Dunkl spectral multipliers with values in UMD lattices

Luc Deleaval - Christoph Kriegler

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Abstract

We show a Hörmander spectral multiplier theorem for $A = A_0 \otimes \mathrm{Id}_Y$ acting on the Bochner space $L^p(\mathbb{R}^d, h_\kappa^2; Y)$, where A_0 is the Dunkl Laplacian, h_κ^2 a weight function invariant under the action of a reflection group and Y is a UMD Banach lattice. We follow hereby a transference method developed by Bonami-Clerc and Dai-Xu, passing through a Marcinkiewicz multiplier theorem on the sphere. We hereby generalize works for $A_0 = -\Delta$ acting on $L^p(\mathbb{R}^d, dx)$ by Girardi-Weis, Hytönen and others before. We apply our main result to maximal regularity for Cauchy problems involving A.

Contents

1 Introduction

Let f be a bounded function on $(0, \infty)$ and $u(f)$ the operator on $L^p(\mathbb{R}^d)$ defined by $[f(-\Delta)g]\hat{} = [u(f)g]\hat{} = f(\|\xi\|^2)\hat{g}(\xi)$. Hörmander's theorem on Fourier multipliers [29, Theorem 2.5] asserts

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that $u(f) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bounded for any any $p \in (1,\infty)$ provided that for some integer α strictly larger than $\frac{d}{2}$, and $q = 2$,

$$\|f\|^q_{\mathcal{H}^\alpha_q} := \max_{k=0,1,\ldots,\alpha} \sup_{R>0} \frac{1}{R} \int_R^{2R} \left| t^k \frac{d^k}{dt^k} f(t) \right|^q dt < \infty. \tag{1.1}$$

This theorem has many refinements and generalizations to various similar contexts. Namely, one can ask for different exponents $q \in [1,\infty)$ in (1.1), and generalize to non-integer α there to get larger (for smaller q and for smaller α) admissible classes $\mathcal{H}^\alpha_q = \{f \in L^1_{loc}(0,\infty) : \|f\|_{\mathcal{H}^\alpha_q} < \infty\}$ of multiplier functions f (see Subsection 2.4). Moreover, it has been a deeply studied question over the last years to know to what extent one can replace the ordinary Laplacian subjacent to Hörmander's theorem by other operators A acting on some $L^p(\Omega)$ space. A theorem of Hörmander type holds true for many elliptic differential operators A, including sub-laplacians on Lie groups of polynomial growth, Schrödinger operators and elliptic operators on Riemannian manifolds, see [3, 12, 25, 26]. More recently, spectral multipliers have been studied for operators acting on $L^p(\Omega)$ only for a strict subset of $(1,\infty)$ of exponents [5, 9, 10, 11, 39, 40], for abstract operators acting on Banach spaces [36], and for operators acting on product sets $\Omega_1 \times \Omega_2$ [50, 57, 58]. A spectral multiplier theorem means then that the linear and multiplicative mapping

$$\mathcal{H}^\alpha_q \to B(X),\ f \mapsto f(A), \tag{1.2}$$

is bounded, where typically $X = L^p(\Omega)$. One important consequence of a spectral multiplier theorem as in (1.2) is the boundedness of Bochner-Riesz means associated with A. Namely, we put for $\beta, R > 0$

$$f^\beta_R(t) = \begin{cases} (1 - t/R)^\beta & 0 < t \leqslant R \\ 0 & t > 0. \end{cases}$$

Then f^β_R belongs to $\mathcal{H}^\alpha_q$ (with uniform norm bound for $R > 0$) if and only if $\beta > \alpha - \frac{1}{q}$, for $q \in [1,\infty)$ and $\alpha > \frac{1}{q}$, see [11, p. 11] and [38]. Thus the boundedness of (1.2) yields boundedness of the Bochner-Riesz means for $f^\beta_R(A)$ if $\beta > \alpha - \frac{1}{q}$. For other applications of a Hörmander spectral multiplier theorem, see at the end of Subsection 3.2.

On the other hand, in the particular case of $A = -\Delta$, another direction of generalization of Hörmander's theorem is possible. Namely, [7, 28, 30, 32, 44, 52, 56, 63] have studied for which Banach spaces Y, the operator $f(-\Delta) \otimes \mathrm{Id}_Y$, initially defined on $L^p(\mathbb{R}^d) \otimes \mathrm{Id}_Y$ extends to a bounded operator on $L^p(\mathbb{R}^d; Y)$ for any f belonging to some Hörmander class $\mathcal{H}^\alpha_q$ (or some Mihlin class, which corresponds essentially to $q = \infty$ in $\mathcal{H}^\alpha_q$). A necessary condition is that Y is a UMD space. Moreover, the Fourier type [28], and Rademacher type/cotype [32] of Y play a role when one strives for better or best possible derivation order α.

In this article, we extend this latter programme partly to Dunkl operators, in place of the pure Laplacian. "Partly" refers to the fact that we treat only radial multipliers (see [58] for multivariate Dunkl spectral multipliers in the case $Y = \mathbb{C}$, but with nevertheless restriction on the underlying reflection group), we restrict to the subclass of Y being a UMD *lattice*, and we do not talk about operator valued spectral multipliers, i.e. f in (1.2) is a function with values in $\{T \in B(X) : T(\lambda - A)^{-1} = (\lambda - A)^{-1}T \text{ for all } \lambda \in \rho(A)\}$ (although this last part would be possible to some extent).

Roughly speaking, Dunkl operators are parameterized (with a continuous set of parameters κ) deformations of the partial derivatives and involve a reflection group W associated with a root system $\mathcal{R}$ (see Subsection 2.5.1. for their definition). A basic motivation for the study of these operators comes from the theory of spherical functions in analysis on Lie groups, which can be, in several situations, regarded as only the W-invariant part of a theory of Dunkl operators. Indeed, these operators play the role of derivatives for a generalized Laplacian Δ_κ (the so-called Dunkl Laplacian), whose restriction to W-invariant functions is given by

$$\operatorname{Res} \Delta_\kappa = \Delta + \sum_{\alpha \in \mathcal{R}} \kappa(\alpha) \frac{\partial_\alpha}{\langle \alpha, \cdot \rangle},$$

and this formula coincides for particular root systems and particular values of κ to the radial part of the Laplace-Beltrami operator of a Riemannian symmetric space of Euclidean type (see [20]). More generally, Dunkl operators have significantly contributed to the development of harmonic analysis associated with a root system and to the theory of multivariable hypergeometric functions. They also naturally appear in various other areas of mathematics, which include for instance, the theory of stochastic processes with values in a Weyl chamber or the theory of integrable quantum many body systems of Calogero-Moser-Sutherland type. As regards the harmonic analysis of Dunkl operators and their related objects, the subjacent analytic structure has a rich analogy with the Fourier analysis. However, there are still many problems to be solved and the theory is still at its infancy. One of the main obstruction is the lack of an explicit formula for the operator V_κ which intertwines the commutative algebra of Dunkl operators with the algebra of standard differential operators with constant coefficients. Apart from the case $W = \mathbb{Z}_2^d$ where the known formula for V_k allows to tackle and bypass some difficulties, many tools of harmonic analysis are not accessible. However, in this paper, we do not restrict ourselves to this particular reflection group, and all our result on Dunkl spectral multipliers are stated and proven for a general reflection group.

Coming back in particular to our $\mathcal{H}_q^\alpha$ Hörmander theorem, the order of derivation α and the integration parameter q that we get are

$$\alpha \in \mathbb{N},\ \alpha > \frac{d}{2} + \gamma_\kappa + \frac{1}{2}, \quad q = 1 \tag{1.3}$$

where $d + 2\gamma_\kappa$ is the doubling dimension of the Dunkl weight h_κ^2 on $\mathbb{R}^d$. Note that usually, one cannot expect to get a Hörmander $\mathcal{H}_2^\alpha$ multiplier theorem for $\alpha < \frac{d}{2} + \gamma_\kappa$ (see e.g. [26]), and that $\mathcal{H}_2^{\frac{d}{2}+\gamma_\kappa+\frac{1}{2}+\epsilon} \hookrightarrow \mathcal{H}_1^{\frac{d}{2}+\gamma_\kappa+\frac{1}{2}+\epsilon} \hookrightarrow \mathcal{H}_2^{\frac{d}{2}+\gamma_\kappa}$, for $\epsilon > 0$, the exponents being sharp in these embeddings (see Lemma 2.7 2.). Our main Theorem, see Theorem 3.13 and the section Preliminaries for precisions, states as follows.

Theorem 1.1 *Let $1 < p < \infty$ and $Y = Y(\Omega')$ be a UMD Banach lattice. Let A be the Dunkl Laplacian on $\mathbb{R}^d$, associated with both a general finite reflection group and a nonnegative multiplicity function κ. Let α be as above in* (1.3). *Assume that $m : (0,\infty) \to \mathbb{C}$ belongs to $\mathcal{H}_1^\alpha$. Then $m(A) \otimes \mathrm{Id}_Y$ extends to a bounded operator on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$.*

One of the features of the vector valued character of this theorem is that an operator of the form $B = \mathrm{Id}_{L^p} \otimes B_0$ will commute with A (or powers of A) and therefore, spectral theory of a *sum* $A^\beta + B$ is at hand. Consequently, we apply Theorem 1.1 to existence, uniqueness and (maximal) regularity of solutions of Cauchy problems or time independent problems involving $A^\beta + B$, where A is the Dunkl Laplacian, $\beta > 0$ is arbitrary and B is as above, see Section 4.

The methods of proof that we use for Theorem 1.1 are:

- Maximal estimates for semigroups associated with the Dunkl operator on $\mathbb{R}^d$ and on the sphere S^{d-1} [16];

- Square function estimates on $L^p(\Omega; Y(\Omega'))$, which is the same as R-boundedness for the space $L^p(\Omega; Y)$ as soon as Y is a UMD lattice;
- H^∞ functional calculus, in particular for vector valued diffusion semigroups [59];
- A reduction method to spherical harmonics, developed in [6, 15, 16, 17], which uses Cesàro means, that is, smoothed approximate identities similar to the Bochner-Riesz means $f_R^\beta(A)$ above.

The H^∞ functional calculus moreover is our starting point upon which we build the $\mathcal{H}_q^\alpha$ functional calculus. This replaces the usually used selfadjoint calculus approach, which defines $f(A)$ in (1.2) on $L^2(\Omega) \cap L^p(\Omega)$, and which by density gives an a priori meaning to $f(A)$ as an operator acting on $L^p(\Omega)$. Note that on $L^p(\Omega; Y)$, there is no selfadjoint calculus at hand, since $L^2(\Omega; Y)$ is not a Hilbert space in general.

We end this introduction with an overview of the following sections. In Section 2, we define and recall the central notions for this article, namely diffusion semigroups, UMD lattices, R-boundedness and square functions, functional calculus and Dunkl analysis. In Section 3, we then develop the proof of the spectral multiplier theorem. In a first place, in Subsection 3.1, we show a Marcinkiewicz multiplier Theorem 3.2, and then in Subsection 3.2, we deduce the Hörmander multiplier Theorem 3.13. We end the article with some illustrative applications to maximal regularity in Section 4.

Note that in the article, the symbol $\lesssim$ means an inequality up to a constant independent of the relevant variables.

2 Preliminaries

In this section, we define and recall the central notions of the article and we prove several lemmas which will be relevant for the sequel.

2.1 Symmetric contraction semigroups

Definition 2.1 *Let (Ω, μ) be a σ-finite measure space. Let $(T_t)_{t \geqslant 0}$ be a family of operators which act boundedly on $L^p(\Omega)$ for any $1 \leqslant p < \infty$. Then $(T_t)_{t \geqslant 0}$ is called symmetric contraction semigroup (on Ω), if*

1. *$(T_t)_{t \geqslant 0}$ is a strongly continuous semigroup on $L^p(\Omega)$ for any $1 \leqslant p < \infty$;*
2. *T_t is selfadjoint on $L^2(\Omega)$ for any $t \geqslant 0$;*
3. *$\|T_t\|_{L^p(\Omega) \to L^p(\Omega)} \leqslant 1$ for any $t \geqslant 0$ and $1 \leqslant p < \infty$.*

If in addition we have

1. *T_t is a positive operator for any $t \geqslant 0$;*
2. *$T_t(1) = 1$ (note that T_t is bounded on $L^\infty(\Omega)$ by selfadjointness and boundedness on $L^1(\Omega)$),*

then $(T_t)_{t \geqslant 0}$ is called a diffusion semigroup on Ω.

For a thorough study of diffusion semigroups, we refer the reader to [51] and [31] in the scalar and the vector valued case respectively.

2.2 UMD lattices

In this article, UMD lattices, i.e. Banach lattices which enjoy the UMD property, play a prevalent role. For a general treatment of Banach lattices and their geometric properties, we refer the reader to [42, Chapter 1]. We recall now definitions and some useful properties. A Banach space Y is called UMD space if the Hilbert transform

$$H : L^p(\mathbb{R}) \to L^p(\mathbb{R}),\ Hf(x) = PV - \int_{\mathbb{R}} \frac{1}{x-y} f(y)dy$$

extends to a bounded operator on $L^p(\mathbb{R}; Y)$, for some (equivalently for all) $1 < p < \infty$ [33, Theorem 5.1]. A UMD space is super-reflexive [2], and hence (almost by definition) B-convex. Let in the following Y be a UMD space which is also a Banach lattice. By B-convexity, Y is order continuous and therefore Y and its dual Y^* can be represented on the same measure space (Ω', μ), and moreover the duality is given simply by

$$\langle y, y^* \rangle = \int_{\Omega'} y(\omega')y^*(\omega')d\mu(\omega'),$$

see [42, 1.a, 1.b]. It is not difficult to show that if Y is UMD, then also its dual is UMD. Hence the dual of a UMD lattice is again a UMD lattice. $L^p(\Omega; Y)$ is reflexive for (Ω, ν) a σ-finite measure space, Y a UMD space and $1 < p < \infty$, since Y is reflexive and thus has the Radon-Nikodym property.

We tacitly shall use several times the following almost trivial observation.

Lemma 2.2 *Let $1 \leqslant p \leqslant \infty$, (Ω, μ) be a measure space and $Y = Y(\Omega')$ a Banach function lattice on (Ω', μ). Let $M : L^p(\Omega; Y) \to L^p(\Omega; Y)$ be a sublinear bounded operator on $L^p(\Omega; Y)$, i.e.*

$$|M(f+g)(\omega, \omega')| \leqslant |M(f)(\omega, \omega')| + |M(g)(\omega, \omega')|$$

for almost all $\omega \in \Omega$ and $\omega' \in \Omega'$. Let further $T : D \subseteq L^p(\Omega; Y) \to L^p(\Omega; Y)$ be a densely defined sublinear operator. If $|Tf(\omega, \omega')| \leqslant c|Mf(\omega, \omega')|$ for $f \in D$ and almost all $\omega \in \Omega$ and $\omega' \in \Omega'$, then $\|Tf\|_{L^p(\Omega;Y)} \leqslant c\|M\|\, \|f\|_{L^p(\Omega;Y)}$ for any $f \in D$.

Proof : This follows immediately from the fact that $L^p(\Omega; Y)$ is a Banach function lattice on $\Omega \times \Omega'$, that $f \leqslant g$ is then given by $f(\omega, \omega') \leqslant g(\omega, \omega')$ almost everywhere, and that $|f| \leqslant |g|$ implies $\|f\| \leqslant \|g\|$ in a Banach lattice. ■

2.3 R-boundedness and square functions

Let X be a Banach space and $\tau \subset B(X)$. Then τ is called R-bounded if there is some $C < \infty$ such that for any $n \in \mathbb{N}$, any $x_1, \ldots, x_n \in X$ and any $T_1, \ldots, T_n \in \tau$, we have

$$\mathbb{E}\Big\| \sum_{k=1}^n \epsilon_k T_k x_k \Big\|_X \leqslant C\mathbb{E}\Big\| \sum_{k=1}^n \epsilon_k x_k \Big\|_X,$$

where the ϵ_k are i.i.d. Rademacher variables on some probability space, that is, $\mathrm{Prob}(\epsilon_k = \pm 1) = \frac{1}{2}$. The least admissible constant C is called R-bound of τ and is denoted by $R(\tau)$. Note that trivially, we always have $R(\{T\}) = \|T\|$ for any $T \in B(X)$.

Let $Y = Y(\Omega')$ be a B-convex Banach lattice. Then we have the norm equivalence

$$\mathbb{E}\Big\| \sum_{k=1}^n \epsilon_k y_k \Big\|_Y \cong \Big\| \Big(\sum_{k=1}^n |y_k|^2 \Big)^{\frac{1}{2}} \Big\|_Y \tag{2.1}$$

uniformly in $n \in \mathbb{N}$ [43]. In particular, this also applies to $L^p(\Omega; Y)$, $1 < p < \infty$, since this will also be a B-convex Banach lattice. We deduce the following lemma.

Lemma 2.3 *Let T be a bounded (linear) operator on a B-convex Banach lattice $Y(\Omega')$. Then its tensor extension $T \otimes \mathrm{Id}_{\ell^2}$, initially defined on $Y(\Omega') \otimes \ell^2 \subset Y(\Omega'; \ell^2)$ is again bounded, on $Y(\Omega'; \ell^2)$. In particular, if $Y(\Omega')$ is a UMD lattice, then $Y(\Omega'; \ell^2)$ is also a UMD lattice.*

Proof : Let $(e_k)_k$ be the canonical basis of ℓ^2. We have

$$\begin{aligned}\Big\|(T \otimes \mathrm{Id}_{\ell^2})\Big(\sum_{k=1}^n y_k \otimes e_k\Big)\Big\|_{Y(\Omega';\ell^2)} &= \Big\|\Big(\sum_{k=1}^n |Ty_k|^2\Big)^{\frac{1}{2}}\Big\|_Y \cong \mathbb{E}\Big\|\sum_{k=1}^n \epsilon_k T y_k\Big\|_Y \\ &\leqslant R(\{T\})\mathbb{E}\Big\|\sum_{k=1}^n \epsilon_k y_k\Big\|_Y \\ &\cong \|T\|\Big\|\Big(\sum_{k=1}^n |y_k|^2\Big)^{\frac{1}{2}}\Big\|_Y .\end{aligned}$$

This shows the first part. For the second part, we note that if $Y(\Omega')$ is UMD, then the Hilbert transform $H : L^p(\mathbb{R}; Y) \to L^p(\mathbb{R}; Y)$ is bounded for all $1 < p < \infty$. Since $L^p(\mathbb{R}; Y)$ is again a B-convex Banach lattice, by the first part, we have that $H : L^p(\mathbb{R}; Y(\Omega'; \ell^2)) \to L^p(\mathbb{R}; Y(\Omega'; \ell^2))$ is bounded. Hence by definition, $Y(\Omega'; \ell^2)$ is a UMD (lattice). ■

2.4 Holomorphic (H^∞) and Hörmander ($\mathcal{H}^\alpha_p$) functional calculus

In this subsection, we recall the necessary background on functional calculus that we will treat in this article. Let $-A$ be a generator of an analytic semigroup $(T_z)_{z \in \Sigma_\delta}$ on some Banach space X, that is, $\delta \in (0, \frac{\pi}{2}]$, $\Sigma_\delta = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \delta\}$, the mapping $z \mapsto T_z$ from Σ_δ to $B(X)$ is analytic, $T_{z+w} = T_z T_w$ for any $z, w \in \Sigma_\delta$, and $\lim_{z \in \Sigma_{\delta'},\, |z| \to 0} T_z x = x$ for any strict subsector $\Sigma_{\delta'}$. We assume that $(T_z)_{z \in \Sigma_\delta}$ is a bounded analytic semigroup, which means $\sup_{z \in \Sigma_{\delta'}} \|T_z\| < \infty$ for any $\delta' < \delta$.

It is well-known [27, Theorem 4.6, p. 101] that this is equivalent to A being pseudo-ω-sectorial for $\omega = \frac{\pi}{2} - \delta$, that is,

1. A is closed and densely defined on X;
2. The spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$ (in $[0, \infty)$ if $\omega = 0$);
3. For any $\omega' > \omega$, we have $\sup_{\lambda \in \mathbb{C}\backslash\overline{\Sigma_{\omega'}}} \|\lambda(\lambda - A)^{-1}\| < \infty$.

We say that A is ω-sectorial if it is pseudo-ω-sectorial and has moreover dense range. In the sequel, we will always assume that A has dense range, to avoid technical difficulties. If A does not have dense range, but X is reflexive, which will always be the case in this article, then we may take the injective part A_0 of A on $\overline{R(A)} \subseteq X$ [41, Proposition 15.2], which then does have dense range. Here, $R(A)$ stands for the range of A. Then $-A$ generates an analytic semigroup on X if and only if so does $-A_0$ on $\overline{R(A)}$. This parallel will continue this section, i.e. the functional calculus for A_0 can be extended to A in an obvious way, see [35, Illustration 4.87].

For $\theta \in (0, \pi)$, let

$$H^\infty(\Sigma_\theta) = \{f : \Sigma_\theta \to \mathbb{C} : f \text{ analytic and bounded}\}$$

equipped with the uniform norm $\|f\|_{\infty,\theta}$. Let further

$$H_0^\infty(\Sigma_\theta) = \big\{ f \in H^\infty(\Sigma_\theta) : \exists\, C, \epsilon > 0 : |f(z)| \leqslant C \min(|z|^\epsilon, |z|^{-\epsilon}) \big\}.$$

For an ω-sectorial operator A and $\theta \in (\omega, \pi)$, one can define a functional calculus $H_0^\infty(\Sigma_\theta) \to B(X)$, $f \mapsto f(A)$ extending the ad hoc rational calculus, by using a Cauchy integral formula. If moreover, there exists a constant $C < \infty$ such that $\|f(A)\| \leqslant C\|f\|_{\infty,\theta}$, then A is said to have bounded $H^\infty(\Sigma_\theta)$ calculus and the above functional calculus can be extended to a bounded Banach algebra homomorphism $H^\infty(\Sigma_\theta) \to B(X)$. This calculus also has the property $f_z(A) = T_z$ for $f_z(\lambda) = \exp(-z\lambda)$, $z \in \Sigma_{\frac{\pi}{2}-\theta}$.

Lemma 2.4 *Let $\omega \in (0,\pi)$ and A be an ω-sectorial operator on X having an $H^\infty(\Sigma_\theta)$ calculus for some $\theta \in (\omega, \pi)$. Let $(f_n)_n$ be a sequence in $H^\infty(\Sigma_\theta)$ such that $f_n(\lambda) \to f(\lambda)$ for any $\lambda \in \Sigma_\theta$ and $\sup_n \|f_n\|_{\infty,\theta} < \infty$. Then for any $x \in X$, $f(A)x = \lim_n f_n(A)x$.*

Proof : See [41, Theorem 9.6] or [14, Lemma 2.1]. ■

We record the following proposition for later use, see [59, Theorem 4].

Proposition 2.5 *Let Y be a UMD lattice. A symmetric contraction semigroup on Ω extends to a bounded analytic semigroup on $L^p(\Omega; Y)$ for any $1 < p < \infty$. Moreover, its negative generator A has a bounded $H^\infty(\Sigma_\theta)$ calculus for some $\theta < \frac{\pi}{2}$.*

For further information on the H^∞ calculus, we refer e.g. to [41]. We now turn to Hörmander function classes and their calculi.

Definition 2.6 *Let $p \in [1,\infty)$ and $\alpha > \frac{1}{p}$. We define the Hörmander class by*

$$\mathcal{H}_p^\alpha = \Big\{ f : (0,\infty) \to \mathbb{C} \text{ bounded and continuous}, \underbrace{\sup_{R>0} \|\phi f(R\,\cdot)\|_{W_p^\alpha(\mathbb{R})}}_{:=\|f\|_{\mathcal{H}_p^\alpha}} < \infty \Big\}.$$

Here ϕ is any $C_c^\infty(0,\infty)$ function different from the constant 0 function (different choices of functions ϕ resulting in equivalent norms) and $W_p^\alpha(\mathbb{R})$ is the classical Sobolev space.

The Hörmander classes have the following properties.

Lemma 2.7
1. *Assume that $\alpha \in \mathbb{N}$ and $1 \leqslant p < \infty$. Then a locally integrable function $f : (0,\infty) \to \mathbb{C}$ belongs to the Hörmander class $\mathcal{H}_p^\alpha$ if and only if*

$$\sum_{k=0}^{\alpha} \sup_{R>0} \int_R^{2R} \Big| t^k \frac{d^k}{dt^k} f(t) \Big|^p dt/t < \infty,$$

if and only if

$$\max_{k=0 \text{ or } k=\alpha} \sup_{R>0} \int_R^{2R} \Big| t^k \frac{d^k}{dt^k} f(t) \Big|^p dt/t < \infty,$$

and the above quantities are equivalent to $\|f\|_{\mathcal{H}_p^\alpha}^p$.

2. *We have the continuous embeddings $H^\infty(\Sigma_\theta) \hookrightarrow \mathcal{H}_q^\alpha \hookrightarrow \mathcal{H}_p^\alpha \hookrightarrow \mathcal{H}_q^\beta$ for $\theta \in (0,\pi)$, $p < q$ and $\alpha \geqslant \beta + \frac{1}{p} - \frac{1}{q}$.*

3. *$\mathcal{H}_p^\alpha$ is a Banach algebra for the pointwise multiplication.*

4. The mapping $\mathcal{H}_p^\alpha \to \mathcal{H}_p^\alpha$, $m \mapsto m((\cdot)^\gamma)$, is an isomorphism for any $\gamma > 0$.

Proof : See [35, Section 4.2.1] for everything except the second claimed equivalence in 1. For the latter, we note that for $0 \leqslant l \leqslant k$, we have

$$\int_1^2 \Big| \frac{d^l}{dt^l} f(t) \Big|^p dt \lesssim \int_1^2 \Big| \frac{d^k}{dt^k} f(t) \Big|^p dt + \int_1^2 |f(t)|^p dt$$

according to [1, Theorem 5.2]. Now for a function $g \in W_p^k(R, 2R)$, take $f(t) = g(Rt)$, and substitute this in the above formula. One readily obtains that the first displayed term in 1. is dominated by the second displayed term. The converse estimate is trivial. ■

We can base a Hörmander functional calculus on the H^∞ calculus by the following procedure.

Definition 2.8 *We say that a 0-sectorial operator has a bounded $\mathcal{H}_p^\alpha$ calculus if for some $\theta \in (0, \pi)$ and any $f \in H^\infty(\Sigma_\theta)$, $\|f(A)\| \leqslant C\|f\|_{\mathcal{H}_p^\alpha} (\leqslant C'\|f\|_{\infty,\theta})$.*

In this case, the $H^\infty(\Sigma_\theta)$ calculus can be extended to a bounded Banach algebra homomorphism $\mathcal{H}_p^\alpha \to B(X)$ in the following way. Let

$$\mathcal{W}_p^\alpha = \{ f : (0, \infty) \to \mathbb{C} : \ f \circ \exp \in W_p^\alpha(\mathbb{R}) \}$$

equipped with the norm $\|f\|_{\mathcal{W}_p^\alpha} = \|f \circ \exp\|_{W_p^\alpha(\mathbb{R})}$. Note that for any $\theta \in (0, \pi)$, the space $H^\infty(\Sigma_\theta) \cap \mathcal{W}_p^\alpha$ is dense in $\mathcal{W}_p^\alpha$ [38]. Since $\mathcal{W}_p^\alpha \hookrightarrow \mathcal{H}_p^\alpha$, by the above density, we get a bounded mapping $\mathcal{W}_p^\alpha \to B(X)$ extending the H^∞ calculus.

Definition 2.9 *Let $(\phi_k)_{k\in\mathbb{Z}}$ be a sequence of functions in $C_c^\infty(0, \infty)$ with the properties that $\text{supp}\phi_k \subset [2^{k-1}, 2^{k+1}]$ and $\sum_{k\in\mathbb{Z}} \phi_k(t) = 1$ for all $t > 0$. Then $(\phi_k)_{k\in\mathbb{Z}}$ is called a dyadic partition of unity.*

Let $(\phi_k)_{k\in\mathbb{Z}}$ be a dyadic partition of unity. For $f \in \mathcal{H}_p^\alpha$, we have that $\phi_k f \in \mathcal{W}_p^\alpha$, hence $(\phi_k f)(A)$ is well-defined. Then it can be shown that for any $x \in X$, $\sum_{k=-n}^n (\phi_k f)(A)x$ converges as $n \to \infty$ and that it is independent of the choice of $(\phi_k)_{k\in\mathbb{Z}}$. This defines the operator $f(A)$, which in turn yields a bounded Banach algebra homomorphism $\mathcal{H}_p^\alpha \to B(X)$, $f \mapsto f(A)$. This is the Hörmander functional calculus. For details of this procedure, we refer to [35, Sections 4.2.3 - 4.2.6].

2.5 Dunkl transform, h-harmonic expansion, weighted space on the unit sphere

2.5.1 The Dunkl transform

We recall some basic concepts of Dunkl operators which will be needed in the article. For more details on Dunkl's analysis, the reader may especially consult [24, 49] and the references therein.

Let $d \in \mathbb{N} \setminus \{0\}$. Let $W \subset \mathcal{O}(\mathbb{R}^d)$ be a finite reflection group associated with a reduced root system $\mathcal{R}$ (not necessarily crystallographic) and let $\kappa : \mathcal{R} \to [0, +\infty[$ be a multiplicity function, that is, a W-invariant function. The (rational) Dunkl operators D_ξ^κ on $\mathbb{R}^d$, introduced in [23], are the following κ-deformations of directional derivatives ∂_ξ by reflections

$$D_\xi^\kappa f(x) = \partial_\xi f(x) + \sum_{\alpha\in\mathcal{R}_+} \kappa(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle x, \alpha\rangle} \langle \xi, \alpha\rangle, \quad x \in \mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product, σ_α denotes the reflection with respect to the hyperplane orthogonal to α and $\mathcal{R}_+$ denotes a positive subsystem of $\mathcal{R}$. The definition is of course independent of the choice of the positive subsystem since κ is W-invariant. These operators map $\mathcal{P}_n^d$ to $\mathcal{P}_{n-1}^d$, where $\mathcal{P}_n^d$ is the space of homogeneous polynomials of degree n in d variables, and they mutually commute. The Dunkl Laplacian is $\Delta_\kappa f = \sum_{i=1}^d (D_{e_i}^\kappa)^2 f$, where $(e_i)_{1\leqslant i\leqslant d}$ is the canonical basis of $\mathbb{R}^d$, and can be written explicitly as follows (see [23])

$$\Delta_\kappa f(x) = \Delta f(x) + 2\sum_{\alpha\in\mathcal{R}_+}\kappa(\alpha)\Big(\frac{\partial_\alpha f(x)}{\langle\alpha,x\rangle} - \frac{\|\alpha\|^2}{2}\frac{f(x)-f(\sigma_\alpha(x))}{\langle\alpha,x\rangle^2}\Big).$$

It generates a semigroup $\mathcal{H}_t^\kappa$ on $L^p(\mathbb{R}^d, h_\kappa^2)$, $1\leqslant p<\infty$, which is a diffusion semigroup in the sense of Definition 2.1 [21, Theorem 2.6] (see also [46, 48]), where the weight h_κ^2 defined on $\mathbb{R}^d$ by

$$h_\kappa^2(x) = \prod_{\alpha\in\mathcal{R}_+}|\langle x,\alpha\rangle|^{2\kappa(\alpha)}$$

is invariant under the action of W and homogeneous of degree $2\gamma_\kappa$, with

$$\gamma_\kappa = \sum_{\alpha\in\mathcal{R}_+}\kappa(\alpha).$$

The Dunkl operators give rise to a rich analytic structure since they are also intertwined with the usual derivatives. Indeed, there exists a unique linear isomorphism V_κ (called intertwining operator) on $\mathcal{P} = \bigoplus_{n\geqslant 0}\mathcal{P}_n^d$ such that

$$V_\kappa(\mathcal{P}_n^d) = \mathcal{P}_n^d, \qquad V_\kappa|_{\mathcal{P}_0^d} = \mathrm{Id}|_{\mathcal{P}_0^d}, \qquad D_\xi^\kappa V_\kappa = V_\kappa\partial_\xi \quad \forall \xi\in\mathbb{R}^d.$$

Unfortunately, the intertwining operator is explicitly known only in some special cases but for a general reflection group, we all the same have the following significant Laplace-type representation due to Rösler (see [47]): for every $x\in\mathbb{R}^d$, there exists a unique probability measure $d\mu_x^\kappa$, compactly supported in the convex hull of the orbit of x under the action of W (among other properties) such that for any $P\in\mathcal{P}$

$$V_\kappa P(x) = \int_{\mathbb{R}^d} P(\xi)d\mu_x^\kappa(\xi),$$

and this formula allows to extend it to various larger function spaces. For $y\in\mathbb{C}^d$, let

$$E_\kappa(x,y) = V_\kappa\big(\mathrm{e}^{\langle\cdot,y\rangle}\big)(x), \quad x\in\mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes the bilinear extension of the Euclidean inner product to $\mathbb{C}^d\times\mathbb{C}^d$. Then $E_\kappa(\cdot,y)$ is the unique real-analytic solution of the spectral problem

$$D_\xi^\kappa f = \langle\xi,y\rangle f \quad \forall\xi\in\mathbb{R}^d, \quad f(0)=1,$$

and moreover, E_κ extends to a holomorphic function on $\mathbb{C}^d\times\mathbb{C}^d$, see [45]. This kernel, the so-called Dunkl kernel, gives rise to an integral transform which generalizes the Euclidean Fourier transform. For every $f\in L^1(\mathbb{R}^d,h_\kappa^2)$, the Dunkl transform of f, denoted by $\mathcal{F}_\kappa f$, is defined by

$$\mathcal{F}_\kappa f(x) = c_\kappa\int_{\mathbb{R}^d} E_\kappa(-ix,y)f(y)h_\kappa^2(y)dy, \quad x\in\mathbb{R}^d,$$

where $c_\kappa^{-1} = \int_{\mathbb{R}^d} e^{-\|x\|^2/2} h_\kappa^2(x)dx$ is a Mehta-type constant. We point out that the Dunkl transform coincides with the Euclidean Fourier transform when $\kappa = 0$ (since $D_\xi^0 = \partial_\xi$ and $V_0 = \mathrm{Id}$) and that it is more or less a Hankel transform when $d = 1$ (and then $W \simeq \mathbb{Z}_2$).

The Dunkl transform has the following properties, where for a given Banach lattice $Y = Y(\Omega')$, we denote by $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ the Bochner space of classes of functions $f : \mathbb{R}^d \to Y$ such that

$$\|f\|_{\kappa,p;Y} = \left(\int_{\mathbb{R}^d} \|f(y)\|_Y^p h_\kappa^2(y)dy \right)^{\frac{1}{p}} < \infty,$$

with the standard modification if $p = \infty$. If $Y = \mathbb{C}$, we usually omit Y in the notations.

Lemma 2.10 *1. If $f \in L^1(\mathbb{R}^d, h_\kappa^2)$ then $\mathcal{F}_\kappa f \in C_0(\mathbb{R}^d)$.*

2. *$\mathcal{F}_\kappa$ is an isomorphism of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ onto itself, and $\mathcal{F}_\kappa^2 f(x) = f(-x)$.*
3. *The Dunkl transform has a unique extension to an isometric isomorphism of $L^2(\mathbb{R}^d, h_\kappa^2)$.*
4. *Let $f \in L^1(\mathbb{R}^d, h_\kappa^2)$. If $\mathcal{F}_\kappa f$ is in $L^1(\mathbb{R}^d, h_\kappa^2)$, then we have the inversion formula*

$$f(x) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix, y)\mathcal{F}_\kappa f(y) h_\kappa^2(y)dy.$$

5. *For $f \in \mathcal{S}(\mathbb{R}^d)$, we have $\Delta_\kappa(f) = \mathcal{F}_\kappa^{-1}[-\|\xi\|^2 \mathcal{F}_\kappa f(\xi)]$, and the semigroup $\mathcal{H}_t^\kappa$ generated by the Dunkl Laplacian satisfies $\mathcal{H}_t^\kappa(f) = \mathcal{F}_\kappa^{-1}[e^{-t\|\xi\|^2}\mathcal{F}_\kappa f(\xi)]$.*
6. *For $m : (0,\infty) \to \mathbb{C}$ a bounded measurable function and Y a Banach space, $T_m(f) = \mathcal{F}_\kappa^{-1}[m(\|\xi\|)\mathcal{F}_\kappa f(\xi)]$ is a well defined element of $C_0(\mathbb{R}^d; Y)$ for $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$.*
7. *Let Y be a UMD lattice and $1 < p < \infty$. Then $-\Delta_\kappa$ is an ω-sectorial operator for some $\omega < \frac{\pi}{2}$ on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$, in particular injective with dense range.*
8. *Let Y be a UMD lattice and $1 < p < \infty$. Let $q \in [1,\infty)$ and $\alpha > \frac{1}{q}$. Let $A = -\Delta_\kappa$ be the negative generator of the Dunkl heat semigroup $\mathcal{H}_t^\kappa$ on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$. Let $\omega < \theta \in (0,\pi)$ such that A is ω-sectorial on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$.*
   - *(a) Suppose that for any $m \in H_0^\infty(\Sigma_\theta)$, the above operator T_m, initially defined on $\mathcal{S}(\mathbb{R}^d) \otimes Y$, extends to a bounded operator on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ and $\|T_m\| \leqslant C\|m\|_{\mathcal{H}_q^\alpha}$. Then A has a Hörmander $\mathcal{H}_q^\alpha$ calculus and $m(A) = T_{\tilde{m}}$ for $m \in \mathcal{H}_q^\alpha$ and $\tilde{m}(t) = m(t^2)$.*
   - *(b) Suppose that there is a $C < \infty$ such that for any $m \in H_0^\infty(\Sigma_\theta)$,*

$$\|m(A)\|_{L^p(\mathbb{R}^d, h_\kappa^2; Y) \to L^p(\mathbb{R}^d, h_\kappa^2; Y)} \leqslant C\|m\|_{\mathcal{H}_q^\alpha}.$$

Then A has a $\mathcal{H}_q^\alpha$ calculus, for any $m \in \mathcal{H}_q^\alpha$, T_m defined above extends to a bounded operator on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ and $m(A) = T_{\tilde{m}}$.

Proof : For parts 1.,2.,3.,4.,5., we refer to [19], [55].

For 6., we note that $\mathcal{F}_\kappa f$ belongs again to $\mathcal{S}(\mathbb{R}^d) \otimes Y$, so $\xi \mapsto m(\|\xi\|)\mathcal{F}_\kappa f(\xi)$ belongs to $L^1(\mathbb{R}^d) \otimes Y$. Now apply part 1.

For 7., note that since Δ_κ generates a diffusion semigroup, it is pseudo-ω-sectorial for some $\omega < \frac{\pi}{2}$ on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ according to Proposition 2.5. Then the fact that $A = -\Delta_\kappa$ is injective on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ (equivalently, has dense range, equivalently is ω-sectorial) can be seen as follows. According to [41, Proposition 15.2], it suffices to show that $t(t+A)^{-1}f \to 0$ for any $f \in L^p(\mathbb{R}^d, h_\kappa^2; Y)$, as $t \to 0$. Since $\sup_{t>0} \|t(t+A)^{-1}\| < \infty$ by pseudo-sectoriality of A, it suffices to consider $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$. For these f, we have

$$t(t+A)^{-1}f = \mathcal{F}_\kappa^{-1}\Big[\frac{t}{t+\|\xi\|^2}\mathcal{F}_\kappa(f)(\xi)\Big] \to 0$$

in $C_0(\mathbb{R}^d; Y)$ by dominated convergence, since $|\frac{t}{t+\|\xi\|^2}| \leqslant 1$ and $\mathcal{F}_\kappa(f) \in \mathcal{S}(\mathbb{R}^d) \otimes Y$ according to part 2. By [41, Proposition 15.2], we already know that $t(t+A)^{-1}f$ converges in $L^p(\mathbb{R}^d, h_\kappa^2; Y)$, so by unicity of the limit, it converges to 0 in $L^p(\mathbb{R}^d, h_\kappa^2; Y)$.

We turn to 8. It follows from 5. and the representation formula

$$(\lambda - A)^{-1} = -\int_0^\infty e^{\lambda t} e^{-tA} dt$$

for $\Re\lambda < 0$ that $T_{\tilde{m}} = m(A)$ for $m(t) = (\lambda - t)^{-1}$. This identity can be extended for $\lambda \in \mathbb{C}\backslash\overline{\Sigma_\omega}$, where $\omega < \theta$ and $\sigma(A) \subset \overline{\Sigma_\omega}$, by analytic continuation. Then the identity follows for any $m \in H_0^\infty(\Sigma_\theta)$ from the Cauchy formula defining the H_0^∞ calculus. We now show step by step that $T_{\tilde{m}} = m(A)$ holds for $m \in H^\infty(\Sigma_\theta)$, for $m \in \mathcal{W}_q^\alpha$ and for $m \in \mathcal{H}_q^\alpha$, under either the assumptions 8. (a) or 8. (b). Each time, it will suffice by linearity and density to show the identity applied to $f \otimes y$ with $f \in \mathcal{S}(\mathbb{R}^d)$ and $y \in Y$. So let $m \in H^\infty(\Sigma_\theta)$. Let

$$\rho_n(\lambda) = \left(\frac{\lambda}{(1+\lambda)^2}\right)^{\frac{1}{n}} \in H_0^\infty(\Sigma_\theta).$$

We have $\rho_n(\lambda) \to 1$ for any $\lambda \in \Sigma_\theta$ and $\sup_n \|\rho_n\|_{\infty,\theta} = \sup_n \|\rho_1\|_{\infty,\theta}^{\frac{1}{n}} < \infty$. The assumption 8. (b) readily implies that A has an $H^\infty(\Sigma_\theta)$ calculus by Lemma 2.7 2., whereas 8. (a) also implies it via the already provided identity $T_{\tilde{m}} = m(A)$ for $m \in H_0^\infty(\Sigma_\theta)$. Thus, by the Convergence Lemma 2.4, we have with $m_n := m\rho_n$,

$$\begin{aligned} m(A)(f \otimes y) &= \lim_n m_n(A)(f \otimes y) \\ &= \lim_n \mathcal{F}_\kappa^{-1}[m_n(\|\xi\|^2)\mathcal{F}_\kappa f(\xi)] \otimes y \\ &= \mathcal{F}_\kappa^{-1}[m(\|\xi\|^2)\mathcal{F}_\kappa f(\xi)] \otimes y \\ &= T_{\tilde{m}}(f \otimes y), \end{aligned}$$

the first limit in $L^p(\mathbb{R}^d, h_\kappa^2; Y)$, the second limit in $C_0(\mathbb{R}^d; Y)$, by 1. and dominated convergence. It follows $T_{\tilde{m}} = m(A)$ for $m \in H^\infty(\Sigma_\theta)$, and that A has a $\mathcal{H}_q^\alpha$ calculus, since

$$\begin{aligned} \|m(A)\| &\leqslant \limsup_n \|m_n(A)\| \\ &\lesssim \limsup_n \|m_n\|_{\mathcal{H}_q^\alpha} \\ &\lesssim \|m\|_{\mathcal{H}_q^\alpha} \limsup_n \|\rho_n\|_{\mathcal{H}_q^\alpha} \\ &\lesssim \|m\|_{\mathcal{H}_q^\alpha} \limsup_n \|\rho_n\|_{\theta,\infty} \\ &\lesssim \|m\|_{\mathcal{H}_q^\alpha}. \end{aligned}$$

Now let $m \in \mathcal{W}^\alpha_q$, and m_n a sequence in $H^\infty(\Sigma_\theta) \cap \mathcal{W}^\alpha_q$ approximating m in $\mathcal{W}^\alpha_q$. We have

$$\begin{aligned} m(A)(f \otimes y) &= \lim_n m_n(A)(f \otimes y) \\ &= \lim_n T_{\tilde{m}_n}(f \otimes y) \\ &= \lim_n \mathcal{F}_\kappa^{-1}[m_n(\|\xi\|^2)\mathcal{F}_\kappa f(\xi)] \otimes y \\ &= \mathcal{F}_\kappa^{-1}[m(\|\xi\|^2)\mathcal{F}_\kappa f] \otimes y, \end{aligned}$$

where the last limit holds by $\mathcal{W}^\alpha_q \hookrightarrow L^\infty(0,\infty)$ and dominated convergence, plus part 1. Thus, $m(A) = T_{\tilde{m}}$ for $m \in \mathcal{W}^\alpha_q$.

Let finally $m \in \mathcal{H}^\alpha_q$. Let $(\phi_k)_{k\in\mathbb{Z}}$ be a dyadic partition of unity as in Definition 2.9. Then

$$m(A)(f \otimes y) = \lim_n \sum_{k=-n}^{n} (\phi_k m)(A)(f \otimes y) = \lim_n \sum_{k=-n}^{n} T_{\widetilde{\phi_k m}}(f \otimes y) = T_{\tilde{m}}(f \otimes y),$$

the second limit holding by almost the same argument as before in the case $\mathcal{W}^\alpha_q$. ■

We now turn to h-harmonic expansions and analysis on the sphere.

2.5.2 h-harmonic expansions and analysis on the sphere.

For more details on h-harmonic expansions and analysis on the sphere, the reader may consult the expertly written book of Dai-Xu [17]. For $d \geqslant 2$, we let $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$, and for $1 \leqslant p < \infty$ and Y a Banach lattice, we let $L^p(S^{d-1}, h_\kappa^2; Y)$ be the Bochner space of equivalence classes of measurable functions $f : S^{d-1} \to Y$ such that

$$\|f\|_{L^p(S^{d-1},h_\kappa^2;Y)} := \left(a_\kappa \int_{S^{d-1}} \|f(y)\|_Y^p h_\kappa^2(y)dy\right)^{\frac{1}{p}} < \infty,$$

with the standard modification if $p = \infty$. Here, the measure on S^{d-1} is surface measure and $a_\kappa^{-1} = \int_{S^{d-1}} h_\kappa^2(y)dy$. By abuse of notation, we also write $\|f\|_{L^p(S^{d-1},h_\kappa^2;Y)} = \|f\|_{\kappa,p;Y}$.

Let $P \in \mathcal{P}^d_n$. Then P is called an h-harmonic polynomial of degree n if $\Delta_\kappa P = 0$. It is well-known (see [22]) that a homogeneous polynomial is an h-harmonic polynomial if it is orthogonal to all polynomials of lower degree with respect to the inner product of $L^2(S^{d-1}, h_\kappa^2)$. For $j \geqslant 0$, we let $\mathrm{proj}_j^\kappa : L^2(S^{d-1}, h_\kappa^2) \to L^2(S^{d-1}, h_\kappa^2)$ be the orthogonal projection with image the space of all h-harmonics of degree j. The projection proj_j^κ has the following integral representation

$$\mathrm{proj}_j^\kappa f(x) = a_\kappa \int_{S^{d-1}} f(y)\mathscr{P}_j^\kappa(x,y)h_\kappa^2(y)dy, \quad x \in S^{d-1},$$

with (see [60, Theorem 3.2, (3.1)])

$$\mathscr{P}_j^\kappa(x,y) = \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}\left(\langle x,\cdot\rangle\right)\Big](y), \quad x,y \in S^{d-1},$$

where

$$\lambda_\kappa = \frac{d}{2} + \gamma_\kappa - 1 \tag{2.2}$$

and where $C_j^{\lambda_\kappa}$ is the standard Gegenbauer (or ultraspherical) polynomial of degree j and index λ_κ (see [53] for instance). Now, let

$$c_{\lambda_\kappa}^{-1} = \int_{-1}^{1} (1-t^2)^{\lambda_\kappa - \frac{1}{2}} dt = \sqrt{\pi}\,\frac{\Gamma(\lambda_\kappa + \frac{1}{2})}{\Gamma(\lambda_\kappa + 1)}.$$

Then, according to [61], we define for $f \in L^1(S^{d-1}, h_\kappa^2)$ and $g \in L^1([-1,1], \omega_{\lambda_\kappa})$, with ω_{λ_κ} the weight for which the Gegenbauer polynomials are orthogonal, that is $\omega_{\lambda_\kappa}(t) = (1-t^2)^{\lambda_\kappa - \frac{1}{2}}$,

$$f * g(x) = a_\kappa \int_{S^{d-1}} f(y) V_\kappa \Big[g\big(\langle x, \cdot \rangle\big) \Big](y) h_\kappa^2 dy,$$

and this generalized convolution, which reduces when $\kappa = 0$ to the spherical convolution [8], satisfies Young-type inequalities [61, Proposition 2.2]. We now state the following lemma, which will be useful in the sequel.

Lemma 2.11 *For $j \geqslant 0$, $1 \leqslant p \leqslant \infty$ and any Banach space Y, the operator* proj_j^κ *extends boundedly to $L^p(S^{d-1}, h_\kappa^2; Y)$, and we have, for j large enough, the norm estimate*

$$\|\mathrm{proj}_j^\kappa\|_{L^p(S^{d-1}, h_\kappa^2; Y) \to L^p(S^{d-1}, h_\kappa^2; Y)} \lesssim j^{2\lambda_\kappa}.$$

Proof : Let $x \in S^{d-1}$. Write for any $j \geqslant 0$

$$\mathrm{proj}_j^\kappa f(x) = a_\kappa \int_{S^{d-1}} f(y) \mathscr{P}_j^\kappa(x,y) h_\kappa^2(y) dy = \left(f * \Big[\frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa} \Big] \right)(x).$$

Thus, by Young-type inequalities for the generalized convolution on the sphere, we have both the inequalities

$$\|\mathrm{proj}_j^\kappa f\|_{L^1(S^{d-1}, h_\kappa^2)} \leqslant \|f\|_{L^1(S^{d-1}, h_\kappa^2)} \Big\| \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa} \Big\|_{L^1([-1,1], \omega_{\lambda_\kappa})}$$
$$\|\mathrm{proj}_j^\kappa f\|_{L^\infty(S^{d-1}, h_\kappa^2)} \leqslant \|f\|_{L^\infty(S^{d-1}, h_\kappa^2)} \Big\| \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa} \Big\|_{L^1([-1,1], \omega_{\lambda_\kappa})}.$$

Besides,

$$\begin{aligned} \Big\| \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa} \Big\|_{L^1([-1,1], \omega_{\lambda_\kappa})} &= c_{\lambda_\kappa} \frac{j+\lambda_\kappa}{\lambda_\kappa} \int_{-1}^1 |C_j^{\lambda_\kappa}(t)| (1-t^2)^{\lambda_\kappa - \frac{1}{2}} dt \\ &\leqslant \frac{j+\lambda_\kappa}{\lambda_\kappa} \frac{(2\lambda_\kappa)_j}{j!}, \end{aligned}$$

where we have used the inequality (see for instance [4, p. 350])

$$|C_j^{\lambda_\kappa}(t)| \leqslant C_j^{\lambda_\kappa}(1) = \frac{(2\lambda_\kappa)_j}{j!},$$

with $(x)_n$ the so-called Pochhammer symbol. Moreover, since we can write $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$, Stirling's formula $\Gamma(a) \simeq \sqrt{2\pi} a^{a-\frac{1}{2}} \mathrm{e}^{-a}$ gives us

$$\frac{(2\lambda_\kappa)_j}{j!} = \frac{\Gamma(2\lambda_\kappa + j)}{\Gamma(2\lambda_\kappa)\Gamma(j+1)} \simeq C(\lambda_\kappa) j^{2\lambda_\kappa - 1}. \tag{2.3}$$

We can conclude, for j large enough, that

$$\Big\| \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa} \Big\|_{L^1([-1,1], \omega_{\lambda_\kappa})} \lesssim j^{2\lambda_\kappa},$$

and therefore, we have both the inequalities

$$\|\mathrm{proj}_j^\kappa\|_{L^1(S^{d-1},h_\kappa^2)\to L^1(S^{d-1},h_\kappa^2)} \lesssim j^{2\lambda_\kappa}$$
$$\|\mathrm{proj}_j^\kappa\|_{L^\infty(S^{d-1},h_\kappa^2)\to L^\infty(S^{d-1},h_\kappa^2)} \lesssim j^{2\lambda_\kappa}.$$

We can tensorise these estimates to get estimates on Bochner spaces, namely

$$\|\mathrm{proj}_j^\kappa\|_{L^p(S^{d-1},h_\kappa^2;Y)\to L^p(S^{d-1},h_\kappa^2;Y)} \lesssim j^{2\lambda_\kappa}$$

holds for $p = 1, \infty$, and then by interpolation also for $p \in [1, \infty]$. ■

We close this section with some facts on both a generalized Poisson and heat semigroup on $(S^{d-1}, h_\kappa^2(y)dy)$. We first recall their definition (see [16, p. 482]).

Definition 2.12 *The Poisson semigroup T_t^κ on $(S^{d-1}, h_\kappa^2(y)dy)$ is defined by*

$$T_t^\kappa f = \sum_{j=0}^\infty e^{-jt}\mathrm{proj}_j^\kappa f, \quad t > 0.$$

The heat semigroup H_t^κ on $(S^{d-1}, h_\kappa^2(y)dy)$ is defined by

$$H_t^\kappa f = \sum_{j=0}^\infty e^{-j(j+2\lambda_\kappa)t}\mathrm{proj}_j^\kappa f, \quad t > 0.$$

The following statement will be of particular interest.

Lemma 2.13 *Both the Poisson and the heat semigroup are diffusion semigroups.*

Proof : That they are semigroups is clear from the fact that the proj_j^κ are projections. The contractivity of H_t^κ on $L^p(S^{d-1}, h_\kappa^2)$ for all $1 \leqslant p \leqslant \infty$ and $t \geqslant 0$ is proved in [16, Proof of Lemma 2.2]. This yields the strong continuity of both semigroups on L^2. Indeed, strong continuity $T_t^\kappa f \to f$ and $H_t^\kappa f \to f$ $(t \to 0)$ is clear for elements of the form $f = \sum_{j=0}^J \mathrm{proj}_j^\kappa g$ (J finite), since $e^{-tj} \to 1$ and $e^{-tj(j+2\lambda_\kappa)} \to 1$ uniformly for $j = 0, \ldots, J$ as $t \to 0$. Now use a 3ϵ argument for general $f \in L^2$. Then the strong continuity extrapolates on L^p by contractivity. It is proved in [16, Proof of Lemma 2.2] that H_t^κ are positive operators, and [16, (2.10)] yields then that T_t^κ is also a positive operator. We finally show that $T_t^\kappa(1) \to 1$ and $H_t^\kappa(1) \to 1$ as $t \to \infty$. This will imply that $T_t^\kappa(1) = 1$, and the same for H_t^κ. Indeed, $T_{t+s}^\kappa(1) = T_t^\kappa T_s^\kappa(1)$, so $1 = \lim_{s\to\infty} T_{t+s}^\kappa(1) = T_t^\kappa \lim_{s\to\infty} T_s^\kappa(1) = T_t^\kappa(1)$. We have

$$T_t^\kappa(1) = \mathrm{proj}_0^\kappa(1) + \sum_{j=1}^\infty e^{-tj}\mathrm{proj}_j^\kappa(1).$$

Now $\mathrm{proj}_0^\kappa(1) = 1$ and by Lemma 2.11

$$\Big\|\sum_{j=1}^\infty e^{-tj}\mathrm{proj}_j^\kappa(1)\Big\|_\infty \leqslant e^{-t}\sum_{j=1}^\infty e^{-t(j-1)}\big\|\mathrm{proj}_j^\kappa(1)\big\|_\infty \lesssim e^{-t} \to 0 \quad (t\to\infty).$$

The same argument applies for H_t^κ. ■

3 Spectral multipliers with values in UMD lattices

3.1 Marcinkiewicz-type multiplier theorem for h-harmonic expansions

In this section, we take $d \in \mathbb{N}$ with $d \geqslant 2$. Let us begin by recalling the definition of the usual difference operator.

Definition 3.1 *Given a sequence $(\mu_j)_{j\geqslant 0}$ of complex numbers, we define recursively*

$$\Delta\mu_j = \mu_j - \mu_{j+1}, \quad \Delta^{n+1}\mu_j = \Delta^n\mu_j - \Delta^n\mu_{j+1}, \quad j \geqslant 0,\, n \geqslant 1.$$

We now state the main result of this section.

Theorem 3.2 *Let $1 < p < \infty$ and $Y = Y(\Omega')$ be a UMD Banach lattice. Let $(\mu_j)_{j\geqslant 0}$ be a scalar sequence. Suppose that for some integer $n_0 > \frac{d}{2} + \gamma_\kappa = \lambda_\kappa + 1$, we have*

(C_0) $\sup_{j\geqslant 0} |\mu_j| \leqslant M < \infty$,

(C_{n_0}) $\sup_{j\geqslant 0} 2^{j(n_0-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^{n_0}\mu_l| \leqslant M < \infty$.

Then $(\mu_j)_{j\geqslant 0}$ defines an $L^p(S^{d-1}, h_\kappa^2; Y)$ multiplier, that is

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j^\kappa f \right\|_{\kappa,p;Y} \leqslant c_p M \|f\|_{\kappa,p;Y},$$

where the constant c_p is independent of f and $(\mu_j)_{j\geqslant 0}$.

This theorem generalizes the scalar case proved by Dai and Xu in [16]. Therefore, if we specialize Theorem 3.2 to $Y = \mathbb{C}$ and $\kappa = 0$, then we recover the famous Marcinkiewicz type theorem for zonal multipliers due to Bonami-Clerc [6].

The proof, which is divided into several lemmas and a proposition, follows the strategy of Bonami-Clerc adapted in the Dunkl setting by Dai-Xu. A crucial role in the proof will be played by several kinds of Littlewood-Paley type g-functions closely related to Cesàro means for h-harmonic expansions. Let us begin with the following notation. Let $(T_t^\kappa)_{t\geqslant 0}$ be the generalized Poisson semigroup on $L^p(S^{d-1}, h_\kappa^2; Y)$. Then we set

$$P_r^\kappa = T_{-\log(r)}^\kappa, \quad 0 < r < 1. \tag{3.1}$$

The first lemma will provide a new equivalent norm on $L^p(S^{d-1}, h_\kappa^2; Y)$, in terms of a well suited g-function.

Lemma 3.3 *Let $1 < p < \infty$ and $Y = Y(\Omega')$ be a UMD Banach lattice. Then, for any $f \in L^p(S^{d-1}, h_\kappa^2; Y)$, we have the two-sided estimate*

$$\frac{1}{c}\|f\|_{\kappa,p;Y} \leqslant \left\| \left(\int_0^1 (1-r) \left| \frac{\partial}{\partial r} P_r^\kappa f \right|^2 dr \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y} \leqslant c\|f\|_{\kappa,p;Y},$$

where in the first inequality we assume that $\int_{S^{d-1}} f(y) h_\kappa^2(y) dy = 0$.

Proof : The semigroup $(T_t^\kappa)_{t\geqslant 0}$ is a diffusion semigroup, so its vector-valued extension on $L^p(S^{d-1},h_\kappa^2;Y)$ has an $H^\infty(\Sigma_\omega)$ calculus of some angle $\omega<\frac{\pi}{2}$, thanks to Proposition 2.5. According to [59, Proposition 9], we have, for any $f\in L^p(S^{d-1},h_\kappa^2;Y)$, the following square function estimate

$$\left\|\left(\int_0^\infty t\left|\frac{\partial}{\partial t}T_t^\kappa f\right|^2 dt\right)^{\frac12}\right\|_{\kappa,p;Y}\leqslant C\|f\|_{\kappa,p;Y}.$$

We next show that we also have the converse estimate to the previous one, under the additional assumption that $\int_{S^{d-1}} f(y)h_\kappa^2(y)dy=0$. Note that

$$\lim_{t\to\infty}T_t^\kappa f=\lim_{t\to\infty}\sum_{j=0}^\infty e^{-jt}\mathrm{proj}_j^\kappa f=\mathrm{proj}_0^\kappa f$$

is the projection onto the kernel of the negative generator A_p of $(T_t^\kappa)_{t\geqslant 0}$ (its version on $L^p(S^{d-1},h_\kappa^2;Y)$), according to the decomposition $L^p(S^{d-1},h_\kappa^2;Y)=\mathrm{Ker}(A_p)\oplus\overline{R(A_p)}$, valid for any negative generator of an analytic semigroup on a reflexive space [41, Proposition 15.2]. Since

$$\mathrm{proj}_0^\kappa f=a_\kappa\int_{S^{d-1}}f(y)h_\kappa^2(y)dy=0,$$

our particular f must lie in $\overline{R(A_p)}$. Consider now the part $\tilde{A}_p$ of A_p on $\overline{R(A_p)}$ [41, Proposition 15.2], which has dense range. Then according to [41, Lemma 9.13], we have the partition

$$f=\int_0^\infty\psi(t\tilde{A}_p)f\frac{dt}{t}=\int_0^\infty\psi(tA_p)f\frac{dt}{t},$$

for $\psi\in H_0^\infty(\Sigma_{\frac{\pi}{2}-\epsilon})$ of $\frac{dt}{t}$-integral 1, as an improper integral, under the additional assumption that $f\in R(A_p)\cap D(A_p)$. Here, $D(A_p)$ stands for the domain of A_p. Apply this partition to $\psi(t)=cte^{-t}\cdot te^{-t}$, we get

$$\langle f,g\rangle=c\int_0^\infty\langle tA_pT_t^\kappa f,tA_{p^*}T_t^{\kappa *}g\rangle\frac{dt}{t},$$

for any $g\in L^{p^*}(S^{d-1},h_\kappa^2;Y^*)$. This implies

$$\begin{aligned}
\left|\langle f,g\rangle\right| &\lesssim \int_0^\infty\left|\langle tA_pT_t^\kappa f,tA_{p^*}T_t^\kappa g\rangle\right|\frac{dt}{t}\\
&=\int_0^\infty\left|\int_{S^{d-1}}\int_{\Omega'}tA_pT_t^\kappa(f)(y,\omega')\cdot\overline{tA_{p^*}T_t^\kappa(g)(y,\omega')}d\omega'h_\kappa^2(y)dy\right|\frac{dt}{t}\\
&\leqslant\int_{S^{d-1}}\int_{\Omega'}\int_0^\infty\left|tA_pT_t^\kappa(f)(y,\omega')\cdot\overline{tA_{p^*}T_t^\kappa(g)(y,\omega')}\right|\frac{dt}{t}d\omega'h_\kappa^2(y)dy\\
&\leqslant\int_{S^{d-1}}\int_{\Omega'}\left(\int_0^\infty\left|tA_pT_t^\kappa(f)(y,\omega')\right|^2\frac{dt}{t}\right)^{\frac12}\cdot\left(\int_0^\infty\left|tA_{p^*}T_t^\kappa(g)(y,\omega')\right|^2\frac{dt}{t}\right)^{\frac12}d\omega'h_\kappa^2(y)dy\\
&\leqslant\left\|\left(\int_0^\infty t\left|\frac{\partial}{\partial t}T_t^\kappa f\right|^2dt\right)^{\frac12}\right\|_{\kappa,p;Y}\left\|\left(\int_0^\infty t\left|\frac{\partial}{\partial t}T_t^\kappa g\right|^2dt\right)^{\frac12}\right\|_{\kappa,p^*;Y^*}.
\end{aligned}$$

Applying now the upper estimate for g and on $L^{p^*}(S^{d-1},h_\kappa^2;Y^*)$, we get

$$\left|\langle f,g\rangle\right|\lesssim\left\|\left(\int_0^\infty t\left|\frac{\partial}{\partial t}T_t^\kappa f\right|^2dt\right)^{\frac12}\right\|_{\kappa,p;Y}\|g\|_{\kappa,p^*;Y^*}.$$

Taking the supremum over all g of norm $\leqslant 1$ yields the desired estimate under the additional assumption $f \in R(A_p) \cap D(A_p)$. Since $R(A_p) \cap D(A_p)$ is dense in $R(A_p)$ [41, Proposition 9.4], we deduce

$$\|f\|_{\kappa,p;Y} \lesssim \left\| \left(\int_0^\infty t \left| \frac{\partial}{\partial t} T_t^\kappa f \right|^2 dt \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y} \lesssim \|f\|_{\kappa,p;Y}, \tag{3.2}$$

the lower estimate under the assumption $\int_{S^{d-1}} f(y) h_\kappa^2(y) dy = 0$.

We next deduce from (3.2) the stated g-function norm equivalence of the lemma. To this end, we proceed by a modification of the proof in [17, p. 38-39]. By the change of variable $r = e^{-t}$, we get

$$g_0(f) := \left(\int_0^\infty t \left| \frac{\partial}{\partial t} P_{e^{-t}}^\kappa f \right|^2 dt \right)^{\frac{1}{2}} = \left(\int_0^1 r |\log(r)| \left| \frac{\partial}{\partial r} P_r^\kappa f \right|^2 dr \right)^{\frac{1}{2}}.$$

Now, we write

$$g(f) = \left(\int_0^1 (1-r) \left| \frac{\partial}{\partial r} P_r^\kappa f \right|^2 dr \right)^{\frac{1}{2}}.$$

We shall show that $\|g(f)\|_{\kappa,p;Y} \lesssim \|f\|_{\kappa,p;Y}$ and $\|g_0(f)\|_{\kappa,p;Y} \lesssim \|g(f)\|_{\kappa,p;Y}$, which completes the proof.

Since $(1-r) \simeq r|\log r|$ for $\frac{1}{2} \leqslant r < 1$, we have

$$g(f) \lesssim \left(\int_0^{\frac{1}{2}} (1-r) \left| \frac{\partial}{\partial r} P_r^\kappa f \right|^2 dr \right)^{\frac{1}{2}} + \left(\int_{\frac{1}{2}}^1 r|\log r| \left| \frac{\partial}{\partial r} P_r^\kappa f \right|^2 \right)^{\frac{1}{2}}. \tag{3.3}$$

The first term on the right hand side, we estimate by

$$\sup_{0 \leqslant r \leqslant \frac{1}{2}} \left| \frac{\partial}{\partial r} P_r^\kappa f \right| \leqslant \sup_{0 \leqslant r \leqslant \frac{1}{2}} \sum_{j=1}^\infty j r^{j-1} |\mathrm{proj}_j^\kappa f| = \sum_{j=1}^\infty j 2^{1-j} |\mathrm{proj}_j^\kappa f|.$$

By Lemma 2.11, we can sum over $j \geqslant 1$ to get

$$\left\| \sup_{0 \leqslant r \leqslant \frac{1}{2}} \left| \frac{\partial}{\partial r} P_r^\kappa f \right| \right\|_{\kappa,p;Y} \lesssim \|f\|_{\kappa,p;Y}.$$

Use now (3.3) to deduce that

$$\|g(f)\|_{\kappa,p;Y} \lesssim \|f\|_{\kappa,p;Y} + \|g_0(f)\|_{\kappa,p;Y} \lesssim \|f\|_{\kappa,p;Y}.$$

We have proved the upper estimate of the lemma. For the lower estimate, we simply use $r|\log r| \lesssim (1-r)$ on $r \in [0,1]$, to deduce $g_0(f) \lesssim g(f)$, and thus,

$$\|f\|_{\kappa,p;Y} \lesssim \|g_0(f)\|_{\kappa,p;Y} \lesssim \|g(f)\|_{\kappa,p;Y}$$

when $\int_{S^{d-1}} f(y) h_\kappa^2(y) dy = 0$. ■

Now, we shall prove that the Cesàro means of h-harmonic expansions are R-bounded on $L^p(S^{d-1}, h_\kappa^2; Y)$. To this end, we recall some definitions.

Definition 3.4 *For $\delta \geqslant 0$ and $l \in \mathbb{N}$, we put*

$$A_l^\delta = \binom{l+\delta}{l} = \frac{(l+\delta)(l+\delta-1)\dots(\delta+1)}{l(l-1)\dots 1} = \frac{\Gamma(l+\delta+1)}{\Gamma(l+1)\Gamma(\delta+1)} = \frac{(\delta+1)_l}{l!}.$$

Then, for $j \geqslant 0$ and $n \geqslant 0$, we put

$$a_j^{\delta,n} = \frac{1}{A_n^\delta} A_{n-j}^\delta \chi_{0 \leqslant j \leqslant n}.$$

We define the Cesàro means of order δ (from now on, just called Cesàro means) by the multiplier

$$S_n^\delta f = \sum_{j=0}^\infty a_j^{\delta,n} \mathrm{proj}_j^\kappa f.$$

We can state the R-boundedness of the Cesàro means of h-harmonic expansions. Recall that we have set $\lambda_\kappa = \frac{d}{2} + \gamma_\kappa - 1$.

Lemma 3.5 *Let $1 < p < \infty$, $Y = Y(\Omega')$ be a UMD Banach lattice. Assume that $\delta > \lambda_\kappa$. Then the Cesàro means $(S_n^\delta)_{n \geqslant 0}$ are R-bounded on $L^p(S^{d-1}, h_\kappa^2; Y)$, that is,*

$$\left\| \left(\sum_{j=0}^\infty |S_{n_j}^\delta f_j|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y} \leqslant c_{p,\delta} \left\| \left(\sum_{j=0}^\infty |f_j|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y}.$$

Proof : By [17, p. 37], we have the estimate

$$\sup_{n \geqslant 0} |S_n^\delta f(x,\omega')| \leqslant c \Big[\mathcal{M}_\kappa f(x,\omega') + \mathcal{M}_\kappa f(-x,\omega') \Big], \quad x \in S^{d-1}, \omega' \in \Omega', \tag{3.4}$$

where $\mathcal{M}_\kappa$ is the following well-suited maximal operator for h-harmonic expansions [62, Proposition 2.3] or [16, (1.5)]

$$\mathcal{M}_\kappa f(x) = \sup_{0 < \theta \leqslant \pi} \frac{\int_{S^{d-1}} |f(y)| V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y) dy}{\int_{S^{d-1}} V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y) dy}, \quad x \in S^{d-1},$$

with $B(x,\theta) = \{y \in \mathbb{R}^d : \langle x, y \rangle \geqslant \cos\theta\} \cap \{y \in \mathbb{R}^d : \|y\| \leqslant 1\}$. It is shown in [16, Proof of Theorem 2.1] that

$$\mathcal{M}_\kappa f(x,\omega') \leqslant c \sup_{t>0} \frac{1}{t} \int_0^t H_s^\kappa |f(\cdot,\omega')|(x) ds,$$

where we recall that H_s^κ is the generalized heat semigroup on $(S^{d-1}, h_\kappa^2(y)dy)$. Since it is a symmetric contraction semigroup, by [59, Theorem 1], the Hopf-Dunford-Schwartz maximal operator

$$M(H)f = \sup_{t>0} \frac{1}{t} \left| \int_0^t H_s^\kappa f ds \right|$$

is bounded on $L^p(S^{d-1},h_\kappa^2;Y(\Omega';\ell^2))$, $Y(\Omega';\ell^2)$ being again a UMD-lattice according to Lemma 2.3. Therefore, we get

$$
\begin{aligned}
\Big\|\Big(\sum_{j=0}^{\infty}|S_{n_j}^{\delta}f_j|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y} &\lesssim \Big\|\Big(\sum_{j=0}^{\infty}|\mathcal{M}_\kappa f_j|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y}\\
&\lesssim \big\|M(H)(f_j)_{j\geqslant 0}\big\|_{L^p(S^{d-1},h_\kappa^2;Y(\Omega';\ell^2))}\\
&\lesssim \big\|(f_j)_{j\geqslant 0}\big\|_{L^p(S^{d-1},h_\kappa^2;Y(\Omega';\ell^2))}\\
&= \Big\|\Big(\sum_{j=0}^{\infty}|f_j|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y}.
\end{aligned}
$$

■

The second lemma we need, under the assumption that the Cesàro means are R-bounded, provides us a crucial norm inequality involving both the Cesàro means and the generalized Poisson semigroup (P_r^κ in fact) on $L^p(S^{d-1},h_\kappa^2;Y)$.

Lemma 3.6 *Let $1<p<\infty$ and $Y=Y(\Omega')$ be a UMD Banach lattice. Let $\delta\geqslant 0$ and assume that the Cesàro means $(S_n^\delta)_{n\geqslant 0}$ are R-bounded on $L^p(S^{d-1},h_\kappa^2;Y)$. If for $j\geqslant 1$, $r_j\in(0,1)$ and I_j is a subinterval of $[r_j,1)$, then*

$$
\Big\|\Big(\sum_{j=1}^{\infty}\big|S_{n_j}^{\delta}\big(P_{r_j}^{\kappa}f_j\big)\big|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y}\leqslant c_p\Big\|\Big(\sum_{j=1}^{\infty}\frac{1}{|I_j|}\int_{I_j}\big|P_r^\kappa f_j\big|^2dr\Big)^{\frac12}\Big\|_{\kappa,p;Y}.
$$

Proof : The proof follows closely the lines of [17, Lemma 4.3.5] which itself follows [6]. We give some details. Firstly, according to [17, p. 41], we have for $j\geqslant 1$, $\delta\geqslant 0$ and $0<r<1$

$$
\big|S_{n_j}^{\delta}P_{r_j}^{\kappa}f_j\big|^2\leqslant c\sum_{l=0}^{n_j}|b_{l,n_j}^{\delta}|\,|S_l^\delta f_j|^2,
$$

where $b_{l,n}^\delta$ are scalars satisfying $\sum_{l=0}^{n}|b_{l,n}^\delta|\leqslant c_\delta$. It follows from the R-boundedness of the Cesàro means

$$
\begin{aligned}
\Big\|\Big(\sum_{j=1}^{\infty}\big|S_{n_j}^{\delta}\big(P_{r_j}^{\kappa}f_j\big)\big|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y} &\leqslant c\Big\|\Big(\sum_{j=1}^{\infty}\sum_{l=0}^{n_j}|b_{l,n_j}^{\delta}|\,|S_l^\delta f_j|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y}\\
&= c\Big\|\Big(\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\Big|S_l^\delta\Big(\chi_{0\leqslant l\leqslant n_j}\sqrt{|b_{l,n_j}^{\delta}|}f_j\Big)\Big|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y}\\
&\lesssim \Big\|\Big(\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\Big|\chi_{0\leqslant l\leqslant n_j}\sqrt{|b_{l,n_j}^{\delta}|}f_j\Big|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y}\\
&= \Big\|\Big(\sum_{j=1}^{\infty}\sum_{l=0}^{n_j}|b_{l,n_j}^{\delta}|\,|f_j|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y}\\
&\leqslant \sqrt{c_\delta}\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^2\Big)^{\frac12}\Big\|_{\kappa,p;Y}.
\end{aligned}
$$

Thus, we have shown

$$\left\| \left(\sum_{j=1}^{\infty} \left| S_{n_j}^{\delta} \left(P_{r_j}^{\kappa} f_j \right) \right|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y}. \tag{3.5}$$

Now let for each $j \geqslant 1$ and $n \geqslant 1$, $(r_{j,i})_{i=0}^{2^n} \subset I_j$ be a finite sequence such that $r_{j,i} - r_{j,i-1} = 2^{-n}|I_j|$ for all $1 \leqslant i \leqslant 2^n$. Then for each $n \in \mathbb{N}$, $R_{j,n} := 2^{-n} \sum_{i=1}^{2^n} |P_{r_{j,i}}^{\kappa} f_j|^2$ is a Riemann sum over Ω' of the integral $\frac{1}{|I_j|} \int_{I_j} |P_r^{\kappa} f_j|^2 dr$. Thus, by dominated convergence, it follows

$$\left\| \left(\sum_{j=1}^{\infty} \frac{1}{|I_j|} \int_{I_j} \left| P_r^{\kappa} f_j \right|^2 dr \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y} = \lim_{n\to\infty} \left\| \left(2^{-n} \sum_{j=1}^{\infty} \sum_{i=1}^{2^n} |P_{r_{j,i}}^{\kappa} f_j|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y}. \tag{3.6}$$

On the other hand, since for each $n \geqslant 1$, $r_j < r_{j,i}$ for all $1 \leqslant i \leqslant n$ and $j \geqslant 1$, we have by (3.5)

$$\begin{aligned} \left\| \left(\sum_{j=1}^{\infty} \left| S_{n_j}^{\delta} \left(P_{r_j}^{\kappa} f_j \right) \right|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y} &= \left\| \left(2^{-n} \sum_{i=1}^{2^n} \sum_{j=1}^{\infty} \left| S_{n_j}^{\delta} P_{r_j/r_{j,i}}^{\kappa} (P_{r_{j,i}}^{\kappa} f_j) \right|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y} \\ &\lesssim \left\| \left(2^{-n} \sum_{i=1}^{2^n} \sum_{j=1}^{\infty} |P_{r_{j,i}}^{\kappa} f_j|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y}. \end{aligned}$$

We conclude with (3.6). ■

Before stating a proposition which is a key step in the proof of Theorem 3.2, we need the following functional which is closely related to the Cesàro means of h-harmonic expansions.

Definition 3.7 *Let $\delta \geqslant 0$. We define the functional $g_\delta(f)$, for given $f \in L^p(S^{d-1}, h_\kappa^2; Y)$, by*

$$g_\delta(f) = \left(\sum_{n=1}^{\infty} \left| S_n^{\delta+1} f - S_n^{\delta} f \right|^2 \frac{1}{n} \right)^{\frac{1}{2}}.$$

Moreover, let $(\nu_k)_{k\geqslant 1}$ be a sequence of nonnegative numbers such that

$$\sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \nu_k = M < \infty.$$

We define the functional $g_\delta^*(f)$, for given $f \in L^p(S^{d-1}, h_\kappa^2; Y)$, by

$$g_\delta^*(f) = \left(\sum_{n=1}^{\infty} \left| S_n^{\delta+1} f - S_n^{\delta} f \right|^2 \frac{\nu_n}{n} \right)^{\frac{1}{2}}.$$

Remark 3.8 *Note that if in particular $\nu_k = 1$ for all $k \geqslant 1$, then $g_\delta^*(f) = g_\delta(f)$.*

The following proposition gives us two important norm inequalities involving the Littlewood-Paley functions $g_\delta(f)$ and $g_\delta^*(f)$.

Proposition 3.9 *Let $1 < p < \infty$ and $Y = Y(\Omega')$ be a UMD Banach lattice. Let $\delta \geqslant 0$. If $f \in L^p(S^{d-1}, h_\kappa^2; Y)$ satisfies $\int_{S^{d-1}} f(y) h_\kappa^2(y) dy = 0$, then*

$$\|f\|_{\kappa,p;Y} \leqslant c_{p,\delta} \|g_\delta(f)\|_{\kappa,p;Y}.$$

Conversely, if the Cesàro means $(S_n^{\delta})_{n\geqslant 0}$ are R-bounded on $L^p(S^{d-1}, h_\kappa^2; Y)$, then

$$\|g_\delta^*(f)\|_{\kappa,p;Y} \leqslant c_{p,\delta} M \|f\|_{\kappa,p;Y},$$

where $M = \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \nu_k$.

Proof : First, recall that we have set in the proof of Lemma 3.3

$$g(f) = \Big(\int_0^1 (1-r)\Big|\frac{\partial}{\partial r} P_r^\kappa f\Big|^2 dr\Big)^{\frac{1}{2}}.$$

It is shown in [17, Section 4.3.2] that $g(f)(x) \leqslant c_\delta g_\delta(f)(x)$. Therefore, it follows immediately with Lemma 3.3

$$\|f\|_{\kappa,p;Y} \leqslant c_p \|g(f)\|_{\kappa,p;Y} \leqslant c_p c_\delta \|g_\delta(f)\|_{\kappa,p;Y},$$

in the case $\int_{S^{d-1}} f(y) h_\kappa^2(y)dy = 0$.

We proceed to the second stated inequality and suppose that the Cesàro means to an order index δ are R-bounded. We follow closely the lines of [17, Section 4.3.2] but, for the readers' convenience, we present the proof all the same. First we may assume that $n \leqslant \sum_{j=1}^n \nu_j \leqslant 2n$, since the desired conclusion for general $(\nu_j)_{j\geqslant 1}$ can be deduced from this case applied to the two sequences $\tilde{\nu}_j = 1$ and $\tilde{\nu}_j = M^{-1}\nu_j + 1$. Now let $\mu_1 = 1$ and $\mu_n = 1 + \sum_{i=1}^{n-1} \nu_i$ for $n \geqslant 2$. Let further $r_n = 1 - \frac{1}{\mu_n}$ and $f_n = P_{r_n}^\kappa f$. It is shown in [17, p. 44] that

$$\big|S_n^{\delta+1} f - S_n^\delta f\big|^2 \leqslant c\big|S_n^{\delta+1} f_n - S_n^\delta f_n\big|^2 + cn^{-3}\sum_{j=1}^{n-1} j^2 \big|S_j^{\delta+1} f_n - S_j^\delta f_n\big|^2.$$

Therefore, in view of Lemma 3.3, we are left with the task of establishing the following inequalities

$$\Big\|\Big(\sum_{n=1}^\infty n^{-1}\big|S_n^{\delta+1} f_n - S_n^\delta f_n\big|^2 \nu_n\Big)^{\frac{1}{2}}\Big\|_{\kappa,p;Y} \leqslant c\|g(f)\|_{\kappa,p;Y} \tag{3.7}$$

and

$$\Big\|\Big(\sum_{n=1}^\infty \frac{\nu_n}{n^4}\sum_{j=1}^{n-1} j^2 \big|S_j^{\delta+1} f_n - S_j^\delta f_n\big|^2\Big)^{\frac{1}{2}}\Big\|_{\kappa,p;Y} \leqslant c\|g(f)\|_{\kappa,p;Y}. \tag{3.8}$$

We start by showing (3.7). To this end, let $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) = 1$ for $|t| \leqslant 1$ and $\eta(t) = 0$ for $|t| \geqslant 2$. Moreover, for $n \geqslant 1$, let L_n^κ and $\tilde{L}_n^\kappa$ be the following multipliers

$$L_n^\kappa f = \sum_{j=0}^\infty \eta\Big(\frac{j}{n}\Big)\mathrm{proj}_j^\kappa f, \quad \tilde{L}_n^\kappa f = -\sum_{j=0}^\infty j\eta\Big(\frac{j}{n}\Big)\mathrm{proj}_j^\kappa f.$$

Comparing symbols of multipliers yields [17, p. 44] for $1 \leqslant j \leqslant n \leqslant N$

$$S_j^{\delta+1} f_n - S_j^\delta f_n = (j+\delta+1)^{-1} P_{r_n}^\kappa\big(S_j^\delta(\tilde{L}_N^\kappa f)\big). \tag{3.9}$$

Using this last equality specializing to $j = n$, we obtain by Lemma 3.6

$$\begin{aligned}\Big\|\Big(\sum_{n=1}^N n^{-1}\big|S_n^{\delta+1} f_n - S_n^\delta f_n\big|^2 \nu_n\Big)^{\frac{1}{2}}\Big\|_{\kappa,p;Y} &\lesssim \Big\|\Big(\sum_{n=1}^N \frac{\nu_n}{n^3}\Big|P_{r_n}^\kappa\Big(S_n^\delta\big(\tilde{L}_N^\kappa(f)\big)\Big)\Big|^2\Big)^{\frac{1}{2}}\Big\|_{\kappa,p;Y} \\ &\lesssim \Big\|\Big(\sum_{n=1}^N \frac{\nu_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}}\Big|P_r^\kappa\big(\tilde{L}_N^\kappa(f)\big)\Big|^2 dr\Big)^{\frac{1}{2}}\Big\|_{\kappa,p;Y}.\end{aligned}$$

Obviously, we have $|P_r^\kappa(\tilde{L}_N^\kappa(f))| = r|L_N^\kappa(\frac{\partial}{\partial r}P_r^\kappa(f))|$. Moreover, the operators L_N^κ are uniformly bounded in $N \in \mathbb{N}$ on $L^p(S^{d-1}, h_\kappa^2; Y)$. Indeed, a straightforward computation gives us

$$L_N^\kappa f = \sum_{j=0}^{2N} \Delta^{l+1}\eta\Big(\frac{j}{N}\Big) A_j^l S_j^l f.$$

Since we have $|\Delta^{l+1}\eta(\frac{j}{N})| \lesssim N^{-l-1}$, then

$$\big|L_N^\kappa f\big| \lesssim \frac{1}{N^{l+1}} \sum_{j=0}^{2N} |A_j^l||S_j^l f| \lesssim \frac{\sup_{j\geqslant 0}|S_j^l f|}{N^{l+1}} \sum_{j=0}^{2N} A_j^l.$$

But $\sum_{j=0}^{2N} A_j^l = A_{2N}^{l+1}$, then we claim (see (2.3)) that

$$N^{-l-1} A_{2N}^{l+1} = N^{-l-1}\frac{(l+2)_{2N}}{(2N)!} \simeq c,$$

and in view of (3.4), choosing $l > \lambda_\kappa$, the operators L_N^κ are uniformly bounded in $N \in \mathbb{N}$ on $L^p(S^{d-1}, h_\kappa^2; Y)$. Now, since they are linear, by Lemma 2.3, a single operator L_N^κ is also bounded on $L^p(S^{d-1}, h_\kappa^2; Y(\Omega'; L^2([0,1]; dr)))$. Therefore,

$$\begin{aligned}&\bigg\|\bigg(\sum_{n=1}^{N} \frac{\nu_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}} \Big|P_r^\kappa\big(\tilde{L}_N^\kappa(f)\big)\Big|^2 dr\bigg)^{\frac12}\bigg\|_{\kappa,p;Y} \\ &\qquad\lesssim \bigg\|\bigg(\sum_{n=1}^{N} \frac{\nu_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}} \Big|\frac{\partial}{\partial r}P_r^\kappa f\Big|^2 dr\bigg)^{\frac12}\bigg\|_{\kappa,p;Y}.\end{aligned}$$

Since $r_{n+1}-r_n = \frac{\nu_n}{\mu_n\mu_{n+1}} \simeq \frac{\nu_n}{n^2}$ and $1-r \simeq \frac{1}{n}$ for all $r \in [r_n, r_{n+1}]$, it follows that

$$\begin{aligned}\bigg\|\bigg(\sum_{n=1}^{N} \frac{\nu_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}} \Big|\frac{\partial}{\partial r}P_r^\kappa f\Big|^2 dr\bigg)^{\frac12}\bigg\|_{\kappa,p;Y} &\lesssim \bigg\|\bigg(\sum_{n=1}^{\infty} \frac{1}{n}\int_{r_n}^{r_{n+1}} \Big|\frac{\partial}{\partial r}P_r^\kappa f\Big|^2 dr\bigg)^{\frac12}\bigg\|_{\kappa,p;Y} \\ &\lesssim \|g(f)\|_{\kappa,p;Y}.\end{aligned}$$

Thus, we have proved that

$$\bigg\|\bigg(\sum_{n=1}^{N} n^{-1}\big|S_n^{\delta+1}f_n - S_n^\delta f_n\big|^2\nu_n\bigg)^{\frac12}\bigg\|_{\kappa,p;Y} \lesssim \|g(f)\|_{\kappa,p;Y}$$

and letting $N \to \infty$ yields (3.7). We now turn to the proof of (3.8), which is similar to the previous one. Indeed, using Lemma 3.6 and (3.9), we have

$$\begin{aligned}\bigg\|\bigg(\sum_{n=1}^{N} \frac{\nu_n}{n^4}\sum_{j=1}^{n-1} j^2\big|S_j^{\delta+1}f_n - S_j^\delta f_n\big|^2\bigg)^{\frac12}\bigg\|_{\kappa,p;Y} &\lesssim \bigg\|\bigg(\sum_{n=1}^{N} \frac{\nu_n}{n^4}\sum_{j=1}^{n-1}\Big|P_{r_n}^\kappa\Big(S_j^\delta\big(\tilde{L}_N^\kappa(f)\big)\Big)\Big|^2\bigg)^{\frac12}\bigg\|_{\kappa,p;Y} \\ &\lesssim \bigg\|\bigg(\sum_{n=1}^{\infty} \frac{\nu_n}{n^4}\sum_{j=1}^{n-1}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}} \Big|\frac{\partial}{\partial r}P_r^\kappa f\Big|^2 dr\bigg)^{\frac12}\bigg\|_{\kappa,p;Y} \\ &\lesssim \bigg\|\bigg(\sum_{n=1}^{\infty} \frac{\nu_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}} \Big|\frac{\partial}{\partial r}P_r^\kappa f\Big|^2 dr\bigg)^{\frac12}\bigg\|_{\kappa,p;Y} \\ &\lesssim \|g(f)\|_{\kappa,p;Y}.\end{aligned}$$

We obtain (3.8) by letting $N \to \infty$. The proof is complete. ■

In view of Remark 3.8, we immediately obtain the following corollary.

Corollary 3.10 *Let $1 < p < \infty$ and $Y = Y(\Omega')$ be a UMD Banach lattice. If $\delta > \lambda_\kappa$, then*

$$\frac{1}{c_p}\|f\|_{\kappa,p;Y} \leqslant \|g_\delta(f)\|_{\kappa,p;Y} \leqslant c_p\|f\|_{\kappa,p;Y},$$

where in the first inequality we assume that $\int_{S^{d-1}} f(y)h_\kappa^2(y)dy = 0$.

We now state the last lemma we shall need for the proof of Theorem 3.2.

Lemma 3.11 *Let δ to be the smallest integer strictly larger than λ_κ and let $n_0 = \delta + 1$. Let $(\mu_j)_{j\geqslant 0}$ be a sequence as in the hypotheses of Theorem 3.2, i.e. satisfying (C_0) and (C_{n_0}), with bound M. Write*

$$M_\mu f = \sum_{j=0}^{\infty} \mu_j \mathrm{proj}_j^\kappa f$$

the associated multiplier. Then we have

$$\|g_\delta(M_\mu f)\|_{\kappa,p;Y} \leqslant C\|g_\delta^*(f)\|_{\kappa,p;Y},$$

where the sequence $(\nu_k)_{k\geqslant 1}$ is $\nu_k = 1 + \sum_{j=1}^{\delta+1} |\Delta^j\mu_k|k^j$ and satisfies $\sup_{n\geqslant 1} \frac{1}{n}\sum_{j=1}^{n} \nu_j \leqslant cM$.

Proof : It is shown in [17, (4.4.2)] that

$$g_\delta(M_\mu f) \leqslant C g_\delta^*(f)$$

holds pointwise, with the sequence $(\nu_k)_{k\geqslant 1}$ given in the lemma. By Lemma 2.2, we immediately deduce

$$\|g_\delta(M_\mu f)\|_{p,\kappa;Y} \leqslant C\|g_\delta^*(f)\|_{\kappa,p;Y}.$$

The statement on $(\nu_k)_{k\geqslant 1}$ is shown in [17, p. 47]. ■

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2 : We can assume that $\mu_0 = 0$. Indeed,

$$\left\|\sum_{j=0}^{\infty} \mu_j \mathrm{proj}_j^\kappa f\right\|_{\kappa,p;Y} \leqslant |\mu_0|\; \|\mathrm{proj}_0^\kappa f\|_{\kappa,p;Y} + \left\|\sum_{j=1}^{\infty} \mu_j \mathrm{proj}_j^\kappa f\right\|_{\kappa,p;Y},$$

and proj_κ^0 is bounded on $L^p(S^{d-1}, h_\kappa^2; Y)$ by Lemma 2.11. Let $\delta = n_0 - 1 > \lambda_\kappa$. Note that

$$a_\kappa \int_{S^{d-1}} M_\mu(f)(y)h_\kappa^2(y)dy = \mathrm{proj}_0^\kappa(M_\mu(f)) = 0$$

if $\mu_0 = 0$. According to Lemma 3.5, the Cesàro means $(S_n^\delta)_{n\geqslant 0}$ are R-bounded. Hence by Proposition 3.9 in conjunction with Lemma 3.11,

$$\|M_\mu(f)\|_{\kappa,p;Y} \lesssim \|g_\delta(M_\mu(f))\|_{\kappa,p;Y} \lesssim \|g_\delta^*(f)\|_{\kappa,p;Y} \lesssim M\|f\|_{\kappa,p;Y}.$$

■

3.2 A multiplier theorem for the Dunkl transform

Bounded vector-valued multipliers on the sphere S^d yield bounded vector-valued spectral multipliers of the Dunkl Laplacian on $\mathbb{R}^d$ by a transference principle, presented in Theorem 3.12 below. In the scalar case, a transference principle from zonal multipliers on S^d to radial multipliers on $\mathbb{R}^d$ was oberved and proved by Bonami-Clerc in [6], and their strategy has been recently adapted by Dai-Wang [15] to obtain bounded multipliers for the Dunkl transform on $\mathbb{R}^d$ from bounded multipliers for h-harmonic expansions on the unit sphere S^d.

Let $W \subset \mathcal{O}(d)$ be a finite reflection group associated with a reduced root system $\mathcal{R}$ and let $\kappa : \mathcal{R} \to [0, +\infty[$ be a multiplicity function with associated weight function h_κ^2. We transfer this to $S^d \subseteq \mathbb{R}^{d+1}$. Namely, for $g \in W$, there exists a unique orthogonal transformation on $\mathbb{R}^{d+1}$, denoted by g' and determined by

$$g'x' = (gx, x_{d+1}), \quad x' = (x, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}.$$

Then $W' = \{g' : g \in W\}$ is a finite reflection group on $\mathbb{R}^{d+1}$ associated with the reduced root system $R' = \{(\alpha, 0) : \alpha \in R\}$. Finally, we let

$$\kappa' : \begin{cases} R' \to \mathbb{R}_+, \\ (\alpha, 0) \mapsto \kappa(\alpha) \end{cases}$$

and associate with it the weight $h_{\kappa'}^2$.

We can now state the following transference principle for the Dunkl transform.

Theorem 3.12 *Let $Y = Y(\Omega')$ be a UMD Banach lattice. Let $m : (0, \infty) \to \mathbb{R}$ be a continuous and bounded function. For $\epsilon > 0$ and $n \geqslant 0$, let $\mu_n^\epsilon = m(\epsilon n)$. Let further $M_\epsilon = M_{\mu^\epsilon}$ be the multiplier*

$$M_\epsilon(f) = \sum_{n=0}^{\infty} m(\epsilon n) \mathrm{proj}_n^{\kappa'} f.$$

Assume that for some $1 \leqslant p \leqslant \infty$ and any $f \in L^p(S^d, h_{\kappa'}^2; Y)$,

$$\sup_{\epsilon>0} \|M_\epsilon f\|_{L^p(S^d, h_{\kappa'}^2; Y)} \leqslant A \|f\|_{L^p(S^d, h_{\kappa'}^2; Y)}.$$

Then m is a radial Dunkl spectral multiplier on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$, that is, for any $f \in L^p(\mathbb{R}^d, h_\kappa^2; Y)$,

$$\|T_m f\|_{L^p(\mathbb{R}^d, h_\kappa^2; Y)} \leqslant c_{d,\kappa} A \|f\|_{L^p(\mathbb{R}^d, h_\kappa^2; Y)},$$

where T_m is a priori defined on $\mathcal{S}(\mathbb{R}^d) \otimes Y$ by

$$T_m(f) = \mathcal{F}_\kappa^{-1}[m(\|\xi\|)\mathcal{F}_\kappa(f)(\xi)].$$

This theorem generalizes the scalar case proved by Dai and Wang in [15]. Therefore, if we specialize Theorem 3.12 to $Y = \mathbb{C}$ and $\kappa = 0$, then we recover the standard result due to Bonami-Clerc [6].

Proof : We follow closely the strategy of [15, Section 3]. Note that continuity of m in 0 is not needed there.

We first assume that for some $c_1,\ c_2 > 0$

$$|m(t)| \leqslant c_1 e^{-c_2 t}, \quad t > 0.$$

In [15, Lemma 3.5], it is shown that the operator T_m has the following integral representation

$$T_m f(x) = \int_{\mathbb{R}^d} f(y)K(x,y)h_\kappa^2(y)dy$$

for a certain $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, the formula holding for $f \in \mathcal{S}(\mathbb{R}^d)$ and a.e. $x \in \mathbb{R}^d$. Then it also holds for $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$. Recall that $Y = Y(\Omega')$ is a function lattice on Ω', and its dual $Y^* = Y^*(\Omega')$ is a function lattice on the same measure space Ω', and the duality is given by

$$\langle y, y^* \rangle = \int_{\Omega'} y(\omega')y^*(\omega')d\mu(\omega')$$

for a certain measure μ on Ω'. Then it is sufficient to prove that

$$\left| \int_{\Omega'} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y,\omega')g(x,\omega')K(x,y)h_\kappa^2(x)h_\kappa^2(y)dxdyd\mu(\omega') \right| \leqslant cA \tag{3.10}$$

holds whenever $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$ and $g \in \mathcal{S}(\mathbb{R}^d) \otimes Y^*$ have compact support and satisfy

$$\|f\|_{L^p(\mathbb{R}^d, h_\kappa^2; Y)} \leqslant 1, \quad \|g\|_{L^{p^*}(\mathbb{R}^d, h_\kappa^2; Y^*)} \leqslant 1.$$

Denote the above triple integral by I and let $\psi : \mathbb{R}^d \to S^d$ be the mapping

$$\psi(x) = \big(\xi \sin\|x\|, \cos\|x\|\big), \qquad \text{for } x = \|x\|\xi \in \mathbb{R}^d \text{ with } \xi \in S^{d-1}.$$

For $N \geqslant 1$, let moreover

$$\psi_N : \begin{cases} \mathbb{R}^d \to NS^d = \{x \in \mathbb{R}^{d+1} : \|x\| = N\} \\ x \mapsto N\psi(\frac{x}{N}). \end{cases}$$

It is shown in [15, Remark 3.1] that given a function $h : B(0,N) = \{x \in \mathbb{R}^d : \|x\| \leqslant N\} \to \mathbb{R}$, there exists a unique function h_N supported in $\{x \in NS^d : \arccos(N^{-1}x_{d+1}) \leqslant 1\}$ such that

$$h_N(\psi_N x) = h(x), \quad x \in B(0,N).$$

Moreover, it is also shown there that

$$\int_{S^d} h_N(Nx)h_{\kappa'}^2(x)dx = N^{-2\lambda_\kappa'-1} \int_{B(0,N)} h(x)h_\kappa^2(x) \left(\frac{\sin(\|x\|/N)}{\|x\|/N}\right)^{2\lambda_\kappa'} dx \tag{3.11}$$

with $\lambda_\kappa' = \lambda_\kappa + \frac{1}{2} = \frac{d}{2} + \gamma_\kappa - \frac{1}{2}$. Let now N be so large that both f and g are supported in $B(0,N)$. Let $h(x) := \|f(x,\cdot)\|_{Y(\Omega')}$ and $f_N : NS^d \to Y$ be the function defined by

$$f_N(\psi_N(x), \omega') = f(x, \omega'),$$

so that we have $h_N(x) = \|f_N(x,\cdot)\|_{Y(\Omega')}$. As mentioned in [15, p. 4064], it follows from (3.11) that

$$\|h_N(N\cdot)\|_{L^p(S^d, h_{\kappa'}^2)} = \|f_N(N\cdot)\|_{L^p(S^d, h_{\kappa'}^2; Y)} \lesssim N^{-\frac{2\lambda_\kappa'+1}{p}}.$$

Similarly, with $h(x) := \|g(x,\cdot)\|_{Y^*(\Omega')}$, it follows that

$$\|g_N(N\cdot)\|_{L^{p^*}(S^d, h_{\kappa'}^2; Y^*)} \lesssim N^{-\frac{2\lambda_\kappa'+1}{p^*}}.$$

Recall in the following that $\mathscr{P}_n^{\kappa'}(x,y)$ is the kernel of $\mathrm{proj}_n^{\kappa'}$. We deduce that

$$\begin{aligned}
&\left|N^{2\lambda_\kappa'+1}\int_{\Omega'}\int_{S^d}\int_{S^d}\left[\sum_{n=0}^\infty m(N^{-1}n)\mathscr{P}_n^{\kappa'}(x,y)\right]f_N(Ny,\omega')g_N(Nx,\omega')h_{\kappa'}^2(x)h_{\kappa'}^2(y)dxdyd\mu(\omega')\right|\\
&\leqslant N^{2\lambda_\kappa'+1}\int_{\Omega'}\int_{S^d}\left|M_{1/N}f_N(Nx,\omega')\right|\cdot|g_N(Nx,\omega')|\,dxd\mu(\omega')\\
&\leqslant N^{2\lambda_\kappa'+1}\|M_{1/N}f_N(N\cdot,\cdot)\|_{\kappa',p;Y}\|g_N(N\cdot,\cdot)\|_{\kappa',p^*;Y^*}\\
&\leqslant N^{2\lambda_\kappa'+1}cAN^{-\frac{2\lambda_\kappa'+1}{p}}N^{-\frac{2\lambda_\kappa'+1}{p^*}}\\
&=cA.
\end{aligned}$$

Denote the expression under the modulus in the first line of the above estimate by I_N. It is shown in [15, Proof of Theorem 3.1] that $\lim_{N\to\infty}I_N=c_{\delta,\kappa}I$ holds in the case $Y=Y^*=\mathbb{C}$. Then it also holds pointwise on Ω', and thus, since we take integrals over Ω' both in the definition of I and I_N, also for general Banach lattices $Y(\Omega')$ and $Y^*(\Omega')$. Then line (3.10) follows immediately, and we have proved Theorem 3.12 in the case where $|m(t)|\leqslant c_1e^{-c_2t}$. We now prove the theorem removing this assumption on m.

Let thus m be a general multiplier function satisfying the hypotheses of Theorem 3.12. For $\delta>0$, put $m_\delta(t)=m(t)e^{-\delta t}$. Then, for any $f\in\mathcal{S}(\mathbb{R}^d)\otimes Y$, we have

$$T_{m_\delta}(f)=\mathfrak{P}_\delta^\kappa T_m(f),$$

where $(\mathfrak{P}_t^\kappa)_{t\geqslant0}$ is the Dunkl Poisson symmetric contraction semigroup on $L^p(\mathbb{R}^d,h_\kappa^2;Y)$, and we have $\mathfrak{P}_\delta^\kappa f\to f$ as $\delta\to0$, because according to Proposition 2.5, $(\mathfrak{P}_t^\kappa)_t$ is strongly continuous on $L^p(\mathbb{R}^d,h_\kappa^2;Y)$. On the one hand, we have by the first part of the proof

$$\begin{aligned}
\|T_{m_\delta}\|_{L^p(\mathbb{R}^d,h_\kappa^2;Y)\to L^p(\mathbb{R}^d,h_\kappa^2;Y)}&\lesssim\sup_{\epsilon>0}\|M_{(m(\epsilon n)e^{-\epsilon\delta n})_{n\geqslant0}}\|_{L^p(S^d,h_{\kappa'}^2;Y)\to L^p(S^d,h_{\kappa'}^2;Y)}\\
&\leqslant A.
\end{aligned}$$

On the other hand, we write

$$\|T_m(f)\|_{L^p(\mathbb{R}^d,h_\kappa^2;Y)}=\lim_{\delta\to0}\|\mathfrak{P}_\delta^\kappa T_m(f)\|_{L^p(\mathbb{R}^d,h_\kappa^2;Y)}=\lim_{\delta\to0}\|T_{m_\delta}(f)\|_{L^p(\mathbb{R}^d,h_\kappa^2;Y)}\lesssim A\|f\|_{L^p(\mathbb{R}^d,h_\kappa^2;Y)}.$$

The proof of the theorem is complete. ■

We shall now prove several important consequences of this theorem. The first one provides us a Hörmander type multiplier theorem for the Dunkl transform.

Theorem 3.13 *Let $1<p<\infty$ and $Y=Y(\Omega')$ be a UMD Banach lattice. Let n_0 be an integer $>\lambda_\kappa'+1=\lambda_\kappa+\frac{3}{2}=\frac{d}{2}+\gamma_\kappa+\frac{1}{2}$. Assume that the multiplier function $m:(0,\infty)\to\mathbb{R}$ belongs to $\mathcal{H}_1^{n_0}$, that means, is bounded with $\|m\|_\infty\leqslant A$ and satisfies the following Hörmander condition*

$$\sup_{R>0}\frac{1}{R}\int_R^{2R}t^{n_0}\Big|\frac{d^{n_0}}{dt^{n_0}}m(t)\Big|dt\leqslant A.$$

Then the spectral multiplier T_m, initially defined for $f\in\mathcal{S}(R^d)\otimes Y$ by

$$T_m(f)=\mathcal{F}_\kappa^{-1}\Big[m\big(\|\xi\|\big)\mathcal{F}_\kappa f(\xi)\Big]$$

extends to a bounded operator on $L^p(\mathbb{R}^d,h_\kappa^2;Y)$ with $\|T_m\|_{L^p(\mathbb{R}^d,h_\kappa^2;Y)\to L^p(\mathbb{R}^d,h_\kappa^2;Y)}\leqslant c_{p,n_0,d}A$.

This theorem, which generalizes the scalar case proved by Dai-Wang [15, Theorem 4.1], only concerns radial multipliers. For scalar valued multivariate (i.e. not necessarily radial) spectral multipliers for Dunkl operators, but only in the particular case where the reflection group W is $(\mathbb{Z}/2\mathbb{Z})^d$, see [58].

Proof : It is shown in [15, Proof of Theorem 4.1] that the above condition on m yields that the sequence $(\mu_n)_{n\geqslant 0}$ defined by $\mu_n = m(\epsilon n)$ satisfies the hypotheses (C_0) and (C_{n_0}) of Theorem 3.2 with a bound $M \leqslant cA$ uniformly in $\epsilon > 0$. Note that in Theorem 3.2, we take S^d in place of S^{d-1} as it is stated there verbatim. Then Theorem 3.2 yields that the hypotheses of Theorem 3.12 are satisfied and we apply it to get the desired conclusion. ■

From Theorem 3.13, we can also immediately deduce the following corollary on Bochner-Riesz means.

Corollary 3.14 *Let $1 < p < \infty$ and $Y = Y(\Omega')$ be a UMD Banach lattice. Let $\alpha > \lfloor \lambda_\kappa \rfloor + 1$. For $R > 0$, let*

$$f_R^\alpha(t) = \begin{cases} (1 - t/R)^\alpha & 0 < t \leqslant R \\ 0 & t > R. \end{cases}$$

Then the Bochner-Riesz means $f_R^\alpha(A)$ associated with the Dunkl Laplacian A are uniformly bounded in $R > 0$ on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$.

Proof : Note that by Lemma 2.10 8. and Theorem 3.13, A has a $\mathcal{H}_1^{n_0}$ calculus on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ for n_0 an integer $> \lambda_\kappa + 1$. It suffices to note that $\|f_R^\alpha\|_{\mathcal{H}_1^{\alpha+1-\epsilon}} < \infty$ for $\epsilon > 0$ (see e.g. [38]) and to apply Theorem 3.13. ■

Note that there is a partial converse of Corollary 3.14. More precisely, if the Bochner-Riesz means $\{f_R^\alpha(A) : R > 0\}$ of a sectorial operator A with H^∞ calculus are R-bounded, then A must have a $\mathcal{H}_1^{\alpha+1}$ calculus (even R-bounded, i.e. $\{f(A) : \|f\|_{\mathcal{H}_1^{\alpha+1}} \leqslant 1\}$ is an R-bounded subset of $B(L^p(\mathbb{R}^d, h_\kappa^2; Y))$), according to [38].

Another application of Theorem 3.13 is the following spectral decomposition of Paley-Littlewood type. We refer e.g. to [37] for applications of this decomposition to the description of complex and real interpolation spaces associated with an abstract operator (the Dunkl Laplacian in our case).

Corollary 3.15 *Let $1 < p < \infty$ and $Y = Y(\Omega')$ be a UMD Banach lattice. Let $(\phi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity (see Definition 2.9). Further let $\psi_n = \phi_n$ for $n \geqslant 1$ and $\psi_0 = \sum_{n=-\infty}^0 \phi_n$, so that $\sum_{n=0}^\infty \psi_n(t) = 1$ for all $t > 0$. Denote $A = -\Delta_\kappa$ the Dunkl Laplacian. Then, for any $f \in L^p(\mathbb{R}^d, h_\kappa^2; Y)$, we have the norm description*

$$\|f\|_{\kappa,p;Y} \cong \left\| \left(\sum_{n\in\mathbb{Z}} \left|\phi_n(A)f\right|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y} \cong \left\| \left(\sum_{n=0}^{\infty} \left|\psi_n(A)f\right|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p;Y}.$$

Proof : Once a Hörmander calculus of A on $L^p(\mathbb{R}^d, h_\kappa; Y)$ is guaranteed by Theorem 3.13, the corollary follows from [37, Theorem 4.1] resp. (2.1), to decompose the norm in Rademacher sums resp. square sums. ■

A Hörmander calculus implies by Lemma 2.7 2. that the Dunkl Laplacian A has an $H^\infty(\Sigma_\omega)$ calculus for any $\omega \in (0, \frac{\pi}{2})$. We deduce the following.

Corollary 3.16 *Let $1 < p < \infty$ and $Y = Y(\Omega')$ be a UMD Banach lattice. Let $\psi \in (0, \frac{\pi}{2})$. For $f \in L^p(\mathbb{R}^d, h_\kappa^2; Y)$, define the maximal function $\mathcal{M}_\psi(f)(x) = \sup_{z\in\Sigma_\psi} \|\exp(-zA)f(x,\cdot)\|_Y$. Then*

$$\|M_\psi(f)\|_{\kappa,p} \leqslant C_\psi \|f\|_{\kappa,p;Y}.$$

Moreover, $\exp(-zA)f(x)$ converges to $f(x)$ in Y for almost every $x \in \mathbb{R}^d$, as z tends to 0 within the sector $\overline{\Sigma_\psi}$.

Proof : Once an $H^\infty(\Sigma_\omega)$ calculus is guaranteed for $\omega < \frac{\pi}{2} - \psi$ according to the above, the maximal function boundedness follows from [54, Theorem 2.4.1]. Then the pointwise convergence follows from the proof in [54, Section 2.6 and Discussion after Corollary 1.3.4], see also [13, Corollary 2]. ■

Remark 3.17 *Note that Theorem 3.13 in the scalar case $Y = \mathbb{C}$, i.e. [15, Theorem 4.1] autoimproves by interpolation with self-adjoint calculus. That is, since the Dunkl Laplacian is self-adjoint, T_m will be bounded on $L^2(\mathbb{R}^d, h_\kappa^2)$ for any $m : (0,\infty) \to \mathbb{C}$ measurable and bounded. Note that $\mathcal{H}_q^\alpha \hookrightarrow L^\infty(0,\infty)$ for $\alpha > \frac{1}{q}$, so T_m is bounded on $L^2(\mathbb{R}^d, h_\kappa^2)$ for $m \in \mathcal{H}_q^\alpha$. Then by bilinear interpolation between $\mathcal{H}_q^\alpha \times L^2(\mathbb{R}^d, h_\kappa^2)$ and $\mathcal{H}_1^{n_0} \times L^p(\mathbb{R}^d, h_\kappa^2)$, one gets that T_m is bounded on $L^t(\mathbb{R}^d, h_\kappa^2)$ for $m \in \mathcal{H}_r^\beta$ for $\frac{1}{r} > 2|\frac{1}{2} - \frac{1}{t}|$ and $\beta > \frac{1}{r}$, $\beta > 2n_0|\frac{1}{2} - \frac{1}{t}|$, $1 < t < \infty$. This interpolation improvement does not work in the Y-valued case, since $L^2(\mathbb{R}^d, h_\kappa^2; Y)$ is not a Hilbert space any more, unless $Y = (H, Z)_\theta$ is a complex interpolation space between a Hilbert space H and another UMD lattice Z, and t above is coupled to the fixed number $\theta \in (0,1)$.*

4 Application to maximal regularity

In this section we apply Theorem 3.13 to abstract Cauchy problems involving the Dunkl Laplacian, and show existence, unicity and regularity results for the solutions.

Definition 4.1 *Let B be an ω-sectorial operator for some $\omega \in (0,\pi)$, acting on some Banach space, and let $\theta \in [\omega, \pi)$. We say that B is R-θ-sectorial if $\{\lambda(\lambda - B)^{-1} : \lambda \in \mathbb{C}\backslash\overline{\Sigma_\theta}\}$ is R-bounded. In this case, we denote by $\omega_R(B)$ the infimum over all θ such that B is R-θ-sectorial.*

According to Lemma 2.7 2. and Lemma 2.10 8., Theorem 3.13 implies that the Dunkl Laplacian A has an $H^\infty(\Sigma_\theta)$ calculus for any $\theta \in (0,\pi)$. Thus, also any fractional power A^β has an $H^\infty(\Sigma_\theta)$ calculus for any $\theta \in (0,\pi)$, for any $\beta > 0$. We then deduce Corollary 4.2 below on maximal regularity. To this end, we let B be an ω-sectorial operator for some $\omega \in (0,\pi)$, acting on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$. We further impose that resolvents of B commute with resolvents of A. For example $B = \mathrm{Id}_{L^p} \otimes B_0$, where B_0 is an ω-sectorial operator acting on Y. In that case, B is (R-)ω-sectorial on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ if and only if B_0 is (R-)ω-sectorial for a given angle $\omega \in (0,\pi)$. Moreover, it is easily checked that $D(B) = L^p(\mathbb{R}^d, h_\kappa^2; D(B_0))$, and the graph norm of $D(B)$ is the norm of $L^p(\mathbb{R}^d, h_\kappa^2; D(B_0))$, $D(B_0)$ itself being equipped with the graph norm.

Corollary 4.2 *Let A be the Dunkl Laplacian on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ associated with some reflection group W and weight h_κ, and B an ω-sectorial operator for some $\omega \in (0,\pi)$, aciting on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ such that resolvents of A and B commute.*

1. *Assume that for some $\omega \in (0,\pi)$, B is R-ω-sectorial. Then, for any $\beta > 0$, $A^\beta + B$ is closed on $D(A^\beta) \cap D(B)$. Moreover, $\|A^\beta f\| + \|Bf\| \leqslant C\|A^\beta f + Bf\|$, for $f \in D(A^\beta) \cap D(B)$.*

2. *Assume that for some $\omega \in (0,\pi)$, B is R-ω-sectorial. If Y has property (α) (see e.g. [41, 4.9]), then $A^\beta + B$ is again R-ω-sectorial and $\omega_R(A^\beta + B) \leqslant \omega(B)$.*

3. *Assume that Y has property (α). If moreover, $\omega_R(B) < \frac{\pi}{2}$, then the abstract Cauchy problem*

$$\begin{cases} \frac{d}{dt}u(t) + A^\beta u(t) + Bu(t) & = f(t) \\ u(0) & = 0 \end{cases} \tag{4.1}$$

has maximal regularity. This means, for any given $T \in (0,\infty], q \in (1,\infty)$ and $f \in L^q([0,T), L^p(\mathbb{R}^d, h_\kappa^2; Y))$, the solution u of (4.1) exists, is almost everywhere differentiable, has values in $D(A^\beta) \cap D(B)$, and there exists $C < \infty$ such that

$$\Big\| \frac{d}{dt} u \Big\|_{L^q([0,T), L^p(\mathbb{R}^d, h_\kappa^2; Y))} + \|A^\beta u(t)\|_{L^q([0,T), L^p(\mathbb{R}^d, h_\kappa^2; Y))} + \|Bu(t)\|_{L^q([0,T), L^p(\mathbb{R}^d, h_\kappa^2; Y))}$$
$$\leqslant C \|f\|_{L^q([0,T), L^p(\mathbb{R}^d, h_\kappa^2; Y))}.$$

Proof : 1. We can apply [34, Theorem 6.3] since A^β has an $H^\infty(\Sigma_\theta)$ calculus such that $\theta + \omega < \pi$.

2. We note that if Y has property (α), then also $L^p(\Omega; Y)$ has. In our situation, $(\Omega, \mu) = (\mathbb{R}^d, h_\kappa^2(x)dx)$. Now apply the second part of [34, Theorem 6.3].

3. This follows from part 2. together with [56, Theorem 4.2]. ■

In the above Corollary 4.2, the domain of A^β is given by general theory of (analytic) semigroups. We give in the next lemma some supplementary information on the domain.

Lemma 4.3 *Let A be a Dunkl Laplacian on $L^p(\mathbb{R}^d, h_\kappa^2; Y)$ as in Corollary 4.2. Then $D = \mathcal{S}(\mathbb{R}^d) \otimes Y$ is a core of A^β for any $\beta > 0$.*

Proof : Assume first that $\beta = n$ belongs to $\mathbb{N}$. According to [18, Theorem 1.9], since D is dense in $L^p(\mathbb{R}^d, h_\kappa; Y)$, it suffices to show that D is invariant under the action of $\exp(-tA^n)$. We clearly have for $f = \sum_{k=1}^K f_k \otimes y_k \in D$,

$$\exp(-tA^n) f = \sum_{k=1}^K \mathcal{F}_\kappa^{-1}[\exp(-t\|\xi\|^{2n})\mathcal{F}_\kappa(f_k)] \otimes y_k,$$

and $\exp(-t\|\xi\|^{2n})$ belongs to $\mathcal{S}(\mathbb{R}^d)$ for any $t > 0$. Thus, also $\exp(-t\|\xi\|^{2n})\mathcal{F}_\kappa(f_k) \in \mathcal{S}(\mathbb{R}^d)$ according to Lemma 2.10 2., and, again by Lemma 2.10 2., $\exp(-tA^n) f \in D$.

Now consider the general case $\beta > 0$. Choose $n \in \mathbb{N}$ with $\beta < n$. Then $D \subset D(A^n) \subset D(A^\beta)$, where the first embedding is dense by the above and the second embedding holds according to [41, Theorem 15.15]. It is also dense. Indeed, let $f \in D(A^\beta)$. Then $f = (1+A)^{-\beta} h$ for some $h \in X := L^p(\mathbb{R}^d, h_\kappa^2; Y)$. By density of $D(A^{n-\beta})$ in X, we can choose a sequence $(g_n)_n$ in $D(A^{n-\beta})$ such that $g_n \to h$ in X. Then $f_n = (1+A)^{-\beta} g_n$ belongs to $D(A^n)$. Moreover, $\|A^\beta f - A^\beta f_n\|_X = \|A^\beta(1+A)^{-\beta}h - A^\beta(1+A)^{-\beta}g_n\|_X \leqslant \|A^\beta(1+A)^{-\beta}\| \, \|h - g_n\|_X \to 0$. Similarly, $\|f - f_n\|_X \to 0$. We have shown the density $D(A^n) \subset D(A^\beta)$ and thus, $D \subset D(A^\beta)$ is dense. ■

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Luc Deleaval
Laboratoire d'Analyse et de Mathématiques Appliquées (UMR 8050)
Université Paris-Est Marne la Vallée
5, Boulevard Descartes
Champs sur Marne
77454 Marne la Vallée Cedex 2
luc.deleaval@u-pem.fr

Christoph Kriegler
Laboratoire de Mathématiques (UMR 6620)
Université Blaise-Pascal (Clermont-Ferrand 2)
Campus Universitaire des Cézeaux
3, place Vasarely
TSA 60026
CS 60026
63 178 Aubière Cedex, France
christoph.kriegler@math.univ-bpclermont.fr