Constructive approaches for the controllability of semilinear wave and heat equations

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Workshop : Simulation of data assimilation under a PDEs constraint - Jussieu - November 2023

with Kuntal Bhandari (Prague), Sue Claret (Clermont-Ferrand), Sylvain Ervedoza (Bordeaux), Irène Gayte (Sevilla), Jérôme Lemoine (Clermont-Ferrand), Emmanuel Trélat (Paris)

GIVEN some semilinear heat/wave equation

$$\begin{cases} \partial_{t(t)} y - \Delta y + f(y) = v 1_\omega, \text{ in } Q_T := \Omega \times (0,T) \\ y = 0 \text{ in } \partial\Omega \times (0,T) \quad + \text{initial conditions} \end{cases} \tag{1}$$

or

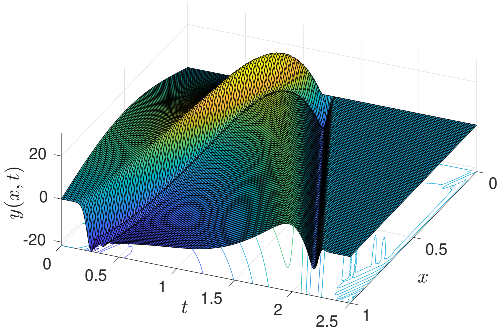
$$\begin{cases} \partial_{t(t)} y - \Delta y + f(y) = 0, \text{ in } Q_T := \Omega \times (0,T), \\ y = v 1_\Gamma \text{ in } \partial\Omega \times (0,T) \quad + \text{initial conditions} \end{cases} \tag{2}$$

ASSUME that there exist control v and $T > 0$ such that $(y(t), y_t(t)) = (0,0)$ forall $t \geq T$.

Litterature Lasiecka-Triggiani'91, Zuazua'93, Fernandez-Cara'00, Barbu'00, Li-Zhang'01, Coron-Trelat'06, Dehman-Lebeau'09, Joly-Laurent'14, Fu-Lu-Zhang'19, Friedman'19, ...

FIND a non trivial sequence $(y_k, v_k)_{k\in\mathbb{N}}$ such that $(y_k, v_k) \to (y, v)$ as $k \to \infty$, with (y, v) a controlled pair for (1) or (2)?

⚠ Non trivial question since in many situations, proofs of controllability are based on non constructive fixed point arguments.



Controlled solution (from the boundary)

$$\begin{cases} y_{tt} - y_{xx} - 3y(1+\ln^{3/2}(2+|y|)) = 0 & \text{in } (0,1)\times(0,2.5), \\ y(0,\cdot) = 0, \quad y(1,\cdot) = v & \text{in } (0,2.5), \\ (y(\cdot,0), y_t(\cdot,0)) = (10\sin(\pi x), 0) & \text{in } (0,1). \end{cases} \quad (3)$$

How do we get such control v ????

Closed connection between control and inverse problems

There are strong connections between
• Exact Controllability problem: find $v \in L^2$ such that

$$\begin{cases} \partial_{tt} y - \Delta y + f(y) = 0, \text{ in } \Omega \times (0,T), \\ y = v 1_\Gamma \text{ in } \partial\Omega \times (0,T) \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) \text{ in } \Omega \end{cases} \tag{4}$$

such that $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1) \text{ in } \Omega$.

• Inverse problem: given a "good" observation $y_{\nu,obs} \in L^2(\Gamma \times (0,T))$, reconstruct y solution of

$$\begin{cases} \partial_{tt} y - \Delta y + f(y) = 0, \text{ in } \Omega \times (0,T), \\ y = 0 \text{ in } \partial\Omega \times (0,T) \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) \text{ in } \Omega \end{cases} \tag{5}$$

such that $\partial_\nu y := y_{\nu,obs}$ on $\Gamma \times (0,T)$.

For the semilinear wave/heat equations

$$\begin{cases} \partial_{t(t)} y - \Delta y + f(y) = 0 \\ + \text{initial conditions and boundary conditions} \end{cases} \tag{6}$$

KNOWN FACT The uniform (w.r.t. initial conditions) exact controllability holds true under the following asymptotic condition on $f \in C^1(\mathbb{R})$

$$\exists \beta > 0 \text{ such that } \limsup_{r\to\infty} \frac{|f(r)|}{r \ln^p(r)} \leq \beta, \qquad p \in (0,2)$$

obtained from non constructive Schauder fixed point arguments [Zuazua'93, Zuazua-Fernandez-Cara'00, Barbu'00, Zhang'07, ...]

CLAIM Under the following assumption,

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Rk: Lack of contraction of the Zuazua's operator

Exact controllability in Zuazua'93 [1] based on a Leray Schauder fixed point argument: Let $\Lambda : L^\infty(Q_T) \to L^\infty(Q_T)$ and $y := \Lambda(\xi)$ is a controlled solution with the control function v_ξ (of minimal L^2-norm)

$$\begin{cases} \partial_{tt} y - \partial_{xx} y + y\,\widehat{f}(\xi) = -f(0) + v_\xi 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \qquad \widehat{f}(r) := \begin{cases} \dfrac{f(r) - f(0)}{r} & \text{if } r \neq 0 \\ f'(0) & \text{if } r = 0 \end{cases} \tag{7}$$

satisfying $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$.

Assume $\omega = (l_1, l_2)$, $T > 2\max(\ell_1, 1 - \ell_2)$, $f \in C^1$ and that $\limsup_{|r|\to\infty} \frac{|f(r)|}{|r| \ln^2 |r|} < \beta$ for some β small enough.

Then, Λ has a fixed point. In particular,

$$\|\Lambda(\xi)\|_\infty \leq C\Big(\|u_0, u_1\|_V + \|f(0)\|_2\Big) \underbrace{(1 + \|\xi\|_\infty)^{(1+C)\sqrt{\beta}}}_{e^{\sqrt{\|\hat{f}(\xi)\|_{L^\infty(Q_T)}}}}, \quad \forall \xi \in L^\infty(Q_T).$$

but Λ is not contracting in general. The sequence $\{y_{k+1}\}_k$ given by $y_{k+1} = \Lambda(y_k)$ is bounded but not convergent.

[1] E. Zuazua, Exact controllability for semilinear wave equations in one space dimension, Ann. Inst. H. Poincaré Anal. Non Linéaire

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Same method for the wave and heat equation

Assuming

$$\exists \beta > 0 \text{ such that } \limsup_{r\to\infty} \frac{|f'(r)|}{\ln^p(r)} \leq \beta, \qquad p \in (0,2)$$

we design convergent sequences (y_k, v_k) from two different approaches :

Method 1 : Least-squares approaches (Newton type linearization)

$$y_{k+1,t(t)} - \Delta y_{k+1} + f'(y_k)y_{k+1} = v_{k+1}1_\omega - f'(y_k)y_{k+1} + f(y_k), k \geq 0$$

where (y_{k+1}, v_{k+1}) is the optimal state-control pair for the cost

$$J(y,v) = \|v\|^2_{L^2(q_T)}$$

Method 2: Zero order linearization and Carleman weights

$$y_{k+1,t(t)} - \Delta y_{k+1} = v_{k+1}1_\omega - f(y_k), k \geq 0$$

where (y_{k+1}, v_{k+1}) is the optimal null state-control pair for the cost

$$J(y,v) = \|\rho(s)v\|^2_{L^2(q_T)} + s\|\rho_0(s)y\|^2_{L^2(q_T)}$$

involving Carleman weights $\rho(x,t,s)$, $\rho_0(x,t,s)$ and parameter $s > 0$

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We consider the nonconvex minimization problem

$$\inf_{(y,v)\in\mathcal{A}} E(y,v), \quad E(y,v) := \frac{1}{2}\left\|\partial_{tt}y - \Delta y + f(y) - v1_\omega\right\|^2_{L^2(Q_T)} \tag{8}$$

$$\mathcal{A} := \Big\{(y,v)\in\mathcal{H} \mid (y(\cdot,0),\partial_t y(\cdot,0)) = (u_0,u_1),\ (y(\cdot,T),\partial_t y(\cdot,T)) = (0,0) \text{ in } \Omega\Big\},$$

$$\mathcal{H} := \Big\{(y,v)\in L^2(Q_T)\times L^2(q_T) \mid \partial_{tt}y - \Delta y \in L^2(Q_T),\ y = 0 \text{ on } \Sigma_T,$$
$$(y(\cdot,0),\partial_t y(\cdot,0)) \in \boldsymbol{V}\Big\}$$

Proposition

$$\forall (y,v)\in\mathcal{A}, \quad \sqrt{E(y,v)} \le C e^{C\|f'(y)\|^2_{L^\infty(L^d)}}\|E'(y,v)\|_{\mathcal{A}_0'}.$$

Consequence:
Any *critical* point $(y,v)\in\mathcal{A}$ of E is a zero of E, and thus is a pair solution of the controllability problem. Moreover:

given any sequence $(y_k,v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|E'(y_k,v_k)\|_{\mathcal{A}_0'} \underset{k\to+\infty}{\longrightarrow} 0$ and such that $\|f'(y_k)\|_{L^\infty(L^d)}$ is uniformly bounded, we have $E(y_k,v_k) \underset{k\to+\infty}{\longrightarrow} 0$.

⚠ A minimizing sequence for E cannot be stuck in a local minimum, even though E fails to be convex (it has multiple zeros).

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A minimizing sequence for E

Let the sequence $(y_k, v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ defined by

$$\boxed{\begin{cases} (y_0, v_0) \in \mathcal{A} \\ (y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k (Y_k, V_k) \qquad \forall k \in \mathbb{N} \\ \lambda_k = \underset{\lambda\in[0,1]}{\text{argmin}}\; E\big((y_k, v_k) - \lambda(Y_k, V_k)\big) \end{cases}} \tag{9}$$

where $(Y_k, V_k) \in \mathcal{A}_0$ is the solution of minimal control norm of

$$\boxed{\begin{cases} \partial_{tt} Y_k - \Delta Y_k + f'(y_k) Y_k = V_k 1_\omega + (\partial_{tt} y_k - \Delta y_k + f(y_k) - v_k 1_\omega) & \text{in } Q_T, \\ Y_k = 0 & \text{on } \Sigma_T, \\ (Y_k(\cdot,0), \partial_t Y_k(\cdot,0)) = (0,0) & \text{in } \Omega. \end{cases}} \tag{10}$$

Lemma $\forall k \geq 0$, $E'(y_k, v_k) \cdot (Y_k, V_k) = 2E(y_k, v_k)$

Theorem (Münch-Trélat 2022, Bottois-Lemoine-Münch 2023)

Assume that $f'' \in L^\infty(\mathbb{R})$ and that $\limsup_{r\to\infty} \frac{|f'(r)|}{ln^p(r)}$ is small enough.
for any $(y_0, v_0) \in \mathcal{A}$, $(y_k, v_k) \to (y, v)$ a controlled pair for the nonlinear wave eq.
The convergence of these sequences is at least linear, and is at least of order 2 after a finite number of iterations.

Let the sequence $(y_k, v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ defined by

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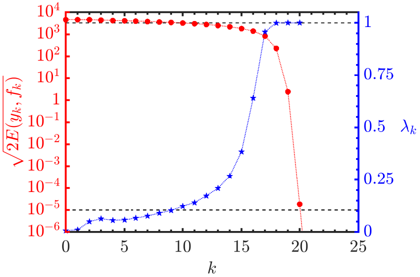
Rk. Link with a Damped Newton method

The sequence $(y_k, v_k)_{k\geq 0}$ coincides with the one associated to a damped Newton method to find a zero of the map $\mathcal{F} : \mathcal{A} \to L^2(Q_T)$ defined by
$\mathcal{F}(y, v) := (\partial_{tt} y - \Delta y + f(y) - v1_\omega)$

$$\begin{cases} (y_0, v_0) \in \mathcal{A} \\ D\mathcal{F}(y_k, v_k) \cdot ((y_{k+1}, v_{k+1}) - (y_k, v_k)) = -\lambda_k \mathcal{F}(y_k, v_k), k \geq 0 \\ \lambda_k = \underset{\lambda\in[0,1]}{\operatorname{argmin}} \|\mathcal{F}((y_k, v_k) - \lambda D\mathcal{F}^{-1}(y_k, v_k)\mathcal{F}(y_k, v_k))\|_{L^2(Q_T)} \end{cases} \tag{11}$$

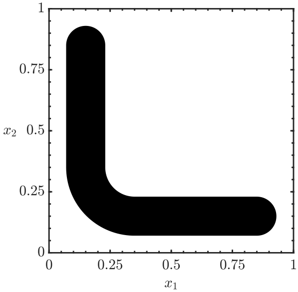
For $\lambda_k = 1$, the least-squares algorithm coincides with the Newton algorithm applied to $\mathcal{F}$ (explaining the quadratic convergence property).

Optimizing the parameter λ_k ensures the global convergence of the algorithm



We consider a two-dimensional case for which $\Omega = (0,1)^2$ and $T = 3$.

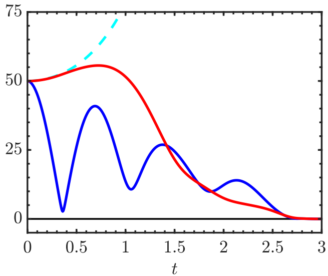
$$\begin{cases} \partial_{tt} y - \Delta y - 10\, y \ln^{1/2}(2 + |y|) = v 1_\omega & \text{in } Q_T := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (100 \sin(\pi x_1) \sin(\pi x_2), 0) & \text{in } \Omega, \end{cases} \tag{12}$$



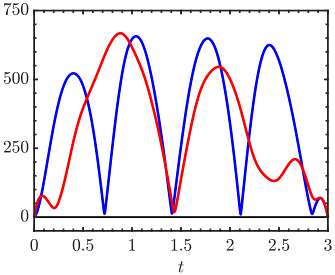
Control domain $\omega \subset \Omega = (0,1)^2$ (black part).

Numerical experiments in the 2d case

$\sharp$iterate k	$\sqrt{2E(y_k, v_k)}$	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\frac{\Vert v_k - v_{k-1}\Vert_{L^2_\chi(q_T)}}{\Vert v_{k-1}\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert v_k\Vert_{L^2_\chi(q_T)}$	λ_k
0	7.44×10^2	–	–	38.116	732.22	1
1	1.63×10^2	1.79×10^0	9.30×10^{-1}	58.691	667.602	1
2	1.62×10^0	8.42×10^{-2}	1.41×10^{-1}	60.781	642.643	1
3	1.97×10^{-3}	1.21×10^{-3}	4.66×10^{-3}	60.745	643.784	1
4	5.11×10^{-10}	6.43×10^{-7}	2.63×10^{-6}	60.745	643.785	–



(red) $\Vert y_4(\cdot,t)\Vert_{L^2(\Omega)}$; (blue) $\Vert y_0(\cdot,t)\Vert_{L^2(\Omega)}$; (dashed) $\Vert y(\cdot,t;0)\Vert_{L^2(\Omega)}$.



(red) $\Vert v_4(\cdot,t)\Vert_{L^2_\chi(\omega)}$; (blue) $\Vert v_0(\cdot,t)\Vert_{L^2_\chi(\omega)}$.

$$\begin{cases} y_{tt} - \Delta y + f(y) = 0 & \text{in } Q_T, \\ y = v\,1_{\Gamma_0} \text{in } \partial\Omega \times (0,T), \quad (y(\cdot,0), y_t(\cdot,0)) = (u_0,u_1) & \text{in } \Omega, \end{cases} \tag{13}$$

Theorem (Bhandari-Lemoine-Münch 2022, Claret-Lemoine-Münch, 2023)

Assume $T > 0$ and $\Gamma_0 \subset \partial\Omega$ large enough. Assume that there exists $\beta^\star > 0$ such that if $f \in \mathcal{C}^1(\mathbb{R})$ satisfies

$$(\mathbf{H_2'}) \quad \exists \alpha > 0, \text{ s.t. } |f'(r)| \leq \alpha + \beta^\star \ln_+^p |r|, \quad \forall r \in \mathbb{R}, \qquad 0<p<3/2$$

then, for any initial state (u_0, u_1) in $\boldsymbol{H} := L^2(\Omega) \times H^{-1}(\Omega)$, the controlled sequence $(y_k, v_k)_{k\in\mathbb{N}^*}$ solution of

$$\begin{cases} \partial_{tt} y_{k+1} - \Delta y_{k+1} = -f(y_k) & \text{in } Q_T, \\ y_{k+1} = v_{k+1}\,1_{\Gamma_0} & \text{in } \partial\Omega \times (0,T), \\ (y_{k+1}(\cdot,0), \partial_t y_{k+1}(\cdot,0)) = (u_0,u_1) & \text{in } \Omega, \end{cases} \tag{14}$$

minimizer of a functional $J_s(y,v) := s\int_{Q_T} \rho^2(s) y^2 + \int_0^T \int_{\Gamma_0} \eta^{-2}\rho_1^2(s) v^2$ converges strongly to a controlled pair (y,v) in $(\mathcal{C}^0([0,T]; L^2(\Omega)) \cap \mathcal{C}^1([0,T]; H^{-1}(\Omega))) \times H_0^1(0,T)$ for the semilinear eq. The convergence holds with a linear rate for the norm $\|\rho(s)\cdot\|_{L^2(Q_T)} + \|\rho_1(s)\cdot\|_{L^2(0,T)}$ and $s \geq \max(s_0, C\ln\|u_0,u_1\|_{\boldsymbol{H}})$.

- For any $s \geq s_0$, we introduce the class $\mathcal{C}(s)$, defined as the closed convex subset of $L^2(Q_T)$

$$\mathcal{C}(s) := \Big\{ y \in L^\infty(0,T;L^2(\Omega)) : \ \|\rho y\|_{L^2(Q)} \leq s, \ \|\rho y\|_{L^\infty(0,T;L^2(\Omega))} \leq s^3 \Big\}. \tag{15}$$

and assume that that the nonlinear function $f' \in \mathcal{C}^0(\mathbb{R})$ in (33) satisfies the logarithmic assumption for some $\beta^\star$ positive precisely chosen later.

- We introduce the operator

$$\Lambda_s : \mathcal{C}(s) \mapsto \mathcal{C}(s), \qquad \Lambda_s(\widehat{y}) = y \tag{16}$$

where y solves

$$\begin{cases} y_{tt} - \Delta y = -f(\widehat{y}) & \text{in } Q_T, \\ y = v\, 1_{\Gamma_0} & \text{on } \Sigma, \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{17}$$

satisfies $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1)$ in Ω, and (y,v) corresponds to the minimizer of a functional J_s

$$J_s(z,u) := s\int_Q \rho^2 |z|^2 \,dxdt + \int_\delta^{T-\delta}\int_{\partial\Omega} \eta^{-2}\Psi^{-1}\rho^2 |u|^2 \,dxdt \tag{18}$$

over the set
$\Big\{ (z,u) \in L^2(Q_T) \times L^2(0,T)$ solution of (17) with $z(\cdot,T) = z_t(\cdot,T) = 0$ in $\Omega \Big\}$.

$$\rho(x,t) := e^{-s\phi(x,t)} \quad \forall (x,t) \in Q. \tag{19}$$

Remark that $e^{-cs} \leq \rho \leq e^{-s}$ in Q with $c := \|\phi\|_{L^\infty(Q)}$ and $\rho, \rho^{-1} \in \mathcal{C}^\infty(\overline{Q})$.
Let then $P := \{w \in \mathcal{C}^0([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)), Lw \in L^2(Q)\}$

Proposition (Baudouin, De Buhan, Ervedoza, 2013)

Assume $T > 0$ and $\Sigma \subset \partial\Omega \times (0,T)$ large enough. There exists $s_0 > 0$, $\lambda > 0$ and $C > 0$, such that for any $s \geq s_0$ and every $w \in P$

$$\begin{aligned} s \int_Q \rho^{-2}(|w_t|^2 + |\nabla w|^2)\,\mathrm{d}x\mathrm{d}t + s^3 \int_Q \rho^{-2}|w|^2\,\mathrm{d}x\mathrm{d}t \\ + s\int_\Omega \rho^{-2}(0)(|w_t(\cdot,0)|^2 + |\nabla w(\cdot,0)|^2)\,\mathrm{d}x + s^3 \int_\Omega \rho^{-2}(0)|w(\cdot,0)|^2\,\mathrm{d}x \\ \leq C\Big(\int_Q \rho^{-2}|w_{tt} - \Delta w|^2\,\mathrm{d}x\mathrm{d}t + s\int_\Sigma \eta^2(t)\Psi(x)\rho^{-2}|\partial_\nu w|^2\,\mathrm{d}x\mathrm{d}t\Big). \end{aligned} \tag{20}$$

[2]

[2] Lucie Baudouin, Maya De Buhan, and Sylvain Ervedoza, Global Carleman estimates for waves and applications. Comm. Partial Differential Equations, 2013.

Lemma

Assume there exists $0 \leq p < 3/2$ *such that* $|f'(r)| \leq \alpha + \beta \ln_+^p |r|$ *with* $\beta > 0$ *small enough. Take* $s \geq \max(s_0, \ln(\|u_0, u_1\|_H))$*. Let* $d(y,z) := \|\rho(s)(y-z)\|_{L^2(Q)}$*. Then,*

$$d(\Lambda_s(\widehat{y}_2), \Lambda_s(\widehat{y}_1)) \leq C(s^{-p}\alpha + \beta^\star c^p) d(\widehat{y}_2, \widehat{y}_1), \qquad \forall \widehat{y}_1, \widehat{y}_2 \in \mathcal{C}(s) \tag{21}$$

- For $s \geq s_0$, $r \in [0,1]$, $r \neq 1/2$ and $(u_0, u_1) \in \boldsymbol{H}$, there exists a constant $C_r > 0$ independent of s such that

$$\|\rho y\|_{L^2(Q)} + s^{-1/2} \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{L^2(\Sigma)} + s^{-2} \|\rho y\|_{L^\infty(0,T;L^2(\Omega))} + s^{-2} \|(\rho y)_t\|_{L^\infty(0,T;H^{-1}(\Omega))}$$
$$\leq C_r \left(s^{r-3/2} \|\rho f(\hat{y})\|_{L^2(0,T;H^{-r}(\Omega))} + s^{-1/2} \|\rho(0) u_0\|_{L^2(\Omega)} + s^{-1/2} \|\rho(0) u_1\|_{H^{-1}(\Omega)} \right). \tag{22}$$

- Let $\widehat{y}_1, \widehat{y}_2 \in \mathcal{C}(s)$. From (22), for all $0 \leq r < 1/2$,

$$d(\Lambda_s(\widehat{y}_2), \Lambda_s(\widehat{y}_1)) \leq C_r s^{r-3/2} \|\rho(f(\widehat{y}_2) - f(\widehat{y}_1))\|_{L^2(0,T;H^{-r}(\Omega))}.$$

Let $r = 3/2 - p > 0$. There exists $1 \leq q < 2$ such that $L^q(\Omega) \hookrightarrow H^{-r}(\Omega)$. We then have

$$d(\Lambda_s(\widehat{y}_2), \Lambda_s(\widehat{y}_1)) \leq C_r s^{-p} \|\rho(f(\widehat{y}_2) - f(\widehat{y}_1))\|_{L^2(0,T;L^q(\Omega))}.$$

$$d(\Lambda_s(\widehat{y}_2),\Lambda_s(\widehat{y}_1)) \leq C_r s^{-p} \|\rho(f(\widehat{y}_2)-f(\widehat{y}_1))\|_{L^2(0,T;L^q(\Omega))}.$$

But, for all $(m_1,m_2)\in\mathbb{R}^2$ there exists $\bar{c}$ such that

$$\begin{aligned}|f(m_1)-f(m_2)| \leq |m_1-m_2||f'(\bar{c})| &\leq |m_1-m_2|(\alpha+\beta^\star \ln_+^p |\bar{c}|)\\ &\leq |m_1-m_2|(\alpha+\beta^\star \ln_+^p(|m_1|+|m_2|))\end{aligned}$$

and therefore, using that $0\leq \ln_+^p \rho \leq c^p s^p$ and that $p=3/2-r$, we get

$$\begin{aligned} d(\Lambda_s(\widehat{y}_2),\Lambda_s(\widehat{y}_1)) &\leq C s^{-p}\|(\alpha+\beta^\star \ln_+^p(|\widehat{y}_1|+|\widehat{y}_2|))\rho(\widehat{y}_2-\widehat{y}_1)\|_{L^2(0,T;L^q(\Omega))}\\ &\leq C s^{-p}\|(\alpha+\beta^\star \ln_+^p(|\widehat{y}_1|+|\widehat{y}_2|))\|_{L^\infty(0,T;L^a(\Omega))}\, d(\widehat{y}_2,\widehat{y}_1)\\ &\leq C s^{-p}\big(\alpha+\beta^\star c^p s^p+\beta^\star\|\ln_+^p(\rho(|\widehat{y}_1|+|\widehat{y}_2|))\|_{L^\infty(0,T;L^a(\Omega))}\big) d(\widehat{y}_2,\widehat{y}_1) \end{aligned} \tag{23}$$

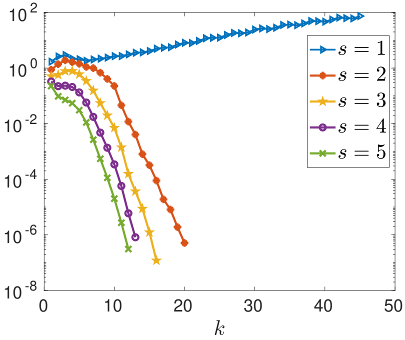
with a such that $1/q=1/2+1/a$. Now, using that, for $\varepsilon=\inf\{\frac{2}{a},\frac{p}{3}\}$

$$\begin{aligned}\|\ln_+^p(\rho(|\widehat{y}_1|+|\widehat{y}_2|))\|_{L^\infty(0,T;L^a(\Omega))} &\leq C\big(\|(\rho\widehat{y}_1)^\varepsilon\|_{L^\infty(0,T;L^a(\Omega))}+\|(\rho\widehat{y}_2)^\varepsilon\|_{L^\infty(0,T;L^a(\Omega))}\big)\\ &\leq C\big(\|\rho\widehat{y}_1\|^\varepsilon_{L^\infty(0,T;L^2(\Omega))}+\|\rho\widehat{y}_2\|^\varepsilon_{L^\infty(0,T;L^2(\Omega))}\big)\leq Cs^p\end{aligned}$$

we infer that

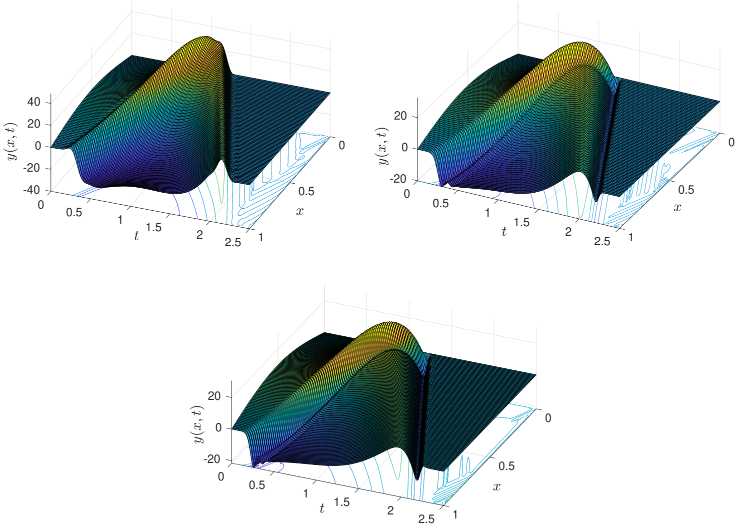
$$d(\Lambda_s(\widehat{y}_2),\Lambda_s(\widehat{y}_1)) \leq C(s^{-p}\alpha+\beta^\star c^p) d(\widehat{y}_2,\widehat{y}_1). \tag{24}$$

$$\begin{cases} y_{tt} - y_{xx} - 5y(1 + \ln^{3/2}(2 + |y|)) = 0 & \text{in } (0,1) \times (0,2.5), \\ y(0,\cdot) = 0, \quad y(1,\cdot) = v & \text{in } (0,2.5), \\ (y(\cdot,0), y_t(\cdot,0)) = (20\sin(\pi x), 0) & \text{in } \Omega, \end{cases} \tag{25}$$



Relative error $\frac{\|\rho(s)y_{k+1} - \rho(s)y_k\|_{L^2(Q_T)}}{\|\rho(s)y_k\|_{L^2(Q_T)}}$ w.r.t. iterations k.

Numerical illustration - 1d case - boundary control



Controlled solution for $s \in \{2, 5, 9\}$.

Let $P := \{w \in \mathcal{C}^0([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)), Lw \in L^2(Q)\}$ and
The optimal pair (y, v) for $J_s(z,u) = s\int_Q \rho^2|z|^2\,\mathrm{dxdt} + \int_\delta^{T-\delta}\int_{\partial\Omega} \eta^{-2}\Psi^{-1}\rho^2|u|^2\,\mathrm{dxdt}$ and solution of

$$\begin{cases} \partial_{tt}y - \Delta y = -f(\hat{y}) & \text{in } Q_T, \\ y = v\,1_{\Gamma_0} & \text{in } \partial\Omega \times (0,T), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{26}$$

is given by $y := \rho^{-2}Lw$ and $v := s\eta^2\Psi\rho^{-2}\partial_\nu w$ where $w \in P$ is the unique solution of the variational formulation

$$\boxed{\begin{aligned} &\int_Q \rho^{-2}LwLz\,\mathrm{dxdt} + s\int_\Sigma \eta^2(t)\Psi(x)\rho^{-2}\partial_\nu w\partial_\nu z\,\mathrm{dxdt} = \\ &\qquad < u_1,\, z(\cdot,0) >_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega u_0\, z_t(\cdot,0)\,\mathrm{dx} \\ &\qquad + < f(\hat{y}), z >_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))}, \quad \forall z \in P \end{aligned}} \tag{27}$$

(recall that $Lw := w_{tt} - \Delta w$)

Let $P_h \subset P$ a finite conformal approximation of P.
Then, $y_h := \rho^{-2} L w_h$ and $v_h := s\eta^2 \Psi \rho^{-2} \partial_\nu w_h$ where $w_h \in P_h$ is the unique solution of the variational formulation

$$\boxed{\begin{aligned}\int_Q \rho^{-2} L w_h L z \,\mathrm{dxdt} + s \int_\Sigma \eta^2(t)\Psi(x)\rho^{-2}\partial_\nu w_h \partial_\nu z \,\mathrm{dxdt} = \\ < u_1, z(\cdot,0) >_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega u_0\, z_t(\cdot,0)\,\mathrm{d}x \\ + < f(\hat{y}), z >_{L^2(0,T;H^{-1}(\Omega)),L^2(0,T;H_0^1(\Omega))}, \quad \forall z \in P_h\end{aligned}} \tag{28}$$

is a convergent approximation of (y, v): $\|\rho(s)(y - y_h)\|_{L^2(Q_T)} \to 0$ and $\|\rho(s)(v - v_h)\|_{L^2(Q_T)} \to 0$.

Rk. This implies an approximation of the strong limit $(y_\star, v_\star)$ of the sequence (y_k, v_k):

$$\|y_\star - y_{k,h}\| \leq \|y_\star - y_k\| + \|y_k - y_{k,h}\|$$

Least-squares approaches (Newton type linearization)

• Münch-Trélat. *Constructive exact control of semilinear 1D wave equations by a least-squares approach*, SICON 2022

• Bottois-Lemoine-Münch. *Constructive proof of the exact controllability for semi-linear multi-dimensional wave equations*, AMSA 2023

Zero order linearization and Carleman weights

• Bhandari-Lemoine-Münch. *Exact boundary controllability of semilinear 1D wave equations through a constructive approach*, MCSS 2023

• Claret-Lemoine-Münch. *- Exact boundary controllability of semilinear wave equations through a constructive approach*, arxiv, 2023

Least-squares approaches (Newton type linearization)

• Lemoine, Marin-Gayte, Münch, *Approximation of null controls for semilinear heat equations using a least-squares approach*, COCV, 2021

• Lemoine-Münch. *Constructive exact control of semilinear 1D heat equations*, MCRF 2023

Zero order linearization and Carleman weights

• Ervedoza-Lemoine-Münch. *Exact controllability of semilinear heat equations through a constructive approach*, EECT 2023

• Bhandari-Lemoine-Münch. *- Global boundary null controllability of one dimensional semi-linear heat equation,* DCDS, 2023

$$\begin{cases} \partial_t y - \Delta y + f(y) = v 1_\omega & \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = u_0 \text{ in } \Omega, \end{cases} \tag{29}$$

Theorem (Lemoine, Münch, 22)

Let $T > 0$ be given. Let $d = 1$. Assume that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f(0) = 0$ and

$$(\mathbf{H_1'}) \quad \exists \alpha > 0, \ \textit{s.t.} \ |f'(r)| \le (\alpha + \beta^\star \ln_+ |r|)^{3/2}, \quad \forall r \in \mathbb{R}$$

for some $\beta^\star > 0$ small enough and

$$(\overline{\mathbf{H}}_\mathbf{p}) \quad \exists p \in [0,1] \ \textit{such that} \ \sup_{\substack{a,b\in\mathbb{R} \\ a\neq b}} \frac{|f'(a) - f'(b)|}{|a-b|^p} < +\infty.$$

Then, for any $u_0 \in H_0^1(\Omega)$, one can construct a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly to a controlled pair for (30) satisfying $y(T) = 0$. Moreover, after a finite number of iterations, the convergence is of order at least $1 + p$.

3

[3] Lemoine, Münch, Constructive exact control of semilinear 1D heat equations. MCRF 2022

$$\begin{cases} \partial_t y - \Delta y + f(y) = v 1_\omega & \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = u_0 \text{ in } \Omega, \end{cases} \tag{30}$$

Theorem (Ervedoza-Lemoine-Münch 2022, Bandhari-Lemoine-Münch, 2023)

Let $T > 0$ be given and $d \leq 5$. Assume that there exists $\beta^\star > 0$ such that if $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f(0) = 0$ and

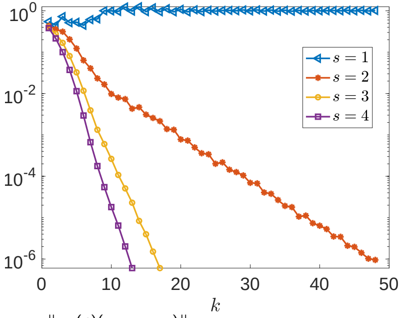
$$(\mathbf{H}_2') \quad \exists \alpha > 0, \ s.t. \ |f'(r)| \leq \alpha + \beta^\star \ln_+^{3/2} |r|, \quad \forall r \in \mathbb{R}$$

Then, for any initial state $u_0 \in L^\infty(\Omega)$, the controlled sequence $(y_k, v_k)_{k \in \mathbb{N}^*}$ solution of

$$\begin{cases} \partial_t y_{k+1} - \Delta y_{k+1} = v_{k+1} 1_\omega - f(y_k) & \text{in } Q_T, \\ y_{k+1} = 0 & \text{in } \partial\Omega \times (0,T), \\ y_{k+1}(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{31}$$

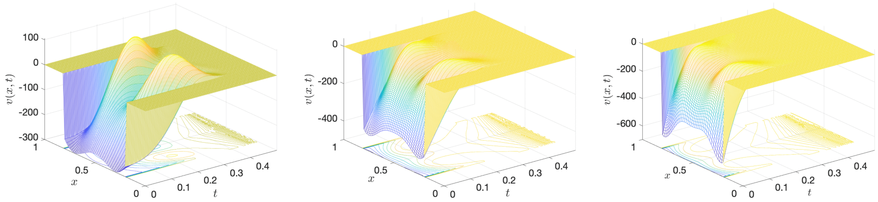
optimal for the cost $J_s(y,v) := s \int_{Q_T} \rho^2(s) y^2 + \int_0^T \int_\omega \rho_0^2(s) v^2$ converges strongly to a controlled pair (y,v) in $L^2(Q_T) \times L^2(q_T)$ for the system (33).
The convergence holds with a linear rate for the norm $\|\rho(s)\cdot\|_{L^2(Q_T)} + \|\rho_1(s)\cdot\|_{L^2(0,T)}$ and s is chosen sufficiently large depending on $\|u_0\|_{L^\infty(\Omega)}$.

$$\begin{cases} y_t - y_{xx} - 5y(1+\ln^{3/2}(2+|y|)) = 1_{(0.2,0.8)}(x)\, v & \text{in } (0,1)\times(0,0.5), \\ y(0,\cdot) = y(1,\cdot) = 0 & \text{in } (0,0.5), \\ y(\cdot,0) = 10\sin(\pi x) & \text{in } (0,1), \end{cases} \tag{32}$$

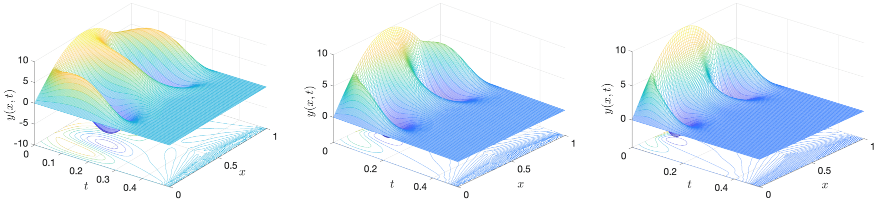


Relative error $\frac{\|\rho_0(s)(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_0(s)y_k\|_{L^2(Q_T)}}$ w.r.t. k for $s \in \{1,2,3,4\}$.

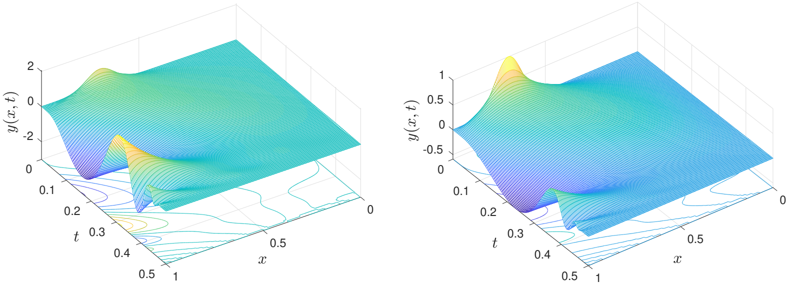
Numerical illustration for the 1d heat equation and distributed control



The distributed control v in $(0,1) \times (0,0.5)$ for $s \in \{2,3,4\}$.



The controlled solution in $(0,1) \times (0,0.5)$ for $s \in \{2,3,4\}$.



Controlled solutions from the boundary with $\nu \in \{0.3, 0.5\}$.

$$\begin{cases} y_t - \nu y_{xx} - 2.5y(1 + \ln^{3/2}(2 + |y|)) = 0 & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(1, \cdot) = v & \text{in } (0, T), \\ y(\cdot, 0) = e^{-100(x-0.7)^2} & \text{in } \Omega, \end{cases} \tag{33}$$

For the semilinear wave/heat equations

$$\boxed{\begin{cases} \partial_{t(t)} y - \Delta y + f(y) = 0 \\ + \text{ initial conditions and boundary conditions} \end{cases}} \qquad (34)$$

assuming mainly $f \in \mathcal{C}^1(\mathbb{R})$ and the growth assumption

$$\exists \beta > 0 \text{ such that } \lim_{r \to \infty} \frac{|f'(r)|}{\ln^p(r)} \leq \beta, \qquad p \in (0,2)$$

one can now construct non trivial sequences $(y_k, v_k)_{k \in \mathbb{N}}$ converging strongly to a state-control pair for the semilinear equation.

Inverse problem: given an observation $y_{\nu,obs} \in L^2(\Gamma \times (0,T))$, reconstruct y solution of

$$\boxed{\begin{cases} \partial_{tt} y - \Delta y + f(y) = 0, \ \text{in } \Omega \times (0,T), \\ y = 0 \text{ in } \partial\Omega \times (0,T) \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) \text{ in } \Omega \end{cases}} \tag{35}$$

such that $\underline{\partial_\nu y := y_{\nu,obs} \text{ on } \partial\Gamma \times (0,T)}$.

Linearization + Least-squares approach : for any
$z \in Y := \{y \in C(0,T; H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)), \rho(s)(y_{tt} - \Delta y) \in L^2(Q)\}$,
• define $\Lambda_s : z \to y$ where (y, ϕ) solves the (well-posed) mixed formulation

$$\sup_{\phi \in L^2(Q_T)} \inf_{y \in Y} \left(\frac{1}{2} \|\rho(s)(y_{\nu,obs} - \partial_\nu y)\|^2_{L^2(Q)} + < \phi, \rho(s)(\partial_{tt} y - \Delta y + f(z)) >_{L^2(Q)} \right)$$

(for some $\rho(s) > 0$ in Q_T well chosen)
• prove that if f' does not grow too fast at infinity and s large enough then Λ_s is a contraction .....

THANK YOU VERY MUCH FOR YOUR ATTENTION

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(for some $\rho(s) > 0$ in Q_T well chosen)
• prove that if f' does not grow too fast at infinity and s large enough then Λ_s is a contraction .....

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