \quad \text{with } J_\eta(\gamma) := J(\gamma) + \frac{\eta}{2}\|\gamma'\|^2_{L^2(0,T)}.$$

The role of η is similar to the one of M in $\mathcal{G}_{\mathrm{ad}}$.

The problem is solved with a fixed-step gradient-descent algorithm.

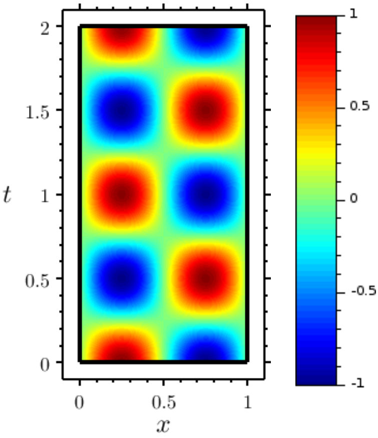
$$\text{For } \rho > 0 \text{ fixed,} \quad \begin{cases} \gamma^0 \in W^{1,\infty}(0,T) \text{ given,} \\ \gamma^{n+1} = \gamma^n - \rho\, j^\eta_{\gamma^n}, \quad \forall n \geq 0. \end{cases}$$

Descent direction

We set $j_\gamma(t) = \int_\Omega \varphi^2(x,t)\chi'_\gamma(x,t)\,\mathrm{d}x$. A descent direction for J_η is found by solving

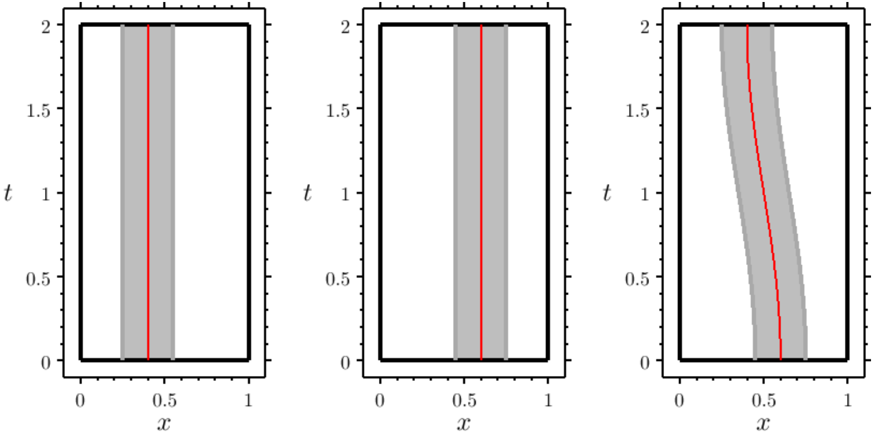
$$\langle j^\eta_\gamma, \widetilde{\gamma}\rangle_{L^2} + \eta\langle {j^\eta_\gamma}', \widetilde{\gamma}'\rangle_{L^2} = \langle j_\gamma, \widetilde{\gamma}\rangle_{L^2} + \eta\langle \gamma', \widetilde{\gamma}'\rangle_{L^2}, \quad \forall \widetilde{\gamma} \in H^1(0,T).$$

$T = 2,\ \delta = 0.15, \qquad y^0(x) = \sin(2\pi x),\ y^1(x) = 0$



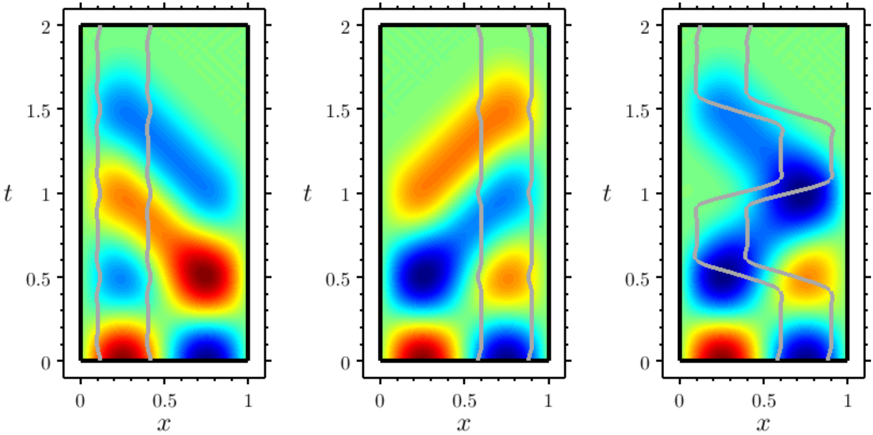
Uncontrolled solution y of (Sta).

$T = 2,\ \delta = 0.15, \qquad y^0(x) = \sin(2\pi x),\ y^1(x) = 0$



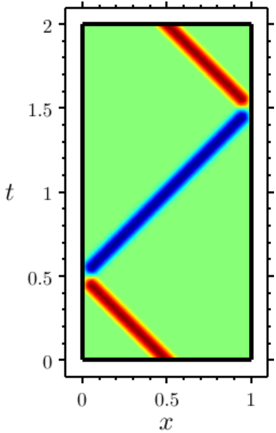
Supports associated with 3 initial curves γ_i^0.

$T = 2$, $\delta = 0.15$, $\quad y^0(x) = \sin(2\pi x)$, $y^1(x) = 0$

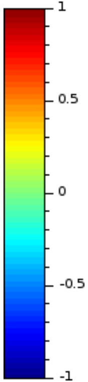


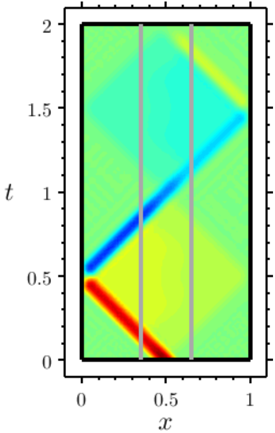
Supports associated with the optimal curves $\gamma_i^\star$.

$T = 2$, $\delta = 0.15$, $\quad y^0(x) = (10x-4)^2(10x-6)^2 \mathbb{1}_{[0.4,0.6]}(x)$, $y^1(x) = (y^0)'(x)$



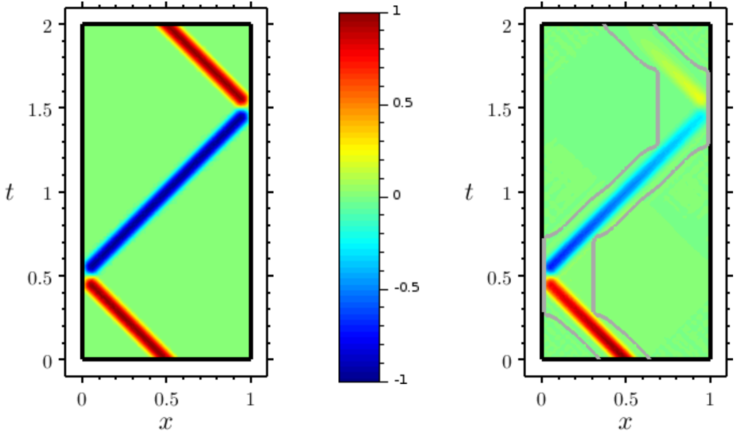
Uncontrolled solution y of (Sta).





Convergence towards the optimal support.

$T = 2,\ \delta = 0.15, \qquad y^0(x) = (10x-4)^2(10x-6)^2\mathbb{1}_{[0.4,0.6]}(x),\ y^1(x) = (y^0)'(x)$



Uncontrolled solution y of (Sta). Support associated with the optimal curve $\gamma^\star$.

Merci pour votre attention

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Optimization of non-cylindrical domains for the exact null controllability of the 1D wave equation

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