Constructive exact controls for semilinear PDEs

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GIVEN some semilinear uniformly exactly controllable PDEs

$$\begin{cases} PDE(y,f) = 0, \\ y = y(x,t) - \text{state}, \quad f = f(x,t) - \text{control function}, \\ + \text{initial conditions and boundary conditions} \end{cases} \tag{1}$$

FIND a sequence $(y_k, f_k)_{k\in\mathbb{N}}$ such that $(y_k, f_k) \to (y, f)$ as $k \to \infty$, with (y, f) a controlled pair for (1) ?

• Largely open issue because in many situations, proofs are based on non constructive fixed point arguments.

• We focus on the case of the wave and heat eq. with distributed controls.

• We design constructive strongly convergent sequence $(y_k, f_k)_{k\in\mathbb{N}}$ using least-squares approaches and extend ideas used for the NS-direct problem in [1], [2]

[1] J. Lemoine, AM, *Resolution of the Implicit Euler scheme for the Navier-Stokes equation through a least-squares method*, Numerische Mathematik 21'

[2] J. Lemoine, AM, *A fully space-time least-squares method for the unsteady Navier-Stokes system*, arxiv.org/abs/1909.05034

- Let $\Omega := (0,1)$, $\omega := (\ell_1, \ell_2)$ with $0 \leq \ell_1 < \ell_2 \leq 1$, $T > 0$. We set $Q_T := \Omega \times (0,T)$, $q_T := \omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$.

$$\boxed{\begin{cases} \partial_{tt} y - \partial_{xx} y + g(y) = f 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases}} \tag{2}$$

- $(u_0, u_1) \in \boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$, $f \in L^2(q_T)$. $g \in C^1(\mathbb{R};\mathbb{R})$
- $|g(r)| \leq C(1+|r|)\ln^2(2+|r|)\ \forall r \in \mathbb{R}$
- $y \in \mathcal{C}^0([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ is unique.

- (2) is exactly controllable in time T IFF for any $(u_0,u_1), (z_0,z_1) \in \boldsymbol{V}$, $\exists$ a control function $f \in L^2(q_T)$ such that $(y(\cdot,T), \partial_t y(\cdot,T)) = (z_0, z_1)$.

Theorem (Zuazua'93)

Assume that $T > 2\max(\ell_1, 1-\ell_2)$. There exists $\beta > 0$ (only depending on Ω and T) such that, if

$$(\mathbf{H}_1) \quad \limsup_{|r|\to\infty} \frac{|g(r)|}{|r|\ln^2|r|} < \beta$$

then (2) is exactly controllable in time T.[3]

[3] E. Zuazua, Exact controllability for semilinear wave equations in one space dimension, Ann. Inst. H. Poincaré Anal. Non Linéaire

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- $|g(r)| \le C(1 + |r|) \ln^2(2 + |r|)\ \forall r \in \mathbb{R}$
- $y \in \mathcal{C}^0([0,T]; H^1_0(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ is unique.

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The proof given in Zuazua'93 is based on a Leray Schauder fixed point argument [that reduces the exact controllability problem to obtaining suitable *a priori* estimates for a linearized wave equation with a potential.]

It is shown that operator $K : L^\infty(Q_T) \rightarrow L^\infty(Q_T)$, where $y := K(\xi)$ is a controlled solution with the control function f_ξ of the linear boundary value problem

$$\begin{cases} \partial_{tt} y - \partial_{xx} y + y\,\widehat{g}(\xi) = -g(0) + f 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \qquad \widehat{g}(r) := \begin{cases} \dfrac{g(r)-g(0)}{r} & \text{if } r \neq 0 \\ g'(0) & \text{if } r = 0 \end{cases} \tag{3}$$

satisfying $(y(\cdot,T), \partial_t y(\cdot,T)) = (z_0, z_1)$ has a fixed point.

The control f is the one of minimal $L^2(q_T)$ norm.

It is shown that if β is small enough, then there exists $M = M(\|(u_0,u_1)\|_{\boldsymbol{V}}, \|(z_0,z_1)\|_{\boldsymbol{V}}) > 0$ such that K maps the ball $B_\infty(0,M)$ into itself.

Lemma (A priori estimates for the linearized eq.)

$A \in L^\infty(Q_T)$, $B \in L^2(Q_T)$, $(z_0, z_1) \in \boldsymbol{V}$. $T > 2\max(\ell_1, 1-\ell_2)$. There exists a unique control of minimal $L^2(q_T)$ norm that the solution of

$$\begin{cases} \partial_{tt} z - \partial_{xx} z + A z = u 1_\omega + B & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ (z(\cdot,0), \partial_t z(\cdot,0)) = (z_0, z_1) & \text{in } \Omega, \end{cases} \tag{4}$$

satisfies $(z(\cdot,T), \partial_t z(\cdot,T)) = (0,0)$ in Ω. Moreover, for $C = C(\Omega, T)$,

$$\|u\|_{2,q_T} + \|(z, \partial_t z)\|_{L^\infty(0,T;\boldsymbol{V})} \le C\Big(\|B\|_2\, e^{(1+C)\sqrt{\|A\|_\infty}} + \|z_0, z_1\|_{\boldsymbol{V}}\Big) e^{C\sqrt{\|A\|_\infty}} \tag{5}$$

Asymptotic condition $(\mathbf{H_1})$ implies $\|\widehat{g}(\xi)\|_\infty \le d + \beta \ln^2(1 + \|\xi\|_\infty)$. The lemma with $B = -g(0)$, $A = \hat{g}(\xi)$ gives that $K(\xi)$ satisfies

$$\boxed{\|K(\xi)\|_\infty \le C\Big(\|u_0, u_1\|_{\boldsymbol{V}} + \|g(0)\|_2\Big) e^{(1+C)\sqrt{d}} (1 + \|\xi\|_\infty)^{(1+C)\sqrt{\beta}}, \quad \forall \xi \in L^\infty(Q_T).}$$

Conclusion: If $(1 + C)\sqrt{\beta} < 1$, then $\exists M > 0$ s.t. $\|\xi\|_\infty \le M \Longrightarrow \|K(\xi)\|_\infty \le M$.

A first idea is to consider the Picard iterations $(y_k)_{k\in\mathbb{N}}$ associated with the operator K:

$$\boxed{\begin{cases} y_0 \in L^\infty(Q_T) \quad \text{given} \\ y_{k+1} = K(y_k),\ k \geq 0 \end{cases}} \tag{6}$$

Such a strategy usually fails since the operator K is in general not contracting, even if g is globally Lipschitz.

Lemma (Contraction property under smallness assumption on g)

Let $M = M(\|u_0, u_1\|_{\boldsymbol{V}}, \beta)$ be such that K maps $B_\infty(0, M)$ into itself and assume that $\hat{g}' \in L^\infty(0, M)$. For any $\xi^i \in B_\infty(0, M)$, $i = 1, 2$, there exists $c(M) > 0$ such that

$$\|K(\xi^2) - K(\xi^1)\|_\infty \leq c(M)\|\hat{g}'\|_{L^\infty(0,M)}\|\xi^2 - \xi^1\|_\infty.$$

We consider the Hilbert space

$$\mathcal{H} := \Big\{(y,f) \in L^2(Q_T) \times L^2(q_T) \mid \partial_{tt}y - \partial_{xx}y \in L^2(Q_T),\ y = 0 \text{ on } \Sigma_T,\ (y(\cdot,0), \partial_t y(\cdot,0)) \in \boldsymbol{V}\Big\}$$

and the subspaces of $\mathcal{H}$ defined by

$$\mathcal{A} := \Big\{(y,f) \in \mathcal{H} \mid (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0,u_1),\ (y(\cdot,T), \partial_t y(\cdot,T)) = (z_0,z_1) \text{ in } \Omega\Big\},$$

$$\mathcal{A}_0 := \Big\{(y,f) \in \mathcal{H} \mid (y(\cdot,0), \partial_t y(\cdot,0)) = (0,0),\ (y(\cdot,T), \partial_t y(\cdot,T)) = (0,0) \text{ in } \Omega\Big\},$$

Note that $\mathcal{A} = (\overline{y}, \overline{f}) + \mathcal{A}_0$ for any $(\overline{y}, \overline{f}) \in \mathcal{A}$.
We define the least-squares functional $E : \mathcal{A} \to \mathbb{R}$ by

$$E(y,f) := \frac{1}{2}\left\|\partial_{tt}y - \partial_{xx}y + g(y) - f1_\omega\right\|^2_{L^2(Q_T)}$$

and consider the nonconvex minimization problem

$$\inf_{(y,f)\in\mathcal{A}} E(y,f) \tag{7}$$

Proposition

$\forall (y,f) \in \mathcal{A}$,

$$\frac{1}{\sqrt{2}\max\left(1, \|g'(y)\|_\infty\right)} \|E'(y,f)\|_{\mathcal{A}_0'} \leq \sqrt{E(y,f)} \leq \frac{1}{\sqrt{2}} C e^{C\sqrt{\|g'(y)\|_\infty}} \|E'(y,f)\|_{\mathcal{A}_0'}. \tag{8}$$

Consequence:
Any *critical* point $(y,f) \in \mathcal{A}$ of E (i.e., $E'(y,f) = 0$) is a zero of E, and thus is a pair solution of the controllability problem. Moreover:

given any sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|E'(y_k,f_k)\|_{\mathcal{A}_0'} \underset{k\to+\infty}{\longrightarrow} 0$ and such that $\|g'(y_k)\|_\infty$ is uniformly bounded, we have $E(y_k, f_k) \underset{k\to+\infty}{\longrightarrow} 0$.

Thanks to this instrumental property, a minimizing sequence for E cannot be stuck in a local minimum, even though E fails to be convex (it has multiple zeros).

For any $(y, f) \in \mathcal{A}$, let $(Y^1, F^1) \in \mathcal{A}_0$ be the solution of

$$\begin{cases} \partial_{tt} Y^1 - \partial_{xx} Y^1 + g'(y) Y^1 = F^1 1_\omega + (\partial_{tt} y - \partial_{xx} y + g(y) - f 1_\omega) & \text{in } Q_T, \\ Y^1 = 0 & \text{on } \Sigma_T, \\ (Y^1(\cdot, 0), \partial_t Y^1(\cdot, 0)) = (0, 0) & \text{in } \Omega, \end{cases} \tag{9}$$

Proposition

For all $(y, f) \in \mathcal{A}$, for some $C = C(\Omega, T)$

- $\|(Y^1, \partial_t Y^1)\|_{L^\infty(0,T;\boldsymbol{V})} + \|F^1\|_{2,q_T} \le C e^{C\sqrt{\|g'(y)\|_\infty}} \sqrt{E(y, f)}$
- $E'(y, f) \cdot (Y^1, F^1) = 2E(y, f)$
- *Assume that some $p \in [0, 1]$, $[g']_p := \sup_{\substack{a,b \in \mathbb{R} \\ a \neq b}} \frac{|g'(a) - g'(b)|}{|a-b|^p} < +\infty$. Then,*

$$\sqrt{E((y, f) - \lambda(Y^1, F^1))} \le \left(|1 - \lambda| + \lambda^{1+p} K(y) \sqrt{E(y, f)}^p\right) \sqrt{E(y, f)} \qquad \forall \lambda \in \mathbb{R}^+ \tag{10}$$

where

$$K(y) := C\,[g']_p \left(C e^{C\sqrt{\|g'(y)\|_\infty}}\right)^{1+p}. \tag{11}$$

Assume that $T > 2\max(\ell_1, 1-\ell_2)$. The vector $-(Y^1, F^1)$, solution of minimal control norm of (9), is a descent direction for E. This leads us to define, for any fixed $m \geq 1$, the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ defined by

$$\begin{cases} (y_0, f_0) \in \mathcal{A} \\ (y_{k+1}, f_{k+1}) = (y_k, f_k) - \lambda_k (Y_k^1, F_k^1) \qquad \forall k \in \mathbb{N} \\ \lambda_k = \underset{\lambda\in[0,m]}{\operatorname{argmin}}\ E\big((y_k, f_k) - \lambda(Y_k^1, F_k^1)\big) \end{cases} \tag{12}$$

where $(Y_k^1, F_k^1) \in \mathcal{A}_0$ is the solution of minimal control norm of

$$\begin{cases} \partial_{tt} Y_k^1 - \partial_{xx} Y_k^1 + g'(y_k) Y_k^1 = F_k^1 1_\omega + (\partial_{tt} y_k - \partial_{xx} y_k + g(y_k) - f_k 1_\omega) & \text{in } Q_T, \\ Y_k^1 = 0 & \text{on } \Sigma_T, \\ (Y_k^1(\cdot, 0), \partial_t Y_k^1(\cdot, 0)) = (0, 0) & \text{in } \Omega. \end{cases} \tag{13}$$

The real number $m \geq 1$ is arbitrarily fixed. It is used in the proof of convergence to bound the sequence of optimal descent steps λ_k.

Strong convergence of the algorithm

Given any $p \in [0,1]$, we set

$$\beta^0(p) := \frac{p^2}{C^2(2p+1)^2} \tag{14}$$

Theorem (Trélat, M 20)

Assume that $T > 2\max(\ell_1, 1-\ell_2)$, that $[g']_p < +\infty$ for some $p \in [0,1]$, and that there exist $\alpha \geq 0$ and $\beta \in [0, \beta^0(p))$ (with the agreement that $\beta = 0$ if $p = 0$), such that

$$\boxed{|g'(r)| \leq \alpha + \beta \ln^2(1+|r|) \qquad \forall r \in \mathbb{R}.} \tag{15}$$

For $p = 0$ (i.e., $g' \in L^\infty(\mathbb{R})$), we assume moreover that $2\|g'\|_\infty C^2 e^{C\sqrt{\|g'\|_\infty}} < 1$. Then, as $k \to \infty$

- *For any $(y_0, f_0) \in \mathcal{A}$, $(y_k, f_k) \to (y, f)$ a controlled pair for the nonlinear wave eq.*
- *$\lambda_k \to 1$.*

Moreover, the convergence of these sequences is at least linear, and is at least of order $1+p$ after a finite number of iterations.

Least-squares approach: assumptions on g

Our assumptions on g :

- For some $p \in [0,1]$

$$(\overline{\mathbf{H}}_\mathbf{p}) \quad [g']_p := \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|g'(a)-g'(b)|}{|a-b|^p} < +\infty$$

and

- For $p \in (0,1]$

$$(\mathbf{H_2}) \quad \exists \alpha \geq 0 \text{ and } \beta \in [0, \beta^\star(p)) \text{ s.t. } |g'(r)| \leq \alpha + \beta \ln^2(1+|r|) \, \forall r \in \mathbb{R}$$

- For $p = 0$,

$$(\mathbf{H_3}) \quad \sqrt{2} C \|g'\|_\infty e^{C\sqrt{\|g'\|_\infty}} < 1$$

are a bit stronger than in Zuazua'93:

$$(\mathbf{H_1}) \quad \limsup_{|r|\to\infty} \frac{|g(r)|}{|r| \ln^2 |r|} < \beta$$

However the function $g(r) = a + br + \beta r \ln^2(1+|r|) \; \forall a, b \in \mathbb{R}$ (which is somehow the limit case in $(\mathbf{H_1})$) satisfies $(\overline{\mathbf{H}}_\mathbf{1})$ and $(\mathbf{H_2})$.

Step 1: We prove by induction that, if β is small enough, then $\{\|y_k\|_{L^\infty(Q_T)}\}_{k\in\mathbb{N}}$ is bounded. Assume $\exists M>0$ such that $\|y_k\|_{L^\infty(Q_T)}\leq M$, $\forall k\leq n$. Then,

$$\begin{aligned}\|y_{n+1}\|_\infty &\leq \|y_0\|_\infty + \sum_{k=1}^{n}|\lambda_k|\|Y_k^1\|_\infty\\ &\leq \|y_0\|_\infty + m\sum_{k=1}^{n} Ce^{C\sqrt{\|g'(y_k)\|_\infty}}\sqrt{E(y_k,f_k)}\end{aligned}$$

But, $e^{C\sqrt{\|g'(y_k)\|_\infty}}\leq e^{C\sqrt{\alpha}}(1+\|y_k\|_\infty)^{C\sqrt{\beta}}\leq e^{C\sqrt{\alpha}}(1+M)^{C\sqrt{\beta}}$, $k\leq n$ so that

$$\boxed{\|y_{n+1}\|_\infty\leq\|y_0\|_\infty + me^{C\sqrt{\alpha}}(1+M)^{C\sqrt{\beta}}\sum_{k=1}^{n}\sqrt{E(y_k,f_k)}}$$

Moreover, from

$$\sqrt{E((y_k,f_k)-\lambda(Y_k^1,F_k^1))}\leq\left(|1-\lambda|+\lambda^{1+p}K(y_k)\sqrt{E(y_k,f_k)}^{p}\right)\sqrt{E(y_k,f_k)}\qquad\forall\lambda\in\mathbb{R}$$

where

$$K(y_k):=C[g']_p\left(Ce^{C\sqrt{\|g'(y_k)\|_\infty}}\right)^{1+p}\leq K(M):=C[g']_p\left(e^{C\sqrt{\alpha}}(1+M)^{C\sqrt{\beta}}\right)^{1+p}\quad(16)$$

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$$\|y_{n+1}\|_\infty \leq \|y_0\|_\infty + \sum_{k=1}^{n} |\lambda_k| \|Y_k^1\|_\infty$$
$$\leq \|y_0\|_\infty + m \sum_{k=1}^{n} Ce^{C\sqrt{\|g'(y_k)\|_\infty}} \sqrt{E(y_k, f_k)}$$

But, $e^{C\sqrt{\|g'(y_k)\|_\infty}} \leq e^{C\sqrt{\alpha}}(1 + \|y_k\|_\infty)^{C\sqrt{\beta}} \leq e^{C\sqrt{\alpha}}(1 + M)^{C\sqrt{\beta}}$, $k \leq n$ so that

$$\boxed{\|y_{n+1}\|_\infty \leq \|y_0\|_\infty + me^{C\sqrt{\alpha}}(1 + M)^{C\sqrt{\beta}} \sum_{k=1}^{n} \sqrt{E(y_k, f_k)}}$$

Moreover, from

$$\sqrt{E((y_k, f_k) - \lambda(Y_k^1, F_k^1))} \leq \Big(|1-\lambda| + \lambda^{1+p} K(y_k) \sqrt{E(y_k, f_k)}^p\Big) \sqrt{E(y_k, f_k)} \qquad \forall \lambda \in \mathbb{R}$$

where

$$K(y_k) := C\,[g']_p \Big(Ce^{C\sqrt{\|g'(y_k)\|_\infty}}\Big)^{1+p} \leq K(M) := C\,[g']_p \Big(e^{C\sqrt{\alpha}}(1 + M)^{C\sqrt{\beta}}\Big)^{1+p} \tag{16}$$

we get that

$$\frac{\sqrt{E(y_{k+1}, f_{k+1})}}{\sqrt{E(y_k, f_k)}} \leq min_{\lambda\in[0,m]}\Big(|1-\lambda| + \lambda^{1+p} K(M)E(y_k, f_k)^{\frac{p}{2}}\Big)$$

and (after computations ....) to

$$\sum_{k=0}^{n} \sqrt{E(y_k, f_k)} \leq \Big(C\,[g']_p\Big(e^{C\sqrt{\alpha}}(1+M)^{C\sqrt{\beta}}\Big)^{1+p}\Big)^{1/p} \sqrt{E(y_0, f_0)}$$

and then to

$$\boxed{\|y_{n+1}\|_\infty \leq \|y_0\|_\infty + [g']_p^{1/p}\Big(Ce^{C\sqrt{\alpha}}(1+M)^{C\sqrt{\beta}}\Big)^{\frac{1+2p}{p}} \sqrt{E(y_0, f_0)}}$$

from which we deduce that $\|y_{n+1}\|_{L^\infty(Q_T)} \leq M$ if $C\sqrt{\beta}\frac{1+2p}{p} < 1$ (for some M large enough)

Step 2: Once we know that $\{\|y_k\|_{L^\infty}\}_{k\in\mathbb{N}}$ is bounded, we get using again

$$\frac{\sqrt{E(y_{k+1},f_{k+1})}}{\sqrt{E(y_k,f_k)}} \leq min_{\lambda\in[0,m]}\Big(|1-\lambda| + \lambda^{1+p} K(y_k)\sqrt{E(y_k,f_k)}^p\Big)^2$$

that the series $\sum_{k=1}^n \|Y_k^1, F_k^1\| \leq C(M)\sum_{k=1}^n \sqrt{E(y_k,f_k)}$ converges and

$$(y_{k+1},f_{k+1}) = (y_0,f_0) - \sum_{k=1}^n \lambda_k(Y_k^1,F_k^1) \to (y_0,f_0) - \sum_{k=1}^\infty \lambda_k(Y_k^1,F_k^1) := (y,f) \quad \text{in} \quad \mathcal{A}$$

with in particular,

$$\boxed{\|(y,f)-(y_k,f_k)\|_{\mathcal{H}} \leq C\sqrt{E(y_k,f_k)}, \quad \forall k\in\mathbb{N}.} \tag{17}$$

Step 3 We pass to the limit w.r.t. k in (using that $\|Y_k^1, F_k^1\|_{\mathcal{A}_0} \to 0$)

$$\begin{cases} \partial_{tt}Y_k^1 - \partial_{xx}Y_k^1 + g'(y_k)Y_k^1 = F_k^1 1_\omega + (\partial_{tt}y_k - \partial_{xx}y_k + g(y_k) - f_k 1_\omega) & \text{in } Q_T, \\ Y_k^1 = 0 & \text{on } \Sigma_T, \\ (Y_k^1(\cdot,0), \partial_t Y_k^1(\cdot,0)) = (0,0) & \text{in } \Omega. \end{cases} \tag{18}$$

Strong convergence of the algorithm

Given any $p \in [0,1]$, we set

$$\beta^0(p) := \frac{p^2}{C^2(2p+1)^2} \qquad (19)$$

Theorem (Trélat, M 20)

Assume that $T > 2\max(\ell_1, 1-\ell_2)$, that $[g']_p < +\infty$ for some $p \in [0,1]$, and that there exist $\alpha \geq 0$ and $\beta \in [0, \beta^0(p))$ (with the agreement that $\beta = 0$ if $p = 0$), such that

$$\boxed{|g'(r)| \leq \alpha + \beta \ln^2(1+|r|) \qquad \forall r \in \mathbb{R}.} \qquad (20)$$

For $p = 0$ (i.e., $g' \in L^\infty(\mathbb{R})$), we assume moreover that $2\|g'\|_\infty C^2 e^{C\sqrt{\|g'\|_\infty}} < 1$.
Then, as $k \to \infty$

- *For any $(y_0, f_0) \in \mathcal{A}$, $(y_k, f_k) \to (y, f)$ a controlled pair for the nonlinear wave eq.*
- *$\lambda_k \to 1$.*

Moreover, the convergence of these sequences is at least linear, and is at least of order $1+p$ after a finite number of iterations.

Defining $F : \mathcal{A} \to L^2(Q_T)$ by $F(y,f) := (\partial_{tt} y - \partial_{xx} y + g(y) - f 1_\omega)$,

$$E(y,f) = \frac{1}{2}\|F(y,f)\|^2_{L^2(Q_T)}$$

For $\lambda_k = 1$, the least-squares algorithm coincides with the Newton algorithm applied to F (explaining the super-linear convergence property).

Optimizing the parameter λ_k gives a global convergence property of the algorithm and leads to the so-called damped Newton method applied to F.

If $E(y_0, f_0)$ is small, assumption $|g'(r)| \leq \alpha + \beta \ln^2(1+|r|) \quad \forall r \in \mathbb{R}$ is not necessary

Proposition

Assume that $[g']_p < \infty$ for some $p \in (0,1]$.
$\exists C([g']_p) > 0$ such that, if $E(y_0, f_0) \leq C([g']_p)$, then $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ converges.
The convergence is at least linear, and is at least of order $1+p$ after a finite number of iterations.

$E(y_0, f_0)$ is notably small if

- $|g(0)|$ is small;
- The initial guess (y_0, f_0) solves the linear controllability problem (i.e. $g \equiv 0$);
- The initial condition (u_0, u_1) and target (z_0, z_1) are small for the norm $\boldsymbol{V}$;

since then

$$E(y_0, f_0) = \frac{1}{2}\|g(y_0)\|_2^2 \leq |g(0)|^2|Q_T| + C(\alpha, \beta, \epsilon)(\|u_0, u_1\|_{\boldsymbol{V}}^{2+\epsilon} + \|z_0, z_1\|_{\boldsymbol{V}}^{2+\epsilon}), \quad \epsilon > 0$$

The proposition is equivalent to the (well-known) local controllability of the wave equation.

Remark 3: Relaxing the condition $(\overline{\mathbf{H}}_{\mathbf{p}})$, $p > 0$

We may weaken the assumption : $\exists p \in (0,1]$ s.t.

$$(\overline{\mathbf{H}}_{\mathbf{p}}) \quad [g']_p := \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|g'(a)-g'(b)|}{|a-b|^p} < +\infty$$

by the following one : $\exists p \in (0,1]$ s.t.

$$(\overline{\mathbf{H}}'_{\mathbf{p}}) \quad \text{There exist } \overline{\alpha}, \overline{\beta}, \gamma \in \mathbb{R}^+ \text{ such that}$$
$$|g'(a) - g'(b)| \leq |a-b|^p\big(\overline{\alpha} + \overline{\beta}(|a|^\gamma + |b|^\gamma)\big), \quad \forall a, b \in \mathbb{R}$$

assuming $\frac{\gamma + C\sqrt{\beta}(1+2p)}{p} < 1$.

Using the observability estimate in [Fu, Yong, Zhang 2007] [4] with the potential $A \in L^\infty(0,T;L^d(\Omega))$

Theorem (Lemoine, M 21)

Let $(u_0,u_1),(z_0,z_1) \in \boldsymbol{V}$. *For any* $x_0 \in \mathbb{R}^d \backslash \overline{\Omega}$, *let* $\Gamma_0 = \{x \in \partial\Omega, (x-x_0)\cdot\nu(x) > 0\}$ *and, for any* $\epsilon > 0$, $\mathcal{O}_\epsilon(\Gamma_0) = \{y \in \mathbb{R}^d \mid |y-x| \leq \epsilon \, for \, x \in \Gamma_0\}$. *Assume*

($\mathbf{H_0}$) $T > 2\max_{x\in\overline{\Omega}}|x-x_0|$ *and* $\omega \subseteq \mathcal{O}_\epsilon(\Gamma_0)\cap\Omega$ *for some* $\epsilon > 0$.

Assume that g' *satisfies* ($\overline{\mathbf{H}}_\mathbf{p}$) *for some* $p \in [0,1]$ *and*

($\mathbf{H_2}$) $\exists \alpha \geq 0, \beta \in [0,\beta^\star(p))$ *s.t.* $|g'(r)| \leq \alpha + \beta \ln^{1/2}(1+|r|)\forall r \in \mathbb{R}$ $(p>0)$

($\mathbf{H_3}$) $\sqrt{2}C\|g'\|_\infty e^{C\|g'\|^2_\infty|\Omega|^{2/d}} < 1$ $(p=0)$

For any $(y_0,f_0) \in \mathcal{A}$, *the sequence* $(y_k,f_k)_{k\in\mathbb{N}}$ *strongly converges to a solution of*

$$\begin{cases} \partial_{tt}y - \Delta y + g(y) = f1_\omega, & in\ Q_T, \\ y = 0, & on\ \Sigma_T, \\ (y(\cdot,0),y_t(\cdot,0)) = (u_0,u_1), (y(\cdot,T),y_t(\cdot,T)) = (z_0,z_1) & in\ \Omega. \end{cases} \tag{21}$$

The convergence is at least linear and is at least of order $1+p$ *after a finite number of iterations.*

[4] X. Fu, J. Yong, X. Zhang *Exact controllability for multidimensional semilinear hyperbolic equations*, SICON 2007

Let $\omega \subset \Omega$.

$$\begin{cases} \partial_t y - \Delta y + g(y) = f 1_\omega & \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{22}$$

where $u_0 \in L^2(\Omega)$ is the initial state of y and $f \in L^2(q_T)$ is a *control* function such that $y(T, \cdot) = 0$.

Theorem (Fernández-Cara, Zuazua, 2000)

Let $T > 0$ be given. Assume that $g : \mathbb{R} \mapsto \mathbb{R}$ is C^1, $g(0) = 0$ and

$$(\mathbf{H_4}) \quad \limsup_{|r| \to \infty} \frac{|g(r)|}{|r| \ln^{3/2} |r|} = 0$$

then (22) *is null controllable in time T.*

[5]

[5] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear,* Ann. Inst. H. Poincaré Anal. Non Linéaire 2000

We introduce, for all $s \geq 0$, the vectorial space $\mathcal{A}_0(s)$

$$\mathcal{A}_0(s) := \Big\{(y,f) : \rho(s)\, y \in L^2(Q_T),\ \rho_0(s) f \in L^2(q_T), \\ \rho_0(s)(\partial_t y - \partial_{xx} y) \in L^2(Q_T),\ y(\cdot,0) = 0 \text{ in } \Omega,\ y = 0 \text{ on } \Sigma_T\Big\} \tag{23}$$

where $\rho_i(s)$ Carleman weights of the form $\boxed{\rho_i(s,x,t) \approx e^{\frac{s\varphi(x)}{T-t}}}$ and the convex space

$$\mathcal{A}(s) := \Big\{(y,f) : \rho(s)\, y \in L^2(Q_T),\ \rho_0(s) f \in L^2(q_T), \\ \rho_0(s)(\partial_t y - \partial_{xx} y) \in L^2(Q_T),\ y(\cdot,0) = u_0 \text{ in } \Omega,\ y = 0 \text{ on } \Sigma_T\Big\} \tag{24}$$

We define the least-squares functional $E : \mathcal{A}(s) \to \mathbb{R}$ by

$$\boxed{E(y,f) := \frac{1}{2}\|\rho_0(s)(\partial_{tt} y - \partial_{xx} y + g(y) - f 1_\omega)\|^2_{L^2(Q_T)}}$$

and consider the nonconvex minimization problem

$$\boxed{\inf_{(y,f)\in\mathcal{A}(s)} E(y,f)}$$

Theorem

Assume $A \in L^\infty(Q_T)$, $s \geq \max(\|A\|_{L^\infty(Q_T)}^{2/3}, s_0)$, $B \in L^2(\rho_0(s), Q_T)$, $z_0 \in L^2(\Omega)$. $\exists v \in L^2(\rho_0(s), q_T)$ *s.t. the solution* z

$$\begin{cases} \partial_t z - \partial_{xx} z + Az = v 1_\omega + B & in \quad Q_T, \\ z = 0 \ on \ \Sigma_T, \quad z(\cdot, 0) = z_0 \ in \ \Omega \end{cases} \tag{25}$$

satisfies $z(\cdot, T) = 0$. *Moreover, the unique control* v *which minimizes together with* z *the functional* $J(z,v) := \frac{1}{2}\|\rho(s)\, z\|^2_{L^2(Q_T)} + \frac{1}{2}\|\rho_0(s)\, v\|^2_{L^2(q_T)}$ *satisfies*

$$\|\rho(s)\, z\|_{L^2(Q_T)} + \|\rho_0(s)\, v\|_{L^2(q_T)} \leq C s^{-3/2}\big(\|\rho_0(s) B\|_{L^2(Q_T)} + e^{cs}\|z_0\|_2\big). \tag{26}$$

and if $z_0 \in H^1_0(\Omega)$

$$\|z\|_{L^\infty(Q_T)} \leq C e^{-\frac{3}{2}s}(1 + \|A\|_{L^\infty(Q_T)})\big(\|\rho_0(s) B\|_{L^2(Q_T)} + e^{cs}\|z_0\|_{H^1_0(\Omega)}\big). \tag{27}$$

Proposition

For any $(y,f)\in\mathcal{A}(s)$ *and* $s\geq \max\left(\|g'(y)\|^{2/3}_{L^\infty(Q_T)}, s_0\right)$,

$$\frac{\|E'(y,f)\|_{\mathcal{A}_0(s)'}}{C\left(1+\|g'(y)\|_{L^\infty(Q_T)}\right)}\leq\sqrt{E(y,f)}\leq C\left(1+s^{-1/2}\|g'(y)\|^{1/2}_{L^\infty(Q_T)}\right)\|E'(y,f)\|_{\mathcal{A}_0(s)'}.$$

Proposition

Assume that g *satisfies* $(\overline{\mathbf{H}}_\mathbf{p})$ *for some* $p\in[0,1]$. *Let* $(y,f)\in\mathcal{A}(s)$, $s\geq \max\left(\|g'(y)\|^{2/3}_{L^\infty(Q_T)}, s_0\right)$. $(Y^1,F^1)\in\mathcal{A}_0(s)$ *solution of*

$$\begin{cases}\partial_t Y^1-\partial_{xx}Y^1+g'(y)Y^1=F^1 1_\omega+\left(\partial_t y-\partial_{xx}y+g(y)-f\,1_\omega\right) & in\quad Q_T,\\ Y^1=0 \text{ on } \Sigma_T,\quad Y^1(\cdot,0)=0 \text{ in } \Omega.\end{cases}\tag{28}$$

satisfies $\|(Y^1,F^1)\|_{\mathcal{A}_0(s)}\leq C\sqrt{E(y,f)}$ *and for any* $\lambda\in\mathbb{R}_+$

$$\sqrt{E((y,f)-\lambda(Y^1,F^1))}\leq\sqrt{E(y,f)}\left(|1-\lambda|+\lambda^{p+1}\frac{C^{1+p}}{1+p}s^{-3/2}e^{-\frac{3p}{2}s}[g']_p\sqrt{E(y,f)}^p\right).\tag{29}$$

Proposition

For any $(y, f) \in \mathcal{A}(s)$ *and* $s \geq \max\left(\|g'(y)\|^{2/3}_{L^\infty(Q_T)}, s_0\right)$,

$$\frac{\|E'(y,f)\|_{\mathcal{A}_0(s)'}}{C\left(1+\|g'(y)\|_{L^\infty(Q_T)}\right)} \leq \sqrt{E(y,f)} \leq C\left(1 + s^{-1/2}\|g'(y)\|^{1/2}_{L^\infty(Q_T)}\right)\|E'(y,f)\|_{\mathcal{A}_0(s)'}.$$

Proposition

Assume that g *satisfies* $(\overline{\mathbf{H}}_\mathbf{p})$ *for some* $p \in [0,1]$. *Let* $(y, f) \in \mathcal{A}(s)$, $s \geq \max\left(\|g'(y)\|^{2/3}_{L^\infty(Q_T)}, s_0\right)$. $(Y^1, F^1) \in \mathcal{A}_0(s)$ *solution of*

$$\begin{cases} \partial_t Y^1 - \partial_{xx} Y^1 + g'(y) Y^1 = F^1 1_\omega + \left(\partial_t y - \partial_{xx} y + g(y) - f\, 1_\omega\right) & in \quad Q_T, \\ Y^1 = 0 \ on \ \Sigma_T, \quad Y^1(\cdot, 0) = 0 \ in \ \Omega. \end{cases} \tag{28}$$

satisfies $\|(Y^1, F^1)\|_{\mathcal{A}_0(s)} \leq C\sqrt{E(y,f)}$ *and for any* $\lambda \in \mathbb{R}_+$

$$\sqrt{E((y,f) - \lambda(Y^1, F^1))} \leq \sqrt{E(y,f)}\left(|1-\lambda| + \lambda^{p+1}\frac{C^{1+p}}{1+p} s^{-3/2} e^{-\frac{3p}{2}s}[g']_p \sqrt{E(y,f)}^p\right). \tag{29}$$

Heat case: Strong convergence of the LS algorithm

Theorem (Lemoine, M 21)

Assume $[g']_p < +\infty$ *for some* $p \in [0,1]$, *and that there exist* $\alpha \geq 0$ *and* $\beta > 0$ *small enough, such that*

$$\boxed{|g'(r)| \leq \alpha + \beta \ln^{3/2}(1+|r|) \qquad \forall r \in \mathbb{R}.} \tag{30}$$

There exists $M > 0$ *such that if*

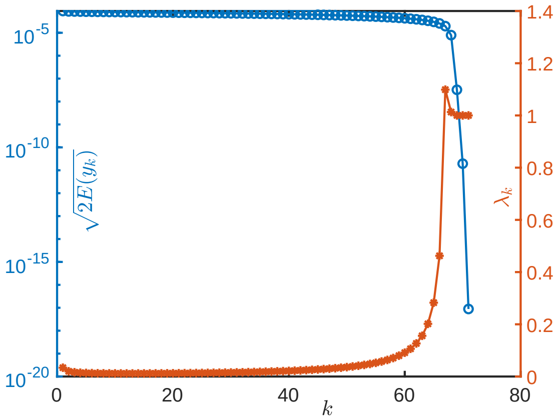
$$s = \max\Big(C(p)\big(\alpha + \beta ln^{3/2}(1+M)\big)^{2/3}, s_0 \Big),$$

then the sequence $(y_k, f_k) \in \mathcal{A}(s)$ *converges, for any* $(y_0, f_0) \in \mathcal{A}(s)$ *to a controlled pair* (y, f) *in* $\mathcal{A}(s)$ *where* f *is a null control for* y *solution of* (22).
The convergence is at least linear, and is at least of order $1+p$ *after a finite number of iterations.*

[6]

[6] The multi-dimensional case is addressed in J. Lemoine, I. Marin-Gayte, AM, *Approximation of null controls for semilinear heat equations using a least-squares approach*. ESAIM:COCV. assuming $g' \in L^\infty(\mathbb{R})$ and taking $E(y,f) = \|\rho_2(s)(\partial_t y - \Delta y + g(y) - f 1_\omega)\|^2_{L^2(0,T;H^{-1}(\Omega))}$

Behavior of $E(y_k, f_k)$ w.r.t. λ_k w.r.t. k



Conclusions

• Constructive proof of controllability (without fixed point arguments) leading to robust algorithm;

• Very likely, can be adapted to any controllable PDEs for which a precise observability estimate is available in the linearized case: boundary case, $g = g(u, \nabla u)$;

• May be possibly adapted to (nonlinear) inverse/assimilation problem;

• The approach allows to get a numerical approximation f_k^h of a nonlinear control f, for k large enough and h (numerical space-time discretization parameter) small enough :

$$\|f - f_k^h\| \leq \|f - f_k\| + \|f_k - f_k^h\|$$

• Ongoing work : improvement of the LS approach to reach the optimal assumptions on g

• Challenging work: Burgers and NS !!?!

THANK YOU VERY MUCH FOR YOUR ATTENTION

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• E. Trélat, AM. *Constructive exact control of semilinear 1D wave equations by a least-squares approach*, arxiv.org/abs/2011.08462

• J. Lemoine, AM. *A fully space-time least-squares method for the unsteady Navier-Stokes system*, arxiv.org/abs/1909.05034

• J. Lemoine, AM. *Resolution of the Implicit Euler scheme for the Navier-Stokes equation through a least-squares method*, Numerische Mathematik 2021.

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• J. Lemoine, I. Marin-Gayte, AM, *Approximation of null controls for semilinear heat equations using a least-squares approach*. ESAIM:COCV.

• E. Zuazua Exact controllability for semilinear wave equations in one space dimension, Ann. Inst. H. Poincaré Anal. Non Linéaire 1993

• E. Fernández-Cara, E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear,* Ann. Inst. H. Poincaré Anal. Non Linéaire 2000

• X. Fu, J. Yong, X. Zhang, *Exact controllability for multidimensional semilinear hyperbolic equations*, SICON 2007