Constructive approaches for the controllability of semilinear heat and wave equations

ARNAUD MÜNCH
Lab. de mathématiques Blaise Pascal - Clermont-Ferrand - France

ICIAM - Tokyo - August 2023

with Kuntal Bhandari (Prague), Sue Claret (Clermont-Ferrand), Sylvain Ervedoza (Bordeaux), Irene Gayte (Sevilla), Jérome Lemoine (Clermont-Ferrand), Emmanuel Trélat (Paris)

General framework and objective

GIVEN some semilinear heat/wave equation

$$\begin{cases} \partial_{t(t)} y - \Delta y + f(y) = v 1_\omega, \text{ in } \Omega \times (0,T) \\ y = 0 \text{ in } \partial\Omega \times (0,T) \quad + \text{initial conditions} \end{cases} \tag{1}$$

or

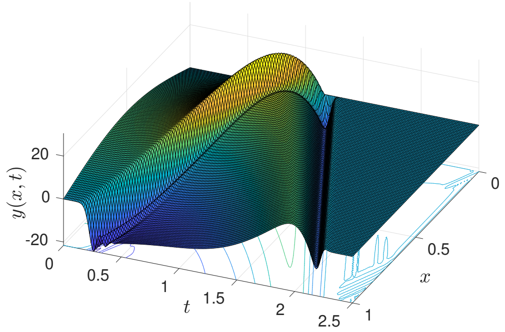
$$\begin{cases} \partial_{t(t)} y - \Delta y + f(y) = 0, \text{ in } \Omega \times (0,T), \\ y = v 1_\Gamma \text{ in } \partial\Omega \times (0,T) \quad + \text{initial conditions} \end{cases} \tag{2}$$

ASSUME that there exist control v and $T > 0$ such that $y(t) = 0$ forall $t \geq T$.

Litterature Lasiecka-Triggiani'91, Zuazua'93, Fernandez-Cara'00, Barbu'00, Li-Zhang'01, Coron-Trelat'06, Dehman-Lebeau'09, Joly-Laurent'14, Fu-Lu-Zhang'19, Friedman'19, ...

FIND a non trivial sequence $(y_k, v_k)_{k\in\mathbb{N}}$ such that $(y_k, v_k) \to (y, v)$ as $k \to \infty$, with (y, v) a controlled pair for (1) or (2)?

⚠ Non trivial question since in many situations, proofs of controllability are based on non constructive fixed point arguments.



Controlled solution (from the boundary)

$$\begin{cases} y_{tt} - y_{xx} - 3y(1 + \ln^{3/2}(2 + |y|)) = 0 & \text{in } (0,1) \times (0, 2.5), \\ y(0,\cdot) = 0, \quad y(1,\cdot) = v & \text{in } (0, 2.5), \\ (y(\cdot,0), y_t(\cdot,0)) = (10\sin(\pi x), 0) & \text{in } (0,1). \end{cases} \quad (3)$$

How do we get such control v ????

For the semilinear wave/heat equations

$$\begin{cases} \partial_{t(t)} y - \Delta y + f(y) = 0 \\ + \text{initial conditions and boundary conditions} \end{cases} \tag{4}$$

KNOWN FACT The uniform (w.r.t. initial conditions) exact controllability holds true under the following asymptotic condition on $f \in C^1(\mathbb{R})$

$$\exists \beta > 0 \text{ such that } \lim\sup_{r \to \infty} \frac{|f(r)|}{r \ln^p(r)} \leq \beta, \qquad p \in (0, 2)$$

obtained from non constructive Schauder fixed point arguments [Zuazua'93, Zuazua-Fernandez-Cara'00, Barbu'00, Zhang'07, ...]

CLAIM Under the following assumption,

$$\exists \beta > 0 \text{ such that } \lim\sup_{r \to \infty} \frac{|f'(r)|}{\ln^p(r)} \leq \beta, \qquad p \in (0, 2)$$

one can construct non trivial sequences $(y_k, v_k)_{k \in \mathbb{N}}$ converging strongly to a state-control pair for the semilinear equation

The wave and heat semilinear equations : known results and claim

For the semilinear wave/heat equations

$$\boxed{\begin{cases} \partial_{t(t)} y - \Delta y + f(y) = 0 \\ + \text{ initial conditions and boundary conditions} \end{cases}} \tag{4}$$

KNOWN FACT The uniform (w.r.t. initial conditions) exact controllability holds true under the following asymptotic condition on $f \in C^1(\mathbb{R})$

$$\exists \beta > 0 \text{ such that } \limsup_{r\to\infty} \frac{|f(r)|}{r \ln^p(r)} \leq \beta, \qquad p \in (0,2)$$

obtained from non constructive Schauder fixed point arguments [Zuazua'93, Zuazua-Fernandez-Cara'00, Barbu'00, Zhang'07, ...]

CLAIM Under the following assumption,

$$\exists \beta > 0 \text{ such that } \limsup_{r\to\infty} \frac{|f'(r)|}{\ln^p(r)} \leq \beta, \qquad p \in (0,2)$$

one can construct non trivial sequences $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly to a state-control pair for the semilinear equation

Same method for the wave and heat equation

Assuming

$$\exists \beta > 0 \text{ such that } \limsup_{r\to\infty} \frac{|f'(r)|}{\ln^p(r)} \le \beta, \qquad p \in (0,2)$$

we design convergent sequences (y_k, v_k) from two different approaches :

Method 1 : Least-squares approaches (Newton type linearization)

$$y_{k+1,t(t)} - \Delta y_{k+1} + f'(y_k)y_{k+1} = v_{k+1}1_\omega - f'(y_k)y_{k+1} + f(y_k), k \ge 0$$

where (y_{k+1}, v_{k+1}) is the optimal state-control pair for the cost

$$J(y,v) = \|v\|^2_{L^2(q_T)}$$

Method 2: Zero order linearization and Carleman weights

$$y_{k+1,t(t)} - \Delta y_{k+1} = v_{k+1}1_\omega - f(y_k), k \ge 0$$

where (y_{k+1}, v_{k+1}) is the optimal null state-control pair for the cost for the cost

$$J(y,v) = \|\rho(s)v\|^2_{L^2(q_T)} + s\|\rho_0(s)y\|^2_{L^2(q_T)}$$

involving Carleman weights $\rho(x,t,s)$, $\rho_0(x,t,s)$ and parameter $s > 0$

Assuming

$$\exists \beta > 0 \text{ such that } \limsup_{r\to\infty} \frac{|f'(r)|}{\ln^p(r)} \le \beta, \qquad p \in (0,2)$$

we design convergent sequences (y_k, v_k) from two different approaches :

Method 1 : Least-squares approaches (Newton type linearization)

$$y_{k+1,t(t)} - \Delta y_{k+1} + f'(y_k)y_{k+1} = v_{k+1}1_\omega - f'(y_k)y_{k+1} + f(y_k), k \ge 0$$

where (y_{k+1}, v_{k+1}) is the optimal state-control pair for the cost

$$J(y,v) = \|v\|^2_{L^2(q_T)}$$

Method 2: Zero order linearization and Carleman weights

$$y_{k+1,t(t)} - \Delta y_{k+1} = v_{k+1}1_\omega - f(y_k), k \ge 0$$

where (y_{k+1}, v_{k+1}) is the optimal null state-control pair for the cost for the cost

$$J(y,v) = \|\rho(s)v\|^2_{L^2(q_T)} + s\|\rho_0(s)y\|^2_{L^2(q_T)}$$

involving Carleman weights $\rho(x,t,s)$, $\rho_0(x,t,s)$ and parameter $s > 0$

We consider the nonconvex minimization problem

$$\inf_{(y,v)\in\mathcal{A}} E(y,v), \quad E(y,v) := \frac{1}{2}\left\|\partial_{tt}y - \Delta y + f(y) - v1_\omega\right\|^2_{L^2(Q_T)} \tag{5}$$

$$\mathcal{A} := \Big\{(y,v)\in\mathcal{H} \;\mid\; (y(\cdot,0),\partial_t y(\cdot,0)) = (u_0,u_1),\; (y(\cdot,T),\partial_t y(\cdot,T)) = (0,0) \text{ in } \Omega\Big\},$$

$$\mathcal{H} := \Big\{(y,v)\in L^2(Q_T)\times L^2(q_T) \;\mid\; \partial_{tt}y - \Delta y \in L^2(Q_T),\; y = 0 \text{ on } \Sigma_T,$$
$$(y(\cdot,0),\partial_t y(\cdot,0)) \in \boldsymbol{V}\Big\}$$

Proposition

$$\forall (y,v)\in\mathcal{A}, \quad \sqrt{E(y,v)} \le C e^{C\sqrt{\|f'(y)\|_\infty}} \|E'(y,v)\|_{\mathcal{A}_0'}.$$

Consequence:
Any *critical* point $(y,v)\in\mathcal{A}$ of E is a zero of E, and thus is a pair solution of the controllability problem. Moreover:

given any sequence $(y_k,v_k)_{k\in\mathbb{N}}$ *in* $\mathcal{A}$ *such that* $\|E'(y_k,v_k)\|_{\mathcal{A}_0'} \underset{k\to+\infty}{\longrightarrow} 0$ *and such that* $\|f'(y_k)\|_\infty$ *is uniformly bounded, we have* $E(y_k,v_k) \underset{k\to+\infty}{\longrightarrow} 0.$

⚠ A minimizing sequence for E cannot be stuck in a local minimum, even though E fails to be convex (it has multiple zeros).

We consider the nonconvex minimization problem

$$\inf_{(y,v)\in\mathcal{A}} E(y,v), \quad E(y,v) := \frac{1}{2}\left\|\partial_{tt}y - \Delta y + f(y) - v1_\omega\right\|^2_{L^2(Q_T)} \tag{5}$$

$$\mathcal{A} := \Big\{(y,v)\in\mathcal{H} \mid (y(\cdot,0),\partial_t y(\cdot,0)) = (u_0,u_1),\ (y(\cdot,T),\partial_t y(\cdot,T)) = (0,0) \text{ in } \Omega\Big\},$$

$$\mathcal{H} := \Big\{(y,v)\in L^2(Q_T)\times L^2(q_T) \mid \partial_{tt}y - \Delta y \in L^2(Q_T),\ y = 0 \text{ on } \Sigma_T,$$
$$(y(\cdot,0),\partial_t y(\cdot,0)) \in \boldsymbol{V}\Big\}$$

Proposition

$$\forall (y,v)\in\mathcal{A}, \quad \sqrt{E(y,v)} \le C e^{C\sqrt{\|f'(y)\|_\infty}} \|E'(y,v)\|_{\mathcal{A}_0'}.$$

Consequence:
Any *critical* point $(y,v)\in\mathcal{A}$ of E is a zero of E, and thus is a pair solution of the controllability problem. Moreover:

given any sequence $(y_k,v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|E'(y_k,v_k)\|_{\mathcal{A}_0'} \underset{k\to+\infty}{\longrightarrow} 0$ and such that $\|f'(y_k)\|_\infty$ is uniformly bounded, we have $E(y_k,v_k) \underset{k\to+\infty}{\longrightarrow} 0$.

⚠ A minimizing sequence for E cannot be stuck in a local minimum, even though E fails to be convex (it has multiple zeros).

Let the sequence $(y_k, v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ defined by

$$\boxed{\begin{cases} (y_0, v_0) \in \mathcal{A} \\ (y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k (Y_k, V_k) & \forall k \in \mathbb{N} \\ \lambda_k = \underset{\lambda\in[0,1]}{\text{argmin}}\ E\big((y_k, v_k) - \lambda(Y_k, V_k)\big) \end{cases}} \quad (6)$$

where $(Y_k, V_k) \in \mathcal{A}_0$ is the solution of minimal control norm of

$$\boxed{\begin{cases} \partial_{tt} Y_k - \Delta Y_k + f'(y_k) Y_k = V_k 1_\omega + (\partial_{tt} y_k - \Delta y_k + f(y_k) - v_k 1_\omega) & \text{in } Q_T, \\ Y_k = 0 & \text{on } \Sigma_T, \\ (Y_k(\cdot,0), \partial_t Y_k(\cdot,0)) = (0,0) & \text{in } \Omega. \end{cases}} \quad (7)$$

Lemma $\forall k \geq 0$, $E'(y_k, v_k) \cdot (Y_k, V_k) = 2E(y_k, v_k)$

Theorem (Münch-Trélat 2022, Bottois-Lemoine-Münch 2023)

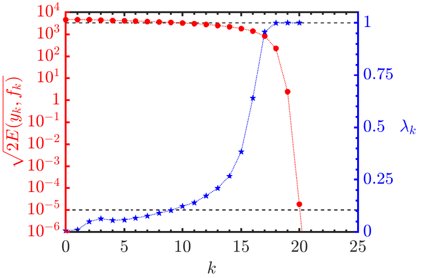
Assume that $f'' \in L^\infty(\mathbb{R})$ and that $\limsup_{r\to\infty} \frac{|f'(r)|}{\ln^p(r)}$ is small enough.
for any $(y_0, v_0) \in \mathcal{A}$, $(y_k, v_k) \to (y, v)$ a controlled pair for the nonlinear wave eq.
The convergence of these sequences is at least linear, and is at least of order 2 after a finite number of iterations.

Actually, the sequence $(y_k, v_k)_{k\geq 0}$ coincides with the one associated to damped Newton method to find a zero of the map $F : \mathcal{A} \to L^2(Q_T)$ defined by $\mathcal{F}(y, v) := (\partial_{tt} y - \Delta y + f(y) - v 1_\omega)$

$$\begin{cases} (y_0, v_0) \in \mathcal{A} \\ D\mathcal{F}(y_k, v_k) \cdot ((y_{k+1}, v_{k+1}) - (y_k, v_k)) = -\lambda_k \mathcal{F}(y_k, v_k), k \geq 0 \\ \lambda_k = \operatorname*{argmin}_{\lambda \in [0,1]} \mathcal{F}((y_k, v_k) - \lambda D\mathcal{F}^{-1}(y_k, v_k)\mathcal{F}(y_k, v_k)) \end{cases} \tag{8}$$

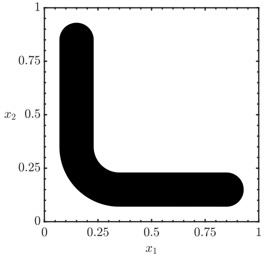
For $\lambda_k = 1$, the least-squares algorithm coincides with the Newton algorithm applied to $\mathcal{F}$ (explaining the quadratic convergence property).

Optimizing the parameter λ_k ensures the global convergence of the algorithm



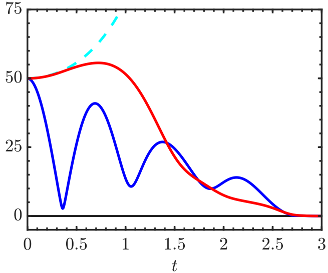
We consider a two-dimensional case for which $\Omega = (0,1)^2$ and $T = 3$.

$$\begin{cases} \partial_{tt} y - \Delta y - 10\, y \ln^{1/2}(2 + |y|) = v 1_\omega & \text{in } Q_T := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (100 \sin(\pi x_1)\sin(\pi x_2), 0) & \text{in } \Omega, \end{cases} \tag{9}$$

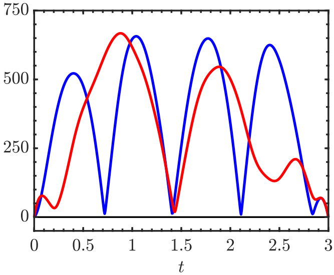


Control domain $\omega \subset \Omega = (0,1)^2$ (black part).

$\sharp$iterate k	$\sqrt{2E(y_k, v_k)}$	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\frac{\Vert v_k - v_{k-1}\Vert_{L^2_\chi(q_T)}}{\Vert v_{k-1}\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert v_k\Vert_{L^2_\chi(q_T)}$	λ_k
0	7.44×10^2	–	–	38.116	732.22	1
1	1.63×10^2	1.79×10^0	9.30×10^{-1}	58.691	667.602	1
2	1.62×10^0	8.42×10^{-2}	1.41×10^{-1}	60.781	642.643	1
3	1.97×10^{-3}	1.21×10^{-3}	4.66×10^{-3}	60.745	643.784	1
4	5.11×10^{-10}	6.43×10^{-7}	2.63×10^{-6}	60.745	643.785	–



(red) $\Vert y_4(\cdot,t)\Vert_{L^2(\Omega)}$; (blue) $\Vert y_0(\cdot,t)\Vert_{L^2(\Omega)}$; (dashed) $\Vert y(\cdot,t;0)\Vert_{L^2(\Omega)}$.



(red) $\Vert v_4(\cdot,t)\Vert_{L^2_\chi(\omega)}$; (blue) $\Vert v_0(\cdot,t)\Vert_{L^2_\chi(\omega)}$.

$$\begin{cases} y_{tt} - \Delta y + f(y) = 0 & \text{in } Q_T, \\ y = v\,1_{\Gamma_0} \text{in } \partial\Omega \times (0,T), \quad (y(\cdot,0), y_t(\cdot,0)) = (u_0,u_1) & \text{in } \Omega, \end{cases} \tag{10}$$

Theorem (Bhandari-Lemoine-Münch 2022, Claret-Lemoine-Münch, 2023)

Assume $T > 0$ and $\Gamma_0 \subset \partial\Omega$ large enough. Assume that there exists $\beta^\star > 0$ such that if $f \in \mathcal{C}^1(\mathbb{R})$ satisfies

$$(\mathbf{H_2'}) \quad \exists \alpha > 0, \ s.t.\ |f'(r)| \leq \alpha + \beta^\star \ln_+^{3/2}|r|, \quad \forall r \in \mathbb{R}$$

then, for any initial state (u_0,u_1) in $\boldsymbol{H} := L^2(\Omega) \times H^{-1}(\Omega)$, the controlled sequence $(y_k, v_k)_{k\in\mathbb{N}^*}$ solution of

$$\begin{cases} \partial_{tt} y_{k+1} - \Delta y_{k+1} = -f(y_k) & \text{in } Q_T, \\ y_{k+1} = v_{k+1}\, 1_{\Gamma_0} & \text{in } \partial\Omega \times (0,T), \\ (y_{k+1}(\cdot,0), \partial_t y_{k+1}(\cdot,0)) = (u_0,u_1) & \text{in } \Omega, \end{cases} \tag{11}$$

minimizer of a functional $J_s(y,v) := s\int_{Q_T} \rho^2(s) y^2 + \int_0^T \int_{\Gamma_0} \eta^{-2}\rho_1^2(s) v^2$ converges strongly to a controlled pair (y,v) in $(\mathcal{C}^0([0,T];H^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))) \times H_0^1(0,T)$ for the system (10).
The convergence holds with a linear rate for the norm $\|\rho(s)\cdot\|_{L^2(Q_T)} + \|\rho_1(s)\cdot\|_{L^2(0,T)}$ and s is chosen sufficiently large depending on $\|(u_0,u_1)\|_{\boldsymbol{H}}$.

Contraction of the operator within a suitable class

• For any $s \geq s_0$ and $R > 0$, we introduce the class $\mathcal{C}_R(s)$, defined as the closed convex subset of $L^\infty(Q_T)$

$$\mathcal{C}_R(s) := \left\{ \widehat{y} \in L^\infty(Q_T) : \ \|\widehat{y}\|_{L^\infty(Q_T)} \leq R, \ \ \|\rho(s)\widehat{y}\|_{L^2(Q_T)} \leq R^{1/2} \right\} \tag{12}$$

and assume that that the nonlinear function $f' \in \mathcal{C}^0(\mathbb{R})$ in (21) satisfies the logarithmic assumption for some $\beta^\star$ positive precisely chosen later.

• We introduce the operator

$$\Lambda_s : L^\infty(Q_T) \mapsto L^\infty(Q_T), \qquad \Lambda_s(\widehat{y}) = y \tag{13}$$

where y solves

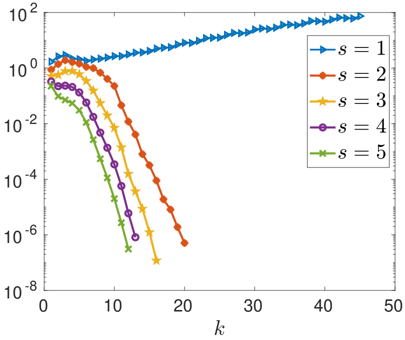
$$\begin{cases} y_{tt} - \Delta y = -f(\widehat{y}) & \text{in } Q_T, \\ y(0,\cdot) = 0, \ y(1,\cdot) = v & \text{in } (0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{14}$$

satisfies $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1)$ in Ω, and (y, v) corresponds to the minimizer of a functional J_s

$$J_s(y,v) := s \int_{Q_T} \rho^2(s) y^2 + \int_0^T \eta^{-2} \rho_1^2(s) v^2 \tag{15}$$

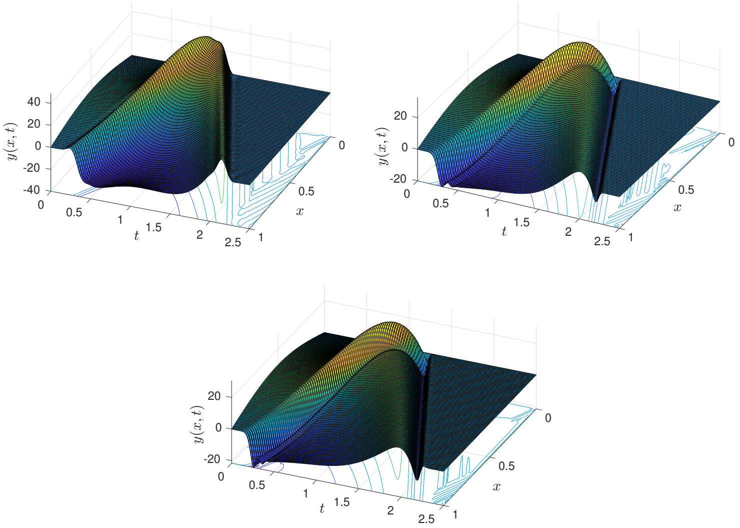
over the set
$\Big\{ (y,v) \in L^2(Q_T) \times L^2(0,T)$ solution of (14) with $y(\cdot,T) = y_t(\cdot,T) = 0$ in $\Omega \Big\}$.

$$\begin{cases} y_{tt} - y_{xx} - 5y(1+\ln^{3/2}(2+|y|)) = 0 & \text{in } (0,1)\times(0,2.5), \\ y(0,\cdot)=0, \quad y(1,\cdot)=v & \text{in } (0,2.5), \\ (y(\cdot,0), y_t(\cdot,0)) = (20\sin(\pi x), 0) & \text{in } \Omega, \end{cases} \tag{16}$$



Relative error $\frac{\|\rho(s)y_{k+1}-\rho(s)y_k\|_{L^2(Q_T)}}{\|\rho(s)y_k\|_{L^2(Q_T)}}$ w.r.t. iterations k.

Numerical illustration - 1d case - boundary control



Controlled solution for $s \in \{2, 5, 9\}$.

References for the wave equation

Least-squares approaches (Newton type linearization)

• Münch-Trélat. *Constructive exact control of semilinear 1D wave equations by a least-squares approach*, SICON 2022

• Bottois-Lemoine-Münch. *Constructive proof of the exact controllability for semi-linear multi-dimensional wave equations*, AMSA 2023

Zero order linearization and Carleman weights

• Bhandari-Lemoine-Münch. *Exact boundary controllability of semilinear 1D wave equations through a constructive approach*, MCSS 2023

• Lemoine-Münch-Sue. *- Exact boundary controllability of semilinear wave equations through a constructive approach*, arxiv, 2023

Least-squares approaches (Newton type linearization)

• Lemoine, Marin-Gayte, Munch, *Approximation of null controls for semilinear heat equations using a least-squares approach*, COCV, 2021

• Lemoine-Munch. *Constructive exact control of semilinear 1D heat equations*, MCRF 2023

Zero order linearization and Carleman weights

• Ervedoza-Lemoine-Munch. *Exact controllability of semilinear heat equations through a constructive approach*, EECT 2023

• Bhandari-Lemoine-Munch. *- Global boundary null controllability of one dimensional semi-linear heat equation,* DCDS, 2023

$$\begin{cases} \partial_t y - \Delta y + f(y) = v 1_\omega & \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{17}$$

Theorem (Lemoine, Munch, 22)

Let $T > 0$ be given. Let $d = 1$. Assume that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f(0) = 0$ and

$$(\mathbf{H}_1') \quad \exists \alpha > 0, \ s.t. \ |f'(r)| \leq (\alpha + \beta^\star \ln_+ |r|)^{3/2}, \quad \forall r \in \mathbb{R}$$

for some $\beta^\star > 0$ small enough and

$$(\overline{\mathbf{H}}_\mathbf{p}) \quad \exists p \in [0, 1] \text{ such that } \sup_{\substack{a,b \in \mathbb{R} \\ a \neq b}} \frac{|f'(a) - f'(b)|}{|a - b|^p} < +\infty.$$

Then, for any $u_0 \in H_0^1(\Omega)$, one can construct a sequence $(y_k, v_k)_{k \in \mathbb{N}}$ converging strongly to a controlled pair for (18) satisfying $y(T) = 0$. Moreover, after a finite number of iterations, the convergence is of order at least $1 + p$.

1

[1] Lemoine, Münch, Constructive exact control of semilinear 1D heat equations. MCRF 2022

Heat eq. and zero order linearization/Carleman setting

$$\begin{cases} \partial_t y - \Delta y + f(y) = v 1_\omega \quad \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{18}$$

Theorem (Ervedoza-Lemoine-Münch 2022, Bandhari-Lemoine-Münch, 2023)

Let $T > 0$ be given and $d \geq 5$. Assume that there exists $\beta^\star > 0$ such that if $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f(0) = 0$ and

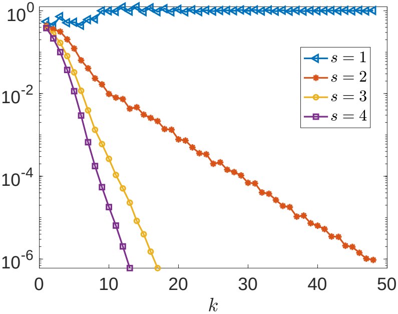
$$(\mathbf{H_2'}) \quad \exists \alpha > 0, \text{ s.t. } |f'(r)| \leq \alpha + \beta^\star \ln_+^{3/2} |r|, \quad \forall r \in \mathbb{R}$$

Then, for any initial state $u_0 \in L^\infty(\Omega)$, the controlled sequence $(y_k, v_k)_{k \in \mathbb{N}^*}$ solution of

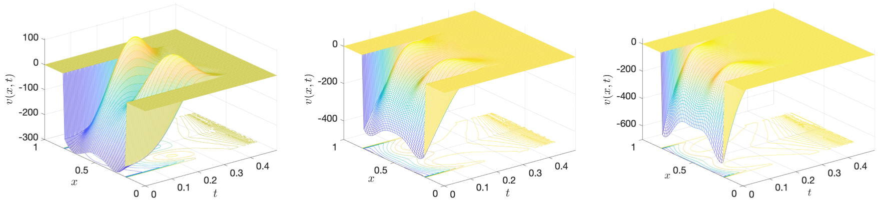
$$\begin{cases} \partial_t y_{k+1} - \Delta y_{k+1} = v_{k+1} 1_\omega - f(y_k) & \text{in } Q_T, \\ y_{k+1} = 0 & \text{in } \partial\Omega \times (0, T), \\ y_{k+1}(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{19}$$

optimal for the cost $J_s(y, v) := s \int_{Q_T} \rho^2(s) y^2 + \int_0^T \int_\omega \rho_0^2(s) v^2$ converges strongly to a controlled pair (y, v) in $L^2(Q_T) \times L^2(q_T)$ for the system (21).
The convergence holds with a linear rate for the norm $\|\rho(s) \cdot\|_{L^2(Q_T)} + \|\rho_1(s) \cdot\|_{L^2(0,T)}$ and s is chosen sufficiently large depending on $\|u_0\|_{L^\infty(\Omega)}$.

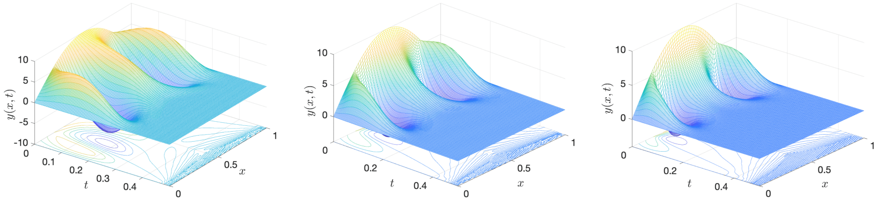
$$\begin{cases} y_t - y_{xx} - 5y(1+\ln^{3/2}(2+|y|)) = 1_{(0.2,0.8)}(x)v & \text{in } (0,1)\times(0,0.5), \\ y(0,\cdot) = y(1,\cdot) = 0 & \text{in } (0,0.5), \\ y(\cdot,0) = 10\sin(\pi x) & \text{in } (0,1), \end{cases} \tag{20}$$



Relative error $\frac{\|\rho_0(s)(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_0(s)y_k\|_{L^2(Q_T)}}$ w.r.t. k for $s \in \{1,2,3,4\}$.

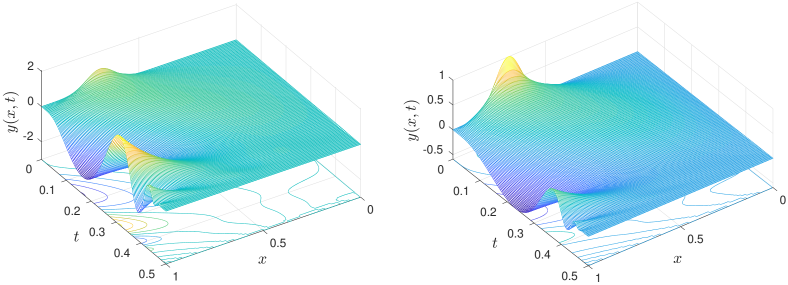


The distributed control v in $(0,1) \times (0,0.5)$ for $s \in \{2,3,4\}$.



The controlled solution in $(0,1) \times (0,0.5)$ for $s \in \{2,3,4\}$.

Numerical illustration for the 1d heat equation and boundary control



Controlled solutions from the boundary with $\nu \in \{0.3, 0.5\}$.

$$\begin{cases} y_t - \nu y_{xx} - 2.5y(1+\ln^{3/2}(2+|y|)) = 0 & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(1,\cdot) = v & \text{in } (0,T), \\ y(\cdot,0) = e^{-100(x-0.7)^2} & \text{in } \Omega, \end{cases} \tag{21}$$

Main conclusion

For the semilinear wave/heat equations

$$\boxed{\begin{cases} \partial_{t(t)} y - \Delta y + f(y) = 0 \\ + \text{ initial conditions and boundary conditions} \end{cases}} \tag{22}$$

assuming mainly $f \in \mathcal{C}^1(\mathbb{R})$ and the growth assumption

$$\exists \beta > 0 \text{ such that } \lim_{r \to \infty} \frac{|f'(r)|}{\ln^p(r)} \leq \beta, \qquad p \in (0, 2)$$

one can now construct non trivial sequences $(y_k, v_k)_{k \in \mathbb{N}}$ converging strongly to a state-control pair for the semilinear equation.

THANK YOU VERY MUCH FOR YOUR ATTENTION