# Boundary controllability of two coupled wave equations with space-time first order coupling in $1-D$

Farid AMMAR KHODJA  Joint work with Yacine MOKHTARI

**Clermont-Ferrand: Contrôle, Problème inverse et Applications**

**04 - 06 october 2021**

# Coupled wave equations

- Problem:

$$\begin{cases} y_{tt} = y_{xx} + M((ay)_t + (by)_x), & \text{in} \quad Q_T := (0,T) \times (0,1), \\ y(t,0) = Bu(t),\ y(t,1) = 0, & \text{in} \quad (0,T), \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x), & \text{in} \quad (0,1), \end{cases}$$

where $y = (y_1, y_2)$ is a vector function and

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathcal{L}\left(\mathbb{R}^2\right),\ B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2,\ a, b \in C^1(\overline{Q_T}; \mathbb{R}),$$

and $u$ a is scalar control function acting at $x = 0$.

- **Issue**: *Exact controllability of this system.*

# Known results in any dimensions

- **Single wave equation**: By now classical results of Bardos-Lebeau-Rauch (microlocal analysis), Fursikov-Imanuvilov (Carleman estimates, without lower order terms), Zhang (Carleman estimates, with lower order terms in time and space) give a complete solution to the exact controllability issue.
- **Systems of wave equations in any space dimension:**
  - **The number of controls equals the number of equations**: Lasiecka-Triggiani ($90'$).
  - **The number of controls lower than the number of equations**: A number of authors (Alabau, Alabau-Léautaud, Dehman-Le Rousseau-Léautaud) gave answers when two wave equations are coupled with *zero-order terms not depending on time*. Common point: coupling coeficients independent of time and with a constant sign. Cui-Laurent-Wang have more general results for a Riemannian manifold without boundary and an arbitrary number of equations with zero order coupling terms.

# Known results in one dimension

- **D. Russell**: a reference paper on the controllability of one-dimensional symmetric hyperbolic systems.
- **More recently**: Avdonin-De Teresa (constant case), Duprez-Olive (cascade systems), Hu-Olive (minimal time of controllability), FAK-Bennour-Teniou (first order coupling independent of time)...
- ...

# The adjoint problem

- It is:

$$\begin{cases} \varphi_{tt} = \varphi_{xx} - M^*(a\varphi_t + b\varphi_x), & \text{in} \quad (0,T)\times(0,1), \\ \varphi_{|x=0,1} = 0, & \text{in} \quad (0,T), \\ (\varphi, \varphi_t)_{|t=T} = (\varphi_0, \varphi_1)\,, & \text{in} \quad (0,1). \end{cases}$$

- Well posed in $H := H_0^1\,(0,1)^2 \times L^2\,(0,1)^2$ and

$$\left\|(\varphi, \varphi_t)\right\|_{C([0,T],H)} + \left\|\varphi_{x|x=0,1}\right\|_{L^2(0,T)^2} \leq C\left\|(\varphi_0, \varphi_1)\right\|_H.$$

# The adjoint problem

As is well-known:

- Exact observability (and thus, exact controllability) amounts to the observability inequality:

$$\left\|(\varphi_0, \varphi_1)\right\|_H^2 \leq C \int_0^T \left|B^* \varphi_x(t,0)\right|^2 dt.$$

- Approximate controllability amounts to:

$$\left(B^* \varphi_x(t,0) = 0,\ t \in (0,T)\right) \Rightarrow \varphi \equiv 0 \text{ in } Q_T.$$

for any solution $\varphi$ of the adjoint problem.

# Definition

## Definition

The adjoint problem is said *weakly exactly obervable (WEO)* if there exists a compact operator $K : H \longrightarrow L^2(0,T)$ such that:

$$\left\|(\varphi_0,\varphi_1)\right\|_H^2 \leq C\int_0^T \left|B^*\varphi_x(t,0)\right|^2 dt + \left\|K(\varphi_0,\varphi_1)\right\|_{L^2(0,T)}^2, \ \forall (\varphi_0,\varphi_1) \in H.$$

(This is a Peetre inequality).

- If this inequality is satisfied, then the observability inequality

$$\left\|(\varphi_0,\varphi_1)\right\|_H^2 \leq C\int_0^T \left|B^*\varphi_x(t,0)\right|^2 dt$$

  is satisfied up to the (finite dimensional) kernel of the linear operator $L : H \longrightarrow L^2(0,T)$ defined by $L(\varphi_0,\varphi_1) = B^*\varphi_x(t,0)$.
- **Thus**: *(WEO/WEC) + (AO/AC)$\Rightarrow$ EO/EC.*

# Notations

To formulate the main results, some notations are necessary: introduce

$$\eta_1 = \frac{a-b}{2}\left(T-t,x\right),\ \eta_2 = \frac{a+b}{2}\left(T-t,x\right),$$

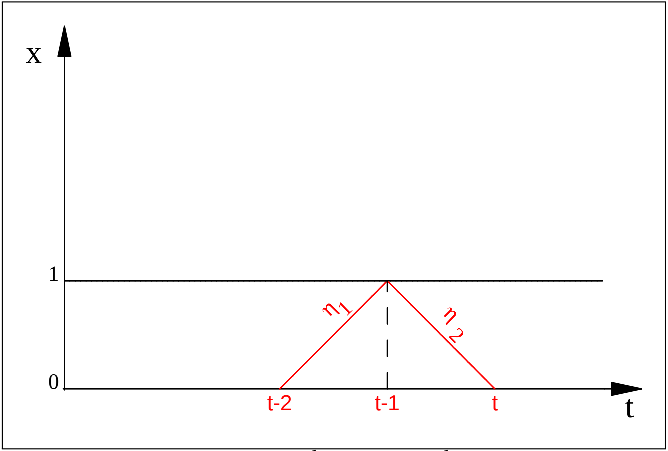
and the associated function: for $t > 0$

$$\phi\left(t\right) = \int_{\max(0,t-2)}^{\max(0,t-1)} \eta_1\left(\tau,\tau-(t-2)\right)d\tau + \int_{\max(0,t-1)}^{t} \eta_2\left(\tau,t-\tau\right)d\tau.$$

If $t \geq 2$, it writes:

$$\phi\left(t\right) = \int_{t-2}^{t-1} \eta_1\left(\tau,\tau-(t-2)\right)d\tau + \int_{t-1}^{t} \eta_2\left(\tau,t-\tau\right)d\tau$$

# Geometric interpretation




$$\eta_1 = \frac{a-b}{2},\ \eta_2 = \frac{a+b}{2}$$

# First result: non observability

**Theorem**

*If $T < 4$, the system is not controllable (neither exactly, nor approximately).*

**Remark**: Indeed, the operator $L\left(\varphi_0, \varphi_1\right) = B^* \varphi_x\left(t, 0\right)$ has an infinite dimensional kernel.

# Controllability: main result

We assume here that $\sigma(M) \subset \mathbb{R}$ and will indicate later the changes if $\sigma(M) \subset \mathbb{C} \backslash \mathbb{R}$.

## Theorem

*Let $n \geq 2$ and $2n \leq T < 2n+2$ and assume the approximate controllability. Exact controllability is equivalent to:*

1. $\operatorname{rank}[B \mid MB] = 2$
2. *For any $x \in [0,1]$, there exist $1 \leq k, \ell \leq n$ such that $2k+2-x \leq T$, $x+2\ell \leq T$ and:*

$$\phi(2k+2-x) \neq 0 \text{ and } \phi(x+2\ell) \neq 0$$

**Remark**: If $\sigma(M) \subset \mathbb{C} \backslash \mathbb{R}$, the second condition must be replaced by:

$$\phi(2k+2-x),\ \phi(x+2\ell) \notin \frac{\pi}{\operatorname{Im}\sigma(M)}\mathbb{Z}.$$

# Geometric interpretation

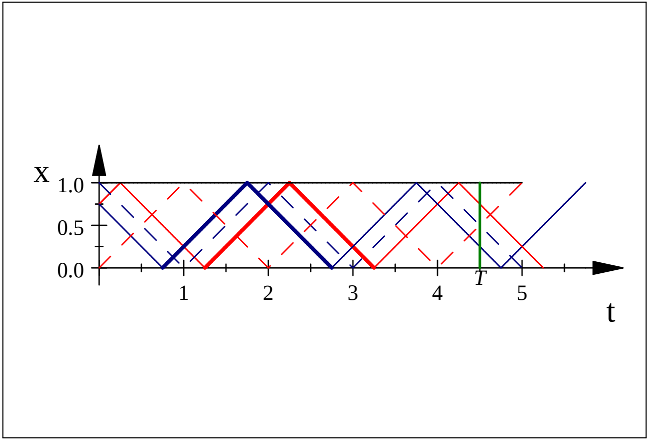



Illustration for $T = 4.5$

The controllability results says:

1. The two characteristics lines issued from any point $(x, 0) \in (0, 1) \times (0, T)$ must touch at least two times the observability boundary before arriving to $t = T$,
2. Along these characteristics lines, the integral $\phi$ must be non zero somewhere (if $\sigma(M) \subset \mathbb{R}$).

For $T < 4$, the first condition is not satisfied for some $(\alpha, b) \subset (0, 1)$. The noncontrollability result is then proved by choosing initial data whose support is close to the characteristics line which does not verify this condition.

# Comments and remarks

- The autonomous case: If $a$ and $b$ do not depend on $t$, then

$$\phi\left(t\right)=\int_0^1 a\left(x\right)dx$$

and thus $b$ does not play any role in the previous result. More generally, if $b$ does not depend on $t$, it does not play any role for the weak observability to hold.

# Comments and remarks

- The autonomous case: If $a$ and $b$ do not depend on $t$, then

$$\phi(t) = \int_0^1 a(x)\,dx$$

  and thus $b$ does not play any role in the previous result. More generally, if $b$ does not depend on $t$, it does not play any role for the weak observability to hold.
- The conditions of the theorem may be satisfied if $a \equiv 0$ and $b$ depends on $t$ : the function $\phi$ will be expressed in function of $b$.

# Comments and remarks

- The autonomous case: If $a$ and $b$ do not depend on $t$, then

$$\phi(t) = \int_0^1 a(x)\, dx$$

  and thus $b$ does not play any role in the previous result. More generally, if $b$ does not depend on $t$, it does not play any role for the weak observability to hold.
- The conditions of the theorem may be satisfied if $a \equiv 0$ and $b$ depends on $t$ : the function $\phi$ will be expressed in function of $b$.
- The weak observability result can be widely generalized: the number of equations may be increased with a number of control functions less than the number of equations.

# Comments and remarks

- The autonomous case: If $a$ and $b$ do not depend on $t$, then

$$\phi(t) = \int_0^1 a(x)\, dx$$

  and thus $b$ does not play any role in the previous result. More generally, if $b$ does not depend on $t$, it does not play any role for the weak observability to hold.
- The conditions of the theorem may be satisfied if $a \equiv 0$ and $b$ depends on $t$ : the function $\phi$ will be expressed in function of $b$.
- The weak observability result can be widely generalized: the number of equations may be increased with a number of control functions less than the number of equations.
- The approximate controllability is much more tricky: we do not know general conditions to insure it.

# Sketch of the proof

## First step

The introdution of the Riemann invariants $p = y_t - y_x$ and $q = y_y + y_x$ makes the adjoint system of wave equations equivalent to the hyperbolic system:

$$\begin{cases} p_t + p_x + M^* \left(\eta_1 p + \eta_2 q\right) = 0, & \text{in} \quad Q_T \\ q_t - q_x + M^* \left(\eta_1 p + \eta_2 q\right) = 0, & \text{in} \quad Q_T \\ (p+q)_{|x=0,1} = 0 & \text{in} \quad (0,T) \\ (p,q)_{|t=0} = (p_0, q_0) & \text{in} \quad (0,1) \end{cases}$$

in the space

$$H = \left\{ (f,g) \in L^2(0,1)^2 \times L^2(0,1)^2 : \int_0^1 (f-g) = 0 \right\}.$$

# Sketch of the proof

## Second step

We then extract the diagonal system:

$$\begin{cases} p_t + p_x + M^*\eta_1 p = 0, & \text{in} \quad Q_T \\ q_t - q_x + M^*\eta_2 q = 0, & \text{in} \quad Q_T \\ (p+q)_{|x=0,1} = 0 & \text{in} \quad (0,T) \\ (p,q)_{|t=0} = (p_0, q_0) & \text{in} \quad (0,1) \end{cases}$$

which is much more easy to deal with. This diagonal system is not equivalent to a wave equations system.
**First key point**: *The difference between the two evolution families is compact.* This is an observation of Neves-Ribeiro-Lopes (1980') extended to this case by FAK-Bader (2000').

# Sketch of the proof
## Third step

**Second key point**: *The difference between the observation operators are also compact*. This was already observed by D. Russell (1977?) but without proof.
**Third key point**: The observation operator operator associated with the diagonal system is identified to a matrix multiplicative operator from $L^2(0,1)^4$ in $L^2(0,1)^m$ with continuous entries depending on $\phi$. Here $m$ depends on $T$. The study of this matrix multiplicative operator leads to the necessary and sufficient conditions of the main result.

# Approximate controllability

- The constant coefficients case can be completely treated by applying the Fattorini-Hautus criteria.
- The cascade system has already been treated in the autonomous case with $b \equiv 0$ by Bennour and al. (2017).
- In our setting, we have considered two special nonautonomous cases:

  $$\eta_1(t,x) = \alpha(t-x),\ \eta_2(t,x) = \beta(t+x);$$

  and

  $$\eta_1(t,x) = \alpha(t+x),\ \eta_2(t,x) = \beta(t-x);$$

  In the two cases, more conditions on $\phi$ are needed.
- The approximate controllability issue remains an open problem in the general case.

# Generalizations

$$\begin{cases} y_{tt} = y_{xx} + (M_1 y)_t + (M_2 y)_x, & \text{in} \quad Q_T := (0,T)\times(0,1), \\ y(t,0) = Bu(t),\ y(t,1) = 0, & \text{in} \quad (0,T), \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x), & \text{in} \quad (0,1), \end{cases}$$

where

$$\begin{aligned} M_i(t,x) &\in C\left(Q_T, \mathcal{L}\left(\mathbb{R}^n\right)\right),\ i = 1,2 \\ B &\in \mathcal{L}\left(\mathbb{R}^m, \mathbb{R}^n\right) \ \ (m < n) \end{aligned}$$

with the same way of proof.
The conditions for *weak observability* should be given by way of the resolvent of the differential systems

$$\theta' = M_i\left(\gamma\left(t,x\right)\right)\theta, \left(i = 1,2\right).$$