Regenerating singular hyperbolic structures from Sol

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1 Introduction

Let M be the mapping torus of an orientation preserving hyperbolic homeomorphism of the 2-torus $\phi : T^2 \to T^2$. That is, M is a compact oriented 3-manifold fibering over S^1 with fiber a torus T^2 and hyperbolic monodromy $\phi : T^2 \to T^2$

$$T^2 \to M \to S^1.$$

This manifold admits a geometric structure modeled on SOL. This structure can not be deformed to hyperbolic structures, but to singular hyperbolic ones. Since ϕ is hyperbolic, it lifts to a linear map of $\mathbb{R}^2 \cong \tilde{T}^2$ with real eigenvalues different from ± 1. Thus T^2 has two invariant foliations by lines parallel to the eigenspaces, which induce two transversely hyperbolic foliations on M. In addition ϕ has a fixed point $* \in T^2$. Let $\Sigma \subset M$ be the curve that is the mapping torus of the identity on the fixed point $*$.

Theorem A *Let M be a torus bundle with hyperbolic monodromy and let $\Sigma \subset M$ be as above. There exist a family of hyperbolic cone structures on M with singular set Σ parametrized by the cone angle $\alpha \in (0, 2\pi)$. When $\alpha \to 2\pi$ this family collapses to a circle, and when $\alpha \to 0$ this family converges to a complete hyperbolic metric on $M - \Sigma$.*

When M is the manifold obtained by 0-surgery on the figure eight knot, this result is well illustrated in the literature. In [Jor] Jørgensen constructed the holonomy representations of these structures in this case. This example was also developed in [HLM] by Hilden, Lozano and Montesinos with an explicit family of Dirichlet polyhedra collapsing to a segment (whose ends are identified to give S^1). The third author of the present paper showed in her thesis that this family of polyhedra can be rescaled to converge to a SOL structure [Sua], which motivated the following remark. The collapsing $M \to S^1$ is precisely the projection of the fibration. We can consider rescalings of the metric in the direction of the fibers so that it does not collapse.

Remark 1.1 *When $\alpha \to 2\pi$, the metric can be rescaled in the direction of the fibers so that the family of Theorem A converges to a SOL structure.*

Remark 1.2 *When collapsing, the developing maps converge to either a projection to the line $\mathbb{R}$ (which is the lift of the projection $M \to S^1$), or to the developing map of a transversely hyperbolic foliation, according to the choice of developing maps (there is no unique choice of developing maps since they can be composed with isometries).*

The existence of the family of hyperbolic cone manifolds in Theorem A will follow from Theorem B below, that describes a space of local deformations, and a theorem of Kojima [Koj], that gives a result of global deformations for hyperbolic cone manifolds with cone angles $\leq \pi$.

The open manifold $N = M - \Sigma$ is a fiber bundle

$$T^2 - \{*\} \to N \to S^1$$

whose monodromy is the restriction of ϕ to the punctured torus $T^2 - \{*\}$. In order to describe the structures on N with *generalized Dehn filling coefficients*, we consider the compact manifold $M - \mathcal{N}(\Sigma)$, where $\mathcal{N}(\Sigma)$ is an open tubular neighborhood of Σ. The manifold $M - \mathcal{N}(\Sigma)$ is a compact core of N and its boundary is a torus. We choose $\{\mu, \lambda\}$ a basis for $H_1(\partial\overline{\mathcal{N}(\Sigma)}, \mathbb{Z})$, so that μ is a meridian of Σ. By using the basis $\{\mu, \lambda\}$ we can define the generalized Dehn filling coefficients $(p, q) \in \mathbb{R}^2 \cup \{\infty\}$ for a given structure on M, which is related to its completion. For instance, the coefficients of the SOL structure on M are $(p, q) = (1, 0)$, because it is the restriction of a SOL structure on M.

Theorem B *Let N be a punctured torus bundle with hyperbolic monodromy. There exists a neighborhood V of $(1, 0)$ in the half-plane $\{(x, y) \in \mathbb{R}^2 \mid x \geq 1\}$ such that every $(p, q) \in V$ are the generalized Dehn filling coefficients of a geometric structure on N of the following kind:*

- *a hyperbolic structure when $p > 1$.*
- *a transverse hyperbolic foliation when $p = 1$.*

Remark 1.3 *The coefficients $(p, q) = (1, 0)$ correspond to the SOL structure and also to two transverse hyperbolic foliations, induced by the foliations on T^2 invariant by ϕ.*

A particular case of these hyperbolic structures on N are structures with coefficients of the form $(p, 0)$ (with $p > 1$), whose completion is a singular hyperbolic metric on M. The singular locus is the curve $\Sigma \subset M$ and the singularity is of cone type with cone angle $2\pi/p$.

To prove Theorem B we will first restrict ourselves to the case where the monodromy matrix of N has positive trace. (In this case, the holonomy group of the complete SOL structure on M is contained in the component of the identity of $\mathrm{Isom}^+(SOL)$, i.e. in SOL itself.) We will work with the space of representations of $\pi_1(N)$ into $SL_2(\mathbb{C})$. We have a canonical exact sequence

$$1 \to \mathbb{R}^2 \to SOL \xrightarrow{\mathrm{LIN}} \mathbb{R} \to 1$$

(see Section 2). If $\mathrm{hol} : \pi_1(M) \to SOL$ denotes the holonomy of the SOL structure on M, then we view the composition $\mathrm{LIN} \circ \mathrm{hol}$ as a representation of $\pi_1(M)$ into the translation group of a geodesic σ in hyperbolic 3-space $\mathbb{H}^3$. Let ρ_0 denote the representation induced on $\pi_1(N)$ by $\mathrm{LIN} \circ \mathrm{hol}$. We shall construct a deformation space

$$U \subset \mathrm{Hom}(\pi_1(N), SL_2(\mathbb{C}))$$

of ρ_0 homeomorphic to a neighborhood of 0 in $\mathbb{C}^2$.

The action of $SL_2(\mathbb{C})$ by conjugation on the representation space restricts on U to the restriction of the following action of $\mathbb{C}^*$ on $\mathbb{C}^2$:

$$e^t \cdot (x,y) = (e^t x, e^{-t} y) \qquad \text{for } e^t \in \mathbb{C}^* \text{ and } (x,y) \in \mathbb{C}^2.$$

The algebra of invariant functions on $\mathbb{C}^2$ is generated by xy. When $c \neq 0$, the level set $xy = c$ is precisely one orbit; moreover the level set $xy = 0$ is the union of three orbits, $\{(0,0)\}$, $\{0\} \times \mathbb{C}^*$ and $\mathbb{C}^* \times \{0\}$. Therefore the quotient $\mathbb{C}^2/\mathbb{C}^*$ is $\mathbb{C}$ with a triple point at the origin.

The quotient $U/\sim$ of our deformation space is a neighborhood of the triple point of $\mathbb{C}^2/\mathbb{C}^*$. This triple point corresponds precisely to the SOL structure and the two transverse hyperbolic foliations induced by the invariant foliation of the torus. Real points in the quotient $U/\sim$ correspond to transverse hyperbolic foliations and the other points correspond to hyperbolic metrics. When we look at generalized Dehn filling coefficients, the triple point of $\mathbb{C}^2/\mathbb{C}^*$ will give a single one, because generalized Dehn filling coefficients are continuous functions. Moreover, complex conjugation in $U/\sim$ corresponds to change of orientation. The orientation conventions may be fixed so that Dehn filling coefficients change the sign or not under change of orientation. In any case, the neighborhood V of Theorem B is homeomorphic to the quotient of a neighborhood of $\mathbb{C}$ by conjugation.

To construct the deformation space we fix a geodesic σ and we use the splitting of the Lie algebra

$$sl_2(\mathbb{C}) = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_-,$$

where $\mathfrak{h}_0$, is the subalgebra of Killing fields that preserve σ and $\mathfrak{h}_+$ and $\mathfrak{h}_-$ are the subspaces of parabolic Killing fields that fix respectively each one of the ends of σ.

The normal bundle $(T\sigma)^\perp$ is the bundle of tangent planes perpendicular to σ

$$\mathbb{R}^2 \to (T\sigma)^\perp \to \sigma.$$

We will work in $(T\sigma)^\perp$, since it is diffeomorphic to hyperbolic 3-space $\mathbb{H}^3$ via the exponential map. As first step for having developing maps, we shall construct a family of maps between SOL and $(T\sigma)^\perp$ by using the plane bundle structure of both SOL and $(T\sigma)^\perp$ and parabolic Killing fields. Then we shall construct a family of holonomy representations and we shall render the maps between SOL and $(T\sigma)^\perp$ equivariant.

Finally, we prove Theorem B in the case where the monodromy matrix A of N has negative trace. Then the two-fold cyclic covering $\widehat{N}$ of N has monodromy matrix A^2 with positive trace. Let Ψ be the involution on $\widehat{N}$ such that $N = \widehat{N}/\Psi$. Then it suffices to show that the proof of Theorem B for $\widehat{N}$ is invariant under Ψ.

The paper is organized as follows: In Section 2 we recall some basic facts about the group SOL, and in Section 3 we study representations of $\pi_1(M)$ into this group. Next in Section 4 we study the splitting of the Lie algebra and some applications of parabolic Killing fields. In Section 5 we construct the deformation space for the representation ρ_0, leaving the proof of some results to the last section. In Section 6 we construct the developing maps associated to the representations of previous section and in Section 7 we compute Dehn filling parameters to conclude the proofs of Theorems A and B. In section 8 we discuss, as an example, the case of torus bundles which are regular coverings of S^3. Finally in Section 9 we prove some technical results about deformations of representations.

2 The group SOL

We recall that SOL is the group of affine transformations in the Minkowski plane $\mathbb{R}^{1,1}$. That is, it is the group of transformations that preserve the Lorentz metric:

$$4\,dx\,dy = d(x+y)^2 - d(x-y)^2.$$

The group SOL is diffeomorphic to $\mathbb{R}^3$, and (x, y, t) are the coordinates of the following transformation:

$$\begin{array}{rcl} (x,y,t) : \mathbb{R}^{1,1} & \to & \mathbb{R}^{1,1} \\ (a,b) & \mapsto & (e^t a + x, e^{-t} b + y). \end{array}$$

Therefore, the product structure on SOL with this coordinate system follows the following rule:

$$(x,y,t)(x',y',t') = (x + e^t x', y + e^{-t} y', t + t').$$

It is clear from this that we have a split exact sequence

$$1 \to \mathbb{R}^2 \to SOL \stackrel{\mathrm{LIN}}{\longrightarrow} \mathbb{R} \to 1 \qquad (1)$$

where $\mathrm{LIN}(x,y,t) = t$ and the kernel of LIN is the translation group. This sequence splits and the action of $\mathbb{R}$ on $\mathbb{R}^2$ is

$$t \cdot (x,y) = (e^t x, e^{-t} y) \qquad \text{for } t \in \mathbb{R},\ (x,y) \in \mathbb{R}^2 \qquad (2)$$

Definition 2.1 *We define $\mathbb{R}_+$ and $\mathbb{R}_-$ to be the $\mathbb{R}$-modules*

$$\begin{array}{rcl} \mathbb{R}\times\mathbb{R}_+ & \to & \mathbb{R}_+ \\ (t,x) & \mapsto & e^t x \end{array} \qquad \begin{array}{rcl} \mathbb{R}\times\mathbb{R}_- & \to & \mathbb{R}_+ \\ (t,x) & \mapsto & e^{-t} x \end{array}$$

The action of $\mathbb{R}$ on $\mathbb{R}^2$ in formula (2) decomposes into two actions:

$$\mathbb{R}^2 = \mathbb{R}_+ \oplus \mathbb{R}_- \qquad \text{as } \mathbb{R}\text{-modules,}$$

A *(set-theoretic) section* to the sequence (1) will be denoted by $\mathrm{trans} : SOL \to \mathbb{R}^2$.

Lemma 2.2 *We have a natural bijection between $\mathbb{R}^{1,1}$ and sections* $\mathrm{trans} : SOL \to \mathbb{R}^2$*. This bijection maps a point $p \in \mathbb{R}^{1,1}$ to the section* trans_p *such that* $\mathrm{trans}_p(\gamma) = \gamma(p) - p$*, for every $\gamma \in SOL$.* □

Remark 2.3 - *Note that using coordinates (x,y,t) corresponds to choosing a section* $\mathrm{trans} : SOL \to \mathbb{R}^2$ *such that* $\mathrm{trans}(x,y,t) = (x,y)$.

- *The choice of a section* $\mathrm{trans} : SOL \to \mathbb{R}^2$ *is equivalent to the choice of a section $\mathbb{R} \to SOL$ to* LIN*, which maps $t \in \mathbb{R}$ to $(0,0,t) \in SOL$*

- *A section* trans : $SOL \to \mathbb{R}^2$ *satisfies the following cocycle condition:*

$$\mathrm{trans}(\gamma_1\gamma_2) = \mathrm{trans}(\gamma_1) + \mathrm{LIN}(\gamma_1)\cdot\mathrm{trans}(\gamma_2) \qquad \textit{for every } \gamma_1,\gamma_2 \in SOL,$$

where $\mathbb{R}$ *acts on* $\mathbb{R}^2$ *as in equation (2) (i.e.* $\mathbb{R}^2 \cong \mathbb{R}_+ \oplus \mathbb{R}_-$*)*

We fix a group Γ and a morphism $\rho : \Gamma \to \mathbb{R}$. We consider the action of Γ on $\mathbb{R}^2 \cong \mathbb{R}_+ \oplus \mathbb{R}_-$ induced by ρ and formula (2). Let $Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)$ and $B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)$ denote respectively the space of cocycles and coboundaries twisted by ρ. That is:

$$Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) = \{\theta : \Gamma \to \mathbb{R}^2 \mid \theta(\gamma_1\gamma_2) = \theta(\gamma_1) + \rho(\gamma_1)\cdot\theta(\gamma_2),\ \forall \gamma_1,\gamma_2 \in \Gamma\}$$

$$B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) = \left\{\theta : \Gamma \to \mathbb{R}^2 \;\middle|\; \begin{array}{c}\text{there exists } a \in \mathbb{R}^2 \text{ such that}\\ \theta(\gamma) = \rho(\gamma)\cdot a - a,\ \forall \gamma \in \Gamma\end{array}\right\}$$

Let $H^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) = Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)/B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho)$ denote the first cohomology group. We have the following:

Lemma 2.4 *Given a representation* $\varphi : \Gamma \to SOL$*,* trans $\circ\varphi$ *is a cocycle twisted by* LIN $\circ\,\varphi$*. Given a fixed representation* $\rho : \Gamma \to \mathbb{R}$*, the map* trans *induces the following bijections, via the identification* $\mathbb{R}_+ \oplus \mathbb{R}_- \cong \mathbb{R}^{1,1}$*:*

- $Z^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) \leftrightarrow \{\varphi \in \mathrm{Hom}(\Gamma, SOL) \mid \mathrm{LIN}\circ\varphi = \rho\}$.
- $B^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) \leftrightarrow \left\{\varphi \in \mathrm{Hom}(\Gamma, SOL) \;\middle|\; \begin{array}{c}\mathrm{LIN}\circ\varphi = \rho \textit{ and } \varphi \textit{has}\\ \textit{a fixed point in } \mathbb{R}^{1,1}\end{array}\right\}$.
- $H^1(\Gamma, (\mathbb{R}_+ \oplus \mathbb{R}_-)^\rho) \leftrightarrow \{\varphi \in \mathrm{Hom}(\Gamma, SOL) \mid \mathrm{LIN}\circ\varphi = \rho\}/\mathbb{R}^2$*, where* $\mathbb{R}^2$ *denotes the translation group acting by conjugation.* □

Remark 2.5 *The section* trans : $SOL \to \mathbb{R}^2$ *is not unique, but two different sections give conjugate bijections.*

3 Representations of $\pi_1(N)$ into SOL

From now on we restrict ourselves to the case where the monodromy matrix of N has positive trace.

In this section we describe the representations of $\pi_1(N)$ into SOL by using the machinery of cohomology described above. This approach will be used when we will study deformations of representations. In particular we shall describe the holonomy representation of the SOL structure.

The fundamental group. As in the introduction, let M denote the mapping torus of a hyperbolic homeomorphism $\phi : T^2 \to T^2$ that preserves orientation, and let N denote the mapping torus of the restriction of ϕ to the punctured torus $T^2 - \{*\}$, where $*$ is a point fixed by ϕ. For the remaining of the paper, we fix $\Gamma = \pi_1(N)$, which has the following presentation:

$$\Gamma \cong \langle\, \lambda, \alpha, \beta \mid \lambda\alpha\lambda^{-1} = f(\alpha), \lambda\beta\lambda^{-1} = f(\beta)\,\rangle,$$

where α and β generate the free group $\pi_1(T^2 - \{*\})$ and $f : \pi_1(T^2 - \{*\}) \to \pi_1(T^2 - \{*\})$ is the isomorphism induced by the restriction of ϕ. Observe that f preserves the commutator $\mu = \alpha\beta\alpha^{-1}\beta^{-1}$. The generators α and β of the free group induce a basis $\{[\alpha], [\beta]\}$ for $H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$. The map f induces also an isomorphism $f_*:\mathbb{Z}^2\to\mathbb{Z}^2$ by abelianization. Let $A\in SL_2(\mathbb{Z})$ denote the matrix of f_* with respect to the basis $\{[\alpha],[\beta]\}$. That is,

$$A=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \quad \text{where} \quad \begin{array}{rl} f_*([\alpha])= & a_{11}[\alpha]+a_{21}[\beta]\\ f_*([\beta])= & a_{12}[\alpha]+a_{22}[\beta]\end{array}.$$

The holonomy representation. By Lemma 2.4 above, to describe a representation into SOL it suffices to give a representation into $\mathbb{R}$ and a twisted cocycle. Since ϕ is hyperbolic and we assume that trace$(A)>0$, the eigenvalues of A are of the form $e^{\pm l}$, with $l\in\mathbb{R}$, $l>0$. We fix the representation $\rho_0:\Gamma\to\mathbb{R}$ defined by $\rho_0(\alpha)=\rho_0(\beta)=0$ and $\rho_0(\lambda)=l$. Let $\begin{pmatrix} b_{11}\\ b_{21}\end{pmatrix}$ and $\begin{pmatrix} b_{12}\\ b_{22}\end{pmatrix}$ be eigenvectors of the transpose matrix A^{t} with respective eigenvalues e^{l} and e^{-l}. We set $B=\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$ and we assume that

$$\det(B)=b_{11}b_{22}-b_{12}b_{21}=1. \tag{3}$$

Remark 3.1 *Observe that all $b_{ij}\neq 0$ because $A\in SL_2(\mathbb{Z})$.*

We consider the cocycles $d_+:\Gamma\to\mathbb{R}_+$ and $d_-:\Gamma\to\mathbb{R}_-$ such that $d_\pm(\lambda)=0$ and

$$\begin{array}{ll} d_+(\alpha)=b_{11}, & d_-(\alpha)=b_{12}\\ d_+(\beta)=b_{21} & d_-(\beta)=b_{22}.\end{array}$$

Since $d_\pm$ are cocycles, they satisfy $d_\pm(\gamma_1\gamma_2)=d_\pm(\gamma_1)+e^{\pm\rho_0(\gamma_1)}d_\pm(\gamma_2)$, for every $\gamma_1,\gamma_2\in\Gamma$. An easy computation proves that d_+ and d_- satisfy the relations of the presentation for Γ and the following lemma:

Lemma 3.2 *The cohomology classes of the cocycles d_+ and d_- form a basis for $H^1(\Gamma,(\mathbb{R}_+\oplus\mathbb{R}_-)^{\rho_0})\cong\mathbb{R}^2$. Equivalently,*

$$Z^1(\Gamma,(\mathbb{R}_+\oplus\mathbb{R}_-)^{\rho_0})=B^1(\Gamma,(\mathbb{R}_+\oplus\mathbb{R}_-)^{\rho_0})\oplus\mathbb{R}d_+\oplus\mathbb{R}d_-.$$

In addition $B^1(\Gamma,(\mathbb{R}_+\oplus\mathbb{R}_-)^{\rho_0})=\{\theta:\Gamma\to\mathbb{R}_+\oplus\mathbb{R}_-\mid\theta(\alpha)=\theta(\beta)=(0,0)\}\cong\mathbb{R}^2$. □

Definition 3.3 *The holonomy representation* hol $:\Gamma\to SOL$ *is the representation such that* LIN ∘ hol $=\rho_0$ *and* trans ∘ hol $=d_+\oplus d_-$

With coordinates this representation is

$$\begin{aligned} hol:\Gamma &\to SOL\\ \gamma &\mapsto (d_+(\gamma),d_-(\gamma),\rho_0(\gamma))\end{aligned}$$

Remark 3.4 *This representation is discrete and it induces a faithful representation of $\pi_1(M)$ the group of the compact manifold. Thus it is the holonomy of a complete SOL structure on M. It can be checked that this is the unique representation with these properties, up to conjugation by automorphisms of SOL.*

4 Parabolic Killing fields

From now on we fix an *oriented* geodesic $\sigma \subset \mathbb{H}^3$.

Parabolic Killing fields associated to the ends of σ and the normal bundle to σ will give us the connection between SOL and hyperbolic space.

Definition 4.1 *An orientation preserving isometry that fixes a geodesic σ is said to be* a hyperbolic translation of complex length $s \in \mathbb{C}$ along σ *when it is a pure translation of length* $\mathrm{Re}(s)$ *composed with a rotation of angle* $\mathrm{Im}(s)$ *along σ. This isometry is denoted by T_s.*

Note that this definition includes rotations, which are elliptic elements. A rotation of angle $\alpha \in \mathbb{R}$ along σ is a hyperbolic translation $T_{i\alpha}$ of complex length $i\alpha$.

The group of hyperbolic translations along σ is isomorphic to $\mathbb{C}^*$, the isomorphism being induced by taking the exponential of the complex length. The identity is also viewed as a hyperbolic translation of complex length in $2\pi i\mathbb{Z}$. Let $\sigma(+\infty)$ and $\sigma(-\infty)$ be the ends of σ. We will consider parabolic elements that fix one of the ends (i.e. isometries whose unique fixed point is one of the ends of σ). Thus, we consider the following subgroups of orientation preserving isometries.

$$
\begin{aligned}
H_0 &= \{\gamma \in \mathrm{Isom}^+(\mathbb{H}^3) \mid \gamma \text{ is a hyperbolic translation along } \sigma\}\\
H_+ &= \{\gamma \in \mathrm{Isom}^+(\mathbb{H}^3) \mid \gamma \text{ is parabolic and fixes } \sigma(+\infty)\}\\
H_- &= \{\gamma \in \mathrm{Isom}^+(\mathbb{H}^3) \mid \gamma \text{ is parabolic and fixes } \sigma(-\infty)\}
\end{aligned}
$$

We remark that H_0 is also the subgroup of orientation preserving isometries that fix both ends of σ.

For example, if we work in the upper half-space model of $\mathbb{H}^3$ and assume that σ is the line joining the points 0 and ∞ , then as subgroups of $PSL_2(\mathbb{C})$,

$$H_0 = \left\{ \pm \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \mid t \in \mathbb{C} \right\},\ H_+ = \{\pm \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \mid t \in \mathbb{C}\} \text{ and } H_- = \{\pm \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) \mid t \in \mathbb{C}\}.$$

The Lie algebra. The algebra of Killing fields of $\mathbb{H}^3$ is also the Lie algebra of the isometry group. This algebra is $sl_2(\mathbb{C})$ and the complex structure is going to be useful. The following proposition is well known.

Proposition 4.2 *We have a splitting of the Lie algebra into one-dimensional complex subspaces*

$$sl_2(\mathbb{C}) = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_-,$$

where $\mathfrak{h}_0$, $\mathfrak{h}_+$ and $\mathfrak{h}_-$ are the respective tangent spaces to H_0, H_+ and H_- at the identity. □

(In the example mentioned above, $\mathfrak{h}_0$, $\mathfrak{h}_+$, $\mathfrak{h}_-$ are the $\mathbb{C}$-vector spaces generated by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, respectively.)

The group $\mathrm{Isom}(\mathbb{H}^3)$ acts on $sl_2(\mathbb{C})$ by the adjoint action, and we consider its restriction to H_0.

Lemma 4.3 *The splitting of Lemma 4.2 is preserved by H_0, the group of hyperbolic translations along σ. A hyperbolic translation T_s of complex length s acts by multiplication by $e^{\pm s}$ on $\mathfrak{h}_\pm$ and as the identity on $\mathfrak{h}_0$.*

Proof. The splitting is preserved because conjugation by elements of H_0 preserves H_0, H_+ and H_-. Since H_0 is commutative it acts trivially on $\mathfrak{h}_0$. The action on $\mathfrak{h}_\pm$ is easily deduced from the action of H_0 on $H_\pm$ by conjugation. □

In this way, if we restrict the action of H_0 to the subgroup of translations T_s with real length $s \in \mathbb{R}$, then $\mathfrak{h}_\pm$ is the complexification of the $\mathbb{R}$-module $\mathbb{R}_\pm$:

$$\mathfrak{h}_\pm \cong \mathbb{R}_\pm \otimes_\mathbb{R} \mathbb{C}.$$

The normal bundle to the geodesic σ. Let $(T\sigma)^\perp$ denote the normal bundle to the geodesic $\sigma \subset \mathbb{H}^3$. It is the bundle of planes orthogonal to σ. The relation between SOL and $(T\sigma)^\perp$ comes from evaluation of Killing fields.

Lemma 4.4 *For every $p \in \sigma$, the evaluation at the point p defines a $\mathbb{R}$-linear isomorphism*

$$\mathfrak{h}_+ \cong (T_p\sigma)^\perp.$$

The same assertion holds for $\mathfrak{h}_-$.

Proof. The elements of $\mathfrak{h}_+$ are parabolic Killing fields that fix the end $\sigma(+\infty)$. Therefore, its evaluation at p is orthogonal to σ, because horospheres centered at $\sigma(+\infty)$ are orthogonal to σ. Since $\dim_\mathbb{R} \mathfrak{h}_+ = \dim_\mathbb{R}(T_p\sigma)^\perp$, it suffices to prove that the evaluation at p is an injection. A Killing field evaluated at p is trivial only when it is tangent to the stabilizer of p, and in the definition of $\mathfrak{h}_+$ we consider only parabolic isometries, excluding elliptic ones. □

Corollary 4.5 *For every $p \in \sigma$, the plane $(T_p\sigma)^\perp$ has two structures of $\mathbb{C}$-vector space, induced respectively by $\mathfrak{h}_+$ and $\mathfrak{h}_-$. These structures are complex conjugated.*

Proof. The corollary is clear from Lemma 4.4 and the only point that needs some explanation is why these complex structures are complex conjugated. This comes naturally from the fact that the two ends of σ give different orientations to σ, hence two different senses or rotation around σ. In the complex plane there is a distinguished sense of rotation, which is inverted by complex conjugation. □

Convention 4.6 *From now on we fix the complex structure on the plane $(T_p\sigma)^\perp$ induced by $\mathfrak{h}_+$.*

With this convention, if $v \in \mathfrak{h}_+$ and $z \in \mathbb{C}$, then the evaluation of $z\,v$ at p is z times the evaluation of v, but if $v \in \mathfrak{h}_-$, then the evaluation of $z\,v$ at p is $\bar{z}$ times the evaluation of v.

The maps from SOL to $(T\sigma)^{\perp}$. We fix now a parametrization $\sigma(t)$ of the geodesic by arc-length. We fix $V_+ \in \mathfrak{h}_+$ and $V_- \in \mathfrak{h}_-$ two non-zero Killing fields such that:

$$V_+(\sigma(0)) = V_-(\sigma(0))$$

(so that $V_+(\sigma(t)) = e^{-2t} V_-(\sigma(t))$ for all t).

Definition 4.7 *Let σ and $V_\pm$ be as above. For $(a,b) \in \mathbb{C}^2$ we define*

$$\begin{array}{rcl} \Delta_{a,\bar{b}} : SOL & \to & (T\sigma)^{\perp} \\ (x,y,t) & \mapsto & a\,x\,V_+(\sigma(t)) + \bar{b}\,y\,V_-(\sigma(t)). \end{array}$$

Recall that by Convention 4.6, we use the complex structure induced by $\mathfrak{h}_+$. Thus, in this definition, $\bar{b}\,y\,V_-(\sigma(t))$ means that first we evaluate V_- at the point $\sigma(t)$ and then we multiply the result by $\bar{b}\,y$. This explains the conjugation $\bar{b}$. The proof of the following two lemmas is straightforward.

Lemma 4.8 *When $ab \notin \mathbb{R}$, the map $\Delta_{a,\bar{b}}$ is a diffeomorphism. It preserves the orientation when $\mathrm{Im}(ab) < 0$ and it reverses it when $\mathrm{Im}(ab) > 0$. When $ab \in \mathbb{R}$ but $ab \neq 0$, the map $\Delta_{a,\bar{b}}$ is a submersion onto $(T_\sigma)^{\perp} \cap T\mathbb{H}^2$ for some hyperbolic plane $\mathbb{H}^2$ containing σ.* □

Lemma 4.9 *Denote by T_s the hyperbolic translation along σ of complex length $s \in \mathbb{C}$. For any $(a,b) \in \mathbb{C}^2$, the following holds:*

(i) for $g_1, g_2 \in SOL$

$$\Delta_{a,\bar{b}}(g_1 \cdot g_2) = \Delta_{a,\bar{b}}(g_1 \cdot (0,0,t_2)) + T_{t_1}(\Delta_{a,\bar{b}}(g_2)),$$

where $g_1 = (x_1, y_1, t_1)$ and $g_2 = (x_2, y_2, t_2)$. In particular (when $g_1 = (0,0,s)$), we have that

$$\Delta_{a,\bar{b}}(x,y,t+s) = T_s(\Delta_{a,\bar{b}}(x,y,t)).$$

(ii) for $g \in SOL$ and $s \in \mathbb{C}$

$$T_s(\Delta_{a,\bar{b}}(g)) = \Delta_{e^s a, \overline{e^{-s} b}}(g \cdot (0,0,\mathrm{Re}(s))).$$

Proof. The proof consists in evaluating each term in the equalities by using the identities:

$$T_s V_\pm = e^{\pm s} V_\pm,$$

and taking care with complex structures. Let us prove for instance (ii), leaving (i) to the reader. Writing in coordinates $g = (x,y,t)$, we have:

$$\begin{array}{lrcl} & \Delta_{a,\bar{b}}(g) & = & a\,x\,V_+(\sigma(t)) + \bar{b}\,y\,V_-(\sigma(t)) \\ \text{and} & T_s(\Delta_{a,\bar{b}}(g)) & = & a\,x\,e^s\,V_+(\sigma(t+\mathrm{Re}(s))) + \bar{b}\,y\,\overline{e^{-s}}\,V_-(\sigma(t+\mathrm{Re}(s))). \end{array}$$

According to Convention 4.6, we use the complex structure induced by V_+ and we write $\overline{e^{-s}}\,V_-(\sigma(t+\mathrm{Re}(s)))$, because first we evaluate V_- at $\sigma(t+\mathrm{Re}(s))$ and then we multiply by $\overline{e^{-s}}$, which is equivalent to evaluate $e^{-s}V_-$ at $\sigma(t+\mathrm{Re}(s))$.

In addition, $g \cdot (0,0,\mathrm{Re}(s)) = (x, y, t+\mathrm{Re}(s))$ and

$$\Delta_{e^s a, \overline{e^{-s} b}}(g \cdot (0,0,\mathrm{Re}(s))) = e^s\,a\,x\,V_+(\sigma(t+\mathrm{Re}(s))) + \overline{e^{-s}\,b}\,y\,V_-(\sigma(t+\mathrm{Re}(s))),$$

which proves point (ii) in the lemma. □

5 Deforming representations

We view $\rho_0 : \Gamma \to \mathbb{R}$ as a representation of Γ into the translation group along a geodesic $\sigma \subset \mathbb{H}^3$. We lift it to a representation into

$$\widetilde{\mathrm{Isom}^+}(\mathbb{H}^3) \cong SL_2(\mathbb{C}),$$

which is the double covering of the orientation preserving isometry group $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$. This representation is still denoted by $\rho_0 : \Gamma \to SL_2(\mathbb{C})$. We shall deform $\rho_0 : \Gamma \to SL_2(\mathbb{C})$ to get the holonomy of hyperbolic structures. We shall construct a nice space of deformations U which will be a neighborhood of ρ_0 in one irreducible component of $\mathrm{Hom}(\Gamma, SL_2(\mathbb{C}))$:

$$\rho_0 \in U \subset \mathrm{Hom}(\Gamma, SL_2(\mathbb{C})).$$

We also describe the restriction to U of the action of $SL_2(\mathbb{C})$ by conjugation. Before constructing U we study varieties of representations. The proof of some technical results is postponed to the end of the section.

The variety of representations The variety of representations

$$R(\Gamma, SL_2(\mathbb{C})) = \mathrm{Hom}(\Gamma, SL_2(\mathbb{C}))$$

is an affine algebraic subset of $\mathbb{C}^9$. The group $SL_2(\mathbb{C})$ acts on $R(\Gamma, SL_2(\mathbb{C}))$ by conjugation and we may consider two quotients, the topological and the algebraic one.

The topological quotient is denoted by

$$R(\Gamma, SL_2(\mathbb{C}))/SL_2(\mathbb{C})$$

and it is not always Hausdorff. The algebraic quotient

$$R(\Gamma, SL_2(\mathbb{C}))//SL_2(\mathbb{C})$$

is called the *variety* of characters and it is a complex affine algebraic set. Since our representation ρ_0 is abelian, we will have to be careful when studying the quotient. In fact a neighborhood of ρ_0 in $R(\Gamma, SL_2(\mathbb{C}))/SL_2(\mathbb{C})$ is not Hausdorff.

The Zariski tangent space $T_{\rho_0} R(\Gamma, SL_2(\mathbb{C}))$ is isomorphic to $Z^1(\Gamma, sl_2(\mathbb{C})^{\rho_0})$, and the tangent space to the orbit by conjugation is isomorphic to the coboundary space $B^1(\Gamma, sl_2(\mathbb{C})^{\rho_0})$. The splitting

$$sl_2(\mathbb{C}) = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_- \tag{4}$$

is preserved by the action of ρ_0. This splitting is quite useful to compute the cocycle space. In Proposition 5.1 below we show that this splitting can be interpreted in the representation space, by looking at abelian and irreducible representations.

We fix $V_\pm \in \mathfrak{h}_\pm$ and $V_0 \in \mathfrak{h}_0$ non-zero elements, so that $V_\pm$ are the same as in the previous section. Thus $\{V_+, V_-, V_0\}$ is a $\mathbb{C}$-basis for $sl_2(\mathbb{C})$.

Irreducible and abelian representations. Recall that a representation $\rho : \Gamma \to SL_2(\mathbb{C})$ is called irreducible if it has no proper invariant subspaces of $\mathbb{C}^2$. The set of irreducible representations is not closed but we will consider its closure:

$$R^{ir}(\Gamma, SL_2(\mathbb{C})) = \overline{\{\rho \in R(\Gamma, SL_2(\mathbb{C})) \mid \rho \text{ is irreducible}\}}$$

In [HPS] we proved that $\rho_0 \in R^{ir}(\Gamma, SL_2(\mathbb{C}))$. A representation is called abelian if its image is an abelian group. The set of abelian representations is closed and we consider:

$$R^{ab}(\Gamma, SL_2(\mathbb{C})) = \{\rho \in R(\Gamma, SL_2(\mathbb{C})) \mid \rho \text{ is abelian}\}.$$

Note that $\rho_0 \in R^{ir}(\Gamma, SL_2(\mathbb{C})) \cap R^{ab}(\Gamma, SL_2(\mathbb{C}))$.

Proposition 5.1 *(a) The analytic germ of $R(\Gamma, SL_2(\mathbb{C}))$ at ρ_0 has two irreducible components, which are precisely $R^{ir}(\Gamma, SL_2(\mathbb{C}))$ and $R^{ab}(\Gamma, SL_2(\mathbb{C}))$.*

(b) In addition

$$\begin{aligned} T_{\rho_0} R^{ir}(\Gamma, SL_2(\mathbb{C})) &= Z^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0}) \\ T_{\rho_0} R^{ab}(\Gamma, SL_2(\mathbb{C})) &= Z^1(\Gamma, \mathfrak{h}_0^{\rho_0}) \oplus B^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0}) \end{aligned}$$

Point (a) is proved in [HPS] and the proof of point b is delayed to Section 8.

Remark 5.2 *As Γ-modules, $\mathfrak{h}_\pm \cong \mathbb{R}_\pm \otimes \mathbb{C}$ (i.e. $\mathfrak{h}_\pm$ is the complexification of $\mathbb{R}_\pm$). Therefore, the computations of cocycles and coboundaries of Section 3 apply here, just by complexifying. As in lemma 3.2, we have:*

$$Z^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0}) = B^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0}) \oplus \mathbb{C} d_+ V_+ \oplus \mathbb{C} d_- V_-,$$

where $d_\pm V_\pm$ is the cocycle $\Gamma \to \mathfrak{h}_\pm$ that maps $\gamma \in \Gamma$ to $d_\pm(\gamma)\, V_\pm$

The slice. We are interested in representations up to conjugation, this is why we consider a slice with respect to the projection of $R^{ir}(\Gamma, SL_2(\mathbb{C}))$ onto the variety of characters. Since our representation is abelian, the slice will not give a parametrization of the space of orbits, but it will be a useful tool to study it. Following [BA], we define:

$$\mathcal{S} = \{\rho \in R^{ir}(\Gamma, SL_2(\mathbb{C})) \mid \rho(\lambda) \text{ is a hyperbolic translation along } \sigma\},$$

where the letter $\mathcal{S}$ stands for slice. Note that $\rho_0(\lambda)$ is a hyperbolic translation along σ of length l, therefore $\rho_0 \in \mathcal{S}$.

The local parametrization for $\mathcal{S}$. Now we want to construct a parametrization for the slice $\mathcal{S}$ in a neighborhood of ρ_0. Given $\rho \in \mathcal{S}$ we consider $\rho(\alpha)\rho_0(\alpha^{-1}) \in SL_2(\mathbb{C})$. We choose ρ sufficiently close to ρ_0, so that $\rho(\alpha)\rho_0(\alpha^{-1}) \in \exp(W)$, where $W \subset sl_2(\mathbb{C})$ is a neighborhood of the origin so that exp restricted to W is injective. Now we look at $\operatorname{pr}(\exp^{-1}(\rho(\alpha)\rho_0(\alpha^{-1}))) \in \mathfrak{h}_+ \oplus \mathfrak{h}_-$, where

$$\operatorname{pr} : sl_2(\mathbb{C}) \to \mathfrak{h}_+ \oplus \mathfrak{h}_-$$

denotes the projection with kernel $\mathfrak{h}_0$.

Proposition 5.3 *There is a neighborhood $\mathcal{U} \subset \mathcal{S}$ of ρ_0 so that the map*

$$\begin{aligned} \mathcal{U} &\to \mathfrak{h}_+ \oplus \mathfrak{h}_- \\ \rho &\mapsto \mathrm{pr}(\exp^{-1}(\rho(\alpha)\rho_0(\alpha^{-1}))) \end{aligned}$$

is a biholomorphism between $\mathcal{U}$ and a neighborhood of the origin in $\mathfrak{h}_+ \oplus \mathfrak{h}_-$.

The proof of proposition 5.3 is delayed until section 9 (see corollary 9.3).

Convention 5.4 *Since $d_\pm(\alpha) \neq 0$ (see remark 3.1), we can identify $\mathfrak{h}_+ \oplus \mathfrak{h}_-$ with $\mathbb{C}^2$ by sending $(a, b) \in \mathbb{C}^2$ to $a \cdot d_+(\alpha)\, V_+ + b \cdot d_-(\alpha) V_- \in \mathfrak{h}_+ \oplus \mathfrak{h}_-$. With this identification, denote by $U \subset \mathbb{C}^2$ the neighborhood of the origin in $\mathbb{C}^2 \cong \mathfrak{h}_+ \oplus \mathfrak{h}_-$ of Proposition 5.3. Given $(a, b) \in U$, we will denote by $\rho_{a,b} \in \mathcal{U} \subset \mathcal{S}$ the representation which is mapped to $a d_+(\alpha)\, V_+ + b\, d_-(\alpha) V_- \in \mathfrak{h}_+ \oplus \mathfrak{h}_-$ under the biholomorphism of Proposition 5.3. In this way, we have a neighborhood of ρ_0 in $\mathcal{S}$ parametrized by $(a, b) \in U \subset \mathbb{C}^2$.*

The action by conjugation The action of $PSL_2(\mathbb{C})$ by conjugation does not preserve the slice $\mathcal{S}$. The subgroup of isometries that preserve it is H_0, which is the group of hyperbolic translations along γ (and also the subgroup of orientation preserving isometries that fix both ends of σ).

Lemma 5.5 *For every $\gamma \in \Gamma$ and every $(a, b) \in U \in \mathbb{C}^2$,*

$$\rho_{e^t a, e^{-t} b}(\gamma) = T_t\, \rho_{a,b}(\gamma)\, T_{-t}$$

for every $t \in \mathbb{C}$ where it makes sense (i.e. for every $t \in \mathbb{C}$ such that $(e^t a, e^{-t} b) \in U$).

Proof. It follows straightforward from the construction of the slice and from Proposition 5.3. ∎

Lemma 5.5 says that the action of this group reads in the parameter space as the action of $\mathbb{C}$ on $\mathbb{C}^2$ defined by

$$t(a, b) = (e^t a, e^{-t} b).$$

Let $U/\sim$ denote the quotient of U by the action of $\mathbb{C}^*$ of Lemma 5.5. This quotient is not Hausdorff and has a triple point, which corresponds to the orbits $\{0\}$, $\{(a, 0) \mid a \neq 0\}$ and $\{(0, b) \mid b \neq 0\}$. If we identified these three points to a single one, then we would obtain an open set of $\mathbb{C}$.

Corollary 5.6 *The quotient of the slice $\mathcal{S}$ by conjugation is isomorphic to $U/\sim$, which is a neighborhood of a triple point in $\mathbb{C}$. In addition, the algebra of invariant functions on $\mathcal{S}$ is generated by*

$$\begin{aligned} U &\to \mathbb{C} \\ (a, b) &\mapsto ab. \end{aligned}$$

Representations into $SL_2(\mathbb{R})$ By identifying $PSL_2(\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$, we view $PSL_2(\mathbb{R})$ as the subgroup of isometries that preserve some *oriented* hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$. Thus $SL_2(\mathbb{R})$ is the group of lifts of such isometries to $SL_2(\mathbb{C})$. In this construction we also assume that the geodesic σ is contained in this plane $\mathbb{H}^2$. Thus the image of ρ_0 belongs to $SL_2(\mathbb{R})$ and we have:

Remark 5.7 *All the results of this section hold true by replacing $\mathbb{C}$ by $\mathbb{R}$.*

For instance the local parametrization of $\mathcal{S}$ in Proposition 5.3 induces a local parametrization of $\mathcal{S} \cap \mathrm{Hom}(\Gamma, SL_2(\mathbb{R}))$ with the neighborhood of the origin $U \cap \mathbb{R}^2$. In addition, even when $a, b \in \mathbb{R}$ and $t \in \mathbb{R} + \pi i \mathbb{Z}$ the formula of Lemma 5.5 has a meaning in $SL_2(\mathbb{R})$, because conjugation by $T_{\pi i}$ preserves $SL_2(\mathbb{R})$. The proofs for the real case are similar to the proofs for the complex one, because $sl_2(\mathbb{C})$ is the complexification of $sl_2(\mathbb{R})$, i.e. $sl_2(\mathbb{C}) = sl_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. For instance we have:

$$\begin{aligned} Z^1(\Gamma, sl_2(\mathbb{C})^{\rho_0}) &= Z^1(\Gamma, sl_2(\mathbb{R})^{\rho_0}) \otimes_{\mathbb{R}} \mathbb{C} \\ B^1(\Gamma, sl_2(\mathbb{C})^{\rho_0}) &= B^1(\Gamma, sl_2(\mathbb{R})^{\rho_0}) \otimes_{\mathbb{R}} \mathbb{C} \\ H^1(\Gamma, sl_2(\mathbb{C})^{\rho_0}) &= H^1(\Gamma, sl_2(\mathbb{R})^{\rho_0}) \otimes_{\mathbb{R}} \mathbb{C}. \end{aligned}$$

6 Deforming developing maps

The following proposition provides a family of developing maps.

Proposition 6.1 *There exists a family of maps $D_{a,\bar{b}} : \tilde{N} \to \mathbb{H}^3$ parametrized by $(a, b) \in U \subset \mathbb{C}^2$ a neighborhood of the origin such that*

(i) *$D_{a,\bar{b}}$ is $\rho_{a,b}$-equivariant.*

(ii) *When $ab \in \mathbb{R} - \{0\}$, $D_{a,\bar{b}}$ is a submersion onto a hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$.*

(iii) *When $ab \notin \mathbb{R}$, $D_{a,\bar{b}}$ is locally a diffeomorphism.*

(iv) *On the end, the structure is described by generalized Dehn Filling coefficients, which are real analytic functions on ab.*

When we say that the structure on the end is described by generalized Dehn Filling coefficients, it means the following. We can write the end of N in the form $(0, 1] \times T^2$ and there exists a geodesic $\sigma \subset \mathbb{H}^3$ such that, for every $x \in \widetilde{T}^2$, if γ_x is a minimizing geodesic between σ and $D(1, x)$, then $D(s, x) = \gamma_x(s)$ (with $\gamma_x(0) \in \sigma$ and $\gamma_x(1) = D(1, x)$). In this case, one can define generalized Dehn filling coefficients and the proposition says that they depend analytically on ab (this will be proved in next section).

Proof. Let σ be a geodesic in hyperbolic 3-space parametrized by arc-length. The restriction of the exponential map

$$\exp_{|(T\sigma)^{\perp}} : (T\sigma)^{\perp} \xrightarrow{\cong} \mathbb{H}^3$$

is a diffeomorphism between the normal bundle $(T\sigma)^{\perp}$ and hyperbolic 3-space.

Convention 6.2 *Along this proof, we will work in* $(T\sigma)^\perp$ *instead of* $\mathbb{H}^3$ *by using the diffeomorphism induced by the exponential map.*

Consider the homeomorphism

$$(T\sigma)^\perp \cong \mathbb{C} \times \mathbb{R}$$

such that

(i) $\mathbb{C} \times \{t\} \cong (T_{\sigma(t)}\sigma)^\perp$ is the orthogonal tangent plane at $\sigma(t)$.

(ii) we fix an isomorphism of $\mathbb{C}$-vector spaces $\mathbb{C} \times \{0\} \cong (T_{\sigma(0)}\sigma)^\perp$ for $t = 0$, where the complex structure on $(T_{\sigma(0)}\sigma)^\perp$ is induced by $\mathfrak{h}_+$, according to Convention 4.6. For example, the isomorphism sending $(z, 0) \in \mathbb{C} \times \{0\}$ to $zV_+(\sigma(0)) \in (T_{\sigma(0)}\sigma)^\perp$.

(iii) parallel transport maps $(z, t_1) \in \mathbb{C} \times \mathbb{R}$ to (z, t_2).

Convention 6.3 *We will use this coordinate system* $\mathbb{C}\times\mathbb{R} \cong (T\sigma)^\perp$. *Thus, given* $x, y \in \mathbb{C}$, *the vector* $xV_+(\sigma(t)) + yV_-(\sigma(t)) \in (T_{\sigma(t)}\sigma)^\perp$ *will have coordinates* $(xe^{-t} + ye^t, t) \in \mathbb{C} \times \mathbb{R}$.

With coordinates in $\mathbb{C}\times\mathbb{R}$, a hyperbolic translation of complex length $s_0+i\, s_1$ (with $s_0, s_1 \in \mathbb{R}$) acts as:

$$T_{s_0+i\, s_1}(z, t) = (e^{i\, s_1} z, t + s_0), \qquad \text{for every } (z, t) \in \mathbb{C} \times \mathbb{R}.$$

For each $t \in \mathbb{R}$, $\mathbb{C} \times \{t\} \cong (T_{\sigma(t)}\sigma)^\perp$ is a $\mathbb{C}$-vector space. We will not use the real vector space structure of $\mathbb{C} \times \mathbb{R}$ but only its $\mathbb{R}$-affine structure.

Construction of $D_{a,\bar{b}}$. Let $\mathcal{D} : \tilde{N} \to SOL$ denote the developing map for the SOL structure on N induced by M. Observe that $\mathcal{D}$ is a covering map of its image. The idea is to use the map $\Delta_{a,\bar{b}} \circ \mathcal{D}$ and to rend it $\rho_{a,b}$-equivariant. We follow the construction of Canary, Epstein and Green, in Lemma 1.7.2 [CEG]. We choose $U_0, U_1, \dots, U_n$ a covering of M such that

- U_0 is a punctured tubular neighborhood of Σ (i.e. $U_0 \cong (D^2-\{*\})\times S^1$)
- U_i is simply connected, for $i \geq 1$.

Let $\pi : \widetilde{M} \to M$ denote the projection of the universal covering, we construct the map $D^1_{a,\bar{b}} : \pi^{-1}(U_1) \to (T\sigma)^\perp$ as follows. We take

$$\pi^{-1}(U_1) = \bigcup_{\gamma\in\Gamma} \gamma V_1 \qquad \text{where } V_1 \text{ is simply connected.}$$

We define $D^1_{a,\bar{b}|V_1} = \Delta_{a,\bar{b}} \circ \mathcal{D}_{|V_1}$, and we take $D^1_{a,\bar{b}}$ to be the $\rho_{a,b}$-equivariant extension of $D^1_{a,\bar{b}|V_1}$.

Next we take a refinement $U^2_0, U^2_1, \dots, U^2_n$ (i.e. a covering of N such that $\overline{U^2_i} \subset U_i$), and we shall construct $D^2_{a,\bar{b}} : \pi^{-1}(U^2_1 \cup U^2_2) \to \mathbb{H}^3$. Let

$$\pi^{-1}(U_2) = \bigcup_{\gamma\in\Gamma} \gamma V_2 \qquad \text{and} \qquad \pi^{-1}(U^2_2) = \bigcup_{\gamma\in\Gamma} \gamma V^2_2,$$

where $\overline{V^2_2} \subset V_2$ and V_2 is simply connected. We define $f : V_2 \to (T\sigma)^\perp$ as:

$$f = \phi\, D^1_{a,\bar{b}} + (1 - \phi)(\Delta_{a,\bar{b}} \circ \mathcal{D}_{|V_2}), \tag{5}$$

where $\phi : V_2 \to [0,1]$ is a $\mathcal{C}^\infty$ function satisfying $\overline{\sup \phi} \subset V_2$ and $\phi_{|\overline{V_2^2} \cap \pi^{-1}(U_1^2)} \equiv 1$. Note that in equality (5) we use the $\mathbb{R}$-affine structure of $(T\sigma)^\perp \cong \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$. By construction:

$$f_{|V_2^2 \cap \pi^{-1}(U_1^2)} = D^1_{a,\bar b | V_2^2 \cap \pi^{-1}(U_1^2)}.$$

Thus we extend $f_{|V_2^2}$ equivariantly to $\pi^{-1}(U_2^2)$ and we glue it to $D^1_{a,\bar b}$ restricted to $\pi^{-1}(U_1^2)$. In this way we obtain $D^2_{a,\bar b} : \pi^{-1}(U_1^2 \cup U_2^2) \to (T\sigma)^\perp$.

We proceed inductively to construct

$$D^n_{a,\bar b} : \pi^{-1}(U_1^n \cup U_2^n \cup \cdots \cup U_n^n) \to (T\sigma)^\perp,$$

where $U_0^n, U_1^n, U_2^n, \ldots, U_n^n$ is a refinement of $U_0, U_1, U_2, \ldots, U_n$. It remains to define the map on $\widetilde{U}_0$. We write $U_0 \cong (0,1) \times T^2$, where 0 points to the end and 1 to the interior of the manifold. We take the closure $\overline{U}_0 \cong (0,1] \times T^2$ and we define a map

$$D^0_{a,\bar b} : \widetilde{\overline{U}_0} \cong (0,1] \times \widetilde{T}^2 \to \mathbb{H}^3$$

such that $D^0_{a,\bar b}(s,x) = s\, D^0_{a,\bar b}(1,x)$, for $x \in \widetilde{T}^2$ and $s \in (0,1]$. This can be done because of the following lemma:

Lemma 6.4 *Let $\varepsilon > 0$ be such that $(1-\varepsilon, 1] \subset U_1^n \cup \cdots \cup U_n^n$. Then:*

$$D^n_{a,\bar b}(s,x) = s D^n_{a,\bar b}(1,x) \qquad \text{for every } s \in (1-\varepsilon, 1] \text{ and } x \in \widetilde{T}^2.$$

Proof. Straightforward, because all constructions are $\mathbb{R}$-linear on fibers, as for instance formula (5). □

For $x \in \widetilde{T}^2$ and $s \in (0,1)$ we define $D^0_{a,\bar b}(1,x) = D^n_{a,\bar b}(1,x)$ and $D^0_{a,\bar b}(s,x) = s D^0_{a,\bar b}(1,x)$. By construction, the structure on the end will be described by Dehn Filling coefficients. In addition, by Lemma 6.4, $D^0_{a,\bar b}$ and $D^n_{a,\bar b}$ coincide in the intersection of their domains. Therefore we define $D_{a,\bar b}$ by gluing $D^0_{a,\bar b}$ and $D^n_{a,\bar b}$.

To check that $D_{a,\bar b}$ satisfies the required properties we need the following two lemmas.

Lemma 6.5 *There exists a one-parameter group of diffeomorphisms $\{h_s : \widetilde{N} \to \widetilde{N}\}_{s \in \mathbb{R}}$ such that*

$$D_{e^t a, \overline{e^{-t} b}} = T_t \circ D_{a,\bar b} \circ h_{-\operatorname{Re}(t)} \qquad \text{for every } t \in \mathbb{C},$$

where T_t is the hyperbolic translation of complex length t along σ.

To be a one-parameter group means that $h_{s+s'} = h_s \circ h_{s'}$ for every $s, s' \in \mathbb{R}$.

Proof. We take h_s the pullback under the covering map $\mathcal{D}$ of right-multiplication by $(0,0,s)$ in SOL (which preserves the image of $\mathcal{D}$). The lemma follows from Lemma 4.9, Lemma 5.5, and linearity of the constructions. □

Lemma 6.6 *For $\gamma \in \Gamma$ and $K \subset SOL$ a compact subset,*

$$\Delta_{a,\bar b}(\operatorname{hol}(\gamma)(x)) = \rho_{a,b}(\gamma)(\Delta_{a,\bar b}(x)) + o(|(a,b)|^2) \qquad \forall x \in K,$$

where the estimation only depends on K and γ.

In this lemma we use again the identification $(T\sigma)^\perp \cong \mathbb{C}\times\mathbb{R}$. We postpone its proof to Section 8.

Corollary 6.7 *For $K \subset \widetilde{N}$ a compact subset, we have*

$$D_{a,\bar{b}}(x) = \Delta_{a,\bar{b}}(\mathcal{D}(x)) + o(|(a,b)|^2) \qquad \forall x \in K,$$

where the estimation only depends on K. □

Lemma 6.8 *If $ab \in \mathbb{R}-\{0\}$, then the image of $D_{a,\bar{b}}$ is contained in $(T\sigma)^\perp \cap T\mathbb{H}^2$ for some hyperbolic plane $\mathbb{H}^2$. In addition the tangent map of $D_{a,\bar{b}} : \tilde{N} \to \mathbb{H}^2$ is surjective.*

Proof. By Lemma 6.5, we can assume that $a, b \in \mathbb{R}-\{0\}$. In this case, by Remark 5.7 $\rho_{a,b}$ is a representation into $SL_2(\mathbb{R})$, and therefore the image of $D_{a,\bar{b}}$ is contained in

$$\mathbb{R}\times\mathbb{R} \subset \mathbb{C}\times\mathbb{R} \cong (T\sigma)^\perp$$

which corresponds to $(T\sigma)^\perp \cap T\mathbb{H}^2$ in $(T\sigma)^\perp$. To study the tangent map, we can restrict to a compact subset $K \subset \tilde{N}$, because of equivariance and Lemma 6.4. In the coordinate system $\mathbb{C}\times\mathbb{R} \cong (T\sigma)^\perp$ of convention 6.3, we have that

$$(\Delta_{a,\bar{b}})(x,y,t) = (axe^{-t} + \bar{b}ye^{t}, t) \in \mathbb{R}\times\mathbb{R} \subset \mathbb{C}\times\mathbb{R}$$

If $(\Delta_{a,\bar{b}})_*$ denotes the tangent map of $\Delta_{a,\bar{b}}$, then

$$\begin{aligned}
(\Delta_{a,\bar{b}})_*(\partial/\partial t) &= \partial/\partial t + (-a\,x\,e^{-t} + b\,y\,e^{t})\,\partial/\partial x,\\
(\Delta_{a,\bar{b}})_*(\partial/\partial x) &= a\,e^{-t}\,\partial/\partial x,\\
(\Delta_{a,\bar{b}})_*(\partial/\partial y) &= b\,e^{t}\,\partial/\partial x.
\end{aligned}$$

By Corollary 6.7, $\Delta_{a,\bar{b}} \circ \mathcal{D}$ and $D_{a,\bar{b}}$ coincide up to some higher order terms on a and b, thus the tangent map of $D_{a,\bar{b}}$ is a surjection for (a,b) close the origin but non-zero. □

Lemma 6.9 *If $ab \in \mathbb{C}-\mathbb{R}$, then the tangent map of $D_{a,\bar{b}} : SOL \to \mathbb{H}^3$ is an isomorphism.*

Proof. By Lemma 6.5 we may assume that $a \in \mathbb{R}$ and $\mathrm{Im}(b) \neq 0$. A similar computation as in the proof of previous lemma works in this case:

$$\begin{aligned}
(\Delta_{a,\bar{b}})_*(\partial/\partial t) &= \partial/\partial t + (-a\,x\,e^{-t} + \mathrm{Re}(b)\,y\,e^{t})\,\partial/\partial x + \mathrm{Im}(b)\,y\,e^{t}\,\partial/\partial y,\\
(\Delta_{a,\bar{b}})_*(\partial/\partial x) &= a\,e^{-t}\,\partial/\partial x,\\
(\Delta_{a,\bar{b}})_*(\partial/\partial y) &= \mathrm{Re}(b)\,y\,e^{t}\,\partial/\partial x + \mathrm{Im}(b)\,e^{t}\,\partial/\partial y.
\end{aligned}$$

Again by Corollary 6.7, the tangent map of $D_{a,\bar{b}}$ is an isomorphism. □

7 Generalized Dehn Filling coefficients

In this section we compute the generalized Dehn filling coefficients of a structure with developing map $D_{a,\bar{b}}$ and holonomy $\rho_{a,b}$.

The Poincaré model. We work with the upper half-space model

$$\mathbb{H}^3 = \{z + e^t j \mid z \in \mathbb{C}, t \in \mathbb{R}\}, \tag{6}$$

with the warped metric $ds^2 = e^{-2t} d|z|^2 + dt^2$. This model contains the hyperbolic plane

$$\mathbb{H}^2 = \{x + e^t j \mid x \in \mathbb{R}, t \in \mathbb{R}\}.$$

In this model, the geodesic σ will be $\sigma(t) = e^t j$, with $t \in \mathbb{R}$. This model shows that the orientation preserving isometry group is

$$\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C}).$$

The hyperbolic translation of complex length $s \in \mathbb{C}$ along σ is

$$T_s = \pm \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix}.$$

The subgroup of isometries that preserve $\mathbb{H}^2$ is the subgroup generated by $PSL_2(\mathbb{R})$ and the order two element $\pm \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. This involution reverses the orientation of $\mathbb{H}^2$.

In this model $V_+ = \left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)$ and $V_- = \left(\begin{smallmatrix} 0 & 0 \\ d & 0 \end{smallmatrix}\right)$ for some non-vanishing constants $c, d \neq 0$. We will assume $c = d = 1$.

Recall that $\pi_1(N)$ has presentation

$$\Gamma \cong \langle\, \lambda, \alpha, \beta \mid \lambda\alpha\lambda^{-1} = f(\alpha), \lambda\beta\lambda^{-1} = f(\beta) \,\rangle,$$

where α and β generate the free group $\pi_1(T^2 - \{*\})$ and $f : \pi_1(T^2 - \{*\}) \to \pi_1(T^2 - \{*\})$ is an isomorphism that preserves the commutator $\mu = \alpha\beta\alpha^{-1}\beta^{-1}$. In particular, μ commutes with λ, so λ and μ generate a peripheral torus group.

Lemma 7.1 *For $(a, b) \in U$,*

$$\rho_{a,b}(\lambda) = \begin{pmatrix} e^l + o(|ab|) & 0 \\ 0 & e^{-l} + o(|ab|) \end{pmatrix}$$

and

$$\rho_{a,b}(\mu) = \begin{pmatrix} 1 + ab + o(|ab|^2) & 0 \\ 0 & 1 - ab + o(|ab|^2) \end{pmatrix}.$$

Proof. The fact that $\rho_{a,b}(\lambda)$ and $\rho_{a,b}(\mu)$ are diagonal comes from the choice $\sigma(t) = e^t j$, because $\rho_{a,b}(\lambda)$ is a translation along σ by construction of the slice, and μ commutes with λ. The formula for $\rho_{a,b}(\lambda)$ is easily deduced from the fact that $\rho_0(\lambda) = \begin{pmatrix} e^l & 0 \\ 0 & e^{-l} \end{pmatrix}$ and the coefficients are analytic functions on the trace of $\rho_{a,b}(\lambda)$ (hence analytic on ab). To compute the coefficients for $\rho_{a,b}(\mu)$, we start with the following formulas, which are proved in Corollary 9.4 in next section.

$$\rho_{a,b}(\alpha) = \begin{pmatrix} 1 & a\, d_+(\alpha) \\ b\, d_-(\alpha) & 1 \end{pmatrix} + o(|(a, b)|^2)$$

and

$$\rho_{a,b}(\beta) = \begin{pmatrix} 1 & a\, d_+(\beta) \\ b\, d_-(\beta) & 1 \end{pmatrix} + o(|(a, b)|^2).$$

Since $\mu = \alpha\beta\alpha^{-1}\beta^{-1}$, an easy computation shows that

$$\rho_{a,b}(\mu) = \begin{pmatrix} 1 + ab + o(|(a,b)|^2) & 0 \\ 0 & 1 - ab + o(|(a,b)|^2) \end{pmatrix}$$

This computation uses the normalization $d_+(\alpha)\, d_-(\beta) - d_-(\alpha)\, d_+(\beta) = 1$, as fixed in equation (3) in Section 3.

The lemma follows from the fact that the coefficients will also be functions on ab. □

Definition 7.2 *We define u and v as the complex functions on ab such that*

$$\rho_{a,b}(\mu) = \pm \begin{pmatrix} e^{u(ab)} & 0 \\ 0 & e^{-u(ab)} \end{pmatrix} \quad \textit{and} \quad \rho_{a,b}(\lambda) = \pm \begin{pmatrix} e^{v(ab)} & 0 \\ 0 & e^{-v(ab)} \end{pmatrix}$$

with $u(0) = \pm 2\pi i$ and $v(0) = l$.

The choice of $u(0)$ and $v(0)$ is necessary to determine the branch of the logarithm. The sign in $u(0) = \pm 2\pi i$ will be determined by the orientation of the developing map we are going to use: we choose the sign $\pm$ according to whether $D_{a,\bar{b}}$ preserves the orientation (when $\operatorname{Im}(ab) < 0$) or reverses the orientation (when $\operatorname{Im}(ab) > 0$).

Definition 7.3 *Following Thurston, we define the* generalized Dehn filling coefficients *as the pair $(p, q) \in \mathbb{R}^2$ such that*

$$pu + qv = 2\pi i$$

or equivalently:

$$\left.\begin{array}{rcl} p \operatorname{Re}(u) + q \operatorname{Re}(v) & = & 0 \\ p \operatorname{Im}(u) + q \operatorname{Im}(v) & = & 2\pi \end{array}\right\}$$

Proposition 7.4 *The generalized Dehn filling coefficients define a homeomorphism between a neighborhood of 0 in $\{ab \in \mathbb{C} \mid \operatorname{Im}(ab) \leq 0\}$ and a neighborhood of $(1,0)$ in $\{(p,q) \in \mathbb{R}^2 \mid p \geq 1\}$.*

Proof. Since we assume $\operatorname{Im}(ab) \leq 0$ we take $u(0) = 2\pi i$, so that when $ab = 0$, $(p,q) = (1,0)$. It follows from Lemma 7.1, that

$$u(ab) = 2\pi i + ab + o(|ab|^2) \qquad \text{and} \qquad v(ab) = l + o(|ab|).$$

In particular, $\operatorname{Re}(u) = o(|ab|)$ and $\operatorname{Im}(v) = o(|ab|)$.

Note that when $ab \in \mathbb{R}$, $\operatorname{Im}(u) = 2\pi$ and $\operatorname{Im}(v) = 0$. Thus

$$p = 1 \qquad \text{and} \qquad q = -\operatorname{Re}(u)/\operatorname{Re}(v) = -(2ab/l) + o(|ab|^2),$$

which defines a homeomorphism between a neighborhood of 0 in the line $\{ab \in \mathbb{C} \mid \operatorname{Im}(ab) = 0\}$ and a neighborhood of $(1,0)$ in the line $\{(p,q) \in \mathbb{R}^2 \mid p = 1\}$. For (a,b) in general we have:

$$p = \frac{-2\pi \operatorname{Re}(v)}{\operatorname{Re}(u)\operatorname{Im}(v) - \operatorname{Im}(u)\operatorname{Re}(v)} = \frac{-2\pi \operatorname{Re} v}{-\operatorname{Im}(u)\operatorname{Re}(v) + o(|ab|^2)}$$

Since $\operatorname{Re}(v) = l + o(|ab|)$ and $\operatorname{Im}(u) = 2\pi + 2\operatorname{Im}(ab) + o(|ab|^2)$, it follows that

$$p = \frac{2\pi}{\operatorname{Im}(u) + o(|ab|^2)} = \frac{2\pi}{2\pi + 2\operatorname{Im}(ab) + o(|ab|^2)} = 1 - \frac{1}{\pi}\operatorname{Im}(ab) + o(|ab|^2).$$

In addition

$$q = \frac{2\pi\operatorname{Re}(u)}{\operatorname{Re}(u)\operatorname{Im}(v) - \operatorname{Im}(u)\operatorname{Re}(v)} = \frac{2\pi\operatorname{Re}(u)}{-2\pi l + o(|ab|^2)} = -\frac{\operatorname{Re}(ab)}{l} + o(|ab|^2).$$

These computations prove that the Dehn filling coefficients defined in a neighborhood of 0 in the half plane $\{ab \in \mathbb{C} \mid \operatorname{Im}(ab) \geq 0\}$ are restriction of a homeomorphism defined in a neighborhood of 0 in the complex plane $\mathbb{C}$. Combining this with the fact that it maps the line $\operatorname{Im}(ab) = 0$ to the line $p = 1$, we obtain the proposition. □

Remark 7.5 *When* $\operatorname{Im}(ab) > 0$, D_{ab} *reverses the orientation and the coefficients that we obtain are the same up to sign, according to the orientation convention.*

This finishes the proof of Theorem B in the case where the monodromy matrix of N has positive trace.

Proof of Theorem B (in the general case). It remains to prove Theorem B when the monodromy matrix A of N has negative trace. Then its eigenvalues are of the form $-e^{\pm l}$, with $l \in \mathbb{R}$, $l > 0$. The two-fold cyclic covering $\widehat{N}$ of N has monodromy matrix A^2 with positive trace, so Theorem B is true for $\widehat{N}$. Let Ψ be the involution on $\widehat{N}$ such that $N = \widehat{N}/\Psi$. In the universal covering $\widetilde{N}$, Ψ lifts to the pull-back $\widetilde{\Psi}$ under the developing map $\mathcal{D} : \widetilde{N} \to SOL$ of the SOL isometry

$$(x, y, t) \to (-e^l x, -e^{-l} y, t + l)$$

We only have to show that the proof of Theorem B for $\widehat{N}$ is invariant under Ψ. This follows from Lemma 6.5 applied to the hyperbolic translation $T_{l+\pi i}$ of complex length $l + \pi i$ along σ. By Lemma 6.5,

$$D_{a,\bar{b}} \circ \widetilde{\Psi} = D_{-e^l a, \overline{-e^{-l} b}} \circ h_l = T_{l+\pi i} \circ D_{a,\bar{b}} \quad \text{for all } a, b \in \mathbb{C}$$

where h_l is the pullback under the covering map $\mathcal{D}$ of right-multiplication by $(0, 0, l)$ in SOL.

Hence the corresponding geometric structures on $\widehat{N}$ are Ψ-invariant. □

Remark 7.6 *When* $q = 0$, *Theorem B provides a family of cone structures on* M *with singular set* Σ *and cone angle* $2\pi/p$. *This family satisfies* $\operatorname{Re}(ab) = 0$.

Proof of Theorem A. Let τ denote the involution of the torus T^2 that has four fixed points. That is, if we view $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, τ lifts to

$$\begin{array}{rcl} \tilde{\tau} : \mathbb{R}^2 & \to & \mathbb{R}^2 \\ x & \mapsto & -x \end{array}$$

Note that τ is central in the mapping class group of T^2, and in particular it commutes with the monodromy $\phi : T^2 \to T^2$. Thus τ extends to an involution of M and also of N, because the fixed point of ϕ is also fixed by τ. We also denote by τ the involution on N.

The quotient N/τ is a three-orbifold with fiber a punctured sphere with three singular points that have ramification index two. The ramification set of N/τ is a link that may have one, two or three components (according to the matrix of ϕ in $SL_2(\mathbb{Z}/2\mathbb{Z})$). The orbifold M/τ is compact and has one singular component more than N/τ, which is the projection of Σ (that is fixed by τ). The key point is that the proof of Theorem B is invariant by τ. Therefore the family of cone structures on M with singular set Σ induces a family of hyperbolic cone structures on M/τ with singular set the projection of Σ. Since the range of cone angles for M is $(2\pi-\varepsilon, 2\pi)$, the range of cone angles for M/τ is $(\pi-\frac{\varepsilon}{2}, \pi)$. Thus, by Kojima's Theorem or the proof of the Orbifold Theorem, we have a family of hyperbolic cone structures on M/τ with cone angles in $(0,\pi)$, that gives a family on M with angles in $(0,2\pi)$.

To check that the proof of Theorem B is invariant by τ, it suffices to apply Lemma 6.5 to the rotation of angle π along σ, which is also the hyperbolic translation $T_{\pi i}$ of complex length πi. According to Lemma 6.5, we have

$$D_{-a,-\bar{b}} = T_{\pi i}\, D_{a,\bar{b}} \qquad \text{for every } a, b \in \mathbb{C}.$$

It follows from this that the structure is τ-invariant. □

Remark 7.7 *When we rescale the metric on $(T\sigma)^{\perp}$, the developing maps converge to the developing map of the SOL structure.*

8 An example: torus bundles which are regular coverings of S^3

Let M be an orientable torus bundle with hyperbolic monodromy which is a regular covering of S^3.

In [Sa2], Sakuma shows that an orientable torus bundle M is a regular covering of S^3 if and only if M admits an orientation-preserving involution h which leaves the base circle Σ invariant and acts on Σ by a reflection. In this case, M is in fact a $\mathbb{Z}_2+\mathbb{Z}_2$ covering of S^3, where the deck-transformation group $\mathbb{Z}_2+\mathbb{Z}_2$ is generated by any such involution h, and the involution τ extending the standard involution of T^2 with four fixed points (see the proof of theorem A), cf. [Sa2], theorem VII.

Let us first consider the quotient M/h. The involution h leaves two fibers T_1, T_2 invariant, whose quotient under h is either a Möbius strip or an annulus. Denote by Σ_i the boundary of T_i/h. Topologically, M/h is the union of two solid tori (tubular neighborhoods of T_1/h and T_2/h), i.e. a lens space. It is easy to check that the involution h is realized as a symmetry of order 2 of any complete SOL structure on M. In the universal cover, h lifts to a SOL isometry of the form

$$\widetilde{h}(x,y,t) = (e^s y, e^{-s} x, -t+s) \quad \text{or} \quad \widetilde{h}(x,y,t) = (-e^s y, -e^{-s} x, -t+s)$$

for some $s \in \mathbb{R}$. The quotient M/h is a SOL orbifold with underlying space a lens space and singularity the link $\Sigma_1 \cup \Sigma_2$, with all cone angles equal to

π. The quotient of the base circle Σ under h is a segment $\overline{\Sigma}$ joining a point of Σ_1 with a point of Σ_2.

Now let us consider the quotient $M/ < h, \tau >$. It is a SOL orbifold with underlying space S^3 and singularity a graph $\mathcal{G}$ that contains $\overline{\Sigma}$ as an edge. All cone angles are equal to π.

Remark 8.1 *The pictures of the graphs $\mathcal{G}$ that appear in this way can be found in [Du], but the distinguished edge $\overline{\Sigma}$ is not marked there.*

Proposition 8.2 *There exist a family of hyperbolic cone structures on S^3 with singular set the graph $\mathcal{G}$, cone angle α varying between 0 and π on the edge $\overline{\Sigma}$, and all other cone angles equal to π. When α tends to π, these structures collapse to a segment.*

Proof. As for Theorem A, it is enough to show that the proof of Theorem B is invariant by the involutions h and τ. Then the family of hyperbolic cone structures on M with singularity Σ and cone angle $2\alpha \in (2\pi - \varepsilon, 2\pi)$, gives rise to a family of hyperbolic cone structures on $M/ < h, \tau >= S^3$ with singularity $\mathcal{G}$, cone angle $\alpha \in (\pi - \varepsilon/2, \pi)$ on $\overline{\Sigma}$ and cone angle π on all other edges. By the proof of the Orbifold Theorem (in the case with vertices), there is a family of hyperbolic cone structures on S^3 with singularity $\mathcal{G}$, cone angle $\alpha \in (0, \pi)$ on $\overline{\Sigma}$ and all other cone angles equal to π.

We already checked that the proof of Theorem B is invariant by τ. For simplicity, let us choose the SOL structure on M so that h lifts in the universal cover to one of the SOL isometries $\widetilde{h}(x, y, t) = (y, x, -t)$ or $\widetilde{h}(x, y, t) = (-y, -x, -t)$. Let us moreover assume that $\widetilde{h}(x, y, t) = (y, x, -t)$. Denote by $r_L : (T\sigma)^{\perp} \to (T\sigma)^{\perp}$ the isometry induced by the rotation of $180°$ around a line L perpendicular to σ at $\sigma(0)$. We want to prove that for every $t \in \mathbb{R} - \{0\}$ close to 0, we can find a pair $(a, b) \in \mathbb{C}^2$ close to 0, such that $ab = it$ and

$$D_{a,\bar{b}} \circ \widetilde{h} = r_L \circ D_{a,\bar{b}}.$$

Let us choose the Killing vector fields $V_{\pm}$ so that $V_+(\sigma(0)) = V_-(\sigma(0))$ is a vector tangent to the line L. Then

$$\begin{aligned} r_L \circ \Delta_{a,\bar{b}}(x, y, t) &= r_L(axV_+(\sigma(t)) + \bar{b}yV_-(\sigma(t))) \\ &= \bar{a}xV_-(\sigma(-t)) + byV_+(\sigma(-t)) \\ &= \Delta_{b,\bar{a}}(y, x, -t) \\ &= \Delta_{b,\bar{a}} \circ \widetilde{h}(x, y, t) \end{aligned}$$

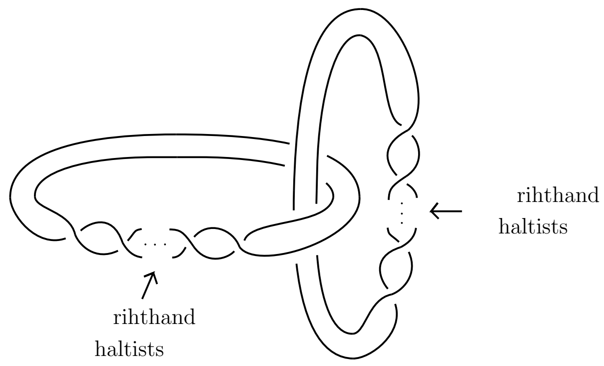
for all $(x, y, t) \in SOL$. A similar computation as in Lemma 5.5 shows that $r_L \circ \rho_{a,b} \circ r_L^{-1} = \rho_{bs,a/s}$, where $s = d_-(\alpha)/d_+(\alpha)$. Equivalently, $r_L \circ \rho_{a,b}(\gamma) \circ r_L^{-1} = \rho_{b,a}(\widetilde{h}_*\gamma)$ for all $\gamma \in \pi_1(N)$. Hence $D_{a,\bar{b}} \circ \widetilde{h} = r_L \circ D_{b,\bar{a}}$. If we choose $a = b = \sqrt{it}$, then $D_{a,\bar{b}} \circ \widetilde{h} = r_L \circ D_{a,\bar{b}}$, hence the corresponding hyperbolic cone structure on M is h-invariant. □

Remark 8.3 *As a corollary, the lens space M/h also has hyperbolic cone structures with singularity the graph $\Sigma_1 \cup \Sigma_2 \cup \overline{\Sigma}$, cone angle $\alpha \in (0, 2\pi)$ on the edge $\overline{\Sigma}$ and all other cone angles equal to π. When $\alpha \to 2\pi$, these structures collapse to a segment.*

As a particular example, let us consider the torus bundles which are two-fold branched cyclic covers of S^3, i.e. those for which the lens space M/h is already S^3. Sakuma shows in [Sa1] that these are exactly the torus bundles $M_{m,n}$ with monodromy matrix of the form

$$A = \begin{pmatrix} -1 & -m \\ n & mn-1 \end{pmatrix}$$

for some pair of integers m, n. The branching locus is the following link in S^3:



The torus bundle $M_{m,n}$ has hyperbolic monodromy when $|mn-2| > 2$. Then proposition 8.2 tells that S^3 has hyperbolic cone structures with singularity the graphs shown in figures 1 and 2, cone angle $\alpha \in (0, 2\pi)$ on the marked edge $\overline{\Sigma}$ in figure 1 (resp. $\alpha \in (0, \pi)$ in figure 2), and all other cone angles equal to π. When $\alpha \to 2\pi$ (resp. $\alpha \to \pi$) these structures collapse to the segment $\overline{\Sigma}$; all edges of the singular graph collapse to points, except the edges drawn thick, which approximate to each other until they all match up in the limit.

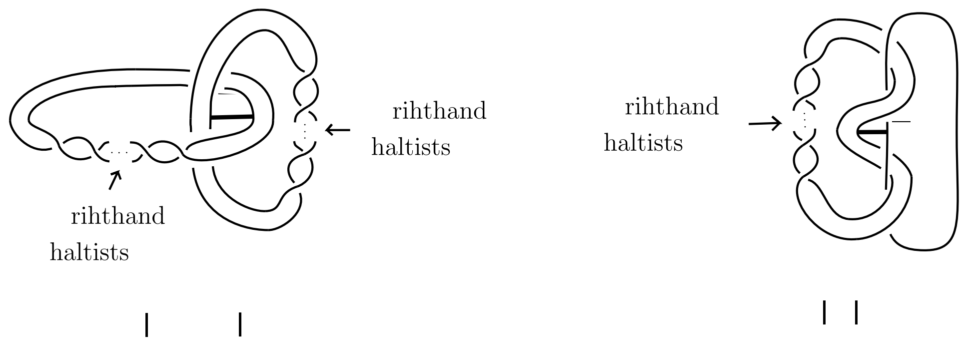


Figure 1: cone angle $\alpha \in (0, 2\pi)$ on the edge $\overline{\Sigma}$

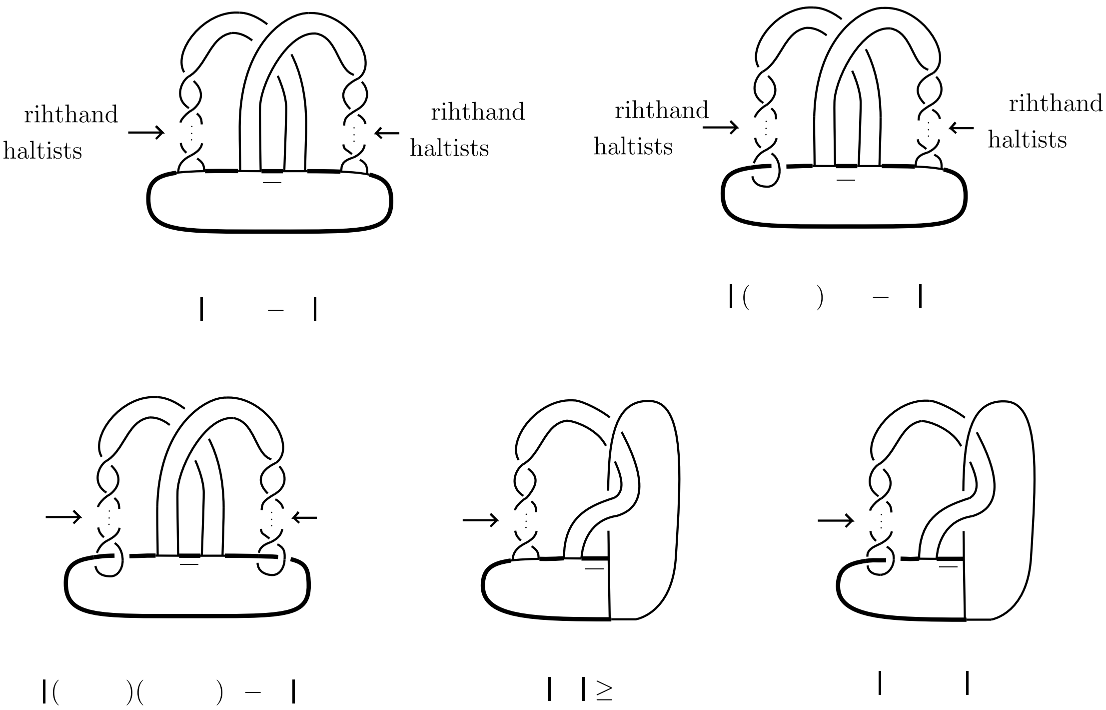


Figure 2: cone angle $\alpha \in (0, \pi)$ on the edge $\overline{\Sigma}$

9 The tangent space to the variety of representations

In this section we prove some results stated in Sections 5 and 6 that involve the tangent space of the space of representations and local parametrizations .

We start by computing $Z^1(\Gamma, sl_2(\mathbb{C})^{\rho_0})$. Since the image of ρ_0 is contained in the translation group along σ, the action of ρ_0 preserves the splitting $sl_2(\mathbb{C}) = \mathfrak{h}_+ \oplus \mathfrak{h}_- \oplus \mathfrak{h}_0$. Thus:

$$Z^1(\Gamma, sl_2(\mathbb{C})^{\rho_0}) = Z^1(\Gamma, (\mathfrak{h}_+)^{\rho_0}) \oplus Z^1(\Gamma, (\mathfrak{h}_-)^{\rho_0}) \oplus Z^1(\Gamma, (\mathfrak{h}_0)^{\rho_0}).$$

Next we study each one of these spaces. Let $V_+ \in \mathfrak{h}_+$, $V_- \in \mathfrak{h}_-$ and $V_0 \in \mathfrak{h}_0$ be the elements defined in Section 5. Consider also the cocycles $d_\pm : \Gamma \to \mathbb{R}_\pm$ defined in section 3 and the morphism $d_0 : \Gamma \to \mathbb{Z} \subset \mathbb{R}$ such that $d_0(\lambda) = 1$ and $d_0(\alpha) = d_0(\beta) = 0$. In the following lemma $d_0 V_+$ denotes the cocycle that maps $\gamma \in \Gamma$ to $d_0(\gamma) V_+ \in \mathfrak{h}_+$, and so on for other cocycles.

Lemma 9.1 *(i)* $Z^1(\Gamma, (\mathfrak{h}_+)^{\rho_0}) \cong \mathbb{C}^2$ *is generated by* $d_0 V_+$ *and* $d_+ V_+$.

(ii) $Z^1(\Gamma, (\mathfrak{h}_-)^{\rho_0}) \cong \mathbb{C}^2$ *is generated by* $d_0 V_-$ *and* $d_- V_-$.

(iii) $B^1(\Gamma, (\mathfrak{h}_\pm)^{\rho_0}) \cong \mathbb{C}$ *is generated by* $d_0 V_\pm$.

(iv) $Z^1(\Gamma, (\mathfrak{h}_0)^{\rho_0}) \cong \mathbb{C}$ *is generated by* $d_0 V_0$.

(v) $B^1(\Gamma, (\mathfrak{h}_\pm)^{\rho_0}) \cong 0$. □

Proof of Proposition 5.1 (b). We prove first that $T_{\rho_0} R^{ir}(\Gamma, SL_2(\mathbb{C})) = Z^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0})$. We know from [HPS] that $T_{\rho_0} R^{ir}(\Gamma, SL_2(\mathbb{C}))$ is four dimensional, hence it suffices to show that the cocycles $d_0 V_\pm$ and $d_\pm V_\pm$ belong to this space. It is clear that $d_0 V_\pm \in T_{\rho_0} R^{ir}(\Gamma, SL_2(\mathbb{C}))$, because they are coboundaries. In addition $d_\pm V_\pm$ is tangent to a path of metabelian representations, that belong to $R^{ir}(\Gamma, SL_2(\mathbb{C}))$, by [HPS]. Thus the equality holds. A similar argument shows that $T_{\rho_0} R^{ab}(\Gamma, SL_2(\mathbb{C})) = Z^1(\Gamma, \mathfrak{h}_0^{\rho_0}) \oplus B^1(\Gamma, (\mathfrak{h}_+ \oplus \mathfrak{h}_-)^{\rho_0})$, because $d_0 V_0$ is tangent to a path of abelian representations.

We defined the slice as:

$$\mathcal{S} = \{\rho \in R^{ir}(\Gamma, SL_2(\mathbb{C})) \mid \rho(\lambda) \text{ is a hyperbolic translation along } \sigma\}$$

Lemma 9.2 *$T_{\rho_0}\mathcal{S}$ is two dimensional generated by d_+V_+ and d_-V_-.*

Proof. It follows from Lemma 9.1 and Proposition 5.1 (b), that d_+V_+, d_-V_-, d_0V_+ and d_0V_- generate $T_{\rho_0} R^{ir}(\Gamma, SL_2(\mathbb{C}))$. Moreover, $d_0(\lambda) \neq 0$ and $d_\pm(\lambda) = 0$. We consider the map

$$\begin{array}{rcl} F : R^{ir}(\Gamma, SL_2(\mathbb{C})) & \to & \hat{\mathbb{C}} \times \hat{\mathbb{C}} \\ \rho & \mapsto & (\rho(\lambda)(\sigma(+\infty)), \rho(\lambda)(\sigma(-\infty))) \end{array}$$

where $\hat{\mathbb{C}} = \partial \mathbb{H}^3$ is the Riemann sphere at the boundary of $\mathbb{H}^3$, and $\sigma(+\infty), \sigma(-\infty) \in \hat{\mathbb{C}}$ are the ends of σ. Note that

$$\mathcal{S} = F^{-1}(\sigma(+\infty), \sigma(-\infty))$$

because hyperbolic translations along σ (with complex length) are the unique orientaion preserving isometries that preserve both ends $\sigma(+\infty)$ and $\sigma(-\infty)$. The tangent map F_* at the point ρ_0 satisfies $F_*(d_0V_+) = (0,1)$, $F_*(d_0V_-) = (1,0)$ and $F_*(d_\pm V_\pm) = 0$, because $d_0(\lambda) = 1$ and $d_\pm(\lambda) = 0$. The lemma follows straighforward from these computations. $\square$

We obtain as a corollary Proposition 5.3.

Corollary 9.3 *(Proposition 5.3) There is a neighborhood $\mathcal{U} \subset \mathcal{S}$ of ρ_0 so that the map*

$$\begin{array}{rcl} \mathcal{U} & \to & \mathfrak{h}_+ \oplus \mathfrak{h}_- \\ \rho & \to & \mathrm{pr}(\exp^{-1}(\rho(\alpha)\rho_0(\alpha^{-1}))) \end{array}$$

is a biholomorphism between $\mathcal{U}$ and a neighborhood of the origin in $\mathfrak{h}_+ \oplus \mathfrak{h}_-$.

Proof. This follows now from Lemma 9.2 and the fact that $d_+(\alpha)$ and $d_-(\alpha)$ do not vanish. $\square$

Corollary 9.4 *For every $\gamma \in \Gamma$ there is a neighborhood of the origin $U_\gamma \subset U$ such that for every $(a,b) \in U$, we have:*

$$\rho_{a,b}(\gamma) = \exp\left\{a\, d_+(\gamma)V_+ + b\, d_+(\gamma)V_- \ + o(|(a,b)|^2)\right\} \rho_0(\gamma)$$

Note that the open set U_γ depends on γ, but we only need to apply it to finitely many elements.

Proof. Use Lemma 9.2. $\square$

Proof of Lemma 6.6 We write $(x,y,t)\in K$ and $\mathrm{hol}(\gamma)=(d_+(\gamma),d_-(\gamma),s)$, where $s=\mathrm{LIN}\,\mathrm{hol}(\gamma)$. By Lemma 4.9:

$$\begin{aligned}\Delta_{a,\bar b}((d_+(\gamma),d_-(\gamma),s)(x,y,t)) &= \Delta_{a,\bar b}(d_+(\gamma),d_-(\gamma),t+s)+T_s\Delta_{a,\bar b}(x,y,t)\\ &= \Delta_{a,\bar b}(d_+(\gamma),d_-(\gamma),t+s)+\rho_0(\gamma)\Delta_{a,\bar b}(x,y,t)\end{aligned}$$

because $T_s=\rho_0(\gamma)$. Next we compute the first order expansion of $\rho_{a,b}(\Delta_{a,\bar b}(x,y,t))$. By Corollary 9.4:

$$\rho_{a,b}(\gamma)=\exp\left\{a\,d_+(\gamma)V_+ + b\,d_-(\gamma)V_- + o(|(a,b)|^2)\right\}\,\rho_0(\gamma).$$

Thus:

$$\begin{aligned}\rho_{a,b}(\gamma)(p) &= \exp\left\{a\,d_+(\gamma)V_+ + b\,d_-(\gamma)V_- + o(|a|^2+|b|^2)\right\}\,\rho_0(\gamma)(p)\\ &= \rho_0(\gamma)(p)+a\,d_+(\gamma)V_+(\rho_0(\gamma)(p))+b\,d_-(\gamma)V_-(\rho_0(\gamma)(p))+o(|(a,b)|^2).\end{aligned}$$

We want to apply this equation to the point $p=\Delta_{a,\bar b}(x,y,t)$. Since $\Delta_{a,\bar b}(x,y,t)=\sigma(t)+o(|a|+|b|)$, we have:

$$\rho_0(\gamma)(\Delta_{a,\bar b}(x,y,t))=\sigma(s+t)+o(|(a,b)|).$$

Thus:

$$V_\pm((\rho_0(\gamma)\Delta_{a,\bar b}(x,y,t))=V_\pm(\sigma(s+t))+o(|(a,b)|).$$

Summarizing, we have:

$$\begin{aligned}\rho_{a,b}(\gamma)(\Delta_{a,\bar b}(x,y,t)) &= \rho_0(\gamma)((\Delta_{a,\bar b}(x,y,t)))+a\,d_+(\gamma)\ \{V_+(\sigma(s+t))+o(|(a,b)|)\}+\\ &\qquad + b\,d_-(\gamma)\ \{V_-(\sigma(s+t))+o(|(a,b)|)\}\\ &= \rho_0(\gamma)((\Delta_{a,\bar b}(x,y,t)))+\Delta_{a,\bar b}(d_+(\gamma),d_-(\gamma),s+t)+o(|(a,b)|^2).\end{aligned}$$

□

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