Space-time method for controllability problems (toward a space-time DDM ???)

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We address the approximation of null controls for a semi-linear wave equation

$$y_{tt} - \Delta y + f(y) = v 1_\omega, \qquad \Omega \times (0,T) \qquad (1)$$

Part 1: Find a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly to a control-state pair for (1)?

Typically, (y_k, v_k) solves a linear controllability problem for

$$z_{tt} - \Delta z + Az = u 1_\omega + B, \qquad \Omega \times (0,T) \qquad (2)$$

Part 2: for each k, find a convergent numerical approximation $(y_{kh}, v_{kh})_{h>0}$ of (y_k, v_k) for (2)?

Rk. $\|v - v_{k,h}\| \leq \|v - v_k\| + \|v_k - v_{k,h}\|$

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- Let $\Omega \subset \mathbb{R}^d$, $\omega \subset \Omega$, $T > 0$. $Q_T := \Omega \times (0,T)$, $q_T := \omega \times (0,T)$.

$$\boxed{\begin{cases} \partial_{tt} y - \Delta y + f(y) = v 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases}} \tag{3}$$

- $(u_0, u_1) \in \boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$, $v \in L^2(q_T)$. $f \in C^1(\mathbb{R};\mathbb{R})$.
- $|f(r)| \leq C(1 + |r|) \ln^2(2 + |r|)\ \forall r \in \mathbb{R}$
- $y \in \mathcal{C}^0([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ is unique.

Definition

(3) is null controllable in time T IFF for any $(u_0, u_1) \in \boldsymbol{V}$, $\exists$ a control function $v \in L^2(q_T)$ such that $(y(\cdot,T), \partial_t y(\cdot,T)) = (0,0)$.

Theorem (Zuazua'93, Zhang '00, ...)

If T and ω are large enough and if f does not grow too fast at infinity

$$(\mathbf{H_1}) \quad \limsup_{|r|\to\infty} \frac{|f(r)|}{|r| \ln^{1/2}|r|} < \beta$$

(for some $\beta > 0$ small enough) then (3) is exactly controllable in time T.

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(for some $\beta > 0$ small enough) then (3) *is exactly controllable in time T.*

Part 1: Construction of a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly to a solution of the semilinear pb.

- Bottois, Lemoine, M. *Constructive exact controls for semi-linear wave equations*, arxiv.
- Trélat, M. *Constructive exact control of semilinear 1D wave equations by a least-squares approach*, SICON 2022
- Bhandari, Lemoine, M. *Exact boundary controllability of 1D semilinear wave equations through a constructive approach*, AIMS EECT 2022
- Lemoine, M, Sue. *Exact boundary controllability of semilinear wave equations through a constructive approach*, arxiv.

The controllability proof given in Zuazua'93, Zhang'00 is based on a Leray Schauder fixed point argument.
Let $\Lambda : L^\infty(0,T;L^d(\Omega)) \to L^\infty(0,T;L^d(\Omega))$, where $y := \Lambda(\xi)$ is a controlled solution with the control v_ξ of the linear problem (assuming $f(0)=0$)

$$\begin{cases} \partial_{tt} y - \Delta y + y\, \dfrac{f(\xi)}{\xi} = v_\xi 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0,u_1) & \text{in } \Omega, \\ (y(\cdot,T), \partial_t y(\cdot,T)) = (0,0) & \text{in } \Omega. \end{cases} \quad (4)$$

Then, Λ has a fixed point.

Useless in practice, since Λ is not contracting: the Picard sequence $y_{k+1} = \Lambda(y_k)$ is bounded but not convergent.

We consider the Hilbert space

$$\mathcal{H} := \Big\{ (y,v) \in L^2(Q_T) \times L^2(q_T) \mid \partial_{tt} y - \Delta y \in L^2(Q_T),\ y = 0 \text{ on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) \in \boldsymbol{V} \Big\}$$

and the subspace

$$\mathcal{A} := \Big\{ (y,v) \in \mathcal{H} \mid (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0,u_1),\ (y(\cdot,T), \partial_t y(\cdot,T)) = (0,0) \text{ in } \Omega \Big\}$$

We define the least-squares functional $E : \mathcal{A} \to \mathbb{R}$ by

$$\boxed{E(y,v) := \frac{1}{2} \big\| \partial_{tt} y - \Delta y + f(y) - v 1_\omega \big\|^2_{L^2(Q_T)}}$$

and consider the nonconvex minimization problem

$$\boxed{\inf_{(y,v) \in \mathcal{A}} E(y,v)} \tag{5}$$

Proposition

$\forall (y,v) \in \mathcal{A}$,

$$\sqrt{E(y,v)} \leq C e^{C\sqrt{\|f'(y)\|_\infty}} \|E'(y,v)\|_{\mathcal{A}_0'}. \tag{6}$$

Consequence:
Any *critical* point $(y,v) \in \mathcal{A}$ of E (i.e., $E'(y,v) = 0$) is a zero of E, and thus is a pair solution of the controllability problem. Moreover:

given any sequence $(y_k, v_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|E'(y_k,v_k)\|_{\mathcal{A}_0'} \underset{k\to+\infty}{\longrightarrow} 0$ and such that $\|f'(y_k)\|_\infty$ is uniformly bounded, we have $E(y_k,v_k) \underset{k\to+\infty}{\longrightarrow} 0$.

Thanks to this instrumental property, a minimizing sequence for E cannot be stuck in a local minimum, even though E fails to be convex (it has multiple zeros).

Strong convergence of the LS method

$(y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k (Y_k, V_k)$ where (Y_k, V_k) solves

$$\begin{cases} \partial_{tt} Y_k - \Delta Y_k + f'(y_k) Y_k = V_k 1_\omega + (\partial_{tt} y_k - \Delta y_k + f(y_k) - v_k 1_\omega) & \text{in } Q_T, \\ Y_k = 0 & \text{on } \Sigma_T, \\ (Y_k(\cdot,0), \partial_t Y_k(\cdot,0)) = (0,0), \quad (Y_k(\cdot,T), \partial_t Y_k(\cdot,T)) = (0,0) & \text{in } \Omega, \end{cases} \tag{7}$$

and $\lambda_k \in (0,1)$ the optimal (damped Newton) parameter.
Here, V_k is the control of minimal $L^2(q_T)$-norm.

Theorem (M-Trélat 2021 ($d = 1$), Bottois-Lemoine-M 22 ($d > 1$))

Assume that T, $\omega \subset \Omega \subset \mathbb{R}^d$ large enough and f Loc Lip satisfies

$$|f'(r)| \leq \alpha + \beta \ln^2(1 + |r|) \qquad \forall r \in \mathbb{R}. \tag{8}$$

For any $(y_0, v_0) \in \mathcal{A}$, the minimizing sequence $(y_k, v_k)_{k \in \mathbb{N}}$ for E converges strongly to a state-control (y, v) for the nonlinear wave eq. The convergence is sur-linear after a finite number of iterations.

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a second constructive method : simpler Linearization leading to contracting prop.

We introduce the operator $\Lambda_s : L^\infty(Q_T) \mapsto L^\infty(Q_T), \qquad \Lambda_s(\widehat{y}) = y$
where y solves

$$\begin{cases} y_{tt} - \Delta y = -f(\widehat{y}) & \text{in } Q_T, \\ y = 0, \ y = v\, 1_{\Gamma_0} & \text{in } \partial\Omega \times (0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \\ (y(\cdot,T), y_t(\cdot,T)) = (0,0) & \text{in } \Omega, \end{cases} \tag{9}$$

and (y, v) corresponds to the minimizer of a functional J_s

$$J_s(y,v) := s \int_{Q_T} \rho^2(s) y^2 + \int_{\Gamma_0} \int_0^T \eta^{-2} \rho_1^2(s) v^2 \tag{10}$$

$\rho(s), \rho_1(s) \approx e^{s\phi(x,t)}$ are Carleman weights; $s > 0$ is a Carleman parameter;

Theorem (Bhandari, Lemoine, M. ($d = 1$), Lemoine Sue M. ($d \geq 1$))

Let $\Omega \subset \mathbb{R}^d$; assume T, Γ_0 are large enough, and f Loc Lip satisfies

$$|f'(r)| \leq \alpha + \beta \ln^{3/2}(1 + |r|) \qquad \forall r \in \mathbb{R}. \tag{11}$$

If $s > 0$ is large enough, then Λ_s is contracting.

For any y_0, The picard iterate $y_{k+1} = \Lambda_s(y_k)$ converges to a controlled solution for the nonlinear pb !!!

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Part 2: (Space-time Numerical) approximation of null controls for linear wave equation

$$\partial_{tt} y - \Delta y + Ay = v 1_\omega + B \quad \text{in } Q_T, \tag{12}$$

- control of minimal $L^2(q_T)$ norm

$$\inf J(y, v) = \int_{q_T} v^2, \quad (y, v) \quad \text{solves} \quad (12)$$

- state-control pair with weighted cost :

$$\inf J(y, v) = \int_{Q_T} \rho^2 y^2 + \int_{q_T} \rho_1^2 v^2, \quad (y, v) \quad \text{solves} \quad (12)$$

Part 2: (space-time Numerical) approximation of null controls for linear wave equation

- Burman, Feizmohammadi, M, Oksanen. *Space-time stabilized finite element methods for a unique continuation problem subject to the wave equation,* M2AN 2021
- Montaner, M. *Approximation of controls for the linear wave equation: a first order mixed formulation*, AIMS MCRF 2019
- Cindea, M. *A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations,* Calcolo 2015
- Cindea, Fernandez-Cara, M, . *Numerical controllability of the wave equation through primal methods and Carleman estimates,* ESAIM COCV 2013
- M, . *A uniformly controllable and implicit scheme for the 1-D wave equation,* ESAIM M2AN 2005

Boundary control of minimal L^2 norm

$$\begin{cases} \text{Minimize } \; J(y,v) = \dfrac{1}{2}\displaystyle\int_0^T \int_{\Gamma_0} |v|^2 \, dt \\ \text{Subject to } \; (y,v) \in \mathcal{C}(y_0, y_1; T) \end{cases} \tag{13}$$

where $\mathcal{C}(y_0, y_1; T)$ denotes the non-empty linear manifold

$$\mathcal{C}(y_0, y_1; T) = \{ (y,v) : v \in L^2(\Gamma_0 \times (0,T)), \; y \text{ solves } (12) \}.$$

Using the Fenchel-Rockafellar theorem [Ekeland-Temam 74], [Brezis 84] we get that

$$\inf_{(y,v)\in\mathcal{C}(y_0,y_1;T)} J(y,v) = - \min_{(\varphi_0,\varphi_1)\in \boldsymbol{V}} J^\star(\varphi_0,\varphi_1)$$

$$\begin{cases} \text{Minimize } \; J^\star(\varphi_0,\varphi_1) = \dfrac{1}{2}\displaystyle\int_0^T\int_{\Gamma_0} \left|\dfrac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < (\varphi_0,\varphi_1),(y_0,y_1) > \\ \text{Subject to } \; (\varphi_0,\varphi_1) \in \boldsymbol{V} := H_0^1(\Omega)\times L^2(\Omega) \quad \text{where} \quad L\varphi := \varphi_{tt} - \Delta\phi = 0 \end{cases} \tag{14}$$

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The stable/consistant centered finite difference scheme with $\Delta t < h$:

$$(\overline{\mathcal{S}}_{h,\Delta t}) \begin{cases} \Delta_{\Delta t} y_{h,\Delta t} - \Delta_h y_{h,\Delta t} = 0, \\ + \text{Initial conditions and Boundary terms} \end{cases} \tag{15}$$

produces a non discrete uniformly bounded and converging control under the condition $\Delta t < h$.

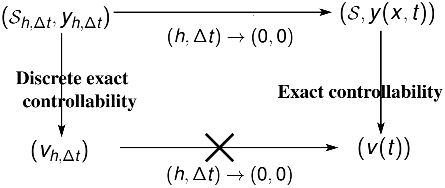

Figure: Non commuting diagram associated to the scheme $(\overline{\mathcal{S}}_{h,\Delta t})$ for $\Delta t < h$.

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } Q_T, \\ y(0,t) = 0, y(1,t) = v(t) & \text{on } (0,T), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (4x 1_{(0,1/2)}(x), 0) & \text{in } \Omega, \end{cases} \quad (16)$$

The control of minimal $L^2(0,T)$ norm is $v(t) = 2(1-t)1_{(1/2,3/2)}(t)$.

The corresponding controlled solution is

$$y(x,t) = \begin{cases} 4x & 0 \le x+t < \frac{1}{2}, \\ 2(x-t) & -\frac{1}{2} < t-x < \frac{1}{2}, \quad x+t \ge \frac{1}{2}, \\ 0 & \text{elsewhere}, \end{cases} \quad (17)$$

The initial condition of the adjoint solution is
$(\phi_0, \phi_1) = (0, -2x\,1_{(0,1/2)}(x)) \in H^1(\Omega) \times H^0(\Omega)$, which gives:

$$\phi(x,t) = \begin{cases} -2xt & 0 \le x+t < \frac{1}{2}, \quad x \ge 0, t \ge 0, \\ \frac{(x-t)^2}{2} - \frac{1}{8} & \frac{1}{2} \le x+t < \frac{3}{2}, \quad -\frac{1}{2} < x-t < \frac{1}{2}, \\ 2(x-1)(1-t) & \frac{3}{2} \le x+t, \quad -\frac{1}{2} < x-t, \\ -\frac{(x+t-2)^2}{2} + \frac{1}{8} & \frac{3}{2} < x+t < \frac{5}{2}, \quad -\frac{3}{2} < x-t \le -\frac{1}{2}, \\ 2x(2-t) & x-t \le -\frac{3}{2}. \end{cases} \quad (18)$$

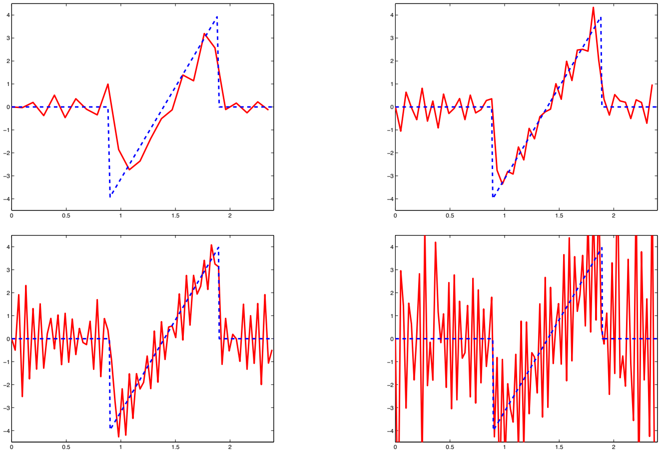

Figure: Control $P(\boldsymbol{v_h})(t)$ vs. $t \in [0, T]$, $\Delta t = 0.98h$, $T = 2.4$ and $h = 1/10$, $h = 1/20$, $h = 1/30$, $h = 1/40$.

We now replace the problem

$$\begin{cases} \text{Min} \;\; J^\star(\varphi_0,\varphi_1) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_{\Gamma_0}\left|\dfrac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < y_0,\, \varphi_1 >_{L^2} - \langle y_1, \varphi_0\rangle_{H^{-1},H_0^1} \\ \text{Subject to} \;\; (\varphi_0,\varphi_1) \in \boldsymbol{V} = H_0^1(\Omega)\times L^2(\Omega) \quad \text{where} \quad L\varphi = 0 \end{cases} \tag{19}$$

by the equivalent problem

$$\begin{cases} min\, J^\star(\varphi) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_{\Gamma_0}\left|\dfrac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < y_0,\, \varphi_t(\cdot,0) >_{L^2} - \langle y_1, \varphi(\cdot\,,0)\rangle_{H^{-1},H_0^1} \\ \text{Subject to}\; \varphi \in \boldsymbol{W} := \left\{\varphi : \varphi \in C^0(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega)),\; L\varphi = 0 \in L^2(Q_T)\right\} \end{cases} \tag{20}$$

Remark- If $\varphi \in \boldsymbol{W}$ then $\frac{\partial\varphi}{\partial\nu} \in L^2(\Gamma_T)$

Remark- $\boldsymbol{W}$ endowed with the norm $\|\varphi\|_{\boldsymbol{W}} := \|\frac{\partial\varphi}{\partial\nu}\|_{L^2(\Gamma_T)}$ is an Hilbert space.

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Remark- If $\varphi\in\boldsymbol{W}$ then $\frac{\partial\varphi}{\partial\nu}\in L^2(\Gamma_T)$

Remark- $\boldsymbol{W}$ endowed with the norm $\|\varphi\|_{\boldsymbol{W}} := \|\frac{\partial\varphi}{\partial\nu}\|_{L^2(\Gamma_T)}$ is an Hilbert space.

We assume T and Γ_0 large enough. We now replace the problem

$$\begin{cases} \text{Min} \;\; J^\star(\varphi_0,\varphi_1) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_{\Gamma_0}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + < y_0,\, \varphi_1 >_{L^2} - \langle y_1, \varphi_0\rangle_{H^{-1},H_0^1} \\ \text{Subject to} \;\; (\varphi_0,\varphi_1)\in \boldsymbol{V} = H_0^1(\Omega)\times L^2(\Omega) \quad \text{where} \quad L\varphi = 0 \end{cases} \quad (21)$$

by the equivalent problem

$$\begin{cases} \min J_r{}^\star(\varphi) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_{\Gamma_0}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\, dt + \frac{r}{2}\|L\varphi\|^2_{L^2(Q_T)} + < y_0,\, \varphi_t(\cdot,0) >_{L^2} - \langle y_1, \varphi(\cdot\,,0)\rangle_{H^{-1},H_0^1} \\ \text{Subject to}\; \varphi\in \boldsymbol{W} := \left\{\varphi : \varphi \in C^0(0,T; H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega)),\; L\varphi = 0 \in L^2(Q_T)\right\} \end{cases} \quad (22)$$

for all $r \geq 0$.

Remark- If $\varphi\in \boldsymbol{W}$ then $\frac{\partial\varphi}{\partial\nu}\in L^2(\Gamma_T)$

Remark- $\boldsymbol{W}$ endowed with the norm $\|\varphi\|_{\boldsymbol{W}} := \|\frac{\partial\varphi}{\partial\nu}\|_{L^2(\Gamma_T)}$ is an Hilbert space.

In order to address the $L^2(Q_T)$ constraint $L\varphi = 0$, we introduce a Lagrange multiplier $\lambda \in L^2(Q_T)$; we consider the saddle point problem :

$$\begin{cases} \sup_{\lambda \in L^2(Q_T)} \inf_{\varphi \in \mathbf{\Phi}} \mathcal{L}_r(\varphi, \lambda), \\ \mathcal{L}_r(\varphi, \lambda) := J_r(\varphi) + < L\varphi, \lambda >_{L^2(Q_T)} \\ \mathbf{\Phi} := \left\{ \varphi : \varphi \in C^0(0, T; H_0^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), L\varphi \in L^2(Q_T) \right\} \supset \boldsymbol{W} \end{cases} \tag{23}$$

Remark- $\mathbf{\Phi}$ is endowed with the inner product,

$$< \varphi, \overline{\varphi} >_{\mathbf{\Phi}} := < \frac{\partial \varphi}{\partial \nu}, \frac{\partial \overline{\varphi}}{\partial \nu} >_{L^2(\Gamma_T)} + < L\varphi, L\overline{\varphi} >_{L^2(Q_T)}, \quad \forall \varphi, \overline{\varphi} \in \mathbf{\Phi}.$$

$\|\varphi\|_{\mathbf{\Phi}} := \sqrt{< \varphi, \varphi >_{\mathbf{\Phi}}}$ is a norm and $(\mathbf{\Phi}, \|\cdot\|_{\mathbf{\Phi}})$ is an Hilbert space.

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$\|\varphi\|_{\mathbf{\Phi}} := \sqrt{< \varphi, \varphi >_{\mathbf{\Phi}}}$ is a norm and $(\mathbf{\Phi}, \|\cdot\|_{\mathbf{\Phi}})$ is an Hilbert space.

Find $(\varphi, \lambda) \in \mathbf{\Phi} \times L^2(Q_T)$ solution of

$$\begin{cases} a_r(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) &= l(\overline{\varphi}), & \forall \overline{\varphi} \in \mathbf{\Phi} \\ b(\varphi, \overline{\lambda}) &= 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{cases} \tag{24}$$

where

$$a_r : \mathbf{\Phi} \times \mathbf{\Phi} \to \mathbb{R}, \quad a_r(\varphi, \overline{\varphi}) = < \frac{\partial \varphi}{\partial \nu}, \frac{\partial \overline{\varphi}}{\partial \nu} >_{L^2(\Gamma_T)} + r < L\varphi, L\overline{\varphi} >_{L^2(Q_T)} \tag{25}$$

$$b : \mathbf{\Phi} \times L^2(Q_T) \to \mathbb{R}, \quad b(\varphi, \lambda) = < L\varphi, \lambda >_{L^2(Q_T)} \tag{26}$$

$$l : \mathbf{\Phi} \to \mathbb{R}, \quad l(\varphi) = - < y_0, \varphi_t(\cdot, 0) >_{L^2} + \langle y_1, \varphi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \tag{27}$$

Rk. The continuity of the linear form l derives from generalized observability ineq.

$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_V^2 \leq C_{obs} \left(\left\| \frac{\partial \varphi}{\partial \nu} \right\|_{L^2(\Gamma_T)}^2 + \|L\varphi\|_{L^2(Q_T)}^2 \right), \qquad \forall \varphi \in \mathbf{\Phi} \tag{28}$$

Let then Φ_h and Λ_h be two finite dimensional spaces parametrized by the variable h such that

$$\Phi_h \subset \mathbf{\Phi}, \quad \Lambda_h \subset L^2(Q_T), \qquad \forall h > 0.$$

Then, we can introduce the following approximated problems : find $(\varphi_h, \lambda_h) \in \Phi_h \times \Lambda_h$ solution of

$$\begin{cases} a_r(\varphi_h, \overline{\varphi}_h) + b(\overline{\varphi}_h, \lambda_h) & = & l(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_h \\ b(\varphi_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{cases} \tag{29}$$

For any $h > 0$, the well-posedness is again a consequence of two properties

the coercivity of the bilinear form a_r on the subset $\mathcal{N}_h(b) = \{\varphi_h \in \Phi_h; b(\varphi_h, \lambda_h) = 0 \quad \forall \lambda_h \in \Lambda_h\}$. From the relation

$$a_r(\varphi, \varphi) \geq \frac{r}{\eta} \|\varphi\|^2_{\mathbf{\Phi}}, \quad \forall \varphi \in \mathbf{\Phi}$$

the form a_r is coercive on the full space $\mathbf{\Phi}$, and so *a fortiori* on $\mathcal{N}_h(b) \subset \Phi_h \subset \mathbf{\Phi}$.

The second property is a discrete inf-sup condition : there exists $\delta > 0$ such that

$$\delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_h} \|\lambda_h\|_{\Lambda_h}} \geq \delta. \tag{30}$$

A necessary condition is: $\dim(\Phi_h) > \dim(\Lambda_h)$

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$$\left\{ \begin{array}{rcll} a_r(\varphi_h, \overline{\varphi}_h) + b(\overline{\varphi}_h, \lambda_h) & = & l(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_h \\ b(\varphi_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in \Lambda_h. \end{array} \right. \tag{29}$$

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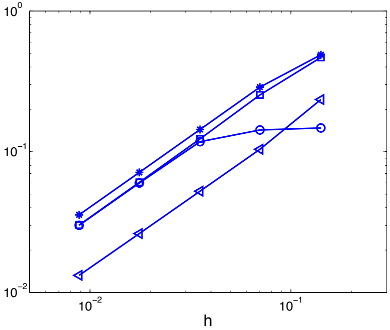

Figure: BFS finite element - Evolution of $\sqrt{r}\delta_{h,r}$ with respect to h for $r = 1$ ($\square$), $r = 10^{-2}$ ($\circ$), $r = h$ ($\star$) and $r = h^2$ ($<$).

$$\delta_h \approx C_r \frac{h}{\sqrt{r}} \quad \text{as} \quad h \to 0^+ \tag{31}$$

[1]

$\alpha > 0$

$$\begin{cases} \sup_{\lambda\in\Lambda} \inf_{\varphi\in\Phi} \mathcal{L}_{r,\alpha}(\varphi,\lambda), \\ \mathcal{L}_{r,\alpha}(\varphi,\lambda) := \mathcal{L}_r(\varphi,\lambda) - \frac{\alpha}{2}\|L\lambda\|^2_{L^2(H^{-1}(\Omega))} - \frac{\alpha}{2}\|\lambda - \partial_\nu\varphi\|^2_{L^2(\Gamma_T)}. \end{cases} \tag{32}$$

$$\Lambda := \Big\{\lambda : \lambda \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega)),$$

$$L\lambda \in L^2([0,T];H^{-1}(\Omega)), \lambda(\cdot,0) = \lambda_t(\cdot,0) = 0, \lambda_{|\Gamma_T} \in L^2(\Gamma_T)\Big\}.$$

Λ is a Hilbert space endowed with the following inner product

$$\langle\lambda,\overline{\lambda}\rangle_\Lambda := \int_0^T \langle L\lambda(t), L\overline{\lambda}(t)\rangle_{H^{-1}(\Omega)} dt + \iint_{\Gamma_T} \lambda\overline{\lambda} d\sigma dt, \qquad \forall\, \lambda,\ \overline{\lambda} \in \Lambda$$

using notably that

$$\|\lambda\|_{L^2(Q_T)} \le C_{\Omega,T}\sqrt{<\lambda,\lambda>_\Lambda}, \quad \forall\lambda\in\Lambda \tag{33}$$

for some positive constant $C_{\Omega,T}$. We denote $\|\lambda\|_\Lambda := \sqrt{<\lambda,\lambda>_\Lambda}$.

[1] H. Barbosa, T. Hugues : The finite element method with Lagrange multipliers on the boundary: circumventing the BabusÃ¿ka-Brezzi condition, 1991

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Illustration for non smooth initial condition - Boundary control - $d = 1$

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } Q_T, \\ y(0,t) = 0, y(1,t) = v(t) & \text{on } (0,T), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (4x1_{(0,1/2)}(x), 0) & \text{in } \Omega, \end{cases} \quad (34)$$

The control of minimal $L^2(0,T)$ norm is $v(t) = 2(1-t)1_{(1/2,3/2)}(t)$.

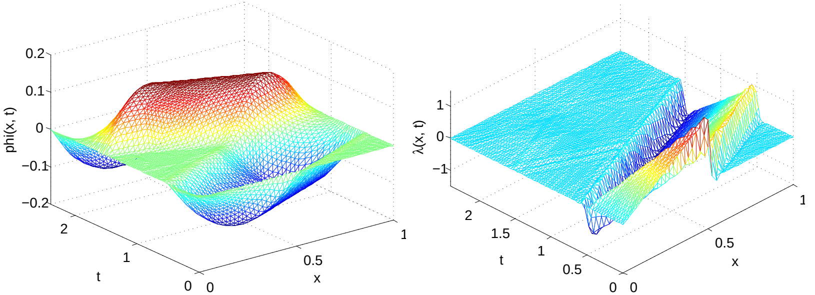


Figure: The dual variable φ_h(Left) and primal variable λ_h (Right) in Q_T; $h = 2.46 \times 10^{-2}$; $r = 10^{-2}$.

Mesh adaptivity

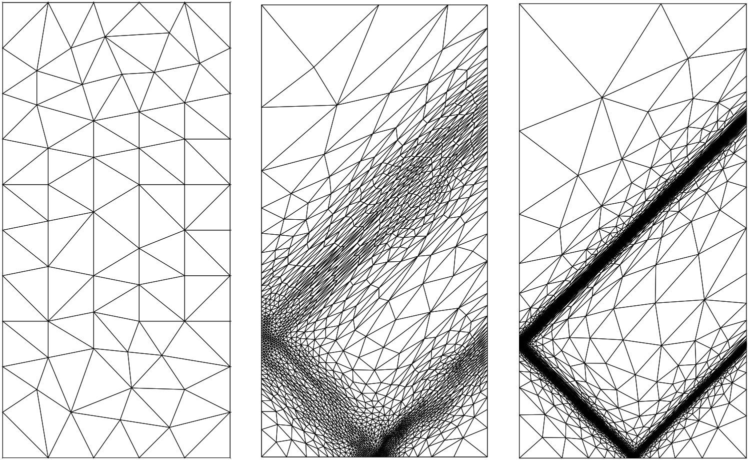


Figure: Iterative refinement of the triangular mesh over Q_T with respect to the variable λ_h: 110, 2 880 and 8 636 triangles.

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } Q_T, \\ y(0,t) = 0, y(1,t) = v(t) & \text{on } (0,T), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (4x 1_{(0,1/2)}(x), 0) & \text{in } \Omega, \end{cases} \tag{35}$$

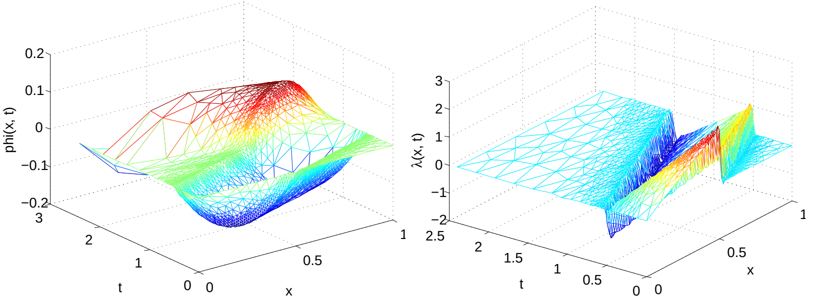


Figure: The variable φ_h and λ_h in Q_T corresponding to the finer mesh; $r = 2 \times 10^{-3}$.

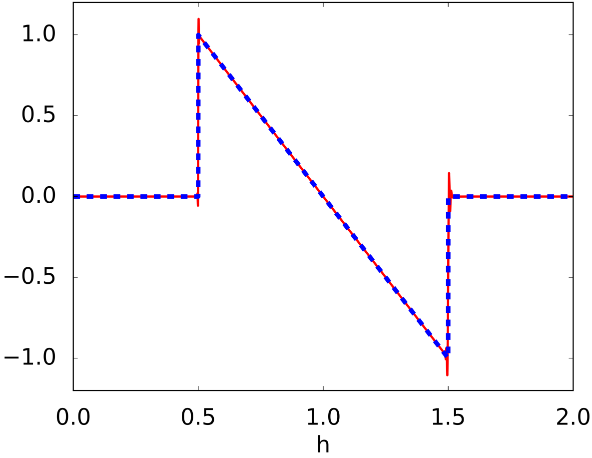


Figure: Control of minimal L^2-norm v (dashed blue line) and its approximation $\lambda_h(1,\cdot)$ (red line) on $(0,T)$. Third adapted mesh , $r=10^{-6}$.

The approach leads to a simple and short Freefem++ code !!

```
real L = 1; // Size of the spatial domain
int N = 10; // Fineness of the mesh
real T = 2; // Final time
mesh Th = square(N,2*N,[L*x,T*y]); //Uniform space-time mesh
fespace Wh(Th,P2); //P2 Finite Element Space for the Dual variables
fespace Mh(Th,P1); //P1 Finite Element Space for the Primal variables
Wh w1, q1;//Declaration of the Dual variables (solution functions)
Wh w2, q2;//Declaration of the Dual variables (test functions)
Mh lambda1,mu1;//Declaration of the Primal variables (solution functions)
Mh lambda2,mu2;//Declaration of the Primal variables (test functions)
func u0 = 4*x*(x>=0 && x<0.5); //Initial data to be controlled
real r = 10e-2; // Augmentation parameter 'r'
real alpha = 10e-2; //  Stabilization parameter 'alpha'

//Definition of the stabilized mixed variational formulation
problem ControlWave([w1,q1,lambda1,mu1],[w2,q2,lambda2,mu2])=
       // Initial conjugate functional terms
       int1d(Th,2)(q1*q2)-int1d(Th,1)(u0*w2)

       // Primal-dual bilinear terms
       +int2d(Th)((dy(w2)-dx(q2))*lambda1+(dy(q2)-dx(w2))*mu1
                 +(dy(w1)-dx(q1))*lambda2+(dy(q1)-dx(w1))*mu2)

       // Augmentation terms
       +int2d(Th)(r*((dy(w1)-dx(q1))*(dy(w2)-dx(q2))+(dy(q1)-dx(w1))*(dy(q2)-dx(w2)))

       // Stabilization terms
       -int1d(Th,2)(alpha*q1*q2+alpha*(q2*lambda1+q1*lambda2))
       -int1d(Th,2,4)(alpha*lambda1*lambda2)
       -int2d(Th)(alpha*((dy(lambda1)-dx(mu1))*(dy(lambda2)-dx(mu2))
                        +(dy(mu1)-dx(lambda1))*(dy(mu2)-dx(lambda2)))
```

```
        // Boundary conditions
        +on(2,w1=0)+on(4,w1=0) + on(4,lambda1=0);

//The following instruction solves the mixed formulation
ControlWave;
//The solution for the dual variables are stored in (w1,q1)
//The solution for the primal variables are stored in (lambda1,mu1)
```

Stabilization technics may also be employed in the context of non-conformal approximations. Let

$$V_h^q = \{p_h \in C(\overline{Q_T}); (p_h)_{|K} \in \mathbb{P}_q(K), \forall K \in \mathcal{T}_h\}$$

and consider the discrete Lagrangian $\mathcal{L}_h : V_h^p \times V_h^q \to \mathbb{R}$, given by

$$\mathcal{L}_h(\phi_h, \lambda_h) := J^\star(\phi_h) + \frac{h^2}{2}\|L\phi_h\|^2_{L^2(Q_T)} + \frac{h}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K}[[\partial_\nu \phi_h]]^2 + h^{-1}\int_{\Sigma_T}\phi_h^2$$
$$+ \int_{Q_T}(-\partial_t\phi_h\partial_t\lambda_h + \nabla\phi_h\nabla\lambda_h) - \frac{h}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K}[[\partial_\nu\lambda_h]]^2 - \frac{h^2}{2}\|L\lambda_h\|^2_{L^2(Q_T)}$$
$$- h^{-1}\int_{\Sigma_T}\lambda_h^2 - \frac{h^2}{2}\|\lambda_h - \chi\partial_\nu\phi_h\|^2_{L^2(Q_T)}$$

$[[\partial_\nu\phi_h]]$ denotes the jump of the normal derivative of ϕ_h across the internal edges of the triangulation.
The terms $h^2\|L\phi_h\|_{L^2(Q_T)}$ and $-h^2\|L\lambda_h\|^2_{L^2(Q_T)}$ play a symmetric role. Both vanish at the continuous level. The jump terms somehow aim to control the regularity of the approximation.

[2]

Moreover, if the saddle-point (λ, ϕ) of $\mathcal{L}_r$ is smooth enough, then the following approximation result holds true

Theorem (Burman, Feizmohammadi, M, Oksanen 2022)

Assume the geometric control condition. Let $p, q \geq 1$ and $h > 0$. Let $(\lambda_h, \phi_h) \in V_h^p \times V_h^q$ be the saddle point of $\mathcal{L}_h$ and assume that the saddle point (λ, ϕ) of $\mathcal{L}_r$ belongs to $H^{p+1}(Q_T) \times H^{q+1}(Q_T)$. Then, there exists a positive constant C independent of h such that

$$\|\chi(\partial_\nu \phi - \partial_\nu \phi_h)\|_{L^2(\Sigma_T)} \leq C(h^{p+\frac{1}{2}} \|u\|_{H^{p+1}(Q_T)} + h^{q-\frac{1}{2}} \|\phi\|_{H^{q+1}(Q_T)}), \tag{36}$$

where χ is a cut-off function $\chi(x, t) = \chi_0(x)\chi_1(t)$, with $\chi_0 \in C_0^\infty(\omega)$, $\chi_1 \in C_0^\infty(0, T)$.

If $(u_0, u_1) \in H^{k+1}(\Omega) \times H^k(\Omega)$ satisfies the compatibility conditions of order k at $\partial\Omega \times \{0\}$. then the solution (u, ϕ) satisfies

$$(u, \phi) \in H^{k+1}(Q_T) \times H^{k+2}(Q_T).$$

[2] Burman, Feizmohammadi, Munch, Oksanen, Spacetime finite element methods for control problems subject to the wave equation, Arxiv2021

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } Q_T, \\ y(0,t) = 0, y(1,t) = v(t) & \text{on } (0,T), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (4x1_{(0,1/2)}(x), 0) & \text{in } \Omega, \end{cases} \quad (37)$$

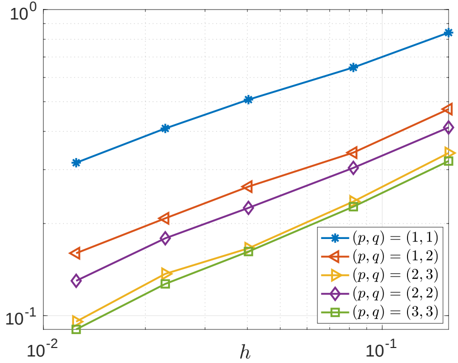


Figure: Relative error on the approximation of the boundary control $\frac{\|\partial_\nu \phi_h(1,\cdot) - v\|_{L^2(0,T)}}{\|v\|_{L^2(0,T)}}$ with respect to h for different approximations.

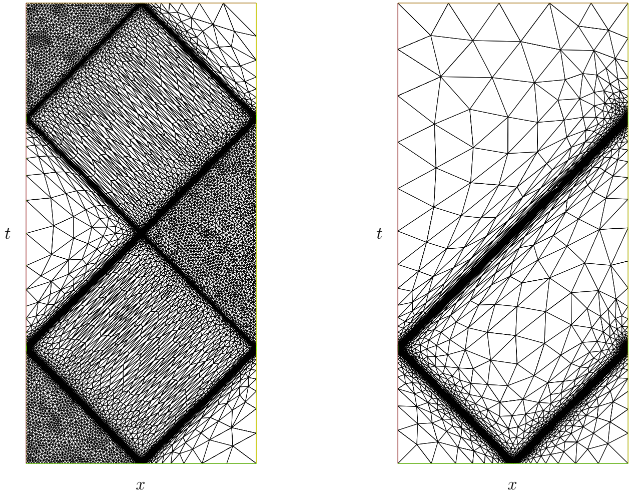


Figure: Locally refine spacetime meshes with respect to ϕ_h (Left) and λ_h (Right).

Dual of the dual - Problem w.r.t. λ

Theorem

$$\sup_{\lambda \in L^2} \inf_{\varphi \in \mathbf{\Phi}} \mathcal{L}_r(\varphi, \lambda) = -\inf_{\lambda \in L^2} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(\varphi_0, 0)$$

where $\varphi_0 \in \mathbf{\Phi}$ *solves* $a_r(\varphi_0, \overline{\varphi}) = l(\overline{\varphi}), \forall \overline{\varphi} \in \mathbf{\Phi}$ *and* $J_r^{\star\star} : L^2 \to \mathbb{R}$ *defined by*

$$J_r^{\star\star}(\lambda) := \frac{1}{2} < \mathcal{P}_r \lambda, \lambda >_{L^2(Q_T)} - b(\varphi_0, \lambda)$$

Lemma

Let $\mathcal{P}_r$ *be the linear operator from* L^2 *into* L^2 *defined by*

$$\mathcal{P}_r \lambda := L\varphi, \quad \forall \lambda \in L^2 \quad \text{where} \quad \varphi \in \mathbf{\Phi} \quad \text{solves} \quad a_r(\varphi, \overline{\varphi}) = b(\overline{\varphi}, \lambda), \quad \forall \overline{\varphi} \in \mathbf{\Phi}.$$

For any $r > 0$, *the operator* $\mathcal{P}_r$ *is a strongly elliptic, symmetric isomorphism from* L^2 *into* L^2.

Rk. The control problem is reduced to the minimization of an unconstrained functional with respect to the control state within a space-time framework!!!

Theorem

$$\sup_{\lambda\in L^2}\inf_{\varphi\in\Phi}\mathcal{L}_r(\varphi,\lambda) = -\inf_{\lambda\in L^2} J_r^{\star\star}(\lambda) \quad + \mathcal{L}_r(\varphi_0,0)$$

where $\varphi_0\in\Phi$ *solves* $a_r(\varphi_0,\overline{\varphi}) = l(\overline{\varphi}), \forall\overline{\varphi}\in\Phi$ *and* $J_r^{\star\star}: L^2\to\mathbb{R}$ *defined by*

$$J_r^{\star\star}(\lambda) := \frac{1}{2} < \mathcal{P}_r\lambda,\lambda >_{L^2(Q_T)} - b(\varphi_0,\lambda)$$

Lemma

Let $\mathcal{P}_r$ *be the linear operator from* L^2 *into* L^2 *defined by*

$$\mathcal{P}_r\lambda := L\varphi, \quad \forall\lambda\in L^2 \quad \textit{where} \quad \varphi\in\Phi \quad \textit{solves} \quad a_r(\varphi,\overline{\varphi}) = b(\overline{\varphi},\lambda), \quad \forall\overline{\varphi}\in\Phi.$$

For any $r>0$*, the operator* $\mathcal{P}_r$ *is a strongly elliptic, symmetric isomorphism from* L^2 *into* L^2*.*

Rk. The control problem is reduced to the minimization of an unconstrained functional with respect to the control state within a space-time framework!!!

The situation is much simpler with cost involving both y and v

$$\begin{cases} \text{Minimize } J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} |y|^2\,dx\,dt + \dfrac{1}{2}\int_0^T\!\!\int_{\Gamma_0} |v|^2\,d\sigma\,dt \\ \text{Subject to } (y,v) \in \mathcal{C}(y_0,y_1;T) \end{cases} \tag{38}$$

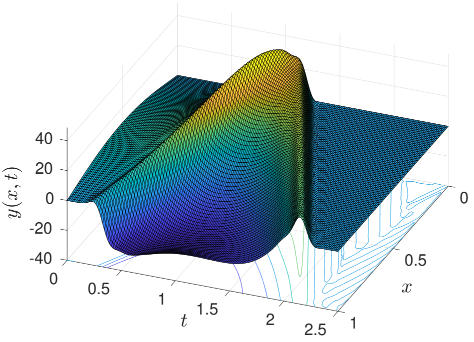
$$v = \frac{\partial\varphi}{\partial\nu} \text{ in } (0,T)\times\Gamma_0 \text{ and } y = L^\star\varphi \text{ in } Q_T.$$

$$\begin{cases} \text{Minimize } J^\star(\varphi) = \dfrac{1}{2}\displaystyle\iint_{Q_T} |L\varphi|^2\,dx\,dt + \dfrac{1}{2}\int_0^T\!\!\int_{\Gamma_0} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma\,dt \\ \qquad\qquad + < (\varphi(\cdot,0),\varphi_t(\cdot,0)),(y_0,y_1) > \\ \text{Subject to } \varphi \in \mathbf{\Phi} \end{cases} \tag{39}$$

$\mathbf{\Phi} := \left\{\varphi : \varphi \in C^0(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega)), L\varphi \in L^2(Q_T)\right\}$ is endowed with the inner product, $< \varphi,\overline{\varphi} >_{\mathbf{\Phi}} := < \frac{\partial\varphi}{\partial\nu}, \frac{\partial\overline{\varphi}}{\partial\nu} >_{L^2(\Gamma_T)} + < L\varphi, L\overline{\varphi} >_{L^2(Q_T)}, \quad \forall \varphi,\overline{\varphi} \in \mathbf{\Phi}.$

$$\begin{cases} \partial_{tt} y - \Delta y - 3y(1 + \ln^{3/2}(2 + |y|)) = 0 & \text{in } (0,1) \times (0,2.5), \\ y(0,t) = 0, y(1,t) = v(t) & \text{on } (0,2.5), \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (10 sin(\pi x), 0) & \text{in } (0,1), \end{cases} \quad (40)$$

$$J(y,v) = s \int_0^T \rho_1(s,t)^2 v^2(t) dt + s \int_{Q_T} \rho_s^2(x,t) y^2(t) dt$$



Controlled solution in $(0,1) \times (0,T)$

Similarly arguments apply for the heat equation

Part 1

• Lemoine, Ervedoza, M. *Exact controllability of semilinear heat equations through a constructive approach*, AIMS EECT, 2023

• Lemoine, M. *Constructive exact control of semilinear 1D heat equations*, AIMS MCRF, 2022

• Gayte-Marin Lemoine, M. *Approximation of null controls for semilinear heat equations using a least-squares appproach*, ESAIM COCV 2021

• Bhandari, Lemoine, M. *Constructive exact control of semilinear 1D heat equations*, arxiv

Part 2

• De Souza, Fernandez-Cara, Lemoine, M. *On the numerical controllability of the two-dimensional heat, Stokes and Navier-Stokes equations,*, J. Scientific computing, 2017

• De Souza, M. *A mixed formulation for the direct approximation of the control of minimal L^2-weighted norm for the linear heat equation,*, Advances in Computational Mathematics, 2016

• Fernandez-Cara, M. *Numerical null controllability of the 1D heat equation: Duality and Carleman weights,*, JOTA 2013

• Fernandez-Cara, M. *Strong convergent approximations of null controls for the heat equation*, SEMA 2013

$$\begin{cases} \partial_t y - \Delta y - 5y(1+\ln^{3/2}(2+|y|)) = v\,1_{(0.2,0.8)} & \text{in } (0,1)\times(0,0.5), \\ y = 0 & \text{on } \{0,1\}\times(0,0.5), \\ y(\cdot,0) = 10sin(\pi x) & \text{in } \Omega, \end{cases} \quad (41)$$

$$J(y,v) = s\int_0^T \rho_1(s,t)^2 v^2(t)dt + s\int_{Q_T} \rho_s^2(x,t)y^2(t)dt$$

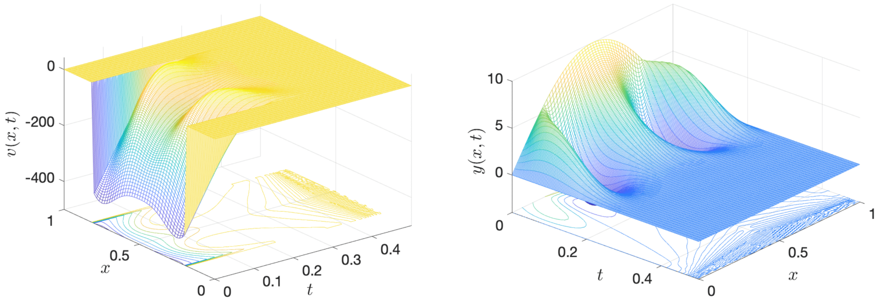


Figure: The control v (Left) and controlled state y (right) in Q_T.

Question to end : Can we use (space-time) DDM to approximate null controls ????

$$\begin{cases} \text{Minimize} \;\; J^\star(\varphi) = \dfrac{1}{2}\displaystyle\iint_{Q_T} |L\varphi|^2\, dx\, dt + \dfrac{1}{2}\int_0^T\int_{\Gamma_0} \left|\dfrac{\partial \varphi}{\partial \nu}\right|^2 d\sigma\, dt \\ \qquad\qquad + < (\varphi(\cdot,0), \varphi_t(\cdot,0)), (y_0, y_1) > \\ \text{Subject to} \;\; \varphi \in \mathbf{\Phi} \end{cases} \tag{42}$$

The corresponding VF is: find $\varphi \in \mathbf{\Phi}$ such that

$$\int_{Q_T} L\varphi L\overline{\varphi} + \int_{0^T}\int_{\Gamma_0} \frac{\partial \varphi}{\partial \nu}\frac{\partial \overline{\varphi}}{\partial \nu} =< (\varphi(\cdot,0), \varphi_t(\cdot,0)), (y_0, y_1) >, \quad \forall \overline{\varphi} \in \mathbf{\Phi}$$

The corresponding boundary value problem is ($L := \partial_{tt} - \partial_{xx}$, $\Omega = (0,1)$)

$$\begin{cases} L(L\varphi) = 0, & Q_T, \\ \varphi(0,t) = 0, L\varphi(0,t) = 0, & (0,T) \\ \varphi(1,t) = 0, L\varphi(1,t) + \varphi_x(1,t) = 0, & (0,T) \\ L\varphi(x,0) = y_0, (L\varphi(x,0))_t = y_1 & (0,1) \\ L\varphi(x,T) = 0, (L\varphi(x,T))_t = 0 & (0,1) \end{cases} \tag{43}$$

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or (equivalently)

$$\begin{cases} Ly = 0, \quad L\varphi = y, & Q_T, \\ \varphi(0,t) = 0, y(0,t) = 0, & (0,T) \\ \varphi(1,t) = 0, y(1,t) - \varphi_x(1,t) = 0, & (0,T) \\ y(x,0) = y_0, y_t(x,0) = y_1 & (0,1) \\ y(x,T) = 0, y_t(x,T) = 0 & (0,1) \end{cases} \tag{45}$$

Thank you for your attention !!!

The corresponding boundary value problem is ($L := \partial_{tt} - \partial_{xx}$)

$$\begin{cases} L(L\varphi) = 0, & Q_T, \\ \varphi(0,t) = 0, L\varphi(0,t) = 0, & (0,T) \\ \varphi(1,t) = 0, L\varphi(1,t) + \varphi_x(1,t) = 0, & (0,T) \\ L\varphi(x,0) = y_0, (L\varphi(x,0))_t = y_1 & (0,1) \\ L\varphi(x,T) = 0, (L\varphi(x,T))_t = 0 & (0,1) \end{cases} \tag{44}$$

or (equivalently)

$$\begin{cases} Ly = 0, \quad L\varphi = y, & Q_T, \\ \varphi(0,t) = 0, y(0,t) = 0, & (0,T) \\ \varphi(1,t) = 0, y(1,t) - \varphi_x(1,t) = 0, & (0,T) \\ y(x,0) = y_0, y_t(x,0) = y_1 & (0,1) \\ y(x,T) = 0, y_t(x,T) = 0 & (0,1) \end{cases} \tag{45}$$

Thank you for your attention !!!