On special values of spinor L-functions of Siegel cusp eigenforms of genus 3

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Abstract.

In this talk I present the result of joint work with Francesco Chiera on explicit computation of the special values of spinor L*-function attached to Siegel cusp form* F_{12} *of weight 12 and genus 3, which was constructed by Miyawaki. To our knowledge, this is the first example, when the critical values can be computed explicitly for the spinor* L*-function of a Siegel cusp form of degree* 3*. The manuscript is available at* `http://www-fourier.ujf-grenoble.fr/~kvankov/`.

Plan of the talk

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6. Expression for $L(s,\Delta)\,L(s-1,\Delta)$
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Motivation: special values

One of bright and motivating examples is the conjecture developed by Birch and Swinnerton-Dyer in the earlier 60s of last century. It relates the value of L-function attached to an elliptic curve $\mathcal{E}$ at point $s = 1$ to the rank of the elliptic curve over the rational numbers, i.e. the number of free generators of its group of rational points. More precisely,

The Tailor expansion of $L(s, \mathcal{E})$ at $s = 1$ has the form

$$L(s, \mathcal{E}) = c(s-1)^r + \text{higher order terms},$$

where $c \neq 0$ and $r = rank(\mathcal{E}(\mathbb{Q}))$.

Therefore, $L(1, \mathcal{E}) = 0 \iff \mathcal{E}(\mathbb{Q})$ is infinite.

Motivation: algebraicity

Let $f = \sum_{n \geqslant 1} a_n q^n \in S_k(\Gamma_0(N), \psi)$ be a primitive cusp eigenform of weight $k \geqslant 2$ with a Dirichlet character ψ modulo N. The associated L-function is

$$L(s, f, \chi) = \sum_{n \geqslant 1} \chi(n) \, a_n \, n^{-s} \, .$$

According to the theorem of Shimura [Shimura:1959] and Manin [Manin:1973], there exist two non-zero complex constants $c^+(f)$ and $c^-(f) \in \mathbb{C}^\times$ called the periods of f, such that for all $s = 1, \ldots, k-1$ and for all Dirichlet characters χ of fixed parity, $(-1)^{k-s}\chi(-1) = \pm 1$, the normalized special values are **algebraic numbers**:

$$L^*(s, f, \chi) = \frac{(2\pi i)^{-s} \, \Gamma(s)}{c^\pm(f)} \, L(s, f, \chi) \in \overline{\mathbb{Q}} \, .$$

In this work we explore the particular case of the cusp form F_{12} of genus 3.

The purpose of this work

Using the conjectured by [Miyawaki:1992] and proved by [Heim:2007] equality

$$L(s, F_{12}, spin) = L(s-9, \Delta)\, L(s-10, \Delta)\, L(s, \Delta \otimes g_{20}),$$

we apply the Rankin-Selberg method for $L(s, \Delta)\, L(s-1, \Delta)$ and $L(s, \Delta \otimes g_{20})$ to obtain the algebraic expression for the special values of the spinor L-function associated to F_{12} in critical points $s = 12, \ldots, 19$ in the form

$$L(s, F_{12}, spin) = R(s)\, \pi^{\alpha(s)}\, \langle \Delta, \Delta \rangle\, \langle g_{20}, g_{20} \rangle$$

where $R(s)$ is an explicit rational number, $\alpha(s)$ is a power of π and $\langle f, g \rangle$ is the Petersson inner product.

We also verify these values numerically using the SAGE software [SAGE] and Dokchitser's ComputeL PARI package [ComputeL].

Spinor L-function: functional equation, critical points

The functional equation for $L(s, F, spin)$ is conjectured by Andrianov [Andrianov:1974, p. 115]:

$$\Lambda(F, kn - \frac{n\,(n+1)}{2} + 1 - s, spin) = \varepsilon(F)\,\Lambda(F, s, spin)\,,$$

where $\Lambda(F, s, spin) = \Gamma_{n,k}(s)\,L(s, F, spin)$, $\varepsilon(F) = (-1)^{k2^{n-2}}$ and

$$\begin{aligned}
\Gamma_{1,k}(s) &= \Gamma_{\mathbb{C}}(s) = 2\,(2\pi)^{-s}\,\Gamma(s)\,,\\
\Gamma_{2,k}(s) &= \Gamma_{\mathbb{C}}(s)\,\Gamma_{\mathbb{C}}(s-k+2)\,,\\
\Gamma_{3,k}(s) &= \Gamma_{\mathbb{C}}(s)\,\Gamma_{\mathbb{C}}(s-k+3)\,\Gamma_{\mathbb{C}}(s-k+2)\,\Gamma_{\mathbb{C}}(s-k+1)\,.
\end{aligned}$$

In particular, for $n = 3$ and $k \geqslant 5$,

$$\Lambda(F, s, spin) = \Lambda(F, 3k-5-s, spin)\,,$$

the critical values (in the sense of [Deligne:1979]) are $s = k, \ldots, 2k-5$, that is the points where $\Gamma_{3,k}(s)$ and $\Gamma_{3,k}(3k-5-s)$ are regular.

Spinor L-function

Suppose that $F \in \mathcal{M}_k^n$ is an eigenform of all Hecke operators $T \in \mathcal{H}_n$:

$$F|T = \lambda_F(T)F\,.$$

Then all the numbers $\lambda_F(T) \in \mathbb{C}$ define a homomorphism $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{n,p}, \mathbb{C})$ given by a $(n+1)$-tuple of complex numbers

$$(\alpha_0, \alpha_1, \dots, \alpha_n) \in (\mathbb{C}^\times)^{n+1}$$

called the Satake p-parameters of F, which are the image of the map $T \mapsto \lambda_F(T)$ under the Satake isomorphism $\mathcal{H}_{n,p} \longrightarrow \mathbb{Q}[x_0^\pm, x_1^\pm, \dots, x_n^\pm]^{W_n}$.

See [Andrianov:1974, Andrianov:1977] for details.

The spinor L-function is

$$L(s, F, spin) = \prod_p \left[Q_{F,p}(p^{-s})\right]^{-1}\,,$$

where $\quad Q_{F,p}(X) = (1 - \alpha_0 X) \prod_{r=1}^{n} \prod_{1 \leqslant i_1 < i_2 \cdots < i_r \leqslant n} (1 - \alpha_0 \alpha_{i_1} \cdots \alpha_{i_r} X)\,.$

What is F_{12}?

F_{12} is the Siegel modular form of weight 12 and of genus 3 with respect to $\mathrm{Sp}_3(\mathbb{Z})$ on the Siegel upper-half plane

$$\mathfrak{H}^3 = \{Z = {}^tZ = X + iY : X, Y \in M_3(\mathbb{R}), Y > 0\},$$

and

$$\mathrm{Sp}_n(\mathbb{Z}) = \left\{M \in M_{2n}(\mathbb{Z}) : M J_n {}^tM = J_n\right\}, \; J_n = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix} .$$

The formal Fourier expansion of a Siegel modular form $F(Z) = \sum_N a(N)\, q^N$ uses the symbol

$$q^N = \exp(2\,\pi\, i\, \mathsf{Tr}(NZ)),$$

where

$$N \in \left\{M \in M_n(\mathbb{Q}) : M = {}^tM,\; M \geqslant 0,\; M \text{ half-integral}\right\} .$$

Definition and Fourier coefficients of F_{12}

Let E_8 be the unique even unimodular lattice of rank 8 i.e.,

$$E_8 = \left\{ {}^t(x_1,\dots,x_8) \in \mathbb{R}^8 \;\middle|\; \begin{array}{l} 2x_i \in \mathbb{Z}\ (i=1,\dots,8), \\ x_1+\cdots+x_8 \in 2\mathbb{Z}, \\ x_i - x_j \in \mathbb{Z} \end{array} \right\},$$

and

$$Q = \begin{pmatrix} 1 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & i & 0 & 0 \end{pmatrix}$$

be 3×8 matrix. Let $Z \in \mathfrak{H}^3$. Then the theta series

$$F_{12} = \sum_{v_1,v_2,v_3 \in E_8} \Re(\det(Q\cdot(v_1,v_2,v_3))^8)\exp(\pi i \operatorname{Tr}((\langle v_i, v_j\rangle)Z))$$

is a cusp form of weight 12 with respect to $\mathrm{Sp}_3(\mathbb{Z})$.

Generalities and notations

Each modular form has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n)\, q^n \in \mathcal{M}_k(N, \chi)\,,$$

where $q = \exp(2\pi i z)$ and $a(n)$ are complex numbers in general. The L-function associated to f is defined as

$$L(s, f) = \sum_{n=1}^{\infty} a(n)\, n^{-s} = \prod_p (1 - a(p)\, p^{-s} + \chi(p)\, p^{k-1-2s})^{-1}\,.$$

The L function associated to two modular forms f and $g = \sum b(n)\, q^n \in \mathcal{M}_l(N, \xi)$ is given by the additive convolution

$$L(s, f, g) = \sum_{n=1}^{\infty} a(n)\, b(n)\, n^{-s}\,.$$

We denote Rankin's product L-function by

$$L(s, f \otimes g) = L_N(2s + 2 - k - l, \chi\xi)\, L(s, f, g)\,,$$

where $L_N(s, \omega) = \sum_{n=1}^{\infty} \omega(n)\, n^{-s}$ with a Dirichlet character ω modulo N.

Generalities and notations: the Petersson inner product

For two elements $f, h \in \mathcal{M}_k(N)$ such that fh is a cusp form, the Petersson inner product $\langle f, h\rangle$ is defined as

$$\langle f, h\rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Phi_N} \overline{f(z)}\, h(z)\, y^{k-2}\, dx\, dy\,,$$

where

$z = x + iy$,
the bar denotes the complex conjugate,
Φ_N is a fundamental domain for $\mathfrak{H}$ modulo $\Gamma_0(N)$,
$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 (\mathrm{mod}\ N) \right\}$.

We also define $\langle f, h\rangle$ by the above formula for nearly holomorphic modular forms f and h on $\mathfrak{H}$ whenever the integral is convergent (see, for example, [Shimura:2007, section 8.2] for definition and properties of the nearly holomorphic modular forms).

Rankin-Selberg method (see [Shimura:1976])

Three principal points:

- ▷ Rankin's Euler product of degree 4:
  $L(s, f \otimes g) = L_N(2s + 2 - k - l, \chi\xi)\, L(s, f, g)$;
- ▷ Integral convolution:
  $$(4\pi)^{-s}\,\Gamma(s)\,L(s,f,g) = \int_0^\infty \int_{-1/2}^{1/2} \overline{f_\rho}\, g\, y^{s-1}\, dx\, dy;$$
- ▷ Passage to the fundamental domain:
  $$(4\pi)^{-s}\,\Gamma(s)\,L(s,f,g) = \int_{\Phi_N} \overline{f_\rho}\, g\, E^*_{k-l,N}(z, s+1-k, \chi\xi)\, y^{s-1}\, dx\, dy,$$

where

$z = x + iy, \quad k > l$ are the weights of f and g,

$f_\rho(z) = \overline{f(-\overline{z})} = \sum \overline{a_n}\, q^n$,

$$E^*_{\lambda,N}(z, s, \omega) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \omega(d)\,(cz+d)^{-\lambda}\,|cz+d|^{-2s}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

Φ_N is a fundamental domain for $\Gamma_0(N)\backslash\mathfrak{H}$ and $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}\}$,

ω is a Dirichlet character modulo N such that $\omega(-1) = (-1)^\lambda$.

Computing $L(s-9,\Delta)\,L(s-10,\Delta)\,L(s,\Delta\otimes g_{20})$

$$L(s-9,\Delta)\,L(s-10,\Delta)=\frac{L(s-9,\Delta\otimes G_{2,2})}{1-\tau(2)\,2^{10-s}+2^{31-2s}}\,.$$

Let $f(z)=\Delta(z)$ be Ramanujan's discriminant modular form of weight 12:

$$f(z)=\Delta(z)=\sum_{n=1}^{\infty}\tau(n)\,q^n=q\prod_{n=1}^{\infty}(1-q^n)^{24}\,,$$

$$L(s,f)=\prod_p\left(1-\tau(p)\,p^{-s}+p^{11-2s}\right)^{-1}=\prod_p\left((1-\alpha_p\,p^{-s})(1-\alpha'_p\,p^{-s})\right)^{-1}$$

i.e. $\alpha_p+\alpha'_p=\tau(p)$, $\alpha_p\,\alpha'_p=p^{11}$. Let $G_2(z)=-\frac{1}{24}+\sum_{n=1}^{\infty}\sigma_1(n)\,q^n$.

Consider $g(z)=G_{2,2}(z)=G_2(z)-2\,G_2(2z)=\frac{1}{24}+\sum_{n=1}^{\infty}\sum_{\substack{d|n\\ 2\nmid d}}d\,q^n$ the Eisenstein series of weight 2 for $\Gamma_0(2)$ (see [Miyake:2006, Lemma 7.2.19]).

$$L(s,g)=(1-2^{1-s})\,\zeta(s-1)\,\zeta(s)=\prod_p\left((1-\beta_p\,p^{-s})\,(1-\beta'_p\,p^{-s})\right)^{-1}\,,$$

i.e. $\beta(p)=1$ for all p, $\beta'(2)=0$ and $\beta'(p)=p$ for all odd primes.

Computing $L(s,\Delta)\,L(s-1,\Delta)$ ...continuation

Using the definition and the series summation lemma we write

$$
\begin{aligned}
L(s,\Delta\otimes G_{2,2}) &= L_2(2s+2-12-2,\psi)\,L(s,f,g)\\
&= \prod_{p\neq 2}(1-p^{12-2s})^{-1}\cdot\sum_{n=1}^{\infty}\tau(n)\,b(n)\,n^{-s}\\
&= \prod_{p\neq 2}(1-p^{12-2s})^{-1}\times\\
&\quad\times\prod_{p}\frac{1-\alpha_p\,\alpha'_p\,\beta_p\,\beta'_p\,p^{-2s}}{(1-\alpha_p\,\beta_p\,p^{-s})\,(1-\alpha'_p\,\beta_p\,p^{-s})\,(1-\alpha_p\,\beta'_p\,p^{-s})\,(1-\alpha'_p\,\beta'_p\,p^{-s})}\\
&= \prod_{p}\frac{1}{(1-\alpha_p\,p^{-s})\,(1-\alpha'_p\,p^{-s})}\prod_{p\neq 2}\frac{1}{(1-\alpha_p\,p^{1-s})\,(1-\alpha'_p\,p^{1-s})}\\
&= (1-\tau(2)\,2^{1-s}+2^{11}\,s^{2-2s})\,L(s,\Delta)\,L(s-1,\Delta)\,.
\end{aligned}
$$

$$
L(s,\Delta)\,L(s-1,\Delta)=\frac{L(s,\Delta\otimes G_{2,2})}{1-\tau(2)\,2^{1-s}+2^{13-2s}}\,.
$$

Expressing $L(s, \Delta \otimes G_{2,2})$ via Petersson product

Using the Rankin-Selberg method,

$$
\begin{aligned}
&L(s, \Delta \otimes G_{2,2}) \\
&= \frac{(4\,\pi)^s}{2\,\Gamma(s)} \int_{\Phi_2} \overline{\Delta(z)}\, G_{2,2}(z)\, E_{10,2}(z, s-11, \xi)\, y^{s-1}\, dx\, dy \\
&= \frac{(4\,\pi)^s\, [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(2)]}{2\,\Gamma(s)} \left\langle \Delta(z), G_{2,2}(z)\, y^{s-11}\, E_{10,2}(z, s-11, \xi) \right\rangle \\
&= \frac{3}{2} \frac{(4\,\pi)^{11}}{\Gamma(s)} \left\langle \Delta(z), \mathcal{H}ol\left(G_{2,2}(z)\, (4\pi y)^{s-11}\, E_{10,2}(z, s-11, \xi)\right) \right\rangle ,
\end{aligned}
$$

where

$$
E_{\lambda,N}(z, s, \xi) = {\sum_{(m,n)}}' \xi(n)\, (mNz+n)^{-\lambda} |mNz+n|^{-2s} ,
$$

$\sum'$ denotes the summation over all $(m, n) \in \mathbb{Z}^2$, $(m, n) \neq (0, 0)$, $\mathcal{H}ol(F)$ is the operator of holomorphic projection [Sturm:1980]. It is defined so, that

$$
\langle f, F \rangle = \langle f, \mathcal{H}ol(F) \rangle \text{ for all } f \in S_k(N, \psi).
$$

Computing $\mathcal{H}ol\left(G_{2,2}(z)\,(4\pi y)^{s-11}\,E_{10,2}(z,s-11,\xi)\right)$

We compute two Fourier coefficients since $\mathcal{H}ol(F) \in S_{12}(2)$, dimension 2. According to [Panchishkin:2003, Prop. 2.2], using Whittaker functions

$$E_{10,2}(z,s-11,\xi) = C_0'(s)\cdot(4\pi y)^{13-2s} + C_0''(s)$$
$$+\, C_1(s)\,\frac{W(4\pi y,s-1,s-11)}{\Gamma(s-1)}\,q + C_2(s)\,\frac{W(8\pi y,s-1,s-11)}{\Gamma(s-1)}\,q^2 + \cdots,$$

where $W(y,s,-r) = \sum_{i=0}^{r} \frac{(-1)^i\binom{r}{i}\Gamma(s)}{\Gamma(s-i)}\,y^{r-i}$ for $r \geqslant 0$, $s \in \mathbb{Z}$ and

$$C_0'(s) = (-1)\frac{2\,\pi^{2s-12}\,\Gamma(2s-13)\,\zeta(2s-13)}{\Gamma(s-11)\,\Gamma(s-1)}, \quad C_0''(s) = (2-2^{13-2s})\,\zeta(2s-12),$$
$$C_1(s) = 2\,\pi^{2s-12}, \qquad C_2(s) = (2-2^{2s-12})\,\pi^{2s-12}.$$

Also recall that $G_{2,2}(z) = 1/24 + q + q^2 + \cdots$. We denote by A_i the Fourier coefficients of $\mathcal{H}ol\left(G_{2,2}(z)\,(4\pi y)^{s-11}\,E_{10,2}(z,s-11,\xi)\right)$.

Computing $\mathcal{H}ol\left(G_{2,2}(z)\,(4\pi y)^{s-11}\,E_{10,2}(z,s-11,\xi)\right)$

The integration according to [Gross-Zagier:1986, Proposition (5.1)] gives:

$$\begin{aligned}
A_1(s) &= \frac{C_0'}{10!}\int_0^{+\infty}(4\pi y)^{2-s}\,\mathrm{e}^{-4\pi y}\,(4\pi y)^{10}\,d(4\pi y)\\
&\quad+\frac{C_0''}{10!}\int_0^{+\infty}(4\pi y)^{s-11}\,\mathrm{e}^{-4\pi y}\,(4\pi y)^{10}\,d(4\pi y)\\
&\quad+\frac{C_1}{24\cdot 10!}\int_0^{+\infty}\frac{W(4\pi y,s-1,s-11)}{\Gamma(s-1)}(4\pi y)^{s-11}\,\mathrm{e}^{-4\pi y}\,(4\pi y)^{10}\,d(4\pi y)\\
&= \frac{\Gamma(13-s)}{10!}\,C_0'+\frac{\Gamma(s)}{10!}\,C_0''+\frac{C_1}{24\cdot 10!}\sum_{i=0}^{11-s}\frac{(-1)^i\binom{11-s}{i}\Gamma(11-i)}{\Gamma(s-1-i)}\,,\\
A_2(s) &= \frac{\Gamma(13-s)}{10!}\,2^{s-2}\,C_0'+\frac{\Gamma(s)}{10!}\,2^{11-s}\,C_0''\\
&\quad+\frac{C_1}{10!}\sum_{i=0}^{11-s}2^i\,\frac{(-1)^i\binom{11-s}{i}\Gamma(11-i)}{\Gamma(s-1-i)}\\
&\quad+\frac{C_2}{24\cdot 10!}\,2^{11-s}\sum_{i=0}^{11-s}\frac{(-1)^i\binom{11-s}{i}\Gamma(11-i)}{\Gamma(s-1-i)}\,.
\end{aligned}$$

Back to Petersson product corresponding to $L(s, \Delta \otimes G_{2,2})$

Recall $\mathcal{H}ol(F) = \alpha \cdot \Delta(z) + \beta \cdot \Delta(2z)$

$$\begin{cases} \Delta(z) = q - 24q^2 + \dots \\ \Delta(2z) = q^2 + \dots \end{cases} \Rightarrow \begin{cases} A_1 = \alpha \cdot 1 + \beta \cdot 0 \\ A_2 = \alpha \cdot (-24) + \beta \cdot 1 \end{cases} \Rightarrow \begin{cases} \alpha = A_1 \\ \beta = 24\, A_1 + A_2\,. \end{cases}$$

Then

$$\begin{aligned} &L(s, \Delta \otimes G_{2,2}) \\ &= \frac{3}{2} \frac{(4\,\pi)^{11}}{\Gamma(s)} \left\langle \Delta(z), \mathcal{H}ol\left(G_{2,2}(z)\,(4\pi y)^{s-11}\,E_{10,2}(z, s-11, \xi)\right)\right\rangle \\ &= \frac{3}{2} \frac{(4\,\pi)^{11}}{\Gamma(s)} \left(A_1(s)\,\langle \Delta(z), \Delta(z)\rangle + (24\,A_1(s) + A_2(s))\,\langle \Delta(z), \Delta(2z)\rangle\right)\,. \end{aligned}$$

Moreover, next we compute $\langle \Delta(z), \Delta(2z)\rangle$ in terms of $\langle \Delta(z), \Delta(z)\rangle$.

Recall that $\Delta(2z) = 2^{-k/2}\,\Delta(z)|_k \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $k = 12$ is the weight of Δ. Then

$$\langle \Delta(z), \Delta(2z)\rangle = 2^{-6}\,\langle \Delta(z), \Delta(z)|_k \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)\rangle\ .$$

Consider $\gamma \in \Gamma_0(2)\backslash\Gamma$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The summation over all γ gives

$$\begin{aligned}\langle \Delta(z), \Delta(2z)\rangle &= 2^{-6}[\Gamma : \Gamma_0(2)]^{-1} \sum_{\gamma} \langle \Delta(z)|\gamma, \Delta(z)|_k \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)\gamma\rangle \\ &= 2^{-6}3^{-1}\left\langle \Delta(z), \mathrm{Tr}^{(2)}\Big(\Delta(z)|_k \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big)\right\rangle\ .\end{aligned}$$

The trace operator $\mathrm{Tr}^{(\mathrm{N})} : \mathcal{M}_k(\Gamma_0(N)) \to \mathcal{M}_k(\Gamma)$ is defined by the action $f \mapsto \displaystyle\sum_{\gamma\in\Gamma_0(N)\backslash\Gamma} f|_k\gamma$. We have (see [Serre:1973])

$$\mathrm{Tr}^{(2)}\left(\Delta(z)|_k \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = 2^{-5}\,T_2(\Delta)\,,$$

where T_2 is the Hecke operator, therefore,

$$\langle \Delta(z), \Delta(2z)\rangle = 2^{-6}3^{-1}\left\langle \Delta(z), 2^{-5}\,T_2\left(\Delta(z)\right)\right\rangle = -\frac{1}{256}\,\langle \Delta(z), \Delta(z)\rangle\ .$$

Intermediate result $L(s-9,\Delta)\,L(s-10,\Delta)\,L(s,\Delta\otimes g_{20})$

Finally, combining the previous formulae together we obtain

$$L(s,\Delta\otimes G_{2,2}) = \frac{3}{2}\frac{(4\pi)^{11}}{\Gamma(s)}\frac{(232\,A_1(s)-A_2(s))}{256}\,\langle\Delta,\Delta\rangle\ .$$

Therefore, for each $s\in\{12,\dots,19\}$ we have:

$$L(s-9,\Delta)\,L(s-10,\Delta) = \frac{3\cdot 2^{13}\pi^{11}(232\,A_1(s-9)-A_2(s-9))}{(1+3\cdot 2^{13-s}+2^{31-2s})\,\Gamma(s-9)}\,\langle\Delta,\Delta\rangle\ .$$

The numerical value of the Petersson inner product for Δ is:

$$\langle\Delta,\Delta\rangle = 0.000001035362056\dots$$

s	R_Δ	P_Δ	$L(s-9,\Delta)\,L(s-10,\Delta)$
12	$\frac{32768}{225} = \frac{2^{15}}{3^2 \cdot 5^2}$	π^5	0.046143339818118
13	$\frac{4096}{81} = \frac{2^{12}}{3^4}$	π^7	0.15813073255203
14	$\frac{2048}{189} = \frac{2^{11}}{3^3 \cdot 7}$	π^9	0.334433094416363
15	$\frac{8192}{4725} = \frac{2^{13}}{3^3 \cdot 5^2 \cdot 7}$	π^{11}	0.528115574483468
16	$\frac{16384}{70875} = \frac{2^{14}}{3^4 \cdot 5^3 \cdot 7}$	π^{13}	0.694972239760782
17	$\frac{8192}{297675} = \frac{2^{13}}{3^5 \cdot 5^2 \cdot 7^2}$	π^{15}	0.816559651925946
18	$\frac{8192}{2679075} = \frac{2^{13}}{3^7 \cdot 5^2 \cdot 7^2}$	π^{17}	0.895457859377812
19	$\frac{65536}{200930625} = \frac{2^{16}}{3^8 \cdot 5^4 \cdot 7^2}$	π^{19}	0.942700248523234

Result for $L(s-9,\Delta)\,L(s-10,\Delta)\,L(s,\Delta\otimes g_{20})$

In the same way, we obtain:

$$L(s,\Delta\otimes g_{20}) = \frac{(4\pi)^{19}}{2\,\Gamma(s)}\left\langle g_{20}, \mathcal{H}ol\left(\Delta\,(4\pi y)^{s-19}\,E_{8,1}(z,s-19)\right)\right\rangle\ ,$$

The result of the holomorphic projection in this case belongs to the one-dimensional space spanned by g_{20}, we need to compute just the first Fourier coefficient. The final result is:

$$L(s,\Delta\otimes g_{20}) = \frac{(4\pi)^{19}}{2\cdot 18!}\left(D_0'' + \frac{\Gamma(31-s)}{\Gamma(s)}\,D_0'\right)\langle g_{20}, g_{20}\rangle\ ,$$

where

$$D_0' = D_0'(s) = 2\,(2\pi)^{2s-30}\frac{\Gamma(2s-31)\,\zeta(2s-31)}{\Gamma(s-11)\,\Gamma(s-19)}\,,$$

$$D_0'' = D_0''(s) = 2\,\zeta(2s-30)\,,$$

and

$$\langle g_{20}, g_{20}\rangle = 0.000008265541531659 70\ldots$$

s	$R_{g_{20}}$	$P_{g_{20}}$	$L(s,\Delta\otimes g_{20})$
12	$\frac{524288}{2338875} = \frac{2^{19}}{3^5\cdot 5^3\cdot 7\cdot 11}$	π^{13}	5.380003562880315
13	$\frac{2097152}{88409475} = \frac{2^{21}}{3^8\cdot 5^2\cdot 7^2\cdot 11}$	π^{15}	5.618889612918517
14	$\frac{4194304}{2791213425} = \frac{2^{22}}{3^8\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17}$	π^{17}	3.513063561721911
15	$\frac{8388608}{97692469875} = \frac{2^{23}}{3^8\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17}$	π^{19}	1.981288433718698
16	$\frac{8388608}{1465387048125} = \frac{2^{23}}{3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 17}$	π^{21}	1.303635536350500
17	$\frac{2097152}{4396161144375} = \frac{2^{21}}{3^{10}\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 17}$	π^{23}	1.072197252248449
18	$\frac{4194304}{92319384031875} = \frac{2^{22}}{3^{11}\cdot 5^4\cdot 7^3\cdot 11\cdot 13\cdot 17}$	π^{25}	1.007825020916877
19	$\frac{2097152}{461596920159375} = \frac{2^{21}}{3^{11}\cdot 5^5\cdot 7^3\cdot 11\cdot 13\cdot 17}$	π^{27}	0.994683426196918

The final result

s	R	P	$L(s, F_{12}, spin)$
12	$\frac{17179869184}{526246875} = \frac{2^{34}}{3^7 \cdot 5^5 \cdot 7 \cdot 11}$	π^{18}	0.248251332624670
13	$\frac{8589934592}{7161167475} = \frac{2^{33}}{3^{12} \cdot 5^2 \cdot 7^2 \cdot 11}$	π^{22}	0.888519130619814
14	$\frac{8589934592}{527539337325} = \frac{2^{33}}{3^{11} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17}$	π^{26}	1.174884717828030
15	$\frac{68719476736}{461596920159375} = \frac{2^{36}}{3^{11} \cdot 5^3 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17}$	π^{30}	1.046349279390801
16	$\frac{137438953472}{1038593070358593750} = \frac{2^{37}}{3^{13} \cdot 5^7 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17}$	π^{34}	0.905990508529256
17	$\frac{17179869184}{130862726865182812 5} = \frac{2^{34}}{3^{15} \cdot 5^6 \cdot 7^4 \cdot 11 \cdot 13 \cdot 17}$	π^{38}	0.875513015091950
18	$\frac{34359738368}{24733055377519551562 5} = \frac{2^{35}}{3^{18} \cdot 5^6 \cdot 7^5 \cdot 11 \cdot 13 \cdot 17}$	π^{42}	0.902464835857626
19	$\frac{137438953472}{9274895766569831835937 5} = \frac{2^{37}}{3^{19} \cdot 5^9 \cdot 7^5 \cdot 11 \cdot 13 \cdot 17}$	π^{46}	0.937688313077777

Numerical computation of the Petersson inner product

It is easy to compute the Petersson inner product $\langle f_k, f_k \rangle$ in terms of special values of the associated L-function for the cusp forms of weights $k = \{12, 16, 18, 20, 22, 26\}$, when the dimension of the space S_k is 1.

Due to [Rankin:1952, Theorem 5]:

$$\langle f_k, f_k \rangle = \frac{(4\pi)^{1-k}\,(k-2)!}{\zeta(l)}\,\frac{\alpha_r}{\alpha_l + \alpha_r - \alpha_k}\,L(k-1, f_k)\,L(l, f_k)\,,$$

where

$$4 \leqslant r \leqslant k/2 - 2\,, \quad l = k - r\,, \quad \alpha_k = -\frac{2k}{B_k}\,,$$

B_k is a Bernoulli number $\left(B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ \ldots\right)$.

Numerical computation of the Petersson inner product

For $f_k = \Delta$ there is only one choice of $l = 8$. To compute the numerical values $L(11, \Delta)$ and $L(8, \Delta)$ we use Dokchitser's L-functions Calculator [ComputeL], which outputs the well known value

$$\langle \Delta, \Delta \rangle = 0.000001035362056804320948209596804\,.$$

For the cusp form of the weight 20 there are three choices $l = 12, 14, 16$. The result is

$$\langle g_{20}, g_{20} \rangle = 0.000008265541531659702744699575969 \text{ for } l = 12,$$
$$\langle g_{20}, g_{20} \rangle = 0.000008265541531659703390644766954 \text{ for } l = 14,$$
$$\langle g_{20}, g_{20} \rangle = 0.000008265541531659703069998511729 \text{ for } l = 16.$$

Note, that in order to achieve the default precision (the functional equation was satisfied to 1E-21) it is necessary to provide just 12 Fourier coefficients in the case of Δ and 14 for g_{20}.

Numerical verification of the main result

We verify the main result numerically by using ComputeL for each term of

$$L(s, F_{12}, spin) = L(s-9, \Delta)\, L(s-10, \Delta)\, L(s, \Delta \otimes g_{20})\,.$$

Let $g_{20} = \sum b(n) q^n$. Using the definition we write

$$L(s, \Delta \otimes g_{20}) = \sum_{d=1}^{\infty} d^{30-2s} \sum_{d_1=1}^{\infty} \tau(d_1) b(d_1) d_1^{-s} = \sum_{n=1}^{\infty} \left(\sum_{d : d^2 | n} d^{30} \tau(\frac{n}{d^2}) b(\frac{n}{d^2}) \right) n^{-s}\,,$$

The functional equation for $L(s, f \otimes g)$ is known, see [Li:1979]. Therefore, the parameters that are used by ComputeL are

$$\Gamma_{\mathbb{C}}(s)\, \Gamma_{\mathbb{C}}(s-l+1) \text{ and } s \mapsto k + l - 1 - s\,,$$

where $k = 20 > l = 12$ are the weights of g_{20} and Δ. In our case (using the Gauss Duplication formula) we have four gamma factors $\Gamma_{\mathbb{R}}(s)\, \Gamma_{\mathbb{R}}(s+1)\, \Gamma_{\mathbb{R}}(s-11)\, \Gamma_{\mathbb{R}}(s-10)$ and the “motivic weight” is 31.

Numerical verification of the main result

Finally, we are able to compare the theoretical result with direct numerical computation:

s	$L(s, F_{12}, spin)$ "theoretical"	$L(s, F_{12}, spin)$ "numerical"	variation
12	0.248251332624670	0.24825133281752	1.98E-10
13	0.888519130619814	0.88851913131006	6.90E-10
14	1.174884717828030	1.17488471874074	9.13E-10
15	1.046349279390801	1.04634928020366	8.13E-10
16	0.905990508529256	0.90599050923308	7.04E-10
17	0.875513015091950	0.87551301577209	6.80E-10
18	0.902464835857626	0.90246483655871	7.01E-10
19	0.937688313077777	0.93768831380622	7.28E-10

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