

Besse Summer School

June 20 2010 - June 26 2010



Royal Poldavian Academy

Besse & Saint-Amastaise

Organised by François Martin, Marusia Rebelledo
& Emmanuel Royer as part of the *Modunombres* project.



Lecturers

YoungJu Choie (POSTECH-Pohang University)
Masanobu Kaneko (Kyushu University)
Min Ho Lee (University of Northern Iowa)
Hossein Movasati (IMPA Rio)
Marios Petropoulos (École polytechnique)
Michaël Pevzner (Université de Reims)

Short Talks

Shaun Cooper (Massey University)
Dominic Lanhier (Western Kentucky University)
Samuel Lelièvre (Université Paris-Sud)
Kamel Mazhouda (Faculté des sciences de Monastir)
Najib Ouled Azaiez (Faculté des sciences de Sfax)
Brundaban Sahu (University College Dublin)
Yi Jun Yao (Penn State University)

Night experimental session

Samuel Lelièvre (Université Paris-Sud)

Guests

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Schedule

	Lundi	Mardi	Mercredi	Jeudi	Vendredi
9 - 10:30	Choié Classroom	Kaneko Discussion Room	Movasati Classroom	Lee Discussion Room	Movasati Classroom
10:30 - 11	Coffee break				
11 - 12:30	Choié Classroom	Kaneko Discussion Room	Movasati Classroom	Lee Discussion Room	Movasati Classroom
12:30 - 14	Lunch				
14 - 15	Cooper Discussion Room	Pevzner Classroom	Yao Discussion Room	Petropoulos Discussion Room	
15 - 16	Lanphier Classroom	Ouled Azaiez Classroom	Hiking		
16 - 16:30	Coffee break				
16:30 - 17:30	Lelièvre Classroom	Pevzner Classroom	Mazhouda Discussion Room	Sahu Classroom	
19	Dinner			Restaurant	Dinner
Night session	Lelièvre Discussion Room				

Coffee breaks, Lunch & Dinner will be in the Eating Room.

Lectures

Young Ju Choie

Three Lectures on Quasimodular forms.

We don't assume much background even on the theory of modular forms. This is for nonexpert who may wish to see the quasimodular forms from the classical point of modular forms.

- 1) Introduction and motivation: we first start with basic definitions and examples and start to introduce various ways to look at Quasimodular forms. Introducing quasipolynomial one can define Hecke actions as well.
- 2) Connection with Jacobi-like forms Rankin-Cohen brackets and L -series: We show how to get Quasimodular forms from Jacobilike forms and also discuss its lifting to the space of Jacobilike forms. Here we show how one get information about modular forms through quasimodular form and see some connection among Dirichlet L -series using Rankin-Cohen brackets.
- 3) Connection with Symmetric Power representation: We explain the old theory of Kuga and Shimura and see how their theory can be extended using quasimodular forms.

Min Ho Lee

- 1) Poincaré series for quasimodular forms. We introduce two types of actions of $SL(2, \mathbb{R})$ on the space of polynomials whose coefficients are holomorphic functions on the upper half plane. There is an automorphism of the space of such polynomials that is $SL(2, \mathbb{R})$ -equivariant, which induces an isomorphism between the space of quasimodular polynomials and that of modular polynomials. This isomorphism can be used establish a correspondence between quasimodular forms and some finite sequences of modular forms, and it allows us to construct Poincaré series for quasimodular forms.

- 2) As is well-known, modular forms for a discrete subgroup Γ of $SL(2, \mathbb{R})$ can be identified with holomorphic sections of certain line bundles over the modular curve U corresponding to Γ . We extend this result to quasimodular forms by introducing vector bundles over U whose sections can be identified with quasimodular forms for Γ . We also discuss connections between quasimodular forms and certain cohomology groups of Γ .
- 3) Heat operators and quasimodular polynomials. We introduce a differential operator \mathcal{D}_λ on quasimodular polynomials that corresponds to the derivative operator on quasimodular forms of weight λ . We discuss the compatibility of such a differential operator with a heat operator on Jacobi-like forms. We also describe linear maps of quasimodular polynomials corresponding to certain linear maps of modular polynomials, which reduce to the operator \mathcal{D}_λ and the formal derivative operator ∂_X in some special cases.

Masanobu Hameko

Certain differential equation related to supersingular j -invariants and quasimodular forms.

In these three hours lectures, I review a specific differential equation first arose in the study of supersingular j -invariants with Don Zagier, and discuss its various modular and quasimodular solutions. Quasimodular solutions supply examples of "extremal" quasimodular forms, of which I explain in some detail together with an example coming from the "mirror symmetry for elliptic curves".

Hossein Movasati

Geometric interpretation of quasi modular forms.

In the framework of Algebraic Geometry modular forms are interpreted as functions from the pairs (E, ω) , where E is an elliptic curve and ω is a regular differential form on E , to the base field which has a functional property with respect to the multiplication of ω by constants. In these lectures we generalize this point of view to the case of quasi modular forms.

The differential form ω is replaced with an element in the first algebraic de Rham cohomology of E and a quasi-modular form has functional properties with respect to the action of an algebraic group. It turns out that the Ramanujan relations between Eisenstein series can be derived from the Gauss-Manin connection of families of elliptic curves. As an application, we show that the problem of finding differential and polynomial equations for modular forms is equivalent to the problem of constructing moduli of elliptic curves with marked elements in de Rham cohomologies and then calculating the corresponding Gauss-Manin connection.

Marios Petropoulos

Gravitational instantons and quasimodular forms.

Modular and quasimodular forms have played an important role in string theory: in the determination of the spectrum and the partition functions, in the description of non-perturbative effects, in higher-order corrections of scalar-field spaces. . . . The latter often appear as "gravitational instantons" i.e. special solutions of Einstein's equations. The course aims at presenting a class of such solutions in four dimensions, obtained by requiring self-duality and Bianchi IX homogeneity. In this case, a vast range of configurations exist, which exhibit interesting modular properties. Examples of other Einstein spaces, without Bianchi IX symmetry, but with similar properties will also be given. The geometrical tools needed to reach and interpret the solutions will be presented in detail in the lectures.

Michaël Pevzner

Generalized Rankin-Cohen brackets and representations of conformal Lie groups.

We shall explain the role that Rankin-Cohen brackets play in branching laws of tensor product representations of conformal groups and will describe the geometric setting that permits to introduce a covariant operator calculus on the so-called causal symmetric spaces. These two ingredients will naturally lead us to the construction of a family of bidifferential operators acting on holomorphic sections of appropriate vector bundles that might be thought of as generalized Rankin-Cohen brackets.

Short talks

Shaun Cooper

Apéry sequences, modular forms, and series for $\frac{1}{\pi}$.

One of the sequences discussed by R. Apéry in his 1978 lecture on the irrationality of $\zeta(3)$ is given by the recurrence relation

$$n^2 u_n = (11n^2 - 11n + 3)u_{n-1} + (n-1)^2 u_{n-2}, \quad n = 2, 3, \dots$$

and initial conditions $u_0 = 1$, $u_1 = 3$. Remarkably, all terms in this sequence are integers. The generating function $f = \sum_{n=0}^{\infty} u_n x^n$ can be uniformized by modular forms. This means there are functions $f = f(\tau)$ and $x = x(\tau)$ that are invariant under a group of transformations, up to some simple factors. Finally, the numbers u_n occur in some series for $\frac{1}{\pi}$ that are similar to some given by Ramanujan. These topics, their interconnections, and the underlying theory will be described. A large number of examples will be presented.

Dominic Lamphier

Combinatorics of differential operators on modular forms.

The derivative of a holomorphic modular form is not a modular form. However, there are differential operators that preserve certain properties of modular forms. There are also bilinear operators which take holomorphic modular forms to holomorphic modular forms. We study combinatorial identities relating these operators and applications to values of L -functions and Fourier coefficients of modular forms.

Samuel Lelièvre

Counting square-tiled surfaces with quasimodular forms.

Square-tiled surfaces are integer points in the strata of moduli spaces of abelian and quadratic differentials. They give a wealth of information on these spaces, allowing to compute their volumes and other important numerical characteristics. Only fairly recently was a classification and geometric description of Teichmüller curves of square-tiled surfaces in the minimal stratum of genus two obtained by McMullen, generalising work of Hubert & Lelièvre. Eskin, Okounkov & Pandharipande have counted the square-tiled surfaces in the strata and proved the quasimodularity of their generating functions, conjectured by Kontsevich. Lelièvre & Royer proved it for countings of square-tiled surfaces by Teichmüller curve in the minimal stratum of genus two. This talk will survey these results focusing on concrete combinatorial and geometric descriptions and examples.

Kamel Mazhouda

Mean squares of L -functions in the Selberg class.

In this talk, we prove some results about mean-square of L -functions in the Selberg class. The main result is, for any function $F(s)$ in the Selberg class of degree $d \geq 2$ satisfying

$$\sum_{n \leq X} |a_F(n)|^2 \leq kX(\log X)^M,$$

where k is any positive constant and M is any arbitrary positive integer, we have

$$\int_0^T \left| F\left(\frac{1}{2} + it\right) \right|^2 dt \ll_{F,k,M} T^{\frac{d}{2}} (\log T)^{M+1}.$$

Rajib Ouled Arzaiez

The compact/non-compact dichotomy.

I will discuss the difference between structures of quasimodular forms spaces in the compact and cocompact cases.

Brundaban Sahu

Rankin-Cohen brackets on quasimodular forms and some applications.

We use Rankin-Cohen brackets for modular forms and quasimodular forms to give a different proof of the results obtained by D. Lanphier and D. Niebur on the van der Pol type identities for the Ramanujan's tau function. As consequences we obtain convolution sums involving the divisor functions. This is a joint work with B. Ramakrishnan.

Yi-Jun Yao

Rankin-Cohen deformations from an outsider's point of view.

We would like to discuss some interactions of number theory with non-commutative geometry via the so-called Rankin-Cohen deformations.

Night experimental session

Samuel Belière

How to use Sage for working in mathematics: the example of square tiled surfaces.

Where is the Royal Poldavian Academy?

By Lieven Lebruyne

From <http://www.neverendingbooks.org/>

Among the items found on Andre Weil at the time of his arrest was “a packet of calling cards belonging to Nicolas Bourbaki, member of the Royal Academy of Poldavia”.

But then, where is the Royal Poldavian Academy situated? Well, surely in the Kingdom of Poldavia, which is a very strange kingdom indeed, its currency unit being the bourbaki and there exist only two types of coins: gold ones (worth n bourbakis) and silver ones (worth m bourbakis). Using gold and silver coins, it is possible to obtain sums such as 10000 bourbakis, 1875 bourbakis, 3072 bourbakis, and so on. Prove that any payment above $mn-2$ bourbakis can be made without the need to receive change.

However, the Kingdom of Poldavia isn't another Bourbaki concoction. The name goes back at least to a joke pulled by the right-wingers of the Action Francaise in May 1929. Here's the TIME article of May 20th 1929:

“When 28 French Republican deputies sat down to their breakfast coffee and croissants early last week, each found a large crinkly letter from Geneva in his morning's mail. Innocent and refreshed after a sound night's sleep, not one Republican deputy saw anything untoward in the fact that the large crinkly letters were embossed on the stationery of “Foreign Minister Lamidaeff, of the Kingdom of Poldavia.” They saw nothing strange in the fact that Poldavians were in financial difficulties, and they found Minister Lamidaeff most thoughtful in not asking for money, but merely for an expression of “moral support” from the Deputies in his campaign to aid Poldavian sufferers. “We believe that our interests were betrayed at the Peace Conference,” wrote Poldavian Lamidaeff. “and we appeal to you as a member of the French Parliament to do your utmost to help us in this our hour of need. The whole nation of Poldavia and its noble monarch who disregarded personal safety in 1916, and joined France in her War for justice and righteousness, pray you to remember our sacrifices.”

What could be fairer than that? Legislators all over the world are always ready to write enthusiastic platitudes in favor of anything that sounds like a good cause. The wronged Poldavians seemed a very good cause. Each of the 28 deputies sat down at his desk and pledged his moral support to “Foreign Minister Lamidaeff of Poldavia.”

None of the 28 deputies noticed that the old Poldavian name of Lamidaeff might read "L'Ami d'A. F."—"the friend of A. F.," "the friend of L'Action Française" famed royalist newspaper of which the editor is Leon Daudet, bon vivant, practical jokester, son of famed Author Alphonse Daudet (Tartarin de Tarascon), exile from the republic he has so consistently lampooned (TIME, June 13, 1927, et seq.). Three days after the 28 gullible deputies replied to the "Poldavian Minister," a special edition of L'Action Française appeared.

"People of France," wrote exiled Editor Daudet, who once escaped from La Santé prison through a hoaxed release order telephoned from the office of the Minister of the Interior, "—People of France, how much longer will you permit such ignorant deputies to represent you before the world? Here are 28 of your elected representatives, and they actually believe there is a Kingdom of Poldavia, and that Lamidaeff is its Foreign Minister. Lamidaeff, c'est moi!"

The consul of Poldavia also appears in the 1936 Tintin-story The Blue Lotus by Hergé. In view of the above AF-connection, it shouldn't come as a surprise that Hergé is often accused of extreme-right sympathies and racism.

To some, Poldavia is a small country in the Balkans, to others it lies in the Caucasus, but has disappeared from the map of Europe. All accounts do agree on one point, namely that Poldavia is a mountainous region.

Today we are pleased to disclose the exact location of the Royal Poldavian Academy, and, thanks to the wonders of Google Earth you can explore the Kingdom of Poldavia at your leisure if you give it the coordinates 45.521082N,2.935495E.

The evidence is based on a letter sent by André Weil to Elie Cartan when the Bourbaki wanted to submit a note for the Comptes Rendus des Séances Hebdomadaires de l'Académie des Sciences under the pseudonyme Nicolas Bourbaki. As the academy requires a biographical note on the author, Weil provided the following information about Bourbaki's life:

"Cher Monsieur,

Je vous envoie ci-joint, pour les C.R., une note que M.Bourbaki m'a chargé de vous transmettre. Vous n'ignorez pas que M.Bourbaki est cet ancien professeur à l'Université Royale de Besse-en-Poldévie, dont j'ai fait la connaissance il y a quelque temps dans un café de Clichy où il passe la plus grande partie de la journée et même de la nuit ; ayant perdu, non seulement sa situation, mais presque toute sa fortune dans les troubles qui firent disparaître de la carte d'Europe la malheureuse nation poldève, il gagne maintenant sa vie en donnant, dans ce café, des leçons de belote, jeu où il est de première force. Il fait profession de ne plus s'occuper de

mathématiques, mais il a bien voulu cependant s'entretenir avec moi de quelques questions importantes et même [ajout manuscrit : me laisser] jeter un coup d'œil sur une partie de ses papiers ; et j'ai réussi à le persuader de publier, pour commencer, la note ci-jointe, qui contient un résultat fort utile pour la théorie moderne de l'intégration, je pense que vous ne verrez pas de difficulté à l'accueillir pour les Comptes-Rendus ; si même les renseignements que je vous donne au sujet de M.Bourbaki ne paraissaient pas suffisamment clairs, j'imagine qu'il n'appartient à l'Académie, et en particulier à celui qui présente la note, que de s'assurer de la valeur scientifique de celle-ci, et non de faire une enquête au sujet de l'auteur. Or j'ai examiné soigneusement le résultat de M.Bourbaki, et son exactitude est hors de doute.

Veillez recevoir, je vous prie, les remerciements de M.Bourbaki et les miens, et croyez toujours à mes sentiments bien affectueusement et respectueusement dévoués.

A.Weil”

That is, “Besse-en-Poldevie“, or simply “Besse“ as in this line from the wedding announcement “Mademoiselle Betti Bourbaki, a former student of the Well-Ordereds of Besse” must be the capital of Poldavia where the Academy is housed.

You may have never heard of Poldavia, but if you are a skiing or cycling enthusiast, the name of its capital sure does ring a bell, or rather so does the name of its sub-part Super Besse. The winter sports resort of Super Besse is located in the commune of Besse-et-Saint-Anastaise in the Parc naturel régional des volcans d'Auvergne in the department of Puy de Dôme, in Auvergne. Situated approximately 50 km from Clermont-Ferrand, it is located at an altitude of 1350 m on the slopes of Puy de Sancy, Puy de la Perdrix and Puy Ferrand. Surely a mountainous region...

Besse-et-Saint-Anastaise, or rather Besse-en-Chandesse as it was formerly called, was the venue of the very first Bourbaki Congress 1935. Surely, they used the “Royal Poldavian Academy“ as their meeting place. But, where is it?

At the Besse meeting were present: Claude Chevalley, Jean Dieudonné, René de Possel, Henri Cartan, Szolem Mandelbrojt, Jean Delsarte, André Weil, the physicist Jean Coulomb, Charles Ehresmann and a “cobaye” called Mirles.

Of these men three held a position at the University Blaise Pascal of Clermont-Ferrand : Mandelbrojt, de Possel and Coulomb and they arranged that the Bourbaki-group could use the universities' biology-outpost in Besse-en-Chandesse. Photographic evidence for this is provided by the man standing apart in the picture in front of the house: the biologist Luc

Olivier.

Concluding : the Royal Poldavian Academy is located at the “Station Biologique de l’Université Blaise Pascal”, Rue du Lavoir, Besse-et-Saint-Anastaise.

On July 12th 2003 a ceremony was held at the Biology-station commemorating the birth of Nicolas Bourbaki (the group), supposedly born July 12th 1935. A plate at an exterior wall of the Biology-station was unveiled.

More information about the mysterious country of Poldavia can be found in the article *La vérité sur la Poldévie* by Michele Audin.