An orientation for the SU(2)–representation space of knot groups

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1 Introduction

In 1985 Casson constructed a new integer valued invariant for homology 3–spheres (see [AM90, GM92]). His construction is based on properties of SU(2)–representation spaces. A surprising and important corollary is that a knot $k \subset S^3$ has Property P if $\Delta_k''(1) \neq 0$ where $\Delta_k(t)$ is the normalized Alexander polynomial of k $(\Delta_k(t^{-1}) = \Delta_k(t)$ and $\Delta_k(1) = 1)$. Here a non trivial knot in the 3–sphere has *Property P* if no non trivial Dehn surgery on the knot yields a homotopy sphere.

The aim of this paper is to study the SU(2)–representation spaces of knot groups. For a given knot $k \subset S^3$ we denote by $\widehat{R}(k)$ the space of equivalence classes of irreducible representations of the knot group $G := \pi_1(S^3 \setminus k)$ in $\mathrm{SU}(2)$. We denote by $\mathrm{Reg}(k) \subset \widehat{R}(k)$ the space of regular representations. Here an irreducible representation $\rho\colon G \to \mathrm{SU}(2)$ is called *regular* if $H^1_\rho(G) \cong \mathbb{R}$ where $H^*_\rho(G) := H^*(G, \mathrm{Ad} \circ \rho)$ denotes the twisted cohomology group of G with coefficients in $\mathfrak{su}(2)$. It follows from [HKl97, Proposition 1] that $\mathrm{Reg}(k) \subset \widehat{R}(k)$ is a real one dimensional manifold. The main result of this paper is to prove that $\mathrm{Reg}(k)$ also carries an orientation:

Theorem 1.1 *Let $k \subset S^3$ be a knot. Then the space $\mathrm{Reg}(k) \subset \widehat{R}(k)$ is a canonically oriented one dimensional manifold. Moreover, we have $\mathrm{Reg}(k^*) = -\mathrm{Reg}(k)$.*

Here k^* denotes the mirror image of k and $-\mathrm{Reg}(k)$ denotes $\mathrm{Reg}(k)$ with the opposite orientation. The construction which enables us to orient the space $\mathrm{Reg}(k)$ is motivated by the definition of Casson's invariant (see Section 3).

Even if $\Delta_k''(1) = 0$ the knot $k \subset S^3$ might still have Property P and in this case the SU(2)–representations might still be useful for proving Property P (see [Bur90, FL92]). A first step in the program of generalizing Burde's proof of Property P for 2–bridge knots (see [Bur90]) is to find knots with a non-trivial SU(2)–representation space. As a corollary of the discussion in Section 5 we obtain:

Corollary 1.2 *Let $k \subset S^3$ be a knot and let $\sigma_k(\omega)$, $\omega \in \mathbb{C}$, be its equivariant signature. If there is an $\alpha \in [0,\pi]$ such that $\Delta_k(e^{2\boldsymbol{i}\alpha}), \sigma_k(e^{2\boldsymbol{i}\alpha}) \neq 0$ then there exists an irreducible* SU(2) *representation $\rho\colon G \to \mathrm{SU}(2)$ and $\dim(\widehat{R}(k)) \geq 1$ in a neighborhood of ρ.*

It is now possible to show that a large class of knots have at least a one dimensional representation space $\widehat{R}(k)$ by using Corollary 1.2 and the results proved in [FL92, HKl97, HKr98]. This also gives some evidence to support the conjecture that every three dimensional manifold with non-trivial fundamental group admits a non-trivial representation into SU(2) (see [Kir93, Probleme 3.105 (A)]).

As a further application we are able to explain a generalization of a result of X.-S. Lin: let G be a knot group and let $m \in G$ be a meridian. A representation $\rho\colon G \to \mathrm{SU}(2)$ is called trace-free if $\operatorname{tr}\rho(m) = 0$. In [Lin92] Lin defined an intersection number for the representation space corresponding to a braid representative of the knot. This number turns out to be a knot invariant denoted by $h(k)$. Roughly speaking, $h(k)$ is the number of conjugacy classes of non-abelian trace-free representations $G \to \mathrm{SU}(2)$ counted with sign. Moreover, Lin established the relation $2h(k) = \sigma(k)$ where $\sigma(k)$ denotes the signature of k. It was suggested by D. Ruberman that the construction could be generalized to representations of knot groups with the trace of the meridians fixed. In [HKr98] we carried out this generalization. More precisely, for a given $\alpha \in (0,\pi)$, there is an integer invariant $h^{(\alpha)}(k)$. This invariant counts the conjugacy classes of non-abelian representations $G \to \mathrm{SU}(2)$, such that $\operatorname{tr}\rho(m) = 2\cos\alpha$ (note that $h(k) = h^{(\pi/2)}(k)$). Moreover the relation $2h^{(\alpha)}(k) = \sigma_k(e^{2\boldsymbol{i}\alpha})$ holds (see [HKr98, Theorem 1.2]), $\sigma_k\colon S^1 \to \mathbb{Z}$ denotes the signature function (note that $\sigma_k(-1) = \sigma(k)$ and see [HKr98, 2.1] for the details).

At first sight it seems mysterious that these two quantities $h^{(\alpha)}(k)$ and $\sigma_k(e^{2\boldsymbol{i}\alpha})$ with apparently different algebraic–geometric contents turn out to be the same. We shall explain this connection in Section 5 using the orientation on the representation space.

This paper is organized as follows. In Section 2 the basic notation and facts are presented. In Section 3 we will describe the main construction and the results. Section 4 contains the proof of Theorem 1.1 and in Section 5 we explain the connection to Lin's result.

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2 Notation and facts

Throughout this paper it will often prove convenient to work with quaternions (we denote this field by $\mathbb{H}$). Therefore, we identify SU(2) with the unit quaternions $\mathrm{Sp}(1) \subset \mathbb{H}$. These two groups are isomorphic via the map

given by
$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto a + b\boldsymbol{j}.$$

The Lie algebra of $\mathrm{Sp}(1)$ is the set $\mathbb{E}$ of pure quaternions and $\mathrm{Sp}(1)$ acts via Ad on $\mathbb{E}$ i.e. $\mathrm{Ad}_q X = qXq^{-1}$ for $q \in \mathrm{Sp}(1)$ and $X \in \mathbb{E}$. We denote by $\delta\colon \mathrm{SU}(2) \to \mathrm{SO}(\mathbb{E}) = \mathrm{SO}(3)$ the 2-fold covering given by $\delta(q) = \mathrm{Ad}_q$. We consider the argument function $\arg\colon \mathrm{SU}(2) \to [0,\pi]$ given by $\arg(A) = \arccos(\mathrm{tr}(A)/2)$. For $\alpha \in (0,\pi)$ we have $\Sigma_\alpha := \arg^{-1}(\alpha)$ is a 2-sphere and $\Sigma_{\pi/2} = \mathbb{E} \cap \mathrm{Sp}(1)$ is the set of pure unit quaternions. From now on we denote by I the open interval $(0,\pi)$.

Given two elements $X, Y \in \mathbb{E}$ there is a product formula: $X \cdot Y = -\langle X, Y\rangle + X \times Y$ where $\langle X, Y\rangle$ denotes the scalar product of X and Y and $X \times Y$ their vector product in $\mathbb{E}$. Note that Ad_q preserves the scalar product. For a given element $X \in \mathbb{E}$ we denote by $X^\perp$ the orthogonal complement of X in $\mathbb{E}$.

For each quaternion $q \in \mathrm{Sp}(1)$ there is an angle α, $0 \le \alpha \le \pi$, and $Q \in \Sigma_{\pi/2}$ such that $q = \cos\alpha + \sin\alpha\, Q$. The pair (α, Q) is unique if and only if $q \neq \pm\mathbf{1}$. Note that $\delta(\alpha, Q)$ is a rotation of angle 2α with fix axis Q. Let G be a group and fix a representation $\rho\colon G \to \mathrm{SU}(2)$. Then for each $g \in G$ such that $\rho(g) \neq \pm\mathbf{1}$ there is an unique $\alpha(g) := \alpha(g,\rho)$, $0 < \alpha(g) < \pi$, and $P_g := P_g(\rho) \in \Sigma_{\pi/2}$ such that $\rho(g) = \cos\alpha(g) + \sin\alpha(g)\, P_g$.

2.1 Representation spaces

Let G be a finitely generated group. The space of all representations of G in $\mathrm{SU}(2)$ is denoted by $R(G) := \mathrm{Hom}(G, \mathrm{SU}(2))$. Note that $R(G)$ is a topological space via the compact open topology where G carries the discrete and $\mathrm{SU}(2)$ the usual topology. A representation $\rho \in R(G)$ is called abelian (resp. central), (resp. trivial) if and only if its image is an abelian (resp. central), (resp. trivial) subgroup of $\mathrm{SU}(2)$. Note that $\rho \in R(G)$ is abelian if and only if it is reducible. The set of abelian representations is denoted by $S(G)$ and the set of central representations by $C(G)$. Two representations $\rho, \rho' \in R(G)$ are said to be conjugate ($\rho \sim \rho'$) if and only if they differ by an inner automorphism of $\mathrm{SU}(2)$. The group $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm\mathbf{1}\}$ acts free on the right on $R(G)$ via conjugation. Two representations are in the same $\mathrm{SO}(3)$–orbit if and only if they are equivalent. Let $\widetilde{R}(G) := R(G) \smallsetminus S(G)$ be the set of non-abelian representations. The space of (non-abelian) conjugacy classes of representations from G into $\mathrm{SU}(2)$ is denoted by $\mathfrak{R}(G)$ ($\widehat{R}(G)$) i.e.
$$\mathfrak{R}(G) := R(G)/\mathrm{SO}(3) \quad \text{and} \quad \widehat{R}(G) := \widetilde{R}(G)/\mathrm{SO}(3).$$

We can think of the map $\widetilde{R}(G) \to \widehat{R}(G)$ as a principal $\mathrm{SO}(3)$–bundle (see [GM92, 3.A] for details).

We present some facts about the algebraic structure of representation spaces which will be used in the sequel: the space $R(G)$ has the structure of a real affine algebraic set i.e. the space $R(G)$ is a subset of $\mathbb{R}^n$ which is defined by polynomial equations (see [AM90]). We can also think of $\Re(G)$ as a subspace of $\mathbb{R}^m$ (see [Kla91]). The map $t\colon R(G) \to \Re(G)$ is a polynomial map. It follows from the Tarski–Seidenberg principle that the image of an algebraic set under a polynomial map is a semi-algebraic set. Here a subset of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of finite intersections of sets defined by a polynomial equation or inequality (see [BCR87] for details). Hence the spaces $\widehat{R}(G)$ and $\Re(G)$ are semi-algebraic sets.

Given a representation $\rho\colon G \to \mathrm{SU}(2)$ the Lie algebra $\mathfrak{su}(2)$ can be viewed as a G–module via $\mathrm{Ad}\circ\rho$ i.e. $g \circ X := \mathrm{Ad}_{\rho(g)}(X)$. We denote by $Z^1_\rho(G)$ (resp. $B^1_\rho(G)$), (resp. $H^1_\rho(G)$) the cocycles (resp. coboundaries), (resp. first cohomology group) of G with coefficients in $\mathfrak{su}(2)$.

Following A. Weil (see [Wei64]) there is an inclusion of the Zariski tangent space $T_\rho(R(G))$ into $Z^1_\rho(G)$ (for details see [Por97]). A cocycle $u \in Z^1_\rho(G)$ is called *integrable* if and only if there exists an analytic path $\rho_t\colon G \to \mathrm{SU}(2)$ such that $\rho_0 = \rho$ and

$$u(g) = \frac{d\rho_t(g)}{dt}\Big|_{t=0} \cdot(\rho(g))^{-1} \quad \text{for all } g \in G.$$

In general it is not true that every element of $Z^1_\rho(G)$ is integrable. However, if the dimension of $R(G)$ at ρ is equal to the dimension of $Z^1_\rho(G)$ then every cocycle is integrable.

The following lemma will be used in the sequel:

Lemma 2.1 *Let G be a group and let $g \in G$. Moreover, let $\rho \in R(G)$ be a representation such that $\rho(g) \neq \pm\mathbf{1}$. If g and $g' \in G$ are conjugate then $\langle u(g), P_g(\rho)\rangle = \langle u(g'), P_{g'}(\rho)\rangle$ for each $u \in Z^1_\rho(G)$. Especially $\langle b(g), P_g(\rho)\rangle = 0$ if $b \in B^1_\rho(G)$.*

Proof. There is a $h \in G$ such that $g' = hgh^{-1}$. We obtain:

$$u(g') = (1 - hgh^{-1}) \circ u(h) + h \circ u(g) \text{ and } P_{g'} = h \circ P_g$$

where $P_g := P_g(\rho)$ and $P_{g'} := P_{g'}(\rho)$. Therefore:

$$\langle u(g'), P_{g'}\rangle = \langle h \circ u(g), h \circ P_g\rangle + \langle (1 - hgh^{-1}) \circ u(h), h \circ P_g\rangle.$$

We obtain: $\langle hgh^{-1} \circ u(h), h \circ P_g\rangle = \langle h^{-1} \circ u(h), P_g\rangle$ from which the first conclusion follows.

If $b \in B^1_\rho(G)$ then there is a $X_0 \in \mathfrak{su}(2)$ such that $b(g) = (1 - g) \circ X_0$ for all $g \in G$. It follows that $\langle b(g), P_g\rangle = \langle X_0, P_g\rangle - \langle g \circ X_0, P_g\rangle = 0$. □

3 The construction

Let M be an oriented homology 3–sphere. The construction of the Casson invariant is based on the fact that a Heegaard splitting $M = H_1 \cup_F H_2$ of M gives rise to embeddings $\widehat{R}(H_i) \hookrightarrow \widehat{R}(F)$ and $\widehat{R}(M) \hookrightarrow \widehat{R}(H_i)$. Here H_i is a handlebody and $F = H_1 \cap H_2$ is a surface of genus g and $\widehat{R}(Y) :=$

$\widehat{R}(\pi_1(Y))$ for any pathwise connected topological space Y. In particular $\widehat{R}(M) = \widehat{R}(H_1) \cap \widehat{R}(H_2)$. The crucial point is that the spaces $\widehat{R}(H_i)$ and $\widehat{R}(F)$ carry a canonical orientation. The Casson invariant $\lambda(M)$ is roughly the "algebraic intersection number" of $\widehat{R}(H_1)$ and $\widehat{R}(H_2)$ in $\widehat{R}(F)$. The two technical difficulties are to make sense of the algebraic intersection number of these proper open submanifolds and to show that it is independent of the Heegaard splitting of M (for this and other details see [AM90, GM92]).

In our construction the Heegaard splitting will be replaced by a plat decomposition of the knot exterior. The main point is to use not only the representation spaces of groups but to consider pairs $(G, \mathcal{S})$ where G is a finitely generated group and $\mathcal{S}$ is a fixed finite set of generators.

Let $k \subset S^3$ be a knot and denote by $X(k) := S^3 \setminus U(k)$ its exterior where $U(k)$ denotes an open regular neighborhood of k. The space $X(k)$ is a three dimensional oriented manifold with torus boundary. We denote by $G := G(k) := \pi_1(X(k))$ the knot group.

Each unoriented knot $k \subset S^3$ can be represented as a $2n$–plat $\widehat{\beta}$. Here $\widehat{\beta}$ is obtained from a $2n$–braid $\beta \in B_{2n}$ by closing it with $2n$ simple arcs (see Figure 1). A $2n$–plat representation $\widehat{\beta}$ of k gives rise to a splitting

$$X(k) = B_1 \cup_{S(2n)} B_2$$

of $X(k)$ where B_i, $i = 1, 2$, is a handlebody of genus n and $S(2n) = B_1 \cap B_2$ is a planar surface with $2n$ boundary components (see Figure 1). We call such a splitting a $2n$–plat decomposition of $X(k)$.

The inclusions $S(2n) \hookrightarrow B_i$ and $B_i \hookrightarrow X(k)$, $i = 1, 2$, give rise to a commutative diagram of epimorphisms

$$\begin{tikzcd} & \pi_1(B_1) \arrow[dr, "p_1"] & \\ \pi_1(S(2n)) \arrow[ur, "\kappa_1"] \arrow[dr, "\kappa_2"'] & & \pi_1(X(k)) \\ & \pi_1(B_2) \arrow[ur, "p_2"'] & \end{tikzcd} \quad . \tag{1}$$

Let $\mathcal{T}_i := \{t_1^{(i)}, \ldots, t_n^{(i)}\}$, $i = 1, 2$, be the special system of generators for $\pi_1(B_i)$ (see Figure 1). Moreover, choose a system $\mathcal{S} := \{s_1, \ldots, s_{2n}\}$ of generators for $\pi_1(S(2n))$ as in Figure 1. The choice of the generators depends in fact from the orientation of S^3 (see Section 4.2 for the details). Each of the generators chosen above is a meridian of $\widehat{\beta}$ and there is a relation $s_1 \cdots s_{2n} = 1$ in $\pi_1(S(2n))$.

Let G be a group and let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a finite system of generators for G. We define the subspace $R^{\mathcal{S}}(G) \subset R(G)$ by

$$R^{\mathcal{S}}(G) := \{\rho \in R(G) \mid \operatorname{tr}\rho(s_i) = \operatorname{tr}\rho(s_j),\ 1 \leq i < j \leq n\} \setminus C(G).$$

For a given $\alpha \in I := (0, \pi)$ we define

$$R^{\mathcal{S}}_\alpha(G) := \{\rho \in R(G)^{\mathcal{S}} \mid \operatorname{tr}\rho(s_i) = 2\cos\alpha,\ 1 \leq i \leq n\}.$$

PSfrag replacements

B_1
B_2
$S(4)$
s_1
s_2
s_3
s_4
$t_1^{(1)}$
$t_2^{(1)}$
$t_1^{(2)}$
$t_2^{(2)}$

Figure 1: plat representation

These spaces depend on the choice of a system of generators. However, they are preserved under the SO(3) action and we are able to define the quotients

$$\widehat{R}^{\mathcal{S}}(G) := (R^{\mathcal{S}}(G) \smallsetminus S(G))/\mathrm{SO}(3) \quad \text{and} \quad \widehat{R}^{\mathcal{S}}_{\alpha}(G) := (R^{\mathcal{S}}(G) \smallsetminus S(G))/\mathrm{SO}(3).$$

Let $G := G(k)$ be a knot group and let $\mathcal{S} = \{s_1, \dots, s_n\}$ be a finite system of generators such that each s_i is a meridian of k. The elements of $\mathcal{S}$ are pairwise conjugate in G and therefore we have $R^{\mathcal{S}}(G) = R(G) \smallsetminus C(G)$.

Let $\phi\colon\ G \to H$ be a homomorphism and let $\mathcal{S}$ (resp. $\mathcal{T}$) be a finite system of generators of G (resp. H). The homomorphism ϕ is called *compatible* with $\mathcal{S}$ and $\mathcal{T}$ if and only if $\phi(s_i)$ is conjugate to an element of $\mathcal{T} \cup \mathcal{T}^{-1}$ for all $s_i \in \mathcal{S}$. It is easy to see that $\phi\colon\ G \to H$ induces a transformation $\hat{\varphi}\colon\ \widehat{R}^{\mathcal{T}}(H) \to \widehat{R}^{\mathcal{S}}(G)$ if it is compatible with $\mathcal{S}$ and $\mathcal{T}$.

It is easy to see that all the epimorphisms in Diagram (1) are compatible with the systems of generators chosen above. For this reason we are interested in the representation spaces $R^{\mathcal{T}_i}(B_i)$ and $R^{\mathcal{S}}(S(2n))$. From (1) we obtain the following diagram of embeddings :

$$\begin{tikzcd}
 & \widehat{R}^{\mathcal{T}_1}(B_1) \arrow[dl, "\widehat{\kappa}_1"'] & \\
\widehat{R}^{\mathcal{S}}(S(2n)) & & \widehat{R}(k) \arrow[ul, "\widehat{p}_1"'] \arrow[dl, "\widehat{p}_2"] \\
 & \widehat{R}^{\mathcal{T}_2}(B_2) \arrow[ul, "\widehat{\kappa}_2"] &
\end{tikzcd} \tag{2}$$

we have: $\widehat{R}(k) = \widehat{Q}_1 \cap \widehat{Q}_2$, where $\widehat{Q}_i := \widehat{\kappa}_i(\widehat{R}^{\mathcal{T}_i}(B_i))$. Diagram (2) is the main tool in the process of defining an orientation on $\widehat{R}(k)$ (in a generic situation).

The next step is to prove that the space $\widehat{R}^{\mathcal{S}}(S(2n))$ is a $(4n-5)$ dimensional manifold. Let F_n be a free group of rank n and let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a basis of F_n. We identify the representation space $R(F_n)$ with $\mathrm{SU}(2)^n$

$$R(F_n) \xrightarrow{\cong} \mathrm{SU}(2)^n, \quad \rho \mapsto (\rho(s_1), \ldots, \rho(s_i)).$$

It is easy to see that $R^{\mathcal{S}}(F_n) \subset R(F_n)$ can be identified with $I \times \Sigma^n_{\pi/2}$. The inclusion $\Phi_n \colon I \times \Sigma^n_{\pi/2} \to \mathrm{SU}(2)^n$ is given by $\Phi_n \colon (\alpha, P_1, \ldots, P_n) \mapsto (\cos\alpha + \sin\alpha\, P_i)^n_{i=1}$. Here and in the sequel $(x_i)^n_{i=1}$ is short for $(x_1, \ldots, x_n)$. The identification $R(F_n) \cong \mathrm{SU}(2)^n$ gives us an isomorphism $T_\rho(R(F_n)) \cong \mathfrak{su}(2)^n$. The latter is induced by the canonical identification $T_A(\mathrm{SU}(2)) \cong \mathfrak{su}(2)$ given by $(A, X) \mapsto XA^{-1}$. Every cocycle $u \in Z^1_\rho(F_n)$ is integrable, since $Z^1_\rho(F_n) \cong \mathfrak{su}(2)^n$.

Let G be a group and let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a finite system of generators for G. Moreover, let $\rho \in R^{\mathcal{S}}_\alpha(G)$ be given and let $P_i := P_{s_i}(\rho) \in \Sigma_{\pi/2}$, i.e. $\rho(s_i) = \cos\alpha + \sin\alpha\, P_i$. We obtain inclusions

$$T_\rho(\widehat{R}^{\mathcal{S}}(G)) \subset Z^1_{\rho,\mathcal{S}}(G) := \{u \in Z^1_\rho(G) \mid \langle u(s_i), P_i\rangle = \langle u(s_j), P_j\rangle,\ 1 \le i, j \le n\}$$

and

$$T_\rho(\widehat{R}^{\mathcal{S}}_\alpha(G)) \subset Z^1_{\rho,\mathcal{S}}(G)_0 := \{u \in Z^1_\rho(G) \mid \langle u(s_i), P_i\rangle = 0,\ 1 \le i \le n\}.$$

We have $B^1_\rho(G) \subset Z^1_{\rho,\mathcal{S}}(G)_0 \subset Z^1_{\rho,\mathcal{S}}(G)$ by Lemma 2.1 and the homology groups $H^1_{\rho,\mathcal{S}}(G)_0 := Z^1_{\rho,\mathcal{S}}(G)_0 / B^1_\rho(G)$ and $H^1_{\rho,\mathcal{S}}(G) := Z^1_{\rho,\mathcal{S}}(G)/B^1_\rho(G)$ are defined.

For a free group F_n with basis $\mathcal{S}$ we get: $R^{\mathcal{S}}_\alpha(F_n)$, $R^{\mathcal{S}}(F_n) \subset R(F_n)$ and

$$T_\rho(R^{\mathcal{S}}(F_n)) \cong Z^1_{\rho,\mathcal{S}}(F_n) \cong \mathbb{R} \oplus P_1^\perp \oplus \cdots \oplus P_n^\perp \tag{3}$$

and

$$T_\rho(R^{\mathcal{S}}_\alpha(F_n)) \cong Z^1_{\rho,\mathcal{S}}(F_n) \cong P_1^\perp \oplus \cdots \oplus P_n^\perp. \tag{4}$$

Let $f_n \colon I \times \Sigma^n_{\pi/2} \to \mathrm{SU}(2)$ be the composition $w_n \circ \Phi_n$ where $w_n \colon \mathrm{SU}(2)^n \to \mathrm{SU}(2)$ is given by $w_n \colon (A_1, \ldots, A_n) \mapsto A_1 \cdots A_n$. The map $f^\alpha_n \colon \{\alpha\} \times \Sigma^n_{\pi/2} \to \mathrm{SU}(2)$ is by definition the restriction $f^\alpha_n := f_n \mid_{\{\alpha\} \times \Sigma^n_{\pi/2}}$ and

$$S_n := \{(\alpha, \mathbf{P}) \in I \times \Sigma^n_{\pi/2} \mid P_i \times P_j = 0,\ 1 \le i, j \le n\}.$$

Lemma 3.1 *Let $n \ge 2$ be an integer. Then the set $f^{-1}_{2n}(\mathbf{1}) \setminus S_{2n}$ is a non empty smooth manifold of dimension $4n-2$ and $(f^\alpha_{2n})^{-1}(\mathbf{1}) \setminus S_{2n}$ is a smooth non empty manifold of dimension $4n-3$.*

Proof. Since $(A, A^{-1}, \mathbf{1}, \ldots, \mathbf{1}, B, B^{-1}, \mathbf{1}, \ldots, \mathbf{1}) \in w^{-1}_{2n}(\mathbf{1})$ for all $A, B \in \mathrm{SU}(2)$ we have $(f^\alpha_{2n})^{-1}(\mathbf{1}) \setminus S_{2n} \neq \emptyset$.

Let $(\alpha, \mathbf{P}) \in I \times \Sigma^n_{\pi/2} \setminus S_n$ be given. We shall show that $D_{(\alpha,\mathbf{P})} f_n$ resp. $D_{(\alpha,\mathbf{P})} f_n^\alpha$ is surjective. Given a $\mathbf{A} \in \mathrm{SU}(2)^n$ there is the following commutative diagram

$$\begin{array}{ccc} T_{\mathbf{A}}(\mathrm{SU}(2)^n) & \xrightarrow{\cong} & \mathfrak{su}(2)^n \\ \big\downarrow D_{\mathbf{A}} w_n & & \big\downarrow \partial_{\mathbf{A}} w \\ T_{w_n(\mathbf{A})}(\mathrm{SU}(2)^n) & \xrightarrow{\cong} & \mathfrak{su}(2) \end{array}$$

where

$$\partial_{\mathbf{A}} w \colon\ (X_1, \dots X_n) \mapsto X_1 + A_1 X_2 A_1^{-1} + \cdots + A_1 \cdots A_{n-1} X_n A_{n-1}^{-1} \cdots A_1^{-1}$$

(see [LM85, 3.7]). Now, $D_{(\alpha,\mathbf{P})} f_n$ resp. $D_{(\alpha,\mathbf{P})} f_n^\alpha$ is surjective if and only if $\partial_{\mathbf{A}} w\mid_V$ resp. $\partial_{\mathbf{A}} w|_{V_0}$ is surjective where $V := \{(X_1, \dots, X_n) \in \mathfrak{su}(2)^n \mid \langle X_i, P_i \rangle = \langle X_j, P_j \rangle,\ 1 \le i, j \le n\}$ and $V_0 := \{(X_1, \dots, X_n) \in \mathfrak{su}(2)^n \mid \langle X_i, P_i \rangle = 0,\ i = 1, \dots, n\}$ (see Equations (3) and (4)). In order to prove the lemma it is sufficient to show that $\partial_{\mathbf{A}} w(V_0) = \mathfrak{su}(2)$ where $A_i = (\alpha, P_i)$.

We choose i_0, $2 \le i_0 \le n$, minimal such that $P_{i_0} \ne \pm P_1$. Let M be the following four dimensional vector space

$$M := \{(X_1, 0, \dots, 0, X_{i_0}, 0, \dots, 0) \mid X_1 \in P_1^\perp \text{ and } X_{i_0} \in P_{i_0}^\perp\} \subset \mathfrak{su}(2)^n.$$

It is obvious that $M \subset V_0$. The matrix $A := A_1 \cdots A_{i_0 - 1}$ commutes with P_1. Therefore, Ad_A is a rotation with fix axis P_1. It is clear that $\mathrm{Ad}_A(P_{i_0}) \ne \pm P_1$ and hence we have $\mathrm{Ad}_A(P_{i_0}^\perp) \ne P_1^\perp$. We obtain

$$\partial_{\mathbf{A}} w(M) = P_1^\perp + \mathrm{Ad}_A(P_{i_0}^\perp) = \mathfrak{su}(2)$$

which proves the lemma. □

Corollary 3.2 *Let $S(2n)$ be a planar surface with $2n$ boundary components and let $\mathcal{S} = \{s_1, \dots, s_{2n}\}$ be a canonical system of generators for $\pi_1(S(2n))$ i.e. $\pi_1(S(2n)) = \langle s_1, \dots, s_{2n} \mid s_1 \cdots s_{2n} = 1 \rangle$. Then the space $\widehat{R}^{\mathcal{S}}(S(2n))$ is a $(4n-5)$ dimensional manifold. and for each $\alpha \in I$ the subset $\widehat{R}^{\mathcal{S}}_\alpha(S(2n)) \subset \widehat{R}^{\mathcal{S}}(S(2n))$ is a submanifold of codimension one with trivial normal bundle. Moreover, for every $\rho \in \widehat{R}^{\mathcal{S}}_\alpha(S(2n))$ we have $T_\rho(\widehat{R}^{\mathcal{S}}(S(2n))) \cong H^1_{\rho,\mathcal{S}}(\pi_1(S(2n)))$ and $T_\rho(\widehat{R}^{\mathcal{S}}_\alpha(S(2n))) \cong H^1_{\rho,\mathcal{S}}(\pi_1(S(2n)))_0$.*

Proof. The corollary follows directly from Lemma 3.1. □

For a given $\rho \in R(X(k))$ we denote by $\rho_i \colon \pi_1(B_i) \to \mathrm{SU}(2)$ the composition $\rho \circ p_i$, $i = 1, 2$, and $\widehat{\rho} := \rho \circ p_i \circ \kappa_i \colon \pi_1(S(2n)) \to \mathrm{SU}(2)$.

Proposition 3.3 *Let $\rho \in \widehat{R}(k)$ be given. With the notation of diagram (2) we have: the representation ρ is regular if and only if $\widehat{Q}_1 \pitchfork_{\widehat{\rho}} \widehat{Q}_2$.*

Proof. Let $G := G(k)$ be the knot group. The set $p_i(\mathcal{T}_i)$ is a system of generators of G (each $p_i(t_l^{(i)})$ is a meridian) and $p_i^*(H^1_\rho(G)) \subset H^1_{\rho_i,\mathcal{T}_i}(\pi_1(B_i))$ by Lemma 2.1. From the Mayer-Vietoris sequence we obtain:

$$\kappa_1^*(H^1_{\rho_1}(B_1)) \cap \kappa_2^*(H^1_{\rho_2}(B_2)) = (p_i \circ \kappa_i)^*(H^1_\rho(X(k)))$$

and hence

$$\kappa_1^*(H^1_{\rho_1,\mathcal{T}_1}(\pi_1(B_1))) \cap \kappa_2^*(H^1_{\rho_2,\mathcal{T}_2}(\pi_1(B_2))) = (p_i \circ \kappa_i)^*(H^1_\rho(G)). \tag{5}$$

For the canonical isomorphism $\Lambda\colon T_{\widehat{\rho}}(\widehat{R}^{\mathcal{S}}(S(2n))) \cong H^1_{\widehat{\rho},\mathcal{S}}(\pi_1(S(2n)))$ we have: $\Lambda(T_{\widehat{\rho}}(\widehat{Q}_i)) = \kappa_i^*(H^1_{\rho_i,\mathcal{T}_i}(\pi_1(B_i)))$ because $\pi_1(B_i)$ is a free group with basis $\mathcal{T}_i$. We obtain from (5):

$$\Lambda(T_{\widehat{\rho}}(\widehat{Q}_1) \cap T_{\widehat{\rho}}(\widehat{Q}_2)) = (p_i \circ \kappa_i)^*(H^1_\rho(G))$$

Since $(\kappa_1 \circ p_1)^*$ is injective we have: $\dim H^1_\rho(G) = \dim(T_{\widehat{\rho}}(\widehat{Q}_1) \cap T_{\widehat{\rho}}(\widehat{Q}_2))$ which proves the proposition. $\square$

As a consequence we get:

Corollary 3.4 *Let $\rho \in \operatorname{Reg}(k)$. Then there is a neighborhood $U = U(\rho) \subset \widehat{R}(k)$ which is diffeomorphic to an open interval. Moreover,* $\operatorname{Reg}(k)$ *is a smooth one dimensional manifold.*

From the orientation convention in Section 4.1 it follows that the manifolds $\widehat{Q}_i \subset \widehat{R}^{\mathcal{S}}(S(2n))$ are oriented. The manifold $\widehat{R}^{\mathcal{S}}(S(2n))$ is oriented too (see Section 4.1). Now, $\widehat{Q}_1 \cap \widehat{Q}_2$ inherits an orientation in a neighborhood of an regular representation $\rho \in \operatorname{Reg}(k)$. As a consequence we see that a plat decomposition of $X(k)$ with plat $\widehat{\beta}$ gives rise to an orientation of $\operatorname{Reg}(\widehat{\beta})$.

Definition 3.5 Let $\beta \in B_{2n}$ be given such that $\widehat{\beta} \subset S^3$ is a knot. We define an orientation for $\operatorname{Reg}(\widehat{\beta})$ by the rule

$$\operatorname{Reg}(\widehat{\beta}) := (-1)^n \widehat{Q}_1 \cap \widehat{Q}_2.$$

It will be proved in Section 4 that the orientation does not depend on the braid β. Therefore each unoriented knot $k \subset S^3$ gives rise to an orientation of $\operatorname{Reg}(k) \subset \widehat{R}(k)$. Moreover, $\operatorname{Reg}(k^*) = -\operatorname{Reg}(k)$ holds (see Lemma 4.7).

Remark 3.6 A construction yielding an orientation for the SU(2)–representation space of a 2-bridge knot was given by the author in [Heu94, Section 5]. But it turns out that this approach does not work in general. However, it is possible to do the explicit calculations for 2–bridge knots and torus knots, i.e. we can orient their SU(2) representation space directly. We shall present the details in a forthcoming paper.

4 Invariance

In this section we shall prove that the orientation of $\mathrm{Reg}(k)$ is independent of the plat decomposition i.e.

Theorem 4.1 *Let $k \subset S^3$ be a knot and let $\beta_i \in B_{2n_i}$ be given, $i = 1, 2$, such that $\widehat{\beta_i} \cong k$. Moreover let $\psi_i \colon \mathrm{Reg}(\widehat{\beta_i}) \to \mathrm{Reg}(k)$ be the identification associated with the plat decomposition of $X(k)$ with respect to $\beta_i \in B_{2n_i}$.*

Then the two orientations of $\mathrm{Reg}(k)$ induced from the identifications ψ_i are the same.

In order to describe the relation between different braids which represent the same plat we need some definitions.

We denote by $\mathbb{R}^3_+$ the upper half space. Let $A \subset \mathbb{R}^3_+$ be a trivial system of n arcs properly embedded into $\mathbb{R}^3_+$. Here the system $A \subset \mathbb{R}^3_+$ is called *trivial* if there are disjoint disks $D_i \subset \mathbb{R}^3_+$, $i = 1, \ldots, n$, and disjoint arcs $\alpha'_i \subset \mathbb{R}^2$ such that $\partial D_i = \alpha_i \cup \alpha'_i$.

PSfrag replacements

α_1
α'_1
D_1
α_2
α'_2
D_2
α_3
α'_3
D_3
$\mathbb{R}^2$
$\mathbb{R}^3_+$

We identify the free group F_{2n} with the fundamental group $\pi_1(\mathbb{R}^2 \smallsetminus \partial A)$ where $(\mathbb{R}^2, \partial A) := \partial(\mathbb{R}^3_+, A)$. For each braid $\beta \in B_{2n}$ there is a diffeomorphism $\varphi_\beta \colon (\mathbb{R}^2, \partial A) \to (\mathbb{R}^2, \partial A)$ which induces the automorphism ϕ_β, i.e. $\phi_\beta = (\varphi_\beta)_* \in \mathrm{Aut}(F_{2n})$.

The braid $\beta \in B_{2n}$ is called a *trivial half braid* if and only if φ_β extends to a diffeomorphism $\bar{\varphi}_\beta \colon (\mathbb{R}^3_+, A) \to (\mathbb{R}^3_+, A)$. We denote by $K_{2n} \subset B_{2n}$ the subgroup of trivial half braids.

Lemma 4.2 (Hilton [Hil75]) *The subgroup $K_{2n} \subset B_{2n}$ is generated by*

$$\{\sigma_1, \sigma_2\sigma_1^2\sigma_2, \sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j} \mid 1 \le j \le n-1\},$$

where $\{\sigma_j \mid 1 \le j \le 2n-1\} \subset B_{2n}$ is the set of elementary braids.

Let $\zeta \in B_{2n}$ and let $\eta_1, \eta_2 \in K_{2n}$ be given. Then it is clear that $\widehat{\zeta}$ and $\widehat{\eta_2\zeta\eta_1}$ are equivalent plats in S^3. This means that two braids in B_{2n} represent the same plat if they are in the same double cosset of B_{2n} modulo the subgroup K_{2n}. Moreover, it is evident that for a given $\zeta \in B_{2n}$ the plats $\widehat{\zeta}$ and $\widehat{\zeta\sigma_{2n}}$ where $\zeta\sigma_{2n} \in B_{2n+2}$ are equivalent plats in S^3. The transformation $\zeta \to \zeta\sigma_{2n}$ is called an *elementary stabilization* (see Figure 4). Two braids are called *stably equivalent* if they represent (after a finite number of elementary stabilizations) the same double cosset modulo the subgroup of trivial half braids. Two braids which represent the same closed braid are stable equivalent. More precisely, we have:

PSfrag replacements

$\dots$

σ_1

$\sigma_2\sigma_1^2\sigma_2$

$\sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j}$

Figure 2: The braids σ_1, $\sigma_2\sigma_1^2\sigma_2$ and $\sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j}$.

Theorem 4.3 (Birman, Reidemeister) *Let $k_i \subset S^3$, $i = 1, 2$, be unoriented knots and let $\beta_i \in B_{2n_i}$ be given such that $\widehat{\beta_i} \cong k_i$. Then $k_1 \cong k_2$ if and only if there exist an integer $t \geq \max(n_1, n_2)$ such that for each $n \geq t$ the braids $\beta_i' = \beta_i\sigma_{2n_i}\sigma_{2n_i+2}\cdots\sigma_{2n} \in B_{2n+2}$, $i = 1, 2$, are in the same double cosset of B_{2n+2} modulo the subgroup K_{2n+2}.*

Proof. The proof can be found in [Bir76b] (see also [Rei60]). □

The proof of Theorem 4.1 splits therefore into two parts. First we prove that the orientation of $\mathrm{Reg}(\beta)$ does not change if we replace the braid β by another braid in the same double cosset (see Section 4.3). In the second step we prove that the orientation does not change under an elementary stabilization (see Section 4.4) .

4.1 Orientations

In this section we introduce the appropriate orientation conventions. In particular we define an orientation on $\widehat{R}^{\mathcal{S}}(S(2n))$. We shall see that certain automorphisms of $\pi_1(S(2n))$ induce orientation preserving (resp. reversing) diffeomorphisms of $\widehat{R}^{\mathcal{S}}(S(2n))$ (see Proposition 4.4 and its proof).

Let M be an oriented manifold. The manifold M with the opposite orientation is denoted by $-M$. The boundary ∂M inherits an orientation by the convention *the inward pointing normal vector in the last position* (see [Hir76]).

From the very beginning we assume that $\mathrm{SU}(2)$ is oriented. We choose the orientation of $\mathrm{SO}(3)$ such that the 2–fold covering $\delta\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a local orientation preserving diffeomorphism. The 2-sphere Σ_α splits $\mathrm{SU}(2)$ into two components. One of these components contains the identity matrix $\mathbf{1}$ and Σ_α is oriented as the boundary of this component. Note that the diffeomorphism $\Sigma_{\pi/2} \to \Sigma_\alpha$ given by $P \mapsto (\alpha, P)$ is orientation preserving. In order to orient the interval $I = (0, \pi)$ we consider the submersion $\mathrm{SU}(2) \setminus \{\pm\mathbf{1}\} \to I$ and we choose an orientation of I such that for each $\alpha \in I$ and each $A \in \Sigma_\alpha$ the orientations of the short exact sequence

$$0 \to T_A\Sigma_\alpha \to T_A\mathrm{SU}(2) \to T_\alpha(I) \to 0$$

fit together. Thus I has the usual orientation.

The manifolds $\{\alpha\} \times \Sigma^n_{\pi/2} \cong \Sigma^n_\alpha$ and $I \times \Sigma^n_{\pi/2}$ carry the product orientations. By Lemma 3.1 we can pull back the orientation of $\mathfrak{su}(2)$ in order to obtain an orientation of the normal bundle $f^*_{2n}(\mathfrak{su}(2))$ of $f^{-1}_{2n}(\mathbf{1}) \setminus S_{2n} \subset I \times \Sigma^{2n}_{\pi/2}$. This enables us to orient the manifold $\widetilde{R}^{\mathcal{S}}(S(2n)) \cong f^{-1}_{2n}(\mathbf{1}) \setminus S_{2n}$ via the convention (*fibre* $\oplus$ *base*) i.e. we choose the orientation for $f^{-1}_{2n}(\mathbf{1}) \setminus S_{2n}$ such that

$$T_{(\alpha,\mathbf{P})}(f^{-1}_{2n}(\mathbf{1}) \setminus S_{2n}) \oplus f^*_{2n}(\mathfrak{su}(2)) = T_{(\alpha,\mathbf{P})}(I \times \Sigma^{2n}_{\pi/2})$$

for all $(\alpha, \mathbf{P}) \in f^{-1}_{2n}(\mathbf{1}) \setminus S_{2n}$. The map $\widetilde{R}^{\mathcal{S}}(S(2n)) \to \widehat{R}^{\mathcal{S}}(S(2n))$ is a principal SO(3) bundle and because SO(3) is connected we have an orientable $(4n-5)$ dimensional manifold $\widehat{R}^{\mathcal{S}}(S(2n))$. We use again the convention (*fibre* $\oplus$ *base*) in order to orient $\widehat{R}^{\mathcal{S}}(S(2n))$ (see [AM90, GM92]).

Let $\beta \in B_n$ be a braid and let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a basis for the free group F_n. The braid β induces an automorphism $\phi_\beta \colon F_n \to F_n$, $\phi_\beta(s_i) = g_i s_{\pi(i)} g_i^{-1}$, where $g_i \in F_n$ and π is a permutation such that $\prod_{i=1}^n \phi_\beta(s_i) = \prod_{i=1}^n s_i$ (see [BZ85]). The automorphism ϕ_β is hence compatible with $\mathcal{S}$. The following fact will be used in the sequel:

Proposition 4.4 *Let $\beta \in B_{2n}$ and let $\phi\colon F_{2n} \to F_{2n}$ be given by $\phi\colon s_i \mapsto s^{-1}_{2n-i+1}$. Then $\widehat{\phi}_\beta\colon \widehat{R}^{\mathcal{S}}(S(2n)) \to \widehat{R}^{\mathcal{S}}(S(2n))$ is orientation preserving and $\widehat{\phi}\colon \widehat{R}^{\mathcal{S}}(S(2n)) \to \widehat{R}^{\mathcal{S}}(S(2n))$ is orientation reversing.*

4.1.1 Proof of Proposition 4.4

Let $F_n := F_n(\mathcal{S})$ be a free group on a given set $\mathcal{S} = \{s_1, \ldots, s_n\}$ of free generators and let $\phi \in \mathrm{Aut}(F_n)$ be a automorphism. Assume that there is a permutation π such that

$$\phi(s_j) = g_j s^{\eta_j}_{\pi(j)} g_j^{-1}, \text{ where } g_j \in F_n \text{ and } \eta_j \in \{\pm 1\}. \tag{6}$$

It follows that $\mathcal{S}$ and $\mathcal{S}' := \phi(\mathcal{S})$ are compatible and we have $R^{\mathcal{S}}(F_n) = R^{\mathcal{S}'}(F_n)$. In this case the automorphism ϕ induces two diffeomorphisms $R(\phi)\colon R(F_n) \to R(F_n)$ and $\phi^{\#}\colon R^{\mathcal{S}}(F_n) \to R^{\mathcal{S}}(F_n)$. We set $N(\phi) := \#\{\eta_j | \eta_j = -1\}$ and

$$s(\phi) := \begin{cases} 0 & \text{if } \pi \text{ is even} \\ 1 & \text{if } \pi \text{ is odd.} \end{cases}$$

The basis $\mathcal{S}$ of F_n gives us an identification $R(F_n) \cong \mathrm{SU}(2)^n$ which carries the product orientation.

Lemma 4.5 *Let $\phi \in \mathrm{Aut}(F_n)$ be given as in Formula 6. We set $N := N(\phi)$ and $s := s(\phi)$.*

Then the map $R(\phi)\colon R(F_n) \to R(F_n)$ is orientation preserving (resp. orientation reversing) if and only if $N + s \equiv 0 \bmod 2$ (resp. $N + s \equiv 1 \bmod 2$).

Moreover, the map $\phi^{\#}\colon R^{\mathcal{S}}(F_n) \to R^{\mathcal{S}}(F_n)$ *is orientation preserving (resp. orientation reversing) if and only if* $N \equiv 0 \bmod 2$ *(resp.* $N \equiv 1 \bmod 2$*).*

Proof. An easy calculation gives the lemma (see also [AM90, Proposition 3.4]). □

Let $\pi_1(S(2n)) = \langle s_1, \ldots, s_{2n} \mid s_1 \cdots s_{2n} = 1 \rangle$ be the fundamental group of $S(2n)$ and let $\phi \in \mathrm{Aut}(F_{2n})$ be an automorphism as in Formula 6 which preserves the normal closure of the element $s_1 \cdots s_{2n} \in F_{2n}$. The automorphism ϕ induces a diffeomorphism $\widehat{\phi}\colon \widehat{R}^{\mathcal{S}}(S(2n)) \to \widehat{R}^{\mathcal{S}}(S(2n))$.

Lemma 4.6 *Let* $\phi \in \mathrm{Aut}(F_n)$ *be an automorphism as in Formula 6. Assume that* $\phi(s_1 \cdots s_{2n}) = g(s_1 \cdots s_{2n})^{\epsilon} g^{-1}$ *where* $g \in F_{2n}$ *and* $\epsilon \in \{\pm 1\}$*. Choose the orientation of* $\widehat{R}^{\mathcal{S}}(S(2n))$ *as above.*

Then the diffeomorphism $\widehat{\phi}\colon \widehat{R}^{\mathcal{S}}(S(2n)) \to \widehat{R}^{\mathcal{S}}(S(2n))$ *is orientation preserving (resp. reversing) if and only if* $N(\phi) + \frac{\epsilon-1}{2} \equiv 0 \bmod 2$ *(resp.* $N(\phi) + \frac{\epsilon-1}{2} \equiv 1 \bmod 2$ *).*

Proof. An inner automorphism of F_n induces the identity on $\widehat{R}^{\mathcal{S}}(S(2n))$. Therefore we might assume that $\phi(s_1 \cdots s_{2n}) = (s_1 \cdots s_{2n})^{\epsilon}$. We obtain the following diagram

$$\begin{tikzcd}
R^{\mathcal{S}}(S(2n)) \arrow[r] \arrow[d, "\phi^{\#}"] & R^{\mathcal{S}}(F_{2n}) \arrow[r, "f_{2n}"] \arrow[d, "\phi^{\#}"] & \mathrm{SU}(2) \arrow[d, "\Psi"] \\
R^{\mathcal{S}}(S(2n)) \arrow[r] & R^{\mathcal{S}}(F_{2n}) \arrow[r, "f_{2n}"] & \mathrm{SU}(2)
\end{tikzcd}$$

where $\Psi\colon A \mapsto A^{\epsilon}$. Now, Ψ is orientation preserving (resp. reversing) if and only if $\epsilon = 1$ (resp. $\epsilon = -1$). This together with Lemma 4.5 proves the lemma. □

Proof of Proposition 4.4. Let $\phi\colon F_{2n} \to F_{2n}$ be given by $\phi\colon s_i \mapsto s_{2n-i+1}$. We have $N(\phi) = 2n$ and $\epsilon = -1$. Hence $\widehat{\phi}$ is orientation reversing by Lemma 4.6. If $\beta \in B_{2n}$ then $N(\phi_\beta) = 0$ and $\epsilon = 1$. Lemma 4.6 implies that $\widehat{\phi}_\beta$ is orientation preserving. □

Note that ϕ_β induces an automorphism of $\pi(S(2n))$ because $\phi_\beta(s_1) \cdots \phi_\beta(s_{2n}) = s_1 \cdots s_{2n}$.

4.2 Choice of the generators (revised)

Let $\beta \in B_{2n}$ be a braid such that $\widehat{\beta}$ is a knot. The aim of this section is to define the special systems of generators corresponding to a plat decomposition of $X(\widehat{\beta})$.

We assume from the very beginning that $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is oriented. We choose $\epsilon \in \{\pm 1\}$ such that (e_1, e_2, e_3) represents the induced orientation of $\mathbb{R}^3$ ($e_1 = (\epsilon, 0, 0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$). For given $n \in \mathbb{N}$ we set

$$p_j := \begin{cases} (j, 0) \in \mathbb{R}^2 & \text{if } \epsilon = 1 \\ (2n - j + 1, 0) \in \mathbb{R}^2 & \text{if } \epsilon = -1 \end{cases} \qquad j = 1, \ldots, 2n.$$

We start with the splitting $\mathbb{R}^3 = H_1 \cup \mathbb{R}^2 \times J \cup H_2$ where $J = [1,2]$ is the closed interval and $H_1 = \{(x,y,z) \in \mathbb{R}^3 \mid z \leq 1\}$ and $H_2 = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 2\}$ are closed half spaces.

We obtain a geometric braid in $\mathbb{R}^2 \times J$ which is also denoted by $\beta \subset \mathbb{R}^2 \times J$ (see [Bir76a]) and we assume that $\beta \cap (\mathbb{R}^2 \times \{i\}) = \mathbf{p} \times \{i\}$, $i = 1, 2$, where $\mathbf{p} := (p_j)_{j=1}^{2n}$. Moreover, we assume that β is contained in a small regular neighborhood of the plane $y = 0$. The $2n$–plat $\widehat{\beta} \subset \mathbb{R}^3$ is obtained from β by closing it with two systems of half circles $A_i = \{a_l^{(i)}\}_{l=1}^{n} \subset H_i \cap (\mathbb{R} \times \{0\} \times \mathbb{R})$ where the endpoints of the half circle $a_l^{(i)}$ are the points $p_{2l} \times \{i\}$ and $p_{2l-1} \times \{i\} \in \partial H_i$ (see Figure 3).

PSfrag replacements

H_1
H_2
$\mathbb{R}^2 \times \{1\}$
$\mathbb{R}^2 \times \{2\}$
x_0
Q
$s_1^{(2)}$
$a_1^{(2)}$
$t_2^{(2)}$
$s_1^{(1)}$
$s_4^{(1)}$
$t_2^{(1)}$
$a_1^{(1)}$

Figure 3: Choice of the generators for $\epsilon = +1$.

Let Q be the cube $Q := [0, 2n+1] \times [-1, 1] \times J \subset \mathbb{R}^2 \times J$ and fix $x_0 := (n, -1, 1) \in \partial Q$. We obtain special systems of generators for the fundamental groups as follows: the generator $s_j^{(i)}$ of $\pi_1((\mathbb{R}^2 \smallsetminus \mathbf{p}) \times \{i\})$ is represented by a loop in $\mathbb{R}^2 \times \{i\}$ consisting of a small circle around $p_j \times \{i\}$ and the shortest arc in ∂Q connecting it to x_0. The circle is oriented according to the following rule: let L_j be the oriented line $p_j \times \mathbb{R}$ (the orientation points in negative z-direction). We orient the circle such that $lk(s_j^{(i)}, L_j) = 1$. With this choice we obtain the presentation $\pi_1((\mathbb{R}^2 \smallsetminus \mathbf{p}) \times \{i\}) = \langle s_1^{(i)}, \ldots, s_{2n}^{(i)} \mid s_1^{(i)} \cdots s_{2n}^{(i)} \rangle$.

In order to proceed we choose an orientation for the plat $\widehat{\beta}$. We shall see later (see Lemma 4.7) that the construction does not depend on this choice. The generators $t_l^{(i)}$, $1 \le l \le n$, of $\pi_1(H_i \smallsetminus A_i)$ are represented by a loop consisting of a small circle around $a_l^{(i)}$ and a shortest arc in $\mathbb{R}^3$ connecting the circle to x_0. The orientation of the circle is given by the condition $lk(\widehat{\beta}, t_l^{(i)}) = 1$ (see Figure 3).

Denote by $\lambda_i \colon \pi_1((\mathbb{R}^2 \smallsetminus \mathbf{p}) \times \{i\}) \to \pi_1(H_i \smallsetminus A_i)$ the homomorphism which is induced by the inclusion. From the choice of the generators it follows that $\lambda_i \colon s_{2l-1}^{(i)} \mapsto (t_l^{(i)})^{\epsilon_l^{(i)}}$ and $\lambda_i \colon s_{2l}^{(i)} \mapsto (t_l^{(i)})^{-\epsilon_l^{(i)}}$ where $\epsilon_l^{(i)} \in \{\pm 1\}$. Note that the $\epsilon_l^{(i)}$ depend on the orientation of $\widehat{\beta}$ and that they change simultaneously if the orientation of $\widehat{\beta}$ is changed.

The braid group B_{2n} may be considered as a subgroup of $\mathrm{Aut}(F_{2n})$ where F_{2n} may be interpreted as the fundamental group $\pi_1(Q \smallsetminus \beta)$. We denote the automorphism determined by $\beta \in B_{2n}$ by ϕ_β i.e. $\phi_\beta \colon \pi_1(Q \smallsetminus \beta) \to \pi_1(Q \smallsetminus \beta)$ is given by $\phi_\beta \colon s_j^{(2)} \mapsto s_j^{(1)}$ (see [BZ85]). Note that $s_j^{(1)} = s_j^{(1)}(s_1^{(2)}, \ldots, s_{2n}^{(2)})$ is a word in the generators $\{s_j^{(2)}\}_{j=1}^{2n}$.

The planar surface $S(2n) := ((\mathbb{R}^2 \smallsetminus U(\mathbf{p})) \times \{1\}) \cup \{\infty\}$ determines a plat decomposition

$$X(\widehat{\beta}) = B_1 \cup_{S(2n)} B_2$$

where $B_1 = (H_1 \smallsetminus U(A_1)) \cup \{\infty\}$ and $B_2 = ((H_2 \cup \mathbb{R}^2 \times J) \smallsetminus U(A_2 \cup \beta)) \cup \{\infty\}$.

It follows that $\kappa_i \colon \pi_1(S(2n)) \to \pi_1(B_i)$ is given by

$$\kappa_1 \colon s_j^{(1)} \mapsto \lambda_1(s_j^{(1)}), \qquad \kappa_2 \colon s_j^{(1)} \mapsto \lambda_2 \circ \phi_\beta(s_j^{(2)}) = \lambda_2(s_j^{(1)}(s_1^{(2)}, \ldots, s_{2n}^{(2)})).$$

We obtain an other plat decomposition by choosing $S'(2n) := ((\mathbb{R}^2 \smallsetminus U(\mathbf{p})) \times \{2\}) \cup \{\infty\}$, $B_1' = ((H_1 \cup \mathbb{R}^2 \times J) \smallsetminus U(A_1 \cup \beta)) \cup \{\infty\}$ and $B_2' = (H_2 \smallsetminus U(A_2)) \cup \{\infty\}$. The epimorphisms κ_i' are than given by

$$\kappa_1' \colon s_j^{(2)} \mapsto \lambda_1 \circ \phi_\beta^{-1}(s_j^{(1)}) = \lambda_1(s_j^{(2)}(s_1^{(1)}, \ldots, s_{2n}^{(1)})) \text{ and } \kappa_2' \colon s_j^{(2)} \mapsto \lambda_2(s_j^{(2)}).$$

We have $\kappa_i = \kappa_i' \circ \phi_\beta$ and we define $\widehat{Q}_i' := \mathrm{Im}(\widehat{\kappa}_i')$. The orientation of $\mathrm{Reg}(\widehat{\beta})$ does not depend on the choice of one of these two splittings: the diffeomorphism $\widehat{\phi}_\beta \colon \widehat{R}^S(S(2n)) \to \widehat{R}^S(S'(2n))$ induces an orientation preserving map from the regular part of $\widehat{Q}_1 \cap \widehat{Q}_2$ to the regular part of $\widehat{Q}_1' \cap \widehat{Q}_2'$ (see Proposition 4.4).

Lemma 4.7 *Let $\beta \in B_{2n}$ be a braid such that $\widehat{\beta}$ is a knot. Then the orientation constructed on* $\mathrm{Reg}(\widehat{\beta})$ *is independent of the orientation of $\widehat{\beta}$.*

A change of the orientation of S^3 changes the orientation of $\mathrm{Reg}(\widehat{\beta})$.

Proof. If we change the orientation of $\widehat{\beta}$ than the $\epsilon_l^{(i)}$ are changing their sign simultaneously. Hence the orientation of $\widehat{Q}_1$ and $\widehat{Q}_2$ are changing simultaneously.

If we change the orientation of S^3 the orientations of $\widehat{Q}_1$ and $\widehat{Q}_2$ are changing simultaneously too. But we have also to change the orientation of $\widehat{R}^{\mathcal{S}}(S(2n))$ because $s_j^{(i)} \mapsto (s_{2n-j+1}^{(i)})^{-1}$ (see Proposition 4.4) and so the orientation of $\widehat{Q}_1 \cap \widehat{Q}_2$ at a regular point changes. $\square$

4.3 Invariance under the change of the double cosset representative

Let $F_{2n} = F(s_1, \dots, s_{2n})$ and $F_n = F(t_1, \dots, t_n)$ be free groups of rank $2n$ and n respectively. For a given $\epsilon_j \in \{\pm 1\}$, $j = 1, \dots, n$, we define an epimorphism $\kappa\colon F_{2n} \to F_n$ by

$$\kappa\colon\ s_{2j-1} \mapsto t^{\epsilon_j}, \text{ and } \kappa\colon\ s_{2j} \mapsto t^{-\epsilon_j}.$$

Let $\zeta \in K_{2n}$ be given. It is proved in [Bir76b] that a given braid is contained in K_{2n} if and only if it leaves the normal closure of $\{s_1 s_2, \dots, s_{2n-1} s_{2n}\}$ in F_{2n} invariant. Therefore we have an automorphism $\zeta^\kappa\colon F_n \to F_n$ such that the following diagram commutes:

$$\begin{array}{ccc} F_{2n} & \xrightarrow{\kappa} & F_n \\ \Big\downarrow \zeta & & \Big\downarrow \zeta^\kappa \\ F_{2n} & \xrightarrow{\kappa} & F_n. \end{array}$$

It is easy to see that

$$\sigma_1^\kappa\colon\ t_1 \mapsto t_1^{-1}, \quad \sigma_1^\kappa\colon\ t_j \mapsto t_j \text{ for } 2 \le j \le n. \tag{7}$$

$$(\sigma_2\sigma_1^2\sigma_2)^\kappa\colon\ t_1 \mapsto t_2^{\epsilon_2} t_1 t_2^{-\epsilon_2}, \quad (\sigma_2\sigma_1^2\sigma_2)^\kappa\colon\ t_j \mapsto t_j \text{ for } 2 \le j \le n \tag{8}$$

and

$$(\sigma_{2k}\sigma_{2k-1}\sigma_{2k+1}\sigma_{2k})^\kappa\colon\ t_j \mapsto t_{\tau_k(j)} \text{ for } 1 \le j \le n. \tag{9}$$

where τ_k, $1 \le k \le n-1$ is the transposition which permutes k and $k+1$.

Let $\beta \in B_{2n}$ be a braid such that $\widehat{\beta}$ is a knot. The plat $\widehat{\beta}$ gives a plat decomposition of $X(k)$. We denote by $\kappa_1 = \lambda_1\colon \pi_1(S(2n)) \to \pi_1(B_1)$ and $\kappa_2 = \lambda_2 \circ \beta\colon \pi_1(S(2n)) \to \pi_1(B_2)$ the induced epimorphisms and $\widehat{Q}_i := \widehat{\kappa}_i(\widehat{R}^{\mathcal{T}_i}(B_i))$. Moreover, assume that $\zeta_i \in K_{2n}$ is given. Then $\widehat{\zeta_2\beta\zeta_1}$ is a knot too. We denote the induced epimorphisms by $\kappa_i'\colon \pi_1(S(2n)) \to \pi_1(B_i)$, $i = 1, 2$, and $\widehat{Q}_i' := \widehat{\kappa}_i'(\widehat{R}^{\mathcal{T}_i}(B_i))$, $i = 1, 2$.

Lemma 4.8 *There is an orientation preserving map $\widehat{\Lambda}(\zeta_i)\colon \widehat{R}^{\mathcal{T}_i}(B_i) \to \widehat{R}^{\mathcal{T}_i}(B_i)$ such that $\widehat{\kappa}_i' = \widehat{\kappa}_i \circ \widehat{\Lambda}(\zeta_i)$.*

Proof. It is sufficient to prove the lemma in the case ζ_i is one of the generators of K_{2n} (see Lemma 4.2). Let $\zeta_i \in \{\sigma_1, \sigma_2\sigma_1^2\sigma_2, \sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j} \mid 1 \le j \le n-1\}$. Note that the epimorphism λ_i' differs from λ_i only if $\zeta_i = \sigma_1$. If $\zeta_i = \sigma_1$ we get $\lambda_i'(s_1^{(i)}) = \lambda_i(s_1^{(i)})^{-1}$ and by equation (7) we obtain $\kappa_i = \kappa_i'$. If $\zeta_i = \sigma_2\sigma_1^2\sigma_2$ we obtain from equation (8) that $\widehat{\zeta}_i^\kappa$ is orientation preserving and an easy calculation gives $\widehat{\kappa}_i' = \widehat{\kappa}_i \circ \widehat{\zeta}_i^\kappa$. The case $\xi_i = \sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j}$ is completely analogous. $\square$

We summarize the results in the following Proposition:

Proposition 4.9 *Let $\beta, \beta' \in B_{2n}$ and assume that β and β' are representing the same double cosset in $K_{2n} \backslash B_{2n} / K_{2n}$. Then we have* $\mathrm{Reg}(\beta) = \mathrm{Reg}(\beta')$ *as oriented manifolds.*

4.4 Invariance under stabilization

Let $\beta \in B_{2n}$ be given. We are interested in the new braid $\beta' := \beta\sigma_{2n} \in B_{2n+2}$ (see Figure 4). We obtain: $\kappa_1 = \lambda_1$, $\kappa_2 = \lambda_2 \circ \beta$ and $\kappa'_1 = \lambda'_1 \circ \sigma_{2n}^{-1}$, $\kappa'_2 = \lambda'_2 \circ \beta$ where $\lambda'_i \colon \pi_1(S'(2n+2)) \to \pi_1(B'_i)$ is given by

$$\lambda'_1(s_j^{(1)}) = \lambda_1(s_j^{(1)}) \text{ if } 1 \leq j \leq 2n \qquad \lambda'_2(s_j^{(2)}) = \lambda_2(s_j^{(2)}) \text{ if } 1 \leq j \leq 2n$$
$$\lambda'_1(s_{2n+1}^{(1)}) = (t_{n+1}^{(1)})^{-\epsilon_n^{(1)}} \qquad \lambda'_2(s_{2n+1}^{(2)}) = (t_{n+1}^{(2)})^{-\epsilon_n^{(1)}}$$
$$\lambda'_1(s_{2n+2}^{(1)}) = (t_{n+1}^{(1)})^{\epsilon_n^{(1)}} \qquad \lambda'_2(s_{2n+2}^{(2)}) = (t_{n+1}^{(2)})^{\epsilon_n^{(1)}}$$

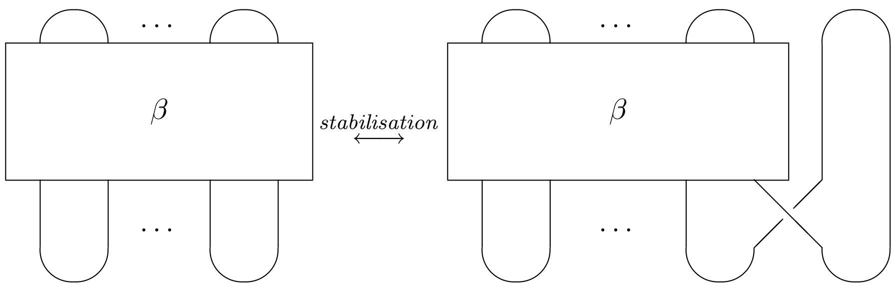

Figure 4: Stabilization

For $\lambda_i^{\#} \colon I \times \Sigma_{\pi/2}^n \to R^{\mathcal{S}}(S(2n))$, we have:

$$\lambda_i^{\#}(\alpha, P_1, \ldots, P_n) = (\alpha, \epsilon_1^{(i)} P_1, -\epsilon_1^{(i)} P_1, \ldots, \epsilon_n^{(i)} P_n, -\epsilon_n^{(i)} P_n)$$

and hence $\kappa_1^{\#} = \lambda_1^{\#}$ and $\kappa_2^{\#} = \beta^{\#} \circ \lambda_i^{\#}$ i.e.

$$\begin{aligned}\kappa_2^{\#}(\alpha, P_1, \ldots, P_n) &= \beta^{\#}(\alpha, \epsilon_1^{(2)} P_1, -\epsilon_1^{(2)} P_1, \ldots, \epsilon_n^{(2)} P_n, -\epsilon_n^{(2)} P_n) \\ &=: (\alpha, P_1^{\beta}, \ldots, P_{2n}^{\beta}).\end{aligned}$$

It follows that

$$\begin{aligned}(\kappa'_1)^{\#}(\alpha, P_j)_{j=1}^{n+1} &= (\sigma_{2n}^{-1})^{\#}(\alpha, \epsilon_1^{(1)} P_1, -\epsilon_1^{(1)} P_1, \ldots, \epsilon_n^{(1)} P_n, -\epsilon_n^{(1)} P_n, -\epsilon_n^{(1)} P_{n+1}, \epsilon_n^{(1)} P_{n+1}) \\ &= (\alpha, \epsilon_1^{(1)} P_1, -\epsilon_1^{(1)} P_1, \ldots, \epsilon_n^{(1)} P_n, -\epsilon_n^{(1)} P_{n+1}, \delta(\alpha, \epsilon_n^{(1)} P_{n+1})(-\epsilon_n^{(1)} P_n), \epsilon_n^{(1)} P_{n+1})\end{aligned}$$

where $\delta \colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is the 2–fold covering (see Section 2). Moreover, we have:

$$(\kappa'_2)^{\#}(\alpha, P_j)_{j=1}^{n+1} = (\alpha, P_1^{\beta}, \ldots, P_{2n}^{\beta}, -\epsilon_n^{(1)} P_{n+1}, \epsilon_n^{(1)} P_{n+1}).$$

It is now easy to show that $Q_1' \cap Q_2' = g(Q_1 \cap Q_2)$ where $g\colon I \times \Sigma^{2n}_{\pi/2} \to I \times \Sigma^{2n+2}_{\pi/2}$ is given by

$$g\colon (\alpha, P_1, \dots, P_{2n}) \mapsto (\alpha, P_1, \dots, P_{2n}, P_{2n}, -P_{2n}).$$

The map g induces an embedding

$$\widehat{g}\colon \widehat{R}^{\mathcal{S}}(S(2n)) \to \widehat{R}^{\mathcal{S}'}(S'(2n+2))$$

where $\mathcal{S}' = \mathcal{S} \cup \{s_{2n+1}, s_{2n+2}\}$. Moreover, we have $f_{2n+2} \circ g = f_{2n}$ which implies that

$$Dg|_{f^*_{2n}(\mathfrak{su}(2))}\colon f^*_{2n}(\mathfrak{su}(2)) \to f^*_{2n+2}(\mathfrak{su}(2)) \tag{10}$$

is an isomorphism. Given $(\alpha, P_1, \dots, P_{2n}) =: (\alpha, \mathbf{P}) \in R^{\mathcal{S}}(S(2n))$ we have

$$T_{g(\alpha,\mathbf{P})}(R^{\mathcal{S}'}(S'(2n+2))) \cong Dg(T_{(\alpha,\mathbf{P})}(R^{\mathcal{S}}(S(2n)))) \oplus T_{P_{2n}}(\Sigma_{\pi/2}) \oplus T_{-P_{2n}}(\Sigma_{\pi/2}) \tag{11}$$

as oriented vector spaces by the orientation convention and equation (10).

Assume that $(\alpha, \mathbf{P}) \in Q_1 \cap Q_2$. Then there are $(\alpha, \mathbf{P}^{(i)}) := (\alpha, P_1^{(i)}, \dots, P_n^{(i)}) \in I \times \Sigma^n_{\pi/2}$ such that $\kappa_i^{\#}(\alpha, \mathbf{P}^{(i)}) = (\alpha, \mathbf{P})$.

Proposition 4.10 *Let $\beta \in B_{2n}$ be given such that $\widehat{\beta}$ is a knot. Moreover let $\beta' := \beta\sigma_{2n} \in B_{2n+2}$. Then $g\colon R^{\mathcal{S}}(S(2n)) \to R^{\mathcal{S}'}(S'(2n+2))$ restricts to an orientation preserving diffeomorphism*

$$g\colon (-1)^n Q_1 \cap Q_2 \to (-1)^{n+1} Q_1' \cap Q_2'$$

in a neighborhood of a regular point.

Proof. Let $(\alpha, \mathbf{P}) \in Q_1 \cap Q_2$ be a regular point. i.e. $Q_1 \pitchfork_{(\alpha,\mathbf{P})} Q_2$. From Proposition 3.3 follows that $Q_1' \pitchfork_{g(\alpha,\mathbf{P})} Q_2'$. We have

$$T_{g(\alpha,P)}(Q_i') \cong Dg(T_{(\alpha,P)}(Q_i)) \oplus \mathcal{U}_i$$

where $\mathcal{U}_i \cong T_{P_n^{(1)}}(\Sigma_{\pi/2})$ as oriented vector spaces. From equation (11) we obtain:

$$T_{g(\alpha,\mathbf{P})}(R(S'(2n+2))) \cong Dg(T_{(\alpha,\mathbf{P})}(R(S(2n)))) \oplus \mathcal{W}$$

where $\mathcal{W} \cong T_{-\epsilon P_n^{(1)}}(\Sigma_{\pi/2}) \oplus T_{\epsilon P_n^{(1)}}(\Sigma_{\pi/2})$.

It is clear that $\mathcal{U}_1 \oplus \mathcal{U}_2 \cong -\mathcal{W}$ as oriented vector spaces. From this it follows that the map $g\colon R^{\mathcal{S}}(S(2n)) \to R^{\mathcal{S}'}(S'(2n+2))$ induces an orientation preserving diffeomorphism

$$g\colon (-1)^n Q_1 \cap Q_2 \to (-1)^{n+1} Q_1' \cap Q_2'$$

in a neighborhood of the regular point $(\alpha, \mathbf{P})$. $\square$

5 Lin's invariant

The aim of this section is to explain why the two quantities $h^{(\alpha)}(k)$ and $\sigma_k(e^{2\boldsymbol{i}\alpha})$ with apparently different algebraic–geometric contents are the same. In order to explain this connection we have to compare Lin's construction with the construction given in Section 3. Lin considered in his paper closed n-braids which are very special $2n$-plats (see Figure 5).

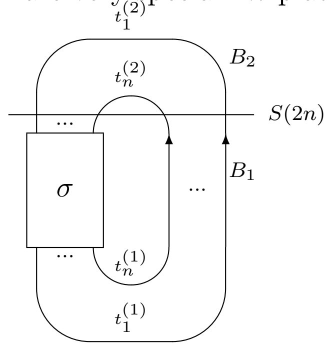

Figure 5: Closed n-braids are special $2n$-plats.

5.1 Outline of Lin's construction

For the convenience of the reader we repeat the notations from [Lin92] and [HKr98].

Let $\sigma \in \mathfrak{B}_n$ be given and denote by $\sigma^\wedge$ the closed n-braid defined by σ. Let F_n be a free group with basis $\mathcal{R} = \{r_1, \ldots, r_n\}$. The braid σ induces a braid automorphism $\phi_\sigma\colon F_n \to F_n$. It follows that σ induces a diffeomorphism of $\mathrm{SU}(2)^n$ i.e.

$$\phi_\sigma^\#(A_1, \ldots, A_n) =: (\phi_\sigma^\#(A_1), \ldots, \phi_\sigma^\#(A_n)).$$

Note that the equation $\prod_{i=1}^n A_i = \prod_{i=1}^n \phi_\sigma^\#(A_i)$ always holds.

It was observed by Lin that the fixed point set of $\phi_\sigma^\#\colon \mathrm{SU}(2)^n \to \mathrm{SU}(2)^n$ can be identified with $R(\sigma^\wedge)$ [Lin92, Lemma 1.2]. Let $(A_1, \ldots, A_n) \in \mathrm{Fix}(\phi_\sigma^\#)$ be given. It follows that $\mathrm{tr}\, A_i = \mathrm{tr}\, A_j$ if $\sigma^\wedge$ is a knot.

For a given $\alpha \in (0, \pi)$ let

$$R_n^\alpha := \{(A_1, \ldots, A_n) \mid \mathrm{tr}(A_i) = 2\cos\alpha,\, 1 \le i \le n\} \subset \mathrm{SU}(2)^n.$$

The space R_n^α carries the canonical product orientation because $R_n^\alpha = \Sigma_\alpha^n$ (see Section 4.1). Since $\phi_\sigma^\#(R_n^\alpha) = R_n^\alpha$ we obtain a diffeomorphism $\phi_\sigma^\#\colon R_n^\alpha \to R_n^\alpha$. Its fixed point set can be identified with $R_\alpha(\sigma^\wedge) := \{\rho \in R(\sigma^\wedge) \mid \mathrm{tr}\, \rho(m) = 2\cos\alpha\}$ where m is a meridian of $\sigma^\wedge$.

Let us consider the following subspaces of $R_n^\alpha \times R_n^\alpha$:

$$\begin{aligned}
H_n^\alpha &:= \{(A_1, \ldots, A_n, B_1, \ldots, B_n) \in R_n^\alpha \times R_n^\alpha | A_1 \cdots A_n = B_1 \cdots B_n\},\\
\Lambda_n^\alpha &:= \{(A_1, \ldots, A_n, A_1, \ldots, A_n) \in R_n^\alpha \times R_n^\alpha\},\\
\Gamma_\sigma^\alpha &:= \{(A_1, \ldots, A_n, \phi_\sigma^\#(A_1), \ldots, \phi_\sigma^\#(A_n)) \in R_n^\alpha \times R_n^\alpha\},\\
S_n^\alpha &:= \{(A_1, \ldots, A_{2n}) \in R_n^\alpha \times R_n^\alpha \mid A_i A_j = A_j A_i,\ 1 \le i, j \le n\}.
\end{aligned}$$

Fix a $\alpha \in I$ such that $\Delta_{\sigma^\wedge}(e^{2\boldsymbol{i}\alpha}) \neq 0$. The intersection $(\Lambda_n^\alpha \smallsetminus S_n^\alpha) \cap (\Gamma_\sigma^\alpha \smallsetminus S_n^\alpha)$ is compact in $H_n^\alpha \smallsetminus S_n^\alpha$ (see [HKr98, 3.6]). Moreover, for $\Theta \in \{H_n^\alpha, \Gamma_\sigma^\alpha, \Lambda_n^\alpha\}$ the quotient $\widehat{\Theta} := (\Theta \smallsetminus S_n^\alpha)/\sim$ is an oriented manifold and the intersection number $h^{(\alpha)}(\sigma) := \langle \widehat{\Lambda}_n^\alpha, \widehat{\Gamma}_\sigma^\alpha \rangle_{H_n^\alpha}$ is defined.

It is proved that for braids σ and τ which are defining equivalent knots $\sigma^\wedge \cong \tau^\wedge \subset S^3$ one gets $h^{(\alpha)}(\alpha) = h^{(\alpha)}(\beta)$ and therefore a knot invariant $h^{(\alpha)}(k)$ is established. Moreover, the equation $h^{(\alpha)}(k) = \frac{1}{2}\sigma_k(e^{2\boldsymbol{i}\alpha})$ holds where $\sigma_k(e^{2\boldsymbol{i}\alpha})$ denotes the Levine–Tristram signature of k (see [HKr98] for details and [Lin92] for the case $\alpha = \pi/2$).

5.2 Comparison with Lin's construction

Let $\sigma^\wedge$ be a knot and choose an orientation as in Figure 5. A closed n-braid is a very special $2n$-plat. Consider the homomorphisms $\lambda_i \colon \pi_1(S(2n)) \to \pi_1(B_i)$ given by

$$\lambda_i \colon s_j \mapsto t_j^{(i)} \text{ and } \qquad \lambda_i \colon s_{2n+1-j} \mapsto (t_j^{(i)})^{-1}, \quad 1 \le j \le n.$$

We obtain the maps $\kappa_i \colon \pi_1(S(2n)) \to \pi_1(B_i)$ given by $\kappa_2 = \lambda_2$ and $\kappa_1 = \lambda_1 \circ \sigma$. Here $\sigma \in B_{2n}$ has the property that $\sigma(s_j) = s_j$ for $n+1 \le j \le 2n$.

Denote by $Q_i \subset R^{\mathcal{S}}(S(2n))$ the image of $\kappa_i^\#$. We are interested in representations with a fixed trace. Therefore we consider the restriction of $\kappa_i^\#$ which gives an embedding $R_\alpha^{\mathcal{T}_i}(B_i) \hookrightarrow R_\alpha^{\mathcal{S}}(S(2n))$ (denote its image by $Q_i^{(\alpha)}$).

Consider the free group F_{2n} with Basis $\mathcal{S} = \{s_1, \ldots, s_{2n}\}$ and let the map $\phi_n^\alpha \colon R_n^\alpha \times R_n^\alpha \to R_\alpha^{\mathcal{S}}(F_{2n})$ be given by

$$\phi_n^\alpha \colon (A_1, \ldots, A_n, B_1, \ldots, B_n) \mapsto (A_1, \ldots, A_n, B_n^{-1}, \ldots, B_1^{-1}).$$

It is clear that ϕ_n^α is orientation preserving (resp. reversing) if and only if n is even (resp. odd). Moreover it is obvious that $\phi_n^\alpha(H_n^\alpha) = R_\alpha^{\mathcal{S}}(S(2n))$.

Lemma 5.1 *We have $\phi_n^\alpha(\Lambda_n^\alpha) = Q_1^{(\alpha)}$ and $\phi_n^\alpha(\Gamma_\sigma^\alpha) = Q_2^{(\alpha)}$.*

Proof. The lemma is proved by an easy calculation. □

Let $F_n^\alpha \colon R_n^\alpha \times R_n^\alpha \to \mathrm{SU}(2)$ be given by $F_n^\alpha \colon (A_1, \ldots, A_n, B_1, \ldots, B_n) \mapsto A_1 \cdots A_n B_n^{-1} \cdots B_1^{-1}$. Note that $F_n^\alpha = \mu \circ (f_n^\alpha \times f_n^\alpha)$ where $\mu \colon \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SU}(2)$ is given by $\mu \colon (A, B) \mapsto AB^{-1}$.

The orientation of $\widetilde{H}_n^\alpha := H_n^\alpha \setminus S_n^\alpha$ is given by the orientation of $R_n^\alpha \times R_n^\alpha \cong \Sigma_\alpha^{2n}$ and the orientation of the normal bundle $(F_n^\alpha)^*(\mathfrak{su}(2))$. Analogously, we have fixed the orientation of $\widetilde{R}_\alpha^{\mathcal{S}}(S(2n))$ by the orientation of $R_\alpha^{\mathcal{S}}(F_{2n}) \cong \Sigma_\alpha^{2n}$ and the orientation of the normal bundle $(f_{2n}^\alpha)^*(\mathfrak{su}(2))$ (see Section 4.1).

Lemma 5.2 *The map $\widehat{\phi}_n^\alpha \colon \widehat{H}_n^\alpha \to \widehat{R}_\alpha^{\mathcal{S}}(S(2n))$ is orientation preserving (resp. reversing) if and only if n is even (resp. odd).*

Proof. There is a commutative diagram

$$\begin{array}{ccc}
R_n^\alpha \times R_n^\alpha & \xrightarrow{\phi_n^\alpha} & R_\alpha^{\mathcal{S}}(F_{2n}) \\
\Big\downarrow{\scriptstyle F_n^\alpha} & & \Big\downarrow{\scriptstyle f_{2n}^\alpha} \\
\mathrm{SU}(2) & \xrightarrow{=} & \mathrm{SU}(2).
\end{array}$$

Therefore, the restriction of the derivative of ϕ_n^α gives an isomorphism between the oriented normal bundles of $\widetilde{H}_n^\alpha$ and $\widetilde{R}_\alpha^{\mathcal{S}}(S(2n))$

$$D\phi_n^\alpha \colon (F_n^\alpha)^*(\mathfrak{su}(2)) \to (f_{2n}^\alpha)^*(\mathfrak{su}(2)).$$

Since ϕ_n^α is orientation preserving (resp. reversing) if and only if n is even (resp. odd) the conclusion of the lemma follows. $\square$

In general $\widehat{Q}_1 \cap \widehat{Q}_2$ is not compact. There might be abelian representations which are the limit of non-abelian representations. However there is a criterion which ensures the compactness of the intersection $\widehat{Q}_1^{(\alpha)} \cap \widehat{Q}_2^{(\alpha)}$.

Lemma 5.3 *Let $k \subset S^3$ be a knot and let $\alpha \in I$ be given. If $\Delta_k(e^{\boldsymbol{i}2t}) \neq 0$ then $\widehat{Q}_1^{(\alpha)} \cap \widehat{Q}_2^{(\alpha)}$ is compact. Moreover, there is an $\epsilon > 0$ such that $\widehat{Q}_1^{(s)} \cap \widehat{Q}_2^{(s)} \subset R_s^{\mathcal{S}}(S(2n))$ is compact for $s \in (\alpha - \epsilon, \alpha + \epsilon)$.*

Proof. The lemma is a consequence of [Kla91, Theorem 19]. $\square$

Since $\widehat{R}_\alpha^{\mathcal{S}}(S(2n)) \subset \widehat{R}^{\mathcal{S}}(S(2n))$ is an oriented codimension one manifold and because $\dim \widehat{R}^{\mathcal{S}}(S(2n)) = 4n - 5$ we obtain that the dimension of $\widehat{Q}_i^{(\alpha)}$ is half the dimension of $\widehat{R}_\alpha^{\mathcal{S}}(S(2n))$. The intersection $Q_1^{(\alpha)} \cap Q_2^{(\alpha)}$ is compact by Lemma 5.3; remember that $\alpha \in I$ is fixed such that $\Delta_k(e^{\boldsymbol{i}2\alpha}) \neq 0$. Hence we are able to define the intersection number

$$\langle \widehat{Q}_1^{(\alpha)}, \widehat{Q}_2^{(\alpha)} \rangle_{\widehat{R}_\alpha^{\mathcal{S}}(S(2n))}.$$

Proposition 5.4 *Let $\sigma \in B_n$ be a braid such that $\sigma^\wedge$ is a knot. Then the map*

$$\widehat{\phi}_n^\alpha \colon \widehat{H}_n^\alpha \to (-1)^n \widehat{R}_\alpha^{\mathcal{S}}(S(2n))$$

is orientation preserving. Moreover we have

$$h^{(\alpha)}(\sigma^\wedge) = \langle \widehat{\Lambda}_n^\alpha, \widehat{\Gamma}_\sigma^\alpha \rangle_{\widehat{H}_n^\alpha} = (-1)^n \langle \widehat{Q}_1^{(\alpha)}, \widehat{Q}_2^{(\alpha)} \rangle_{\widehat{R}_\alpha^{\mathcal{S}}(S(2n))}.$$

Proof. The proof follows from Lemma 5.1 and Lemma 5.2. □

For given $\alpha_1, \alpha_2 \in I$ we denote by $\widehat{F}(\alpha_1, \alpha_2)$ the following subspace of $\widehat{R}^{\mathcal{S}}(S(2n))$

$$\widehat{F}(\alpha_1, \alpha_2) := \bigcup_{\beta \in [\alpha_1, \alpha_2]} \widehat{R}^{\mathcal{S}}_{\beta}(S(2n)).$$

There is an $\epsilon > 0$ such that

$$\widehat{F}(\alpha - \eta, \alpha + \eta) \cap \widehat{Q}_1 \cap \widehat{Q}_2$$

is compact for all $0 \leq \eta < \epsilon$.

We fix $\eta > 0$ such that $\widehat{F}_\eta \cap \widehat{Q}_1 \cap \widehat{Q}_2$ is compact where $\widehat{F}_\eta := \widehat{F}(\alpha - \eta, \alpha + \eta)$. In general we have $\widehat{Q}_i \pitchfork \widehat{R}^{\mathcal{S}}_{\alpha}(S(2n))$ for all $\alpha \in I$. Choose an isotopy $\widehat{Q}^{(\alpha)}_2 \rightsquigarrow \widetilde{Q}^{(\alpha)}_2$ with compact support such that $\widehat{Q}^{(\alpha)}_1 \pitchfork \widetilde{Q}^{(\alpha)}_2$. Extent this isotopy to an isotopy $\widehat{Q}_2 \rightsquigarrow \widetilde{Q}_2$ with compact support such that $\widehat{Q}_1 \pitchfork_{\widehat{F}_\eta} \widetilde{Q}_2$ and $\widetilde{Q}_2 \pitchfork R^{\mathcal{S}}_{\alpha}(S(2n))$ for all $\alpha \in I$. This is possible because the normal bundle of $\widehat{R}^{\mathcal{S}}_{\alpha}(S(2n)) \subset \widehat{R}^{\mathcal{S}}(S(2n))$ is trivial.

Remember that $\widehat{Q}_1 \cap \widetilde{Q}_2$ is an oriented one dimensional manifold in a neighborhood of $\widehat{R}^{\mathcal{S}}_{\alpha}(S(2n))$.

Lemma 5.5 *Let $\widehat{Q}_1$ and $\widetilde{Q}_2$ be given as above. Then the intersection number $\langle \widehat{Q}_1 \cap \widetilde{Q}_2, \widehat{R}^{\mathcal{S}}_{\alpha}(S(2n)) \rangle_{\widehat{R}^{\mathcal{S}}(S(2n))}$ is defined and the following equation holds:*

$$\langle \widehat{Q}^{(\alpha)}_1, \widehat{Q}^{(\alpha)}_2 \rangle_{\widehat{R}^{\mathcal{S}}_{\alpha}(S(2n))} = \langle \widehat{Q}_1 \cap \widetilde{Q}_2, \widehat{R}^{\mathcal{S}}_{\alpha}(S(2n)) \rangle_{\widehat{R}^{\mathcal{S}}(S(2n))}.$$

Proof. The manifold $\widehat{R}^{\mathcal{S}}_{\alpha}(S(2n)) \subset \widehat{F}_\eta$ is of codimension one. The intersection $\widehat{Q}_1 \cap \widetilde{Q}_2 \cap \widehat{F}_\eta$ is compact oriented and one dimensional. Therefore the intersection number is defined. The intersection numbers are equal which follows from the orientation convention (see Section 4.1). □

It is now possible to explain Lin's result: let $K \subset \widehat{R}(k)$ be a compact component and let $\widetilde{K}$ be a smooth oriented one dimensional approximation for K. It is obvious that

$$\langle \widetilde{K}, \widehat{R}^{\mathcal{S}}_{\alpha}(S(2n)) \rangle_{\widehat{R}^{\mathcal{S}}(S(2n))} = 0.$$

Therefore only a non-compact component of $\widehat{R}(k)$ can give a contribution to the intersection number.

Let $N \subset \widehat{R}(k) \subset \Re(k)$ be a non-compact component and let $\bar{N} \subset \Re(k)$ be its closure. The difference $\bar{N} \setminus N$ consists of finitely many abelian representations which are the limit of non-abelian representations. For that reason counting $\langle \widehat{\Delta}^{\alpha}_n, \widehat{\Gamma}^{\alpha}_\sigma \rangle_{\widehat{H}^{\alpha}_n}$ of representations with multiplicity is equivalent to counting the zeros of the Alexander polynomial on the unit circle with multiplicity. On the other hand, the signature $\sigma_k(e^{2i\alpha})$ is also a weighted sum of zeros of the Alexander polynomial (see [Kau87, Chapter XII]).

A further consequence of the connection is the following:

Theorem 5.6 *Let $k \subset S^3$ be a knot and denote by m its meridian and let $\alpha \in I$ be given such that $\Delta_k(e^{2\boldsymbol{i}\alpha})$, $\sigma_k(e^{2\boldsymbol{i}\alpha}) \neq 0$.*

Then there is a non abelian representation $\rho \in \widehat{R}(k)$ such that $\operatorname{tr}\rho(m) = 2\cos\alpha$. Moreover, there is an arc $\rho_t \in \Re(k)$, $\alpha \in [-\epsilon, \epsilon]$ through $\rho = \rho_0$ such that $\rho_{\pm\epsilon}$ are abelian and $\operatorname{tr}\rho_{-\epsilon}(m) < 2\cos\alpha$ and $\operatorname{tr}\rho_{\epsilon}(m) > 2\cos\alpha$.

Proof. Let $\widehat{Q}_1$ and $\widetilde{Q}_2$ as above. Since $\sigma_k(e^{2\boldsymbol{i}\alpha}) \neq 0$ there is an arc in $\widehat{Q}_1 \cap \widetilde{Q}_2 \cap \widehat{F}_\eta$ which connects $\widehat{R}^{\mathcal{S}}_{\alpha-\eta}(S(2n))$ and $\widehat{R}^{\mathcal{S}}_{\alpha+\eta}(S(2n))$. We have to conclude that there is already such an arc in $\widehat{Q}_1 \cap \widehat{Q}_2 \cap \widehat{F}_\eta$.

Now $\widehat{Q}_1 \cap \widehat{Q}_2 \subset \widehat{R}^{\mathcal{S}}(S(2n))$ can be identified with $\widehat{R}(k)$ which has the structure of a semi–algebraic set (see Section 2). Therefore we can think of $\widehat{Q}_1 \cap \widehat{Q}_2 \cap \widehat{F}_\eta$ as an compact semi–algebraic set. Each compact semi–algebraic set has a triangulation (see [BCR87, Théorème 9.2.1]).

Assume there is no path in $\widehat{Q}_1 \cap \widehat{Q}_2 \cap \widehat{F}_\eta$ connecting $\widehat{R}^{\mathcal{S}}_{\alpha-\eta}$ and $\widehat{R}^{\mathcal{S}}_{\alpha+\eta}$. We can choose an open regular neighborhood U of $\widehat{Q}_1 \cap \widehat{Q}_2 \cap \widehat{F}_\eta$ in $\widehat{R}^{\mathcal{S}}(S(2n))$. Of course there is no path in U connecting $\widehat{R}^{\mathcal{S}}_{\alpha-\eta}$ and $\widehat{R}^{\mathcal{S}}_{\alpha+\eta}$. It is possible to choose an isotopy $\widehat{Q}_2 \rightsquigarrow \widetilde{Q}_2$ with support contained in U. Since there is no path in U connecting $\widehat{R}^{\mathcal{S}}_{\alpha-\eta}$ and $\widehat{R}^{\mathcal{S}}_{\alpha+\eta}$ there can not be such a path in $\widehat{Q}_1 \cap \widetilde{Q}_2 \cap \widehat{F}_\eta$. By Lemma 5.5, Proposition 5.4 and [HKr98, Theorem 1.2] we have $\sigma_k(e^{2\boldsymbol{i}\alpha}) = 0$ which contradicts our assumption.

It is easy to see that in addition we may assume that $\rho_{\pm\epsilon}$ are reducible. □

Corollary 1.2 is an immediately consequence of theorem above.

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