Deformations of reducible representations of 3-manifold groups into $\mathrm{SL}_2(\mathbb{C})$

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Abstract

Let M be a 3-manifold with torus boundary which is a rational homology circle. We study deformations of reducible representations of $\pi_1(M)$ into $\mathrm{SL}_2(\mathbb{C})$ associated to a simple root of the Alexander polynomial. We also describe the local structure of the representation and character varieties.

1 Introduction

Let M be a connected, compact, orientable, irreducible, 3-manifold such that ∂M is a torus. We assume that the first Betti number $\beta_1(M)$ is one, i.e. M is a rational homology circle. A good class of examples arises from knots in S^3. For a given tame knot in S^3 the complement $M(k)$ of an open tubular neighborhood of k in S^3 satisfies all conditions. Since the rank of $H_1(M,\mathbb{Z})$ is one we have a canonical surjection $\phi\colon \pi_1(M) \to \Lambda$ where $\Lambda := H_1(M,\mathbb{Z})/tors(H_1(M,\mathbb{Z}))$ is an infinite cyclic group. Moreover, the Alexander polynomial $\Delta_M(t) \in \mathbb{Q}[t,t^{-1}]$ is well defined (see Section 2.1).

A representation $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ of a group Γ is called *irreducible* if the only subspaces of $\mathbb{C}^2$ which are invariant under $\rho(\Gamma)$ are $\{0\}$ and $\mathbb{C}^2$. According to a result of Thurston (see [Thu, Chapter 5] and [CS83, Proposition 3.2.1]) it is possible to deform an irreducible representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ non-trivially if $\rho(\mathrm{Im}(\pi_1(\partial M) \to \pi_1(M))) \not\subset \{\pm E\}$ where E denotes the unit matrix.

There is no general theorem which allows the deformation of reducible representations $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$. In [FK91] the authors proved that every abelian representation of a classical knot group which corresponds to a simple root of the Alexander polynomial on the complex unit circle is a limit point of an arc of irreducible representations $\rho_t\colon \pi_1(M) \to \mathrm{SU}(2)$. This result is generalized in [Her97] and [HK98] by replacing the condition of the simple root by a condition on the signature operator. An other generalization of the result of Frohman and Klassen recently established in [BA98b] (see also [BA98a]).

The first aim of this paper is to prove a deformation result for certain reducible, non abelian representations (Theorem 1.1). In a second step we shall use this result to study the local structure of the $\mathrm{SL}_2(\mathbb{C})$–representation variety (Theorem 1.2 and Corollary 1.3).

Every representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ which factors through $\phi\colon \pi_1(M) \to \Lambda$ is determinated by the image of a generator t of $\Lambda = \langle t | - \rangle$. For a given $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ we denote by ρ_λ the abelian representation which is given by $t \mapsto \mathrm{diag}(\lambda^1, \lambda^{-1}) \in \mathrm{SL}_2(\mathbb{C})$. Note that the representation ρ_λ depends on the choice of a generator of Λ but its character $\chi_{\rho_\lambda}\colon \pi_1(M) \to \mathbb{C}$, $\chi_{\rho_\lambda}(\gamma) := \mathrm{tr}\,\rho_\lambda(\gamma)$, is well defined. There exists a reducible, non abelian representation $\varphi_\lambda\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ such that $\chi_{\varphi_\lambda} = \chi_{\rho_\lambda}$ if and only if $\Delta_M(\lambda^2) = 0$. This is a well known result of Burde and de Rham if M is a complement of a knot in S^3, (see [Bur67, dR67] and Section 4.1).

We denote by $R(M) := R(M, \mathrm{SL}_2(\mathbb{C})) = \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))$ the $\mathrm{SL}_2(\mathbb{C})$–representation variety of the fundamental group $\pi_1(M)$ of M (see Section 2.2). The set of all representations $\rho \in R(M)$ which factor through $\phi\colon \pi_1(M) \to \Lambda$ is denoted by $S(M)$. Note that $S(M) \subset R(M)$ is an irreducible algebraic component (see Section 2.2). We shall prove of the following theorem:

1.1 Theorem *Let $\lambda \in \mathbb{C}^*$ be given such that $\rho_\lambda \in R(M)$ is not ∂-central and let $\varphi_\lambda\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a reducible, non abelian representation such that $\chi_{\varphi_\lambda} = \chi_{\rho_\lambda}$.*

If λ^2 is a simple root of the Alexander polynomial $\Delta_M(t)$, then the representation φ_λ is the limit of irreducible representations. More precisely, φ_λ is a smooth point of the representation variety $R(M)$; it is contained in a unique irreducible four–dimensional component $R_\lambda(M)$ of the $\mathrm{SL}_2(\mathbb{C})$–representation variety $R(M)$.

Here a representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is called *∂-central* iff the image of ρ restricted to $\mathrm{Im}(\pi_1(\partial M) \to \pi_1(M))$ is contained in the center $\{\pm E\}$ of $\mathrm{SL}_2(\mathbb{C})$. Note that ρ_λ, $\lambda \in \mathbb{C}^* \smallsetminus \{\pm 1\}$, is never ∂-central if $\phi(\mathrm{Im}(\pi_1(\partial M) \to \pi_1(M))) = \Lambda$ or when λ is not a root of unity (see Lemma 2.3). Hence ρ_λ is not ∂-central if $H_1(M, \mathbb{Z})$ has no torsion (i.e. M is the exterior of a knot in an integer homology sphere) or if M is the exterior of a zero homotop knot in a manifold with finite fundamental group.

The main ingredients of the proof of Theorem 1.1 are the existence of a set of natural obstructions which control the deformations of a given representation (see Section 3) and the calculation of the dimension of the space of cocycles $Z^1(M, \mathfrak{sl}_2^{\varphi_\lambda})$ (Section 4.1).

The *variety of characters* $X(\Gamma)$ of a finitely generated group Γ is the quotient in the algebraic category of the action of $\mathrm{SL}_2(\mathbb{C})$ by conjugation on the variety of representations $R(M)$ (see [MS84, II.4.]). We denote the projection by $\pi\colon R(\Gamma) \to X(\Gamma)$. Following Culler and Shalen (see [CS83]), $X(\Gamma)$ is a complex affine variety, but it is not necessarily irreducible. For a representation $\rho \in R(\Gamma)$, its projection onto $X(\Gamma)$, denoted by χ_ρ, is called the character of ρ. The character χ_ρ may be interpreted as a map:

$$\chi_\rho\colon \Gamma \to \mathbb{C}, \qquad \chi_\rho\colon \gamma \mapsto \mathrm{tr}(\rho(\gamma)).$$

Note that two irreducible representations ρ and ρ' are conjugate if and only if $\chi_\rho = \chi_{\rho'}$. Let $R^{irr}(\Gamma) \subset R(\Gamma)$ be the subset of irreducible representations and denote $X^{irr}(\Gamma) := \pi(R^{irr}(\Gamma))$. The subsets $R^{irr}(\Gamma) \subset R(\Gamma)$ and $X^{irr}(\Gamma) \subset X(\Gamma)$ are Zariski-open (see [CS83, 1.3.2]). The subset $R^{red}(\Gamma)$ of reducible representation is Zariski-closed and $X^{red}(\Gamma) := \pi(R^{red}(\Gamma))$.

We denote by $Y(M)$ the projection of $S(M)$ to the character variety $X(M)$. It is clear that $Y(M)$ is an irreducible component of $X^{red}(M)$. We also use the notation $\chi_\lambda := \chi_{\varphi_\lambda} = \chi_{\rho_\lambda}$ and $X_\lambda(M) := \pi(R_\lambda(M))$.

1.2 Theorem *Let ρ_λ be a representation as in Theorem 1.1. The curves $X_\lambda(M)$ and $Y(M)$ are the unique irreducible components of $X(M)$ that contain χ_λ. In addition χ_λ is a smooth point of both curves and*

$$T^{\mathrm{Zar}}_{\chi_\lambda}(X_\lambda(M)) \cap T^{\mathrm{Zar}}_{\chi_\lambda}(S(M)) = \{0\}$$

We obtain:

1.3 Corollary *Let ρ_λ be a representation as in Theorem 1.1. The varieties $R_\lambda(M)$ and $S(M)$ are the unique irreducible components of $R(M)$ that contain ρ_λ. In addition ρ_λ is a smooth point of both varieties and*

$$T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M)) \cap T^{\mathrm{Zar}}_{\rho_\lambda}(S(M)) = T^{\mathrm{Zar}}_{\rho_\lambda}(\mathcal{O}(\rho_\lambda))$$

where $\mathcal{O}(\rho_\lambda)$ is the orbit of ρ_λ.

Note that the orbit $\mathcal{O}(\rho_\lambda)$ of ρ_λ is a non-singular *proper* component of the intersection of $S(M)$ and $R_\lambda(M)$ i.e. $\dim(S(M)\cap R_\lambda(M)) = \dim \mathcal{O}(\rho_\lambda) = 2$. It follows from the proof of Theorem 1.2 and Corollary 1.3 that the kernel of the differential mapping of π at ρ_λ is the tangent space $T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M))$ (see Remark 5.16) i.e.

$$\mathrm{Ker}(d_{\rho_\lambda}\pi\colon T^{\mathrm{Zar}}_{\rho_\lambda}(R(M)) \to T^{\mathrm{Zar}}_{\chi_\lambda}(X(M))) = T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M)).$$

A character $\chi \in X(M)$ is said to be *real* if it is real valued (i.e. if $\chi(\gamma)\in\mathbb{R}$ for every $\gamma\in\Gamma$). Since $X(M)$ is defined over $\mathbb{Q}$ it makes sense to consider the variety $X(M)^{\mathbb{R}}$, which is the set of real points of $X(M)$. Points in $X(M)^{\mathbb{R}}$ are precisely the real characters, because the function algebra is generated by evaluation functions.

An irreducible representation $\rho\colon \Gamma\to \mathrm{SL}_2(\mathbb{C})$ is conjugate to a real representation (i.e. into $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SU}(2)$) if and only if its character $\chi_\rho\colon \Gamma\to\mathbb{C}$ is a real-valued function (see [HK97, Lemma 1]). It is clear that the character χ_λ is real-valued iff λ is real or on the complex unit circle.

When $|\lambda| = 1$, the next corollary is the theorem of Frohman and Klassen.

1.4 Corollary *Let $\lambda\in\mathbb{C}^*$ be given as in Theorem 1.1. If χ_λ is real, then χ_λ is a smooth point of the curve of real characters in $X_\lambda(M)$. (i.e. a neighborhood of χ_λ in $X_\lambda(M)\cap X(M)^{\mathbb{R}}$ is a smooth arc).*

In addition this smooth arc can be parametrized as $\{\chi_t \mid t\in(-\varepsilon,\varepsilon)\}$ such that $\chi_0=\chi_\lambda$, χ_t is irreducible for $t\neq 0$, and

(i) if $\lambda\in\mathbb{R}$, then χ_t is the character of a representation into $\mathrm{SL}_2(\mathbb{R})$;

(ii) if $|\lambda|=1$ then χ_t is the character of a representation into $\mathrm{SU}(2)$ for $t>0$ and $\mathrm{SU}(1,1)$ for $t<0$.

The group $\mathrm{SU}(1,1)$ is conjugate to $\mathrm{SL}_2(\mathbb{R})$. In the statement of the corollary we write both $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}(1,1)$ because, when $\lambda\in\mathbb{R}$ then $\rho_\lambda\in\mathrm{SL}_2(\mathbb{R})$, and when $|\lambda|=1$ then $\rho_\lambda\in \mathrm{SU}(1,1)\cap\mathrm{SU}(2)$.

After finishing this paper, we learned from E. P. Klassen that there is an overlap with the 1991 thesis of D. Shors [Sho91]. He obtained similar results in the case of the exterior of a knot in S^3, but unfortunately none of these results has been published.

If λ^2 is not a simple root of the Alexander polynomial the situation is more complicated even if we assume that λ^2 is no root of the second Alexander polynomial. In Section 6 we shall present the following examples which arise from knots in S^3.

Let $k\subset S^3$ be the knot 8_{20}. We have that $\Delta_k(t)=(t^2-t+1)^2$ and we denote $\xi := \exp(i\pi/6)$. The character χ_{ρ_ξ} is not a smooth point of the variety of irreducible representations. More precisely, there are at least two irreducible components of $\overline{X^{irr}(M(k))}$ passing through χ_{ρ_ξ}.

Let $k \subset S^3$ be the 2–bridge knot $\mathfrak{b}(49,17)$. We have $\Delta_k(t) = (2t^2-3t+2)^2$ and we denote by ζ a complex number on the unit circle such that $\Delta_k(\zeta^2)=0$. In this case every reducible but non abelian representation φ_ζ such that $\chi_{\varphi_\zeta} = \chi_{\rho_\zeta}$ is a smooth point of the representation variety but the transversality statements of Theorem 1.2 and Corollary 1.3 are not satisfied. The statement of Corollary 1.4 is also not valid : there is a real arc χ_t such that $\chi_0 = \chi_\lambda$ and χ_t is the character of a representation into SU(2) for all t in a neighborhood of 0.

The paper is organized as follows: In Section 2 the basic notation and facts are presented. In Section 3 we recall some results about deformation of representations. The proof of Theorem 1.1 is presented in section Section 4. Section 5 includes the proofs of Theorem 1.2 and Corollaries 1.3 and 1.4. The last section is devoted to the examples above.

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2 Notation and facts

Let $\Gamma := \langle S_1,\dots,S_n | R_1,\dots,R_m\rangle$ be a finitely presented group i.e. $\Gamma \cong F_n/R$ where $F_n := F(S_1,\dots,S_n)$ is the free group of rank n and $R = \langle R_1,\dots,R_m\rangle$ is the normal subgroup of F_n generated by the relations $R_j = R_j(S_1,\dots,S_n)$. We denote the canonical projection by $\psi\colon F_n \to \Gamma$.

2.1 Lemma *Let $\Lambda = \langle t|-\rangle$ be an infinite cyclic group and let $\Gamma = \langle S_1,\dots,S_n|R_1,\dots,R_m\rangle$ be a finitely presented group. For every surjective homomorphism $\varphi\colon \Gamma \to \Lambda$ there is a presentation $\langle S_1',\dots,S_n'|R_1',\dots,R_m'\rangle$ of Γ such that $\varphi(S_i') = t$.*

Proof. The lemma follows from the Euclidean algorithm (see example 2.2). □

2.2 Example Let Γ be given by $\Gamma = \langle S_1, S_2 | S_1^2 = S_2^3\rangle$ and let $\phi\colon \Gamma \to \Lambda$ be given by $\phi(S_1) = t^3$, $\phi(S_2) = t^2$. We define $S_1' := S_1S_2^{-1}$, $S_2' := S_2(S_1S_2^{-1})^{-1}$ and we obtain $\Gamma = \langle S_1', S_2' | S_1'S_2'S_1' = S_2'S_1'S_2'\rangle$.

2.3 Lemma *Let $\lambda \in \mathbb{C}^* \smallsetminus \{\pm 1\}$ be given. Then the representation ρ_λ is ∂-central iff and only if there is a integer $n>1$ such that $\phi(\mathrm{Im}(\pi_1(\partial M) \to \pi_1(M))) \subset n\mathbb{Z}$ and λ^2 is a root of unity of order n.*

Proof. If $\phi(\mathrm{Im}(\pi_1(\partial M) \to \pi_1(M))) = \Lambda$ or if λ is not a root of unity then ρ_λ is never ∂-central.

On the other hand if $\phi(\mathrm{Im}(\pi_1(\partial M) \to \pi_1(M))) \subset n\mathbb{Z}$ and if λ^2 is a root of unity of order n then $\rho_\lambda(\gamma) = \pm E$ for all $\gamma \in \mathrm{Im}(\pi_1(\partial M) \to \pi_1(M))$. □

2.1 The Alexander polynomial of M and Fox calculus

The standard reference for this section is [Mil68] all proofs and details can be found there. Let M be as in the introduction and denote its fundamental group $\pi_1(M)$ by Γ. We denote by $\widetilde{M}$ the infinite cyclic covering determined by the the epimorphism $\phi\colon \Gamma \to \Lambda \cong \mathbb{Z}$. The vector space $H_1(\widetilde{M},\mathbb{Q})$ is a torsion $\mathbb{Q}\Lambda$-module and a generator of its order ideal is called the Alexander polynomial of M; denoted by Δ_M (note that Δ_M depends only on the fundamental group Γ).

In order to proceed we choose a generator t of Λ. The group algebra $\mathbb{Q}\Lambda$ can be identified with the ring of Laurent polynomials $\mathbb{Q}[t,t^{-1}]$. Since the choice of a generator of Λ is not canonical there are in fact two polynomials $f(t)$ and $f(t^{-1})$ which correspond to the same element of the group algebra $\mathbb{Q}\Lambda$.

We obtain a presentation matrix $A(t)$ for $H_1(\widetilde{M},\mathbb{Q})$ over $\mathbb{Q}[t,t^{-1}]$ from a presentation of Γ as follows. Note first that the deficiency of $\Gamma := \pi_1(M)$ is one. Moreover, every presentation of Γ, obtained from a cell decomposition of M, has deficiency one ([Jac80, Chapter V]), i.e. we have a presentation

$$\Gamma = \langle S_1,\dots,S_n | R_1,\dots,R_{n-1}\rangle.$$

By Lemma 2.1 we assume that $\phi(S_i) = t$. The matrix $A(t)$ is obtained from the Jacobian $J(t) = (J_{ji}(t))$, $J_{ji}(t) = \phi\psi(\partial R_j/\partial S_i) \in \mathbb{Q}[t,t^{-1}]$, by omitting one of its columns (see [BZ85, Chapter 9.C]). It follows from the fundamental formula of the Fox calculus that $R_j - 1 = \sum_{i=1}^{n}(\partial R_j/\partial S_i)(S_i-1)$ and hence

$$\sum_{i=1}^{n} J_{ji}(t) = \sum_{i=1}^{n} \phi\psi(\partial R_j/\partial S_i) = 0, \qquad \text{in } \mathbb{Q}[t,t^{-1}]. \tag{1}$$

i.e. the columns of $J(t)$ are linear dependent (see [BZ85, 9.12]).

The Alexander polynomial $\Delta_M(t) \in \mathbb{Q}[t,t^{-1}]$ is the determinant of the $(n-1)\times(n-1)$ matrix $A(t)$,

$$\Delta_M(t) = \det A(t).$$

By the Blanchfield duality theorem (see [Bla57]) the Alexander polynomial is unique up to multiplication with elements of the form $\{\alpha t^n | \alpha \in \mathbb{Q},\, n \in \mathbb{Z}\}$. Note that $\Delta_M(1) \neq 0$.

In the sequel we use the following notations: the partial derivations $\partial/\partial S_i\colon \mathbb{Q}F_n \to \mathbb{Q}F_n$ are denoted by ∂_i. For a given non zero complex number $\lambda \in \mathbb{C}^*$ and an element $f(t) \in \mathbb{Q}[t,t^{-1}]$ we denote by $f(\lambda) \in \mathbb{C}$ the valuation of $f(t)$ at $t = \lambda$. For every $\eta \in \mathbb{Q}F_n$ we denote by $\overline{\eta}(t)$ its image in $\mathbb{Q}[t,t^{-1}]$ i.e. $\overline{\eta}(t) := \phi\psi(\eta)$.

2.4 Example $J(\lambda^2)$ denotes the $(n-1)\times n$ matrix over $\mathbb{C}$ with entries $J_{ji}(\lambda^2) = \overline{\partial_i R_j}(\lambda^2)$. In $\mathbb{Q}F_n$ we have $\partial_l(\eta_1\eta_2) = \partial_l(\eta_1)\overline{\eta_2}(1) + \eta_1\partial_l(\eta_2)$.

In the sequel we shall use the following lemma which connects the Fox derivations with the usual derivative. We denote by $\frac{d}{dt}\colon \mathbb{C}[t,t^{-1}] \to \mathbb{C}[t,t^{-1}]$ the usual differential operator i.e. $\frac{dt^n}{dt} = nt^{n-1}$. The Fox derivations of higher order are denoted by $\partial_{kl} := \frac{\partial^2}{\partial S_k \partial S_l}$.

2.5 Lemma *Let $F_n := F_n(S_1,\dots,S_n)$ be a free group of rank n and consider the epimorphism $\phi\colon F_n \to \langle t|-\rangle \cong \mathbb{Z}$ given by $\phi(S_i) = t$. For every element $R \in \operatorname{Ker}\phi$ we have:*

$$\frac{d}{dt}(t\overline{\partial_l R}(t)) + \sum_{k=1}^{n} \overline{\partial_{lk} R}(t) \equiv 0.$$

Proof. Note that the subgroup $\operatorname{Ker}\phi$ is normally generated by $\{e_2,\dots,e_n\}$ where $e_i := S_1 S_i^{-1}$. The set $\mathcal{S} := \{\gamma e_i \gamma^{-1} \mid \gamma \in F_n\}$ generates therefore $\operatorname{Ker}\phi$ as a subgroup.

For given l, $1 \le l \le n$ we define $D_l, G_l\colon \operatorname{Ker}\phi \to \mathbb{C}[t,t^{-1}]$ as follows:

$$D_l(R) := -t\frac{d}{dt}(\overline{\partial_l R}(t)) \quad \text{and} \quad G_l(R) := \sum_{k=1}^{n} \overline{\partial_{lk} R}(t) + \overline{\partial_l R}(t).$$

We show that $D_l, G_l\colon \operatorname{Ker}\phi \to (\mathbb{C}[t,t^{-1}],+)$ are homomorphisms which coincide on $\mathcal{S}$. This proves the lemma.

Let $V, W \in \operatorname{Ker}\phi$. We use the formula $\partial_l(VW) = \partial_l V + V\partial_l W$ and we obtain:

$$\begin{aligned} D_l(VW) &= -t\frac{d}{dt}(\overline{\partial_l(VW)}(t)) \\ &= -t\frac{d}{dt}(\overline{\partial_l V + V\partial_l W}(t)) \\ &= -t\frac{d}{dt}(\overline{\partial_l V}(t)) + -t\frac{d}{dt}(\overline{\partial_l W}(t)) \quad \text{because } \overline{V}(t)=1 \Leftrightarrow V \in \operatorname{Ker}\phi \\ &= D_l(V) + D_l(W) \end{aligned} \tag{2}$$

and

$$\begin{aligned} G_l(VW) &= \sum_{k=1}^{n} \overline{\partial_{lk}(VW)}(t) + \overline{\partial_l(VW)}(t) \\ &= \sum_{k=1}^{n} \overline{\partial_l(\partial_k V + V\partial_k W)}(t) + \overline{\partial_l V + V\partial_l W}(t) \\ &= \sum_{k=1}^{n} \overline{\partial_{lk} V}(t) + \overline{\partial_l V}(t) + \sum_{k=1}^{n} \overline{\partial_l V}(t)\overline{\partial_k W}(1) + \sum_{k=1}^{n} \overline{V\partial_{lk} W}(t) + \overline{V\partial_l W}(t) \\ &= G_l(V) + G_l(W) \end{aligned} \tag{3}$$

The last equation follows now because $\overline{V}(t) = 1$ and $\sum_{k=1}^{n} \overline{\partial_k W}(1) = 0$ if $V \in \operatorname{Ker}\phi$. Note that $\overline{\partial_k W}(1)$ is the exponent sum of S_k in W.

We have $\epsilon_{li} := \overline{\partial_l e_i}(t) \in \{-1, 0, 1\}$. A short calculation (details are left to the reader) gives:

$$D_l(\gamma e_i \gamma^{-1}) = G_l(\gamma e_i \gamma^{-1}) = -t^{n_\gamma} n_\gamma \epsilon_{li} \quad \text{if} \quad \phi(\gamma) = t^{n_\gamma}.$$

We have therefore $D_l(R) = G_l(R)$ for all $R \in \operatorname{Ker}\phi$. □

2.2 Representation spaces

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. In general we call a representation $\rho\colon \Gamma \to G$ *abelian* (resp. *central*) iff its image is contained in an abelian subgroup (resp. in the center) of G. Note that $G = \mathrm{SU}(2)$ the notations reducible and abelian coincide. The space of all representations of a finitely presented group $\Gamma = \langle S_1, \ldots, S_n | R_1, \ldots, R_m \rangle$ to G, denoted by $R(\Gamma, G)$ and equipped with the compact open topology, can be identified with the following space:

$$R(\Gamma, G) := \{(g_1, \ldots, g_n) \in G^n \mid R_j(g_1, \ldots, g_n) = E,\ j = 1, \ldots, m\}$$

where E is the unity in G. Hence we have a smooth map $\mathbf{R}\colon G^n \to G^m$ and $R(\Gamma, G)$ can be identified with $\mathbf{R}^{-1}(E, \ldots, E)$. The element $(g_1, \ldots, g_n)$ is identified with the representation ρ if and only if $\rho(S_i) = g_i$.

Let $\rho\colon \Gamma \to G$ be a representation. The Lie algebra $\mathfrak{g}$ can be viewed as a Γ–module, denoted by $\mathfrak{g}^\rho$, via $\operatorname{Ad}\circ\rho$, i.e. $\gamma \circ X = \operatorname{Ad}_{\rho(\gamma)}(X)$ for all $\gamma \in \Gamma$ and $X \in \mathfrak{g}$. We denote by $C^n := C^n(\Gamma, \mathfrak{g}) := \{u\colon \Gamma^n \to \mathfrak{g}\}$ the space of n–cochains and the coboundary operator is denoted by

$\delta\colon C^n \to C^{n+1}$. Let $B^*(\Gamma, \mathfrak{g}^\rho)$ (resp. $Z^*(\Gamma, \mathfrak{g}^\rho)$, resp. $H^*(\Gamma, \mathfrak{g}^\rho)$) be the coboundaries (resp. cocycles, resp. cohomology group) of Γ with coefficients in $\mathfrak{g}^\rho$.

Let $f\colon \Gamma' \to \Gamma$ be a homomorphism. We obtain a representation $f^*\rho := \rho \circ f$, $f^*\rho\colon \Gamma' \to G$, and the Lie algebra $\mathfrak{g}$ can be viewed as a Γ'–module. The cochain map $f^*\colon C^n(\Gamma, \mathfrak{g}) \to C^n(\Gamma', \mathfrak{g})$ induces a homomorphism $f^*\colon H^n(\Gamma, \mathfrak{g}^\rho) \to H^n(\Gamma', \mathfrak{g}^{f^*\rho})$.

The cohomology class of a cocycle u is denoted by $[u]$. By composing the cup product with the Lie bracket we obtain the *cup-bracket* $C^p \otimes C^q \xrightarrow{\sqcup} C^{p+q}$ given by

$$(u \sqcup v)(\gamma_1, \dots, \gamma_{p+q}) := [u(\gamma_1, \dots, \gamma_p), (\gamma_1 \cdots \gamma_p) \circ v(\gamma_{p+1}, \dots, \gamma_{p+q})]$$

(see [Bro82] for the details).

It was observed by Weil (see [Wei64, LM85]) that the space of cocycles $Z^1(\Gamma, \mathfrak{g}^\rho)$ can be identified with the kernel of the derivative $D_{\mathbf{g}}\mathbf{R}$ where $\mathbf{g} = (g_1, \dots, g_n)$ corresponds to the representation ρ i.e. $\rho(S_i) = g_i$. This observation is based on the fact that every element $W \in F_n$ gives an evaluation $e_W\colon G^n \to G$ and that we have the following commutative diagram

$$\begin{array}{ccc} T_{\mathbf{g}}(G^n) & \xrightarrow{\cdot(g_1^{-1}, \dots, g_n^{-1})} & \mathfrak{g}^n \\ {\scriptstyle D_{\mathbf{g}}(e_W)}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ T_{e_W(\mathbf{g})}(G) & \xrightarrow{\cdot W(\mathbf{g})^{-1}} & \mathfrak{g} \end{array}$$

where $\Phi(X_1, \dots, X_n) = \sum_{i=1}^n \partial_i W \circ X_i$ (the action of F_n on $\mathfrak{g}$ is given by $S_i \circ X = \mathrm{Ad}_{g_i}(X)$). Hence we have:

$$Z^1(\Gamma, \mathfrak{g}^\rho) \cong \{(X_1, \dots, X_n) \in \mathfrak{g}^n \mid \sum_{i=1}^n \partial_i R_j \circ X_i = 0, \quad \text{for } j = 1, \dots, m\}. \tag{4}$$

The space $R(\Gamma, G)$ inherits an algebraic structure if G is an algebraic group. We are mainly interested in the case $G = \mathrm{SL}_2 := \mathrm{SL}_2(\mathbb{C})$ and we shall write $R(\Gamma) := R(\Gamma, \mathrm{SL}_2)$. We choose the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ for the Lie algebra $\mathfrak{sl}_2$ of SL_2 where

$$\mathbf{e}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

It is easy to see that the map $\mathbf{R}\colon \mathrm{SL}_2^n \to \mathrm{SL}_2^m$ is polynomial in the ambient coordinates ($\mathrm{SL}_2 \subset \mathbb{C}^4$). The set $\mathbf{R}^{-1}(E, \dots, E) \subset \mathbb{C}^{4n}$ is therefore an affine algebraic set and the induced algebraic structure on $R(\Gamma)$ does not depend from the presentation. The space $R(\Gamma)$ carries two topologies, the *Zariski* and the *complex* or *classical* topology (see [Sha77, Ch. II, § 2.3]). If we refer to the Zariski topology we shall use in the sequel the addition Zariski, e.g. Zariski–open.

It follows that the Zariski tangent space of $R(\Gamma)$ at ρ, denoted by $T_\rho^{\mathrm{Zar}}(R(\Gamma))$, can be identified with a subspace of the cocycles $Z^1(\Gamma, \mathfrak{sl}_2^\rho)$ (see [Wei64, LM85, Por97]). For every $\rho \in R(G)$ we have $\dim_\rho R(\Gamma) \leq T_\rho^{\mathrm{Zar}}(R(\Gamma))$ where $\dim_\rho R(\Gamma)$ denotes the local dimension of $R(G)$ at ρ (see [Sha77, Ch. II, § 1.4]). Here and in the sequel we shall call a representation $\rho \in R(\Gamma)$ *regular* if $\dim_\rho R(\Gamma) = \dim Z^1(\Gamma, \mathfrak{sl}_2^\rho)$. This notation is justified by the following lemma:

2.6 Lemma *Let $\rho \in R(\Gamma)$ be given. If ρ is regular then ρ is a smooth point of the representation variety $R(\Gamma)$ and ρ is contained in a unique component of dimension* $\dim Z^1(\Gamma, \mathfrak{sl}_2^\rho)$.

Proof. For every $\rho \in R(\Gamma)$ we have $\dim_\rho R(\Gamma) \leq T_\rho^{\mathrm{Zar}}(R(\Gamma)) \leq \dim \operatorname{Ker} D_{\mathbf{g}}\mathbf{R} = \dim Z^1(\Gamma, \mathfrak{sl}_2^{\rho_1})$. The representation ρ is therefore a *simple* point of $R(\Gamma)$ (see [Sha77, Ch. II, § 1.4]). The conclusion follows from Theorem 6 of [Sha77, Ch. II, § 2.2]. □

Let now $\Gamma = \pi_1(M)$ be the fundamental group of a three dimensional manifold as in the introduction. We choose a generator t of Λ and a presentation $\Gamma = \langle S_1, \ldots, S_n | R_1, \ldots, R_{n-1}\rangle$ such that $\phi(S_i) = t$. For a given $\lambda \in \mathbb{C}^*$ let $\rho_\lambda\colon \Gamma \to \mathrm{SL}_2$ be given by $\rho_\lambda\colon S_i \mapsto \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{smallmatrix}\right)$. It follows from Proposition 3.4 of [LM85] that the trivial representation ρ_1 is a smooth point of $R(M) := R(\pi_1(M))$ and that the unique irreducible component $S(M) \subset R(M)$ which contains ρ_1 is the union of all representations which factor through $\phi\colon \Gamma \to \Lambda$. This is the special case of the of the following result which will be need in the sequel:

2.7 Theorem (Klassen [Kla91]) *Let $\lambda \in \mathbb{C}^*$ be given. If $\Delta_M(\lambda^2) \neq 0$ then there is a neighborhood of ρ_λ in $R(\Gamma)$ consisting entirely of points of the component $S(M)$. Moreover, $\rho_\lambda \in R(\Gamma)$ is a smooth point and $S(M)$ is the unique component through ρ_λ.*

Proof. It is clear that $\rho_\lambda \in S(M)$ and that $\dim S(M) = 3$. By Lemma 2.6 we have to show that $\dim Z^1(\Gamma, \mathfrak{sl}_2^{\rho_\lambda}) = 3$.

The action of Γ on $\mathfrak{sl}_2$ is given by $\mathrm{Ad}_{\rho_\lambda(S_i)} = \mathrm{diag}(\lambda^2, 1, \lambda^{-2})$ with respect to the basis $\{\mathbf{e}_i\}$. Hence we have

$$Z^1(\Gamma, \mathfrak{sl}_2^{\rho_\lambda}) = Z^1(\Gamma, \mathbb{C}_{\lambda^2}) \oplus Z^1(\Gamma, \mathbb{C}) \oplus Z^1(\Gamma, \mathbb{C}_{\lambda^{-2}})$$

where $\mathbb{C}_\alpha$, $\alpha \in \mathbb{C}^*$, denotes the Γ–module $\mathbb{C}$ (the action is given by $S_i \circ z = \alpha z$). We have the identification

$$Z^1(\Gamma, \mathbb{C}_{\lambda^2}) \cong \operatorname{Ker} J(\lambda^2)$$

where we think of the valuation of the Jacobian $J(\lambda^2)\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ as a linear mapping.

It follows from the Blanchfield duality theorem that $\operatorname{rk} J(\lambda^2) = \operatorname{rk} J(\lambda^{-2})$. Since the Alexander polynomial is a principal minor of $J(\lambda^{\pm 2})$ and since $\Delta_M(\lambda^{\pm 2}) \neq 0$ we obtain $\operatorname{rk} J(\lambda^{\pm 2}) = n-1$. From which the lemma follows. Note that $\mathbb{C} = \mathbb{C}_1$ is the trivial Γ–module and that $\operatorname{rk} J(1) = n-1$ since $\Delta_M(1) \neq 0$. □

3 Review on the deformations of representations

In order to construct deformations of a given representation we use the classical approach, i.e. we first solve the corresponding formal problem and apply then a deep theorem of Artin (see Proposition 3.6). The formal deformations of a representation $\rho\colon \Gamma \to \mathrm{SL}_2 := \mathrm{SL}_2(\mathbb{C})$ are in general determined by infinite series of obstructions (see [BA98b, Gol84]). This obstructions where first studied by Kodaira and Spencer in a different context. The point of view presented here is motivated by Douady (see [Dou61]).

Let Γ be a finitely presented group and let $\rho\colon \Gamma \to \mathrm{SL}_2$ be a representation. A *formal deformation* of ρ is a representation $\rho_\infty\colon \Gamma \to \mathrm{SL}_2(\mathbb{C}[[t]])$ such that $p_0 \circ \rho_\infty = \rho$ where $p_0\colon \mathrm{SL}_2(\mathbb{C}[[t]]) \to \mathrm{SL}_2$ is the evaluation homomorphism at $t = 0$. Here we denote by $\mathbb{C}[[t]]$ the ring of formal power series.

Every formal deformation ρ_∞ of ρ can be written in the form

$$\rho_\infty(\gamma) = \exp(\sum_{i=1}^{\infty} t^i u_i(\gamma))\rho(\gamma)$$

where $u_i\colon \Gamma \to \mathfrak{sl}_2$ are elements of $C^1(\Gamma, \mathfrak{sl}_2^\rho)$ and an easy calculation gives that $u_1 \in Z^1(\Gamma, \mathfrak{sl}_2^\rho)$ is a cocycle (see also Lemma 3.3). We call a cocycle $u_1 \in Z^1(\Gamma, \mathfrak{sl}_2^\rho)$ *integrable* if there is a formal deformation of ρ with leading term u_1.

For every $k \in \mathbb{Z}$, $k \geq 0$, we define the ring $A_k := \mathbb{C}[[t]]/(t^{k+1})$ and $A_\infty := \mathbb{C}[[t]]$. We are interested in the following Lie group $G_k := \mathrm{SL}_2(A_k)$ and in its Lie algebra $\mathfrak{g}_k := \mathfrak{sl}_2(A_k)$ (see [Ser92]). Note that $G_0 = \mathrm{SL}_2$, $\mathfrak{g}_0 = \mathfrak{sl}_2$ and $\mathfrak{g}_k = \{\sum_{i=0}^k t^i X_i \mid X_i \in \mathfrak{sl}_2\}$. For every $k > l$ we have a projection $\pi_{k,l}\colon G_k \to G_l$. The projection $\pi_{k+1,k}$ is denoted by π_k and $\pi_{\infty,k}$ is denoted by p_k.

Let $\rho \in R(\Gamma)$ and $u_i\colon \Gamma \to \mathfrak{sl}_2$, $i = 1, \dots, k$, be given. We define a map $\tilde\rho_k := \tilde\rho_k^{(\rho;u_1,\dots,u_k)}\colon \Gamma \to G_\infty$ by

$$\tilde\rho_k(\gamma) := \exp(tu_1(\gamma) + \cdots + t^k u_k(\gamma))\rho(\gamma). \tag{6}$$

For all $i \geq 0$ we obtain a map $\rho_i\colon \Gamma \to G_i$ given by $\rho_i := \rho_i^{(\rho;u_1,\dots,u_k)} := p_i \circ \tilde\rho_k$.

In the sequel we shall denote by δ the coboundary operator of $C^*(\Gamma, \mathfrak{sl}_2^\rho)$. We shall prove the following proposition:

3.1 Proposition *Let $\rho \in R(\Gamma)$ and $u_i \in C^1(\Gamma, \mathfrak{sl}_2^\rho)$, $1 \leq i \leq k$ be given. If $\rho_k^{(\rho;u_1,\dots,u_k)}$ is a homomorphism then there is an obstruction class $\zeta_{k+1} := \zeta_{k+1}^{(u_1,\dots,u_k)} \in H^2(\Gamma, \mathfrak{sl}_2^\rho)$ with the following properties:*

(i) *There is a cochain $u_{k+1}\colon \Gamma \to \mathfrak{sl}_2$ such that $\rho_{k+1}^{(\rho;u_1,\dots,u_{k+1})}$ is a homomorphism if and only if $\zeta_{k+1} = 0$.*

(ii) *The obstruction ζ_{k+1} is natural i.e. if $f\colon \Gamma' \to \Gamma$ is a homomorphism then*

$$f^*\rho_k^{(\rho;u_1,\dots,u_k)} := \rho_k^{(\rho;u_1,\dots,u_k)} \circ f = \rho_k^{(f^*\rho;f^*u_1,\dots,f^*u_k)}$$

is also a homomorphism and $f^*(\zeta_{k+1}^{(u_1,\dots,u_k)}) = \zeta_{k+1}^{(f^*u_1,\dots,f^*u_k)}$.

As a consequence we obtain:

3.2 Corollary *Let $\rho \in R(G)$ be given. An infinite sequence $u_i \in C^1(\Gamma, \mathfrak{sl}_2^\rho)$, $i \in \mathbb{N}$, defines a representation $\rho_\infty\colon \Gamma \to \mathrm{SL}_2(\mathbb{C}[[t]])$,*

$$\rho_\infty(\gamma) = \exp(\sum_{i=1}^\infty t^i u_i(\gamma))\rho(\gamma),$$

if and only if $u_1 \in Z^1(\Gamma, \mathfrak{sl}_2^\rho)$ is a cocycle and $\zeta_{k+1}^{(u_1,\dots,u_k)} = 0$ for all $k \geq 1$.

Proof. If ρ_∞ is a homomorphism then $\rho_k := p_k \circ \rho_\infty$ is a homomorphism for all k. Since $\rho_k = \rho_k^{(\rho;u_1,\dots,u_k)}$ and $\rho_{k+1} = \rho_{k+1}^{(\rho;u_1,\dots,u_{k+1})}$ we have $\zeta_{k+1}^{(u_1,\dots,u_k)} = 0$ by Proposition 3.1.

If u_1 is a cocycle and if $\zeta_{k+1}^{(u_1,\dots,u_k)} = 0$ for all $k \geq 1$ then by Proposition 3.1 $\rho_k := \rho_k^{(\rho;u_1,\dots,u_k)}$ is a homomorphism for all $k \geq 1$ and hence ρ_∞ is a homomorphism. □

For given $u_i\colon \Gamma \to \mathfrak{sl}_2$, $i = 1, \dots, k$, we define $\tilde U_{k-1} := \tilde U_{k-1}^{(u_1,\dots,u_k)}\colon \Gamma \to \mathfrak{g}_\infty$ as follows:

$$\tilde U_{k-1}(\gamma) = u_1(\gamma) + 2tu_2(\gamma) + \ldots + kt^{k-1}u_k(\gamma) \tag{7}$$

For all $i \geq 0$ we obtain a map $U_i\colon \Gamma \to \mathfrak{g}_i$ given by $U_i := U_i^{(u_1,\dots,u_k)} := p_i \circ \tilde U_{k-1}$.

We fix from now on a representation $\rho \in R(G)$.

3.3 Lemma *Let* $u_i\colon \Gamma \to \mathfrak{sl}_2$, $i = 1, \dots, k+1$ *be given and define* $\tilde\rho_{k+1}$ *(resp.* $\tilde U_k$*) as in equation (6) (resp. equation (7)). Assume that* $\rho_k := p_k \circ \tilde\rho_{k+1}\colon \Gamma \to G_k$ *is a homomorphism. Then* $\rho_{k+1} := p_{k+1} \circ \tilde\rho_{k+1}\colon \Gamma \to G_{k+1}$ *is a homomorphism if and only if* $U_k := p_k \circ \tilde U_k \in Z^1(\Gamma, \mathfrak{g}_k^{\rho_k})$ *is a cocycle.*

Proof. The map ρ_{k+1} is a homomorphism if and only if

$$\tilde\rho_{k+1}(\gamma_1)\tilde\rho_{k+1}(\gamma_2) \equiv \tilde\rho_{k+1}(\gamma_1\gamma_2) \bmod t^{k+2}.$$

If we apply the usual differential operator $\frac{d}{dt}$ to this equation we obtain:

$$\tilde U_k(\gamma_1)\tilde\rho_{k+1}(\gamma_1)\tilde\rho_{k+1}(\gamma_2) + \tilde\rho_{k+1}(\gamma_1)\tilde U_k(\gamma_2)\tilde\rho_{k+1}(\gamma_2) \equiv \tilde U_k(\gamma_1\gamma_2)\tilde\rho_{k+1}(\gamma_1\gamma_2) \bmod t^{k+1}.$$

Since ρ_k is a homomorphism this is equivalent to the following equation in $\mathfrak{g}_k$:

$$U_k(\gamma_1) + \rho_k(\gamma_1)U_k(\gamma_2)\rho_k(\gamma_1)^{-1} = U_k(\gamma_1\gamma_2)$$

hence $U_k \in Z^1(\Gamma, \mathfrak{g}_k^{\rho_k})$ is a cocycle.

If $U_k \in Z^1(\Gamma, \mathfrak{g}_k^{\rho_k})$ is a cocycle then we use the same calculation and we obtain

$$\tilde\rho_{k+1}(\gamma_1)\tilde\rho_{k+1}(\gamma_2) - \tilde\rho_{k+1}(\gamma_1\gamma_2) \equiv C \bmod t^{k+2}$$

where $C \in \mathrm{M}_2(\mathbb{C})$ is a matrix. We obtain $C = 0$ by evaluating the equation at $t = 0$. □

Let $\rho_k := \rho_k^{(\rho;u_1,\dots,u_k)}\colon \Gamma \to G_k$ be a homomorphism. In order to find a cochain $u_{k+1}\colon \Gamma \to \mathfrak{sl}_2$ such that $\rho_{k+1} := \rho_{k+1}^{(\rho;u_1,\dots,u_{k+1})}$ is a homomorphism we consider the following exact sequence of Γ–modules

$$0 \to \mathfrak{sl}_2^\rho \xrightarrow{\alpha_k} \mathfrak{g}_k^{\rho_k} \xrightarrow{\pi_{k-1}} \mathfrak{g}_{k-1}^{\rho_{k-1}} \to 0$$

where $\alpha_k(X) = t^k X$ and $\rho_{k-1} = \pi_{k-1} \circ \rho_k$. This sequence gives rise to the following exact sequence in cohomology (see Proposition 6.1 of [Bro82, Ch. III]):

$$H^1(\Gamma, \mathfrak{g}_k^{\rho_k}) \xrightarrow{(\pi_{k-1})_*} H^1(\Gamma, \mathfrak{g}_{k-1}^{\rho_{k-1}}) \xrightarrow{\beta_{k-1}} H^2(\Gamma, \mathfrak{sl}_2^\rho). \tag{8}$$

3.4 Definition Let u_i, $i = 1, \dots, k$, be given. If $\rho_k := \rho_k^{(\rho;u_1,\dots,u_k)}\colon \Gamma \to G_k$ is a homomorphism then by Lemma 3.3 $U_{k-1}^{(u_1,\dots,u_k)} \in Z^1(\Gamma, \mathfrak{g}_{k-1}^{\rho_{k-1}})$ where $\rho_{k-1} = \pi_{k-1} \circ \rho_k$. We define

$$\zeta_{k+1} = \zeta_{k+1}^{(u_1,\dots,u_k)} := \beta_{k-1}([U_{k-1}^{(u_1,\dots,u_k)}]) \in H^2(\Gamma, \mathfrak{sl}_2^\rho).$$

Note that we have the following explicit construction for ζ_{k+1}. Denote by δ_k the coboundary operator of $C^*(\Gamma, \mathfrak{g}_k^{\rho_k})$ and let $\tilde U_{k-1} := \tilde U_{k-1}^{(u_1,\dots,u_k)}$ be given as in equation (7). Then we get

$$(\alpha_k)_*(\zeta_{k+1}) = [\delta_k(p_k \circ \tilde U_{k-1})].$$

3.5 Example Let $\rho\colon \Gamma \to \mathrm{SL}_2$ be a representation and let $u_1 \in Z^1(\Gamma, \mathfrak{sl}_2^\rho)$ be given. We have a homomorphism $\rho_1\colon \Gamma \to G_1$ given by

$$\rho_1(\gamma) = (E + t u_1(\gamma))\rho(\gamma).$$

We consider u_1 as a map $u_1\colon \Gamma \to \mathfrak{g}^1$ and an easy calculation gives $\delta_1(u_1) = t(u_1 \sqcup u_1)$. We have therefore $\zeta_2 = \zeta_2^{(u_1)} = [u_1 \sqcup u_1]$. If $0 = \zeta_2 \in H^2(\Gamma, \mathfrak{sl}_2^\rho)$ we can choose $u_2 \in C^1(\Gamma, \mathfrak{sl}_2^\rho)$ such that $2\,\delta(u_2) + u_1 \sqcup u_1 = 0$. The map $U_1^{(u_1,u_2)} \in Z^1(\Gamma, \mathfrak{g}_1^{\rho_1})$ is a cocycle and $\rho_2^{(u_1,u_2)}\colon \Gamma \to G_2$ a homomorphism.

Proof of Proposition 3.1. Let $\rho_k := \rho_k^{(\rho;u_1,\dots,u_k)}\colon \Gamma \to G_k$ be a homomorphism. By Lemma 3.3 we have $U_{k-1}^{(u_1,\dots,u_k)} \in Z^1(\Gamma, \mathfrak{g}_{k-1}^{\rho_{k-1}})$ where $\rho_{k-1} = \pi_{k-1} \circ \rho_k$.

Form the exactness of the sequence (8) it follows that $\beta_{k-1}([U_{k-1}^{(u_1,\dots,u_k)}]) = 0$ if and only if $[U_{k-1}^{(u_1,\dots,u_k)}] \in \operatorname{Im}(\pi_{k-1})_*$. This is equivalent to the existence of a cocycle $U_k \in Z^1(\Gamma, \mathfrak{g}^{\rho_k})$ such that $U_{k-1}^{(u_1,\dots,u_k)} = \pi_{k-1} \circ U_k$. It follows that $U_k = U_k^{(u_1,\dots,u_{k+1})}$ for a map $u_{k+1}\colon \Gamma \to \mathfrak{sl}_2$ and $\rho_{k+1}^{(u_1,\dots,u_{k+1})}$ is a homomorphism by Lemma 3.3.

The naturality assertion follows from the definition of the connection homomorphism (see Proposition 6.1 of [Bro82, Ch. II]). □

We denote by $\mathbb{C}\{t\} \subset \mathbb{C}[[t]]$ the ring of convergent power series. Starting from a formal deformation of ρ we obtain a *convergent* deformation as follows:

3.6 Proposition *Let $\rho_\infty\colon \Gamma \to \mathrm{SL}_2(\mathbb{C}[[t]])$ be a formal deformation of $\rho \in R(\Gamma)$. Then for every $N \in \mathbb{N}$ there exists a convergent deformation $\widehat{\rho}_\infty\colon \Gamma \to \mathrm{SL}_2(\mathbb{C}\{t\})$ such that $\widehat{\rho}_\infty(\gamma) \equiv \rho_\infty(\gamma) \bmod t^N$ for all $\gamma \in \Gamma$.*

Proof. Let $\Gamma = \langle S_1, \dots, S_n \mid R_1, \dots, R_m \rangle$ be a finite presentation. We have $R(\Gamma) \subset \mathrm{SL}_2^n$ and we fix $(A_1, \dots, A_n) \in \mathrm{SL}_2^n$ such that $\rho(S_i) = A_i$. It is easy to see that we can identify the space $R(\Gamma)$ with the following subset of $\mathbb{C}^{4n}$:

$$\{(Y_1, \dots, Y_n) \in \mathrm{M}_2(\mathbb{C}) \mid (E + Y_i) \in \mathrm{SL}_2,\ R_j((E + Y_1)A_1, \dots, (E + Y_n)A_n) = E\}.$$

Hence there is a system of polynomial equations $\mathbf{F}(\mathbf{y}) = \mathbf{0}$ such that

$$R(\Gamma) \cong V(\mathbf{F}) := \{\mathbf{y} \in \mathbb{C}^{4n} \mid \mathbf{F}(\mathbf{y}) = \mathbf{0}\}. \tag{9}$$

Note that the solution $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ corresponds to the representation ρ. A formal deformation of ρ corresponds to a formal solution $\mathbf{y}(t) \in \mathbb{C}[[t]]$, $\mathbf{y}(0) = \mathbf{0}$, of the system $\mathbf{F}(\mathbf{y}(t)) = \mathbf{0}$. By a theorem of Artin (see [Art68]) there is for a given $N \in \mathbb{N}$ a convergent solution $\widehat{\mathbf{y}}(t) \in \mathbb{C}\{t\}$ such that $\widehat{\mathbf{y}}(t) \equiv \mathbf{y}(t) \bmod t^N$. □

The following lemma will be used in the sequel:

3.7 Lemma *Let $\rho \in R(\Gamma)$ be regular and let $u_i \in C^1(\Gamma, \mathfrak{sl}_2^\rho)$ be given such that $\rho^{(\rho;u_1,\dots,u_k)}\colon \Gamma \to G_k$ is a homomorphism. Then there exists for every $v \in Z^1(\Gamma, \mathfrak{sl}_2^\rho)$ a cochain $u_{k+1} \in C^1(\Gamma, \mathfrak{sl}_2^\rho)$ such that $\rho^{(u_1,\dots,u_k,u_{k+1}+v)}\colon \Gamma \to G_{k+1}$ is a homomorphism.*

Proof. Recall that $\rho \in R(\Gamma)$ is regular if and only if $\dim_\rho R(\Gamma) = \dim Z^1(\Gamma, \mathfrak{sl}_2^\rho)$. We have the identification $R(\Gamma) \cong V := V(\mathbf{F}) \subset \mathbb{C}^M$ where the solution $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ corresponds to the representation ρ (see equation (9)). The representation $\rho^{(u_1,\dots,u_k)}\colon \Gamma \to G_k$ corresponds to a polynomial vector $\mathbf{y}_k(t) \in (\mathbb{C}[t])^M$ of degree k such that $\mathbf{F}(\mathbf{y}_k(t)) \equiv \mathbf{0} \bmod t^{k+1}$. The element $v \in Z^1(\Gamma, \mathfrak{sl}_2^\rho)$ gives us a vector $\mathbf{v} \in T_{\mathbf{0}}(V)$. It follows from Lemma 2.6 that $\mathbf{0} \in V$ is a smooth point.

It is now easy to see (using the formal implicit function theorem, see [Mum95]) that we can extend $\mathbf{y}_k(t)$ i.e. there is a $\mathbf{w} \in \mathbb{C}^M$ such that $\mathbf{y}_{k+1}(t) := \mathbf{y}_k(t) + t^k(\mathbf{v} + \mathbf{w})$ satisfies

$$\mathbf{F}(\mathbf{y}_{k+1}(t)) \equiv \mathbf{0} \bmod t^{k+2}.$$

This gives us the existence of the representation $\rho^{(\rho;u_1,\dots,u_k,u_{k+1}+v)}\colon \Gamma \to G_{k+1}$ claimed in the lemma. □

4 The deformation of reducible metabelian representations

Let $\Gamma = \pi_1(M)$ be the fundamental group of a three dimensional manifold as in the introduction. We choose a generator t of Λ and a presentation $\Gamma = \langle S_1, \ldots, S_n | R_1, \ldots, R_{n-1}\rangle$ such that $\phi(S_i) = t$.

Let $\varphi_\lambda \colon \Gamma \to \mathrm{SL}_2$ be a reducible, non abelian representation such that $\chi_{\varphi_\lambda} = \chi_{\rho_\lambda}$. Note that φ_λ is metabelian. The proof of Theorem 1.1 relies on the calculation of the dimension of the space of cocycles $Z^1(M, \mathfrak{sl}_2^{\varphi_\lambda})$ which will be presented in Section 4.1. It is there where we use the condition that λ^2 is the simple root of the Alexander polynomial. If the dimension of $H^1(M, \mathfrak{sl}_2^{\varphi_\lambda})$ is one we are able to use the the inclusion $R(M) \hookrightarrow R(\partial M)$ in order to prove that every element of $Z^1(M, \mathfrak{sl}_2^{\varphi_\lambda})$ is integrable. Theorem 1.1 follows then from Lemma 2.6.

Let $\rho\colon \pi_1(M) \to \mathrm{SL}_2$ be a representation such that $\rho(\mathrm{Im}(\pi_1(\partial M) \to \pi_1(M))) \subset \mathrm{SL}_2$ contains a non parabolic element. First note that the inclusion $\partial M \hookrightarrow M$ induces an injection $\iota\colon \pi_1(\partial M) \to \pi_1(M)$. If ι is not an injection then M is homeomorphic to a solid torus and every representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2$ would be abelian. We denote by $\Gamma_0 := \iota(\pi_1(\partial M)) \subset \Gamma$ the image of ι.

4.1 Lemma *Let $\rho\colon \Gamma \to \mathrm{SL}_2$ be a non abelian representation such that $\rho(\Gamma_0)$ contains a non parabolic element. If $\dim Z^1(\Gamma, \mathfrak{sl}_2^\rho) = 4$ then we have an injection $\iota^*\colon H^1(\Gamma, \mathfrak{sl}_2^\rho) \to H^1(\Gamma_0, \mathfrak{sl}_2^\rho)$ and an isomorphism $\iota^*\colon H^2(\Gamma, \mathfrak{sl}_2^\rho) \to H^2(\Gamma_0, \mathfrak{sl}_2^\rho)$.*

Proof. Since ρ is non abelian we have $\dim B^1(\Gamma, \mathfrak{sl}_2^\rho) = 3$ from which $\dim H^1(\Gamma, \mathfrak{sl}_2^\rho) = 1$ follows (see [Por97, Prop. 3.12]). We consider the exact sequence in cohomology for the pair $(M, \partial M)$:

$$\begin{aligned} H^1(M, \partial M; \mathfrak{sl}_2^\rho) &\xrightarrow{i_1^*} H^1(M; \mathfrak{sl}_2^\rho) \xrightarrow{i_2^*} H^1(\partial M; \mathfrak{sl}_2^\rho) \xrightarrow{\Delta} H^2(M, \partial M; \mathfrak{sl}_2^\rho) \\ &\xrightarrow{i_3^*} H^2(M; \mathfrak{sl}_2^\rho) \xrightarrow{i_4^*} H^2(\partial M; \mathfrak{sl}_2^\rho) \to 0. \end{aligned} \tag{10}$$

It follows from Poincaré duality that $\mathrm{rk}(i_2^*) = \frac{1}{2}\dim H^1(\partial M; \mathfrak{sl}_2^\rho)$. Since $\rho(\Gamma_0)$ contains a non parabolic element we have $\dim H^1(\partial M; \mathfrak{sl}_2^\rho) = 2$ (see [Por97, Prop. 3.18]). This together with $\dim H^1(\Gamma, \mathfrak{sl}_2^\rho) = 1$ gives that i_1^* is an injection (see [Hod86, HK97]).

It follows again from Poincaré duality that Δ is a surjection and hence i_4^* is an isomorphism. The manifold M and the boundary torus ∂M are Eilenberg–Mac Lane spaces and hence the lemma follows. □

Let now $\varphi_\lambda \in R(M)$ be a reducible non abelian representation such that $\chi_{\varphi_\lambda} = \chi_{\rho_\lambda}$. We can assume, up to conjugation, that $\varphi_\lambda(\gamma)$ is an upper triangular matrix for all $\gamma \in \Gamma$. There are $\lambda_i \in \mathbb{C}^*$ and $a_i \in \mathbb{C}$, $i = 1, \ldots, n$, such that $\varphi_\lambda(S_i) = \left(\begin{smallmatrix} \lambda_i & a_i \\ 0 & \lambda_i^{-1} \end{smallmatrix}\right)$. From $\chi_{\varphi_\lambda}(S_i) = \lambda + \lambda^{-1}$ it follows by an easy calculation that $\lambda_i \in \{\lambda, \lambda^{-1}\}$. Since $\chi_{\varphi_\lambda}(S_iS_j) = \chi_{\rho_\lambda}(S_iS_j)$ we obtain that all λ_i are equal. By exchanging λ and λ^{-1} we can assume that

$$\varphi_\lambda(S_i) = \begin{pmatrix} \lambda & \lambda^{-1}a_i \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot \begin{pmatrix} \lambda^2 & a_i \\ 0 & 1 \end{pmatrix}.$$

The a_i are not all equal (φ_λ is non abelian), i.e. the vectors $\mathbf{a}$ and $\mathbf{e}$ are linear independent where

$$\mathbf{a} := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad \mathbf{e} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

For a given $\lambda \in \mathbb{C}^*$ and for given $\mathbf{a} \in \mathbb{C}^n$, we denote by $\rho_\lambda^{\mathbf{a}}\colon F_n \to \mathrm{SL}_2$ the representation given by $\rho_\lambda^{\mathbf{a}}\colon S_i \mapsto \left(\begin{smallmatrix} \lambda & \lambda^{-1}a_i \\ 0 & \lambda^{-1}\end{smallmatrix}\right)$.

4.2 Lemma *Let $W = W(S_1,\dots,S_n) \in F_n$ be given. Then we have*

$$\rho_\lambda^{\mathbf{a}}(W) = \begin{pmatrix} \overline{W}(\lambda) & \overline{W}(\lambda^{-1}) \sum_{i=1}^n \overline{(\partial_i W)}(\lambda^2) a_i \\ 0 & \overline{W}(\lambda^{-1}) \end{pmatrix}.$$

Proof. For given $V, W \in F_n$ we have: $\partial_i(VW) = \partial_i V + V\partial_i W$ in $\mathbb{C}F_n$. It follows from this equation that

$$\tilde\rho\colon W \mapsto \begin{pmatrix} \overline{W}(\lambda) & \overline{W}(\lambda^{-1}) \sum_{i=1}^n \overline{(\partial_i W)}(\lambda^2) a_i \\ 0 & \overline{W}(\lambda^{-1}) \end{pmatrix}.$$

defines a homomorphism. The lemma follows since $\tilde\rho(S_i) = \rho_\lambda^{\mathbf{a}}(S_i)$. □

The homomorphism $\rho_\lambda^{\mathbf{a}}\colon F_n \to \mathrm{SL}_2$ factors through Γ if and only if $\sum_{i=1}^n \overline{\partial_i R_j}(\lambda^2) a_i = 0$ for all $j = 1,\dots,n-1$. This system of equations can be written in the form $J(\lambda^2)\mathbf{a} = 0$ where $J(t)$ is the Jacobian of the presentation of Γ (see Section 2.1).

4.3 Corollary (Burde [Bur67], de Rham [dR67]) *There is a reducible, non abelian representation $\varphi_\lambda\colon \Gamma \to \mathrm{SL}_2$ such that $\chi_{\rho_\lambda} = \chi_{\varphi_\lambda}$ if and only if $\Delta_M(\lambda^2) = 0$.*

Proof. Let $\varphi_\lambda \in R(\Gamma)$ be a reducible, non abelian representation we have (up to conjugation and the exchange of λ and λ^{-1}) that $\varphi_\lambda = \rho_\lambda^{\mathbf{a}}$ for a vector $\mathbf{a} \in \mathbb{C}^n$ which is not a multiple of $\mathbf{e}$. It follows that $J(\lambda^2)\mathbf{a} = J(\lambda^2)\mathbf{e} = 0$ and hence $\operatorname{rk} J(\lambda^2) \le n-2$ which implies $\Delta_M(\lambda^2) = 0$.

If $\Delta_M(\lambda^2) = 0$ then we have a vector $\mathbf{a} \in \mathbb{C}^n$ such that $J(\lambda^2)\mathbf{a} = 0$ and $\mathbf{a}$ is not a multiple of $\mathbf{e}$. The representation $\rho_\lambda^{\mathbf{a}}\colon F_n \to \mathrm{SL}_2$ factors through Γ. □

In order to prove Theorem 1.1 we will show that $\dim Z^1(\Gamma, \mathfrak{sl}_2^{\varphi_\lambda}) = 4$ if λ^2 is a simple root of $\Delta_M(t)$ and that every cocycle is integrable.

Let us assume for the moment the following proposition which will be proved in the next subsection.

4.4 Proposition *Let $\lambda \in \mathbb{C}^*$ be given such that λ^2 is a simple root of the Alexander polynomial $\Delta_M(t)$. For every reducible, non abelian representation φ_λ such that $\chi_{\varphi_\lambda} = \chi_{\rho_\lambda}$ we have $\dim Z^1(\Gamma, \mathfrak{sl}_2^{\varphi_\lambda}) = 4$.*

Proof of Theorem 1.1. We shall prove first that every element of $Z^1(\Gamma, \mathfrak{sl}_2^{\varphi_\lambda})$ is integrable.

We fix the notation $\rho := \varphi_\lambda$. Let $u_1,\dots,u_k\colon \Gamma \to \mathfrak{sl}_2$ be given such that $\rho_k^{(\rho;u_1,\dots,u_k)}$ is a homomorphism. We shall show first that $\zeta_{k+1}^{(\rho;u_1,\dots,u_k)} = 0$. The representation

$$\iota^*\rho_k^{(\rho;u_1,\dots,u_k)} = \rho_k^{(\iota^*\rho;\iota^*u_1,\dots,\iota^*u_k)}$$

can be extended to a representation $\iota^*\rho_{k+1}\colon \Gamma_0 \to G_{k+1}$ by Lemma 3.7. Note that $\iota^*\rho(\Gamma_0)$ contains a non parabolic element because ρ_λ is not ∂-central. Hence $\iota^*\rho$ is a non singular point of the representation variety $R(\Gamma_0)$ (see [Por97, 3.3.2]). It follows from Lemma 3.7 and Proposition 3.1 that $\zeta_{k+1}^{(\iota^*u_1,\dots,\iota^*u_k)} = 0$. The injectivity of ι^* and the naturality of the obstruction give $\zeta_{k+1}^{(u_1,\dots,u_k)} = 0$. We obtain a cochain $u_{k+1}\colon \Gamma \to \mathfrak{g}$ such that $\rho_{k+1}^{(\rho;u_1,\dots,u_{k+1})}$ is a homomorphism by Proposition 3.1.

This process gives us an infinite sequence $(u_k)_{k\geq 1}$, $u_k\colon \Gamma \to \mathfrak{g}$, such that $\zeta_{k+1}^{(u_1,\dots,u_k)} = 0$ for all $k \geq 1$. This shows that we can solve all obstructions and hence by Corollary 3.2 every cocycle $u_1 \in Z^1(\Gamma, \mathfrak{sl}_2^{\rho})$ is integrable. Hence we have $\dim_\rho R(G) = \dim Z^1(\Gamma, \mathfrak{sl}_2^{\rho})$ by Proposition 3.6 i.e. $\rho = \varphi_\lambda$ is a regular representation. The theorem follows now from Lemma 2.6. □

4.1 Proof of Proposition 4.4

We assume from now on that λ^2 is a simple root of $\Delta_M(t)$. As before we choose a vector $\mathbf{a} \in \mathbb{C}^n$ with $J(\lambda^2)\mathbf{a} = \mathbf{0}$ which is not a multiple of $\mathbf{e}$

$$\mathbf{a} := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad \mathbf{e} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

The following result of Frohman and Klassen will be needed in the sequel:

4.5 Lemma (Frohman and Klassen [FK91]) *Let $\lambda \in \mathbb{C}^*$ and $\mathbf{a}$ be given as above. Then the linear inhomogeneous system of equations*

$$J(\lambda^2)\mathbf{x} = J'(\lambda^2)\mathbf{a}$$

has no solution.

Proof. Assume that $\mathbf{x}$ is a solution of the system $J(\lambda^2)\mathbf{x} = J'(\lambda^2)\mathbf{a}$. Then

$$J(\lambda^2)(\mathbf{x} - x_1\mathbf{e}) = J'(\lambda^2)(\mathbf{a} - a_1\mathbf{e})$$

and $\mathbf{a} - a_1\mathbf{e} \neq \mathbf{0}$. Note that $J(t)\mathbf{e} = 0$ by equation (1) and hence $J'(t)\mathbf{e} = 0$.

Let $A(t)$ be the matrix obtained from the Jacobian $J(t)$ by omitting the first column and let $\tilde{\mathbf{x}}$ (resp. $\tilde{\mathbf{a}}$) be obtained from $\mathbf{x}$ (resp. $\mathbf{a}$) by omitting the first entry. Hence we have solution of the system $A(\lambda^2)\tilde{\mathbf{x}} = A'(\lambda^2)\tilde{\mathbf{a}}$ where $\tilde{\mathbf{a}} \neq \mathbf{0}$ satisfies $A(\lambda^2)\tilde{\mathbf{a}} = \mathbf{0}$. Such a solution can not exist by case 2 of Lemma 8.1 of [FK91]. □

We already saw that every reducible, non abelian representation $\varphi_\lambda \in R(\Gamma)$ such that $\chi_{\varphi_\lambda} = \chi_{\rho_\lambda}$ is conjugate to a representation $\rho^{\mathbf{a}}_{\lambda^{\pm 1}}$. Proposition 4.4 follows from the following:

4.6 Lemma *Let $\mathbf{a} \in \mathbb{C}^n$ be given such that $\rho^{\mathbf{a}}_\lambda\colon \Gamma \to \mathrm{SL}_2$ is non abelian. Then we have:* $\dim Z^1(\Gamma, \mathfrak{sl}_2^{\rho^{\mathbf{a}}_\lambda}) = 4$.

We have $\rho^{\mathbf{a}}_\lambda(S_i) = \left(\begin{smallmatrix} \lambda & \lambda^{-1}a_i \\ 0 & \lambda^{-1} \end{smallmatrix}\right) =: A_i$. The free group $F_n = F_n(S_1, \dots, S_n)$ acts on $\mathfrak{sl}_2$ via $\mathrm{Ad} \circ \rho^{\mathbf{a}}_\lambda$ i.e. $S_i \circ X = A_i X A_i^{-1}$ for all $X \in \mathfrak{sl}_2$. By choosing the basis (5) of $\mathfrak{sl}_2$ the linear map $\mathrm{Ad}_{A_i} \in \mathrm{Aut}(\mathfrak{sl}_2)$ is given by

$$\begin{pmatrix} \lambda^2 & -2a_i & -\lambda^{-2}a_i^2 \\ 0 & 1 & \lambda^{-2}a_i \\ 0 & 0 & \lambda^{-2} \end{pmatrix}.$$

This defines clearly a homomorphism $\psi\colon\ F_n \to \mathrm{Aut}(\mathfrak{sl}_2) \cong \mathrm{SL}_3(\mathbb{C})$ which extends the group algebra $\psi\colon\ \mathbb{C}F_n \to \mathrm{End}(\mathfrak{sl}_2) \cong \mathrm{SL}_3(\mathbb{C})$. An easy calculation (see [Bir76, 3.2] and Lemma 4.2) gives

$$\psi(\eta) = \begin{pmatrix} \overline{\eta}(\lambda^2) & -2\sum_{i=1}^n \overline{\partial_i \eta}(\lambda^2) a_i & * \\ 0 & \overline{\eta}(1) & * \\ 0 & 0 & \overline{\eta}(\lambda^{-2}) \end{pmatrix}$$

and hence

$$\psi(\frac{\partial R_j}{\partial S_i}) = \begin{pmatrix} \overline{\partial_i R_j}(\lambda^2) & -2\sum_{l=1}^n \overline{\partial_{li} R_j}(\lambda^2) a_l & c_{ij} \\ 0 & \overline{\partial_i R_j}(1) & d_{ij} \\ 0 & 0 & \overline{\partial_i R_j}(\lambda^{-2}) \end{pmatrix}$$

where $\partial_{li} R_j := \partial^2 R_j / \partial S_l \partial S_i$ denotes the second derivation and the c_{ij}, d_{ij} are complex numbers.

By writing $X_i = \left(\begin{smallmatrix} x_i \\ y_i \\ z_i \end{smallmatrix}\right)$ the equation (4) is equivalent to the following system of $3n-3$ equations.

$$\begin{aligned} \sum_{i=1}^n \overline{\partial_i R_j}(\lambda^2)\, x_i - 2\sum_{i,l=1}^n \overline{\partial_{li} R_j}(\lambda^2) a_l\, y_i + \sum_{i=1}^n c_{ji}\, z_i &= 0 \\ \sum_{i=1}^n \overline{\partial_i R_j}(1)\, y_i + \sum_{i=1}^n d_{ji}\, z_i &= 0 \\ \sum_{i=1}^n \overline{\partial_i R_j}(\lambda^2)\, z_i &= 0 \end{aligned} \tag{11}$$

where $j = 1, \dots, n-1$.

Note that $\overline{\partial_i R_j}(\lambda^2) = J_{ji}(\lambda^2)$ are the entries of the Jacobian matrix. Hence the system (11) can be written as

$$\begin{pmatrix} J(\lambda^2) & K & C \\ 0 & J(1) & D \\ 0 & 0 & J(\lambda^{-2}) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{12}$$

where $K = (K_{ji})$ is given by $K_{ji} := -2\sum_{l=1}^n \overline{\partial_{li} R_j}(\lambda^2)\, a_l$ and C, D are $(n-1) \times n$ complex matrices.

The rank of the coefficient-matrix $\mathcal{A} := \mathcal{A}(\lambda^2, \mathbf{a})$ in (12) satisfies:

$$3n - 5 \leq \operatorname{rk} \mathcal{A} \leq 3n - 4.$$

The upper bound follows from Poincaré duality (see Lemma 4.1) and the lower bound from $\operatorname{rk} J(\lambda^{\pm 2}) = n-2$ and $\operatorname{rk} J(1) = n-1$.

Note that a solution $\left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{smallmatrix}\right)$ of (12) gives a cocycle $v\colon\ \Gamma \to \mathfrak{sl}_2$, $v(S_i) = \left(\begin{smallmatrix} y_i & x_i \\ z_i & -y_i \end{smallmatrix}\right)$. The space of coboundaries is three dimensional and is spanned by the following three elements $v_i\colon\ \Gamma \to \mathfrak{sl}_2$:

$$v_1(S_i) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v_2(S_i) = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}, \quad v_3(S_i) = \begin{pmatrix} a_i & -a_i^2 \\ 1-\lambda^2 & -a_i \end{pmatrix}. \tag{13}$$

Proof of Lemma 4.6. In order to prove the lemma we will show that $\operatorname{rk} \tilde{\mathcal{A}} = 2n-2$ where

$$\tilde{\mathcal{A}} := \tilde{\mathcal{A}}(\lambda^2, \mathbf{a}) := \begin{pmatrix} J(\lambda^2) & K \\ 0 & J(1) \end{pmatrix}.$$

Since the coboundaries v_1 and v_2 from (13) are given by solutions of the system $\tilde{\mathcal{A}}\binom{\mathbf{x}}{\mathbf{y}} = \binom{\mathbf{0}}{\mathbf{0}}$ we have $\operatorname{rk}\tilde{\mathcal{A}} \leq 2n-2$.

If $\operatorname{rk}\tilde{\mathcal{A}} < 2n-2$ then there must be a solution $\mathbf{x} \in \mathbb{C}^n$ of the inhomogeneous system

$$J(\lambda^2)\mathbf{x} = -K\mathbf{e}$$

since the non trivial solutions of $J(1)\mathbf{y} = \mathbf{0}$ are spanned by the vector $\mathbf{e}$.

By a Lemma 4.5 we know that the system

$$J(\lambda^2)\mathbf{x} = J'(\lambda^2)\mathbf{a}$$

has no solution. In order to connect these two equations we need to look at $K\mathbf{e}$:

$$-K\mathbf{e} = 2\begin{pmatrix} \sum_{k,l=1}^n \overline{\partial_{kl}R_1}(\lambda^2)a_k \\ \vdots \\ \sum_{k,l=1}^n \overline{\partial_{kl}R_{n-1}}(\lambda^2)a_k \end{pmatrix}.$$

By applying Lemma 2.5 to the relations R_j we obtain:

$$\sum_{k=1}^n \Big(\sum_{l=1}^n \overline{\partial_{kl}R_j}(\lambda^2) + \overline{\partial_k R_j}(\lambda^2)\Big)a_k = -\lambda^2 \sum_{k=1}^n \frac{d}{dt}\big(\overline{\partial_k R_j}(t)\big)_{t=\lambda^2} a_k.$$

Note that $J(\lambda^2)\mathbf{a} = \mathbf{0}$ which is equivalent to $\sum_{k=1}^n \overline{\partial_k R_j}(\lambda^2)a_k = 0$ for all $j = 1, \ldots, n-1$. Using all this we obtain:

$$\begin{aligned} -K\mathbf{e} &= 2\begin{pmatrix} \sum_{k,l=1}^n \overline{\partial_{kl}R_1}(\lambda^2)a_k \\ \vdots \\ \sum_{k,l=1}^n \overline{\partial_{kl}R_{n-1}}(\lambda^2)a_k \end{pmatrix} = 2\begin{pmatrix} -\lambda^2 \sum_{k=1}^n \frac{d}{dt}\big(\overline{\partial_k R_1}(t)\big)_{t=\lambda^2} a_k \\ \vdots \\ -\lambda^2 \sum_{k=1}^n \frac{d}{dt}\big(\overline{\partial_k R_{n-1}}(t)\big)_{t=\lambda^2} a_k \end{pmatrix} \\ &= -2\lambda^2 J'(\lambda^2)\mathbf{a}. \end{aligned}$$

We can hence apply the result of Frohman and Klassen and we obtain $\operatorname{rk}\tilde{\mathcal{A}} = 2n-2$ from which $\dim Z^1(\Gamma, \mathfrak{sl}_2^{\rho_\lambda^{\mathbf{a}}}) = 4$ follows. □

5 The structure of the representation space

This section is divided into five subsections. In the first one we study the set of representations with character χ_λ. In the second one we prove Theorem 1.2 about the local geometry of $X(M)$ at the character χ_λ. Then we prove Corollary 1.3 about the local geometry of $R(M)$ at the abelian representation ρ_λ. In the fourth subsection we prove Corollary 1.4 and in the last subsection we prove some technical lemmas.

5.1 The set of representations with character χ_λ

We want to study the set of representations that have character χ_λ, i.e. the set $\pi^{-1}(\chi_\lambda)$, where $\pi : R(M) \to X(M)$ denotes the natural projection. Besides the abelian representation ρ_λ, in

Section 4 we showed two metabelian representations belonging to $\pi^{-1}(\chi_\lambda)$, that we describe next. Choose vectors $\mathbf{a},\mathbf{b}\in\mathbb{C}^n$ with $J(\lambda^2)\mathbf{a}=J(\lambda^{-2})\mathbf{b}=0$, which are not multiples of $\mathbf{e}$, where

$$\mathbf{e}:=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix},\quad \mathbf{a}:=\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}\quad\text{and}\quad \mathbf{b}:=\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}.$$

We use $\rho_\lambda^{\mathbf{a}}$ and $\rho_{\lambda^{-1}}^{\mathbf{b}}$ to denote the representations defined as:

$$\rho_\lambda^{\mathbf{a}}(S_i)=\begin{pmatrix}\lambda & \lambda^{-1}a_i\\ 0 & \lambda^{-1}\end{pmatrix}\qquad\text{and}\qquad \rho_{\lambda^{-1}}^{\mathbf{b}}(S_i)=\begin{pmatrix}\lambda^{-1} & \lambda b_i\\ 0 & \lambda\end{pmatrix}.$$

If $\mathcal{O}$ denotes the orbit by conjugation then have the inclusion

$$\mathcal{O}(\rho_\lambda)\cup\mathcal{O}(\rho_\lambda^{\mathbf{a}})\cup\mathcal{O}(\rho_{\lambda^{-1}}^{\mathbf{b}})\subseteq\pi^{-1}(\chi_\lambda).$$

5.1 Proposition *We have the equality $\pi^{-1}(\chi_\lambda)=\mathcal{O}(\rho_\lambda)\cup\mathcal{O}(\rho_\lambda^{\mathbf{a}})\cup\mathcal{O}(\rho_{\lambda^{-1}}^{\mathbf{b}})$. In addition, $\overline{\mathcal{O}(\rho_\lambda^{\mathbf{a}})}=\mathcal{O}(\rho_\lambda)\cup\mathcal{O}(\rho_\lambda^{\mathbf{a}})$ and $\overline{\mathcal{O}(\rho_{\lambda^{-1}}^{\mathbf{b}})}=\mathcal{O}(\rho_\lambda)\cup\mathcal{O}(\rho_{\lambda^{-1}}^{\mathbf{b}})$ are both irreducible non-singular varieties.*

Proof. The equality $\pi^{-1}(\chi_\lambda)=\mathcal{O}(\rho_\lambda)\cup\mathcal{O}(\rho_\lambda^{\mathbf{a}})\cup\mathcal{O}(\rho_{\lambda^{-1}}^{\mathbf{b}})$ follows from the fact that every representation in $\pi^{-1}(\chi_\lambda)$ is abelian or metabelian by Proposition 1.5.5. in [CS83] and from Corollary 4.3.

To show that $\mathcal{O}(\rho_\lambda)\cup\mathcal{O}(\rho_\lambda^{\mathbf{a}})$ is a nonsingular variety we construct the explicit equations. The ambient space is $\mathbb{C}^{4n}$ and we choose the embedding $R(M)\subset\mathbb{C}^{4n}$ induced by the following generating system of $\pi_1(M)$:

$$\{S_1, S_2S_1^{-1}, S_3S_1^{-1},\ldots,S_nS_1^{-1}\}.$$

In other words,the coordinates of a representation $\rho\in R(M)$ are the entries of $\rho(S_1)$, $\rho(S_2S_1^{-1})$, $\rho(S_3S_1^{-1}),\ldots,$ $\rho(S_nS_1^{-1})$. We assume that $a_1=0$, after replacing $\mathbf{a}$ by $\mathbf{a}-a_1\mathbf{e}$, so that

$$\rho_\lambda^{\mathbf{a}}(S_1)=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\qquad\text{and}\qquad \rho_\lambda^{\mathbf{a}}(S_iS_1^{-1})=\begin{pmatrix}1 & a_i\\ 0 & 1\end{pmatrix}.$$

We consider the affine subspace $E\subset\mathbb{C}^{4n}$ of elements of the following form:

$$\begin{aligned}\rho_\lambda^{\mathbf{a}}(S_1) &= \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}+(\lambda-\lambda^{-1})\begin{pmatrix}x_1 & -x_2\\ x_3 & -x_1\end{pmatrix}\quad\text{and}\\ \rho_\lambda^{\mathbf{a}}(S_iS_1^{-1}) &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+a_i\begin{pmatrix}-y_1 & y_2\\ -y_3 & y_1\end{pmatrix},\quad\text{with } x_1,x_2,x_3,y_1,y_2,y_3\in\mathbb{C}.\end{aligned}$$

The affine space E has dimension 6 and we work with the coordinates $(x_1,x_2,x_3,y_1,y_2,y_3)\in\mathbb{C}^6$. We remark that $\rho_\lambda\in E$ has coordinates $(0,0,0,0,0,0)$ and that $\rho_\lambda^{\mathbf{a}}\in E$ has coordinates $(0,0,0,0,1,0)$. The choice of the affine space E is explained by the following fact: if ρ is a representation conjugate to $\rho_\lambda^{\mathbf{a}}$, then $\rho\in E$. In addition, if $\pm\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{PSL}_2()2(\mathbb{C})$ is the conjugation matrix, then ρ has coordinates

$$(x_1,x_2,x_3,y_1,y_2,y_3)=(bc,ab,cd,ac,a^2,c^2). \tag{14}$$

By looking at the identities satisfied by coordinates of the form (14), we consider the variety $W \subset E$ defined by

$$\begin{cases} x_1y_1 = x_2y_3 \\ x_1y_2 = x_2y_1 \\ y_1^2 = y_2y_3 \end{cases} \qquad \begin{cases} x_3y_2 - y_1x_1 = y_1 \\ y_1x_3 - y_3x_1 = y_3 \\ x_2x_3 - x_1^2 = x_1 \end{cases} \tag{15}$$

We claim that $W = \mathcal{O}(\rho_\lambda) \cup \mathcal{O}(\rho_\lambda^{\mathbf{a}})$. The inclusion $\mathcal{O}(\rho_\lambda^{\mathbf{a}}) \subset W$ is clear by construction, because a point in $\mathcal{O}(\rho_\lambda^{\mathbf{a}})$ has coordinates of the form (14) and therefore satisfies equations (15). The inclusion

$$\mathcal{O}(\rho_\lambda) \subseteq W \cap \{y_1 = y_2 = y_3 = 0\},$$

also follows easily. We next show the other inclusion $W \subseteq \mathcal{O}(\rho_\lambda) \cup \mathcal{O}(\rho_\lambda^{\mathbf{a}})$. If a point $\rho \in W$ satisfies $y_3(\rho) \neq 0$ then, by setting $c = \sqrt{y_3}$, $b = x_1/c$, $d = x_3/c$ and $a = y_1/c$, we deduce that ρ is conjugate to $\rho_\lambda^{\mathbf{a}}$ with conjugation matrix $\pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. In a similar way, if $\rho \in W$ satisfies $y_2(\rho) \neq 0$ then also $\rho \in \mathcal{O}(\rho_\lambda^{\mathbf{a}})$. In the last case, if $\rho \in W$ satisfies $y_2(\rho) = y_3(\rho) = 0$ then $y_1(\rho) = 0$, which implies that $\rho(S_i) = \rho(S_1)$. In addition, equation $x_2(\rho)x_3(\rho) - x_1(\rho)^2 = x_1(\rho)$ implies that $\rho(S_1)$ belongs to $\mathrm{SL}_2(\mathbb{C})$ and that $\operatorname{tr}\rho(S_1) = \operatorname{tr}\rho_\lambda(S_1)$. Therefore $\rho \in \mathcal{O}(\rho_\lambda)$.

To prove that W is non-singular, we first remark that $\dim \mathcal{O}(\rho_\lambda) = 2$ and $\dim \mathcal{O}(\rho_\lambda^{\mathbf{a}}) = 3$ which implies that $\dim W = 3$. In order to prove that every point in $\mathcal{O}(\rho_\lambda^{\mathbf{a}})$ (resp. $\mathcal{O}(\rho_\lambda)$) is smooth, it suffices to check it for a single point $\rho_\lambda^{\mathbf{a}}$ (resp. ρ_λ) by homogeneity. This can be done by using the explicit equations (15), (since the coordinates of ρ_λ and $\rho_\lambda^{\mathbf{a}}$ are particularly simple, this computation is straightforward). □

5.2 Lemma *The germ of $\overline{\mathcal{O}(\rho_\lambda^{\mathbf{a}})}$ and the germ of $\overline{\mathcal{O}(\rho_{\lambda^{-1}}^{\mathbf{b}})}$ are contained in the same irreducible component of the germ of $R(M)$ at ρ_λ.*

Proof. We fix $\varepsilon > 0$ and we choose $U_\varepsilon \subset R(M)$ a neighborhood of ρ_λ as follows:

$$U_\varepsilon = \{\rho \in R(M) \mid \|\rho(S_i)\rho_\lambda(S_i^{-1}) - \mathrm{Id}\,\| < \varepsilon, \text{ for } i = 1, \ldots, n\},$$

where $\| \; \|$ denotes the Euclidean norm in $M_{2\times 2}(\mathbb{C}) \cong \mathbb{C}^4$. We also choose $\mathbf{a}$ and $\mathbf{b}$ such that $\|\mathbf{a}\| < \varepsilon/2$ and $\|\mathbf{b}\| < \varepsilon/2$. In particular $\rho_\lambda^{\mathbf{a}} \in U_{\varepsilon/2}$.

By Theorem 1.1, $\rho_\lambda^{\mathbf{a}}$ is a smooth point of $R(M)$, with local dimension 4. In addition, by using the description of the tangent space, we can find a path $[0,\delta] \to R(M)$, with $\delta > 0$, that maps $t \in [0,\delta]$ to a representation ρ'_t that satisfies:

$$\rho'_t(S_i) = \begin{pmatrix} \lambda + O(t) & \lambda^{-1}a_i + O(t) \\ \lambda b_i t + O(t^2) & \lambda^{-1} + O(t) \end{pmatrix}.$$

We take $\delta > 0$ sufficiently small so that $\rho'_t \in U_\varepsilon \;\; \forall t \in [0,\delta]$. In particular ρ'_δ and $\rho_\lambda^{\mathbf{a}}$ belong to the same component of U_ε.

Next we construct a path of representations conjugated to ρ'_δ. For $t \in [\delta, 1]$ we define ρ''_t to be the representation conjugated to ρ'_δ by $\pm \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}$. Thus:

$$\rho''_t(S_i) = \begin{pmatrix} \lambda + O(\delta) & t\big(\lambda^{-1}a_i + O(\delta)\big) \\ \lambda b_i \frac{\delta}{t} + \frac{O(\delta^2)}{t} & \lambda^{-1} + O(\delta) \end{pmatrix}.$$

Since $\|\mathbf{a}\| < \varepsilon/2$ and $\|\mathbf{b}\| < \varepsilon/2$, we may choose δ sufficiently small so that this path belongs to U_ε. As the action by conjugation is algebraic and invertible, it follows that ρ''_δ stays in the same component of U_ε than $\rho^{\mathbf{a}}_\lambda$. Moreover ρ''_δ satisfies:

$$\rho''_\delta(S_i) = \begin{pmatrix} \lambda + O(\delta) & O(\delta) \\ \lambda b_i + O(\delta) & \lambda^{-1} + O(\delta) \end{pmatrix}.$$

We consider a sequence $\delta(n) \to 0$ converging to zero, and we have that $\rho''_{\delta(n)} \to \rho''_0$, where ρ''_0 is defined by

$$\rho''_0(S_i) = \begin{pmatrix} \lambda & 0 \\ \lambda b_i & \lambda^{-1} \end{pmatrix}.$$

Thus ρ''_0 is conjugate to $\rho^{\mathbf{b}}_{\lambda^{-1}}$ by $\pm\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. Since $\overline{\mathcal{O}(\rho^{\mathbf{b}}_{\lambda^{-1}})}$ and $\overline{\mathcal{O}(\rho^{\mathbf{a}}_{\lambda})}$ are both smooth, the lemma follows easily. □

5.2 The local geometry of $X(M)$ at the abelian character χ_ρ

In this subsection we prove Theorem 1.2, that asserts that $X_\lambda(M)$ and $Y(M)$ are the unique irreducible components of the variety of characters containing χ_λ, that both are curves smooth at χ_λ and that the intersection of tangent spaces is zero.

Proof of Theorem 1.2. The proof is organized as follows. First we prove that the analytic germ of $R(M)$ at the abelian representation ρ_λ has only two irreducible components (Proposition 5.3). Using this proposition and Lemma 5.4, we show that the analytic germ of $X(M)$ at the character χ_λ has also two irreducible components (Corollary 5.5). Next we show that χ_λ is a smooth point of $Y(M)$ and of $X_\lambda(M)$ (Propositions 5.6 and 5.9). Finally, in Proposition 5.11 we show the property about the intersection of Zariski tangent spaces.

5.3 Proposition *The analytic germ of $R(M)$ at ρ_λ has only two irreducible components, which are precisely the germ of $R_\lambda(M)$ and the germ of $S(M)$. In addition, ρ_λ is a smooth point of $S(M)$.*

Proof. Let $U \subset R(M)$ be a neighborhood of ρ_λ. The analytic variety U has at least two irreducible components U_0 and U_1, where $U_0 = S(M) \cap U(M)$ and U_1 is the component that contains the germ of $\pi^{-1}(\chi_\lambda)$, which exists by Proposition 5.1 and Lemma 5.2. The variety U_0 is irreducible and smooth, because the map

$$\begin{array}{rcl} S(M) & \to & \mathrm{SL}_2(\mathbb{C}) \\ \rho & \mapsto & \rho(S_1) \end{array}$$

is an isomorphism.

Let U_j be any irreducible component of U, we claim that either $U_j = U_0$ or $U_j = U_1$. First at all, if all the representations of U_j are abelian, we shall see that $U_j = U_0$ by using the structure of the set of abelian representations. One can easily check that the set of abelian representations is a collection of *pairwise disjoint* irreducible varieties (each one isomorphic to $\mathrm{SL}_2(\mathbb{C})$). The set of those varieties is in bijection with $\mathrm{Hom}(tors(H_1(M,\mathbb{Z})), S^1)$, where the component $S(M)$ corresponds to the trivial map. In particular, since the varieties of abelian representations do not intersect each other, $U_j = U_0$.

Now we assume that U_j contains non-abelian representations. We claim that U_j contains a non-abelian representation with character χ_λ. We consider the restriction of the projection

$\pi|_{U_j} : U_j \to X(M)$ and we want to study the fibers of $\pi|_{U_j}$. If $\mathcal{O}(\rho)$ denotes the orbit by conjugation of ρ, then $\mathcal{O}(\rho) \cap U_j$ is contained in a fiber of $\pi|_{U_j}$. Since $\mathcal{O}(\rho) \cap U_j$ is an open subset of $\mathcal{O}(\rho)$, it follows that

$$\dim((\pi|_{U_j})^{-1}(\chi_\rho)) \geq \dim(\mathcal{O}(\rho) \cap U_j) = \dim(\mathcal{O}(\rho)).$$

Therefore the generic dimension of the fibers of $\pi|_{U_j}$ is at least 3 because being non-abelian is an open property in the space of representations, and if ρ is non-abelian then $\dim(\mathcal{O}(\rho)) = 3$. In particular, 3 is the lower bound for the dimension of *all* the fibers of $\pi|_{U_j}$, and since $\dim(\mathcal{O}(\rho_\lambda)) = 2$, U_j must contain a non-abelian representation with character χ_λ. By Proposition 5.1 and Lemma 5.2, $U_j = U_1$. □

In order to use Proposition 5.3 to study the germ of $X(M)$ at χ_λ, we need the following lemma:

5.4 Lemma *The projection $\pi : R(M) \to X(M)$ is open at the abelian representation ρ_λ i.e. if U is a classical neighborhood of ρ_λ then $\pi(U)$ is a classical neighborhood of χ_λ.*

This lemma is rather technical and its proof is postponed to the end of the section. We state the following corollary of this lemma and Proposition 5.3.

5.5 Corollary *The analytic germ of $X(M)$ at χ_λ has only two irreducible components, which are the respective analytic germs of $X_\lambda(M) = \pi(R_\lambda(M))$ and of $Y(M) = \pi(S(M))$.* □

5.6 Proposition *The character χ_λ is a smooth point of $Y(M)$.*

Proof. We recall that the function algebra $\mathbb{C}[X(M)]$ is finitely generated by the evaluation functions

$$\begin{array}{rccc} I_\gamma \colon & X(M) & \to & \mathbb{C} \\ & \chi & \mapsto & \chi(\gamma) \end{array} \qquad \text{where } \gamma \in \Gamma.$$

Given our system of generators $S_1, \ldots, S_n$, such that $\phi(S_i) = 1 \in \mathbb{Z}$, every $\chi \in Y(M)$ satisfies $I_{S_i}(\chi) = I_{S_j}(\chi)$. In addition, $\forall \gamma \in \Gamma$ and $\forall \chi \in Y(M)$, $I_\gamma(\chi) = I_{S_1^r}(\chi)$, where $r = \phi(\gamma) \in \mathbb{Z}$. The function $I_{S_1^r}$ is a polynomial on I_{S_1}, and it follows from this that $\mathbb{C}[I_{S_1}] = \mathbb{C}[Y(M)]$ is the ring of polynomials in one variable. Hence $Y(M)$ is a curve isomorphic to the complex line $\mathbb{C}$. □

The following is the key lemma for concluding the proof of Theorem 1.2.

5.7 Lemma *There exist a disk $\Delta \subset \mathbb{C}$ centered at 0, an analytic map $f \colon \Delta \to X_\lambda(M)$ and a rational function $g \in \mathbb{C}(X(M))$ such that $g \circ f = Id_\Delta$.*

This lemma is based in the following one, whose proof is postponed to the end of the section. Up to conjugation, a metabelian representation is either $\rho_\lambda^{\mathbf{a}}$ or $\rho_{\lambda^{-1}}^{\mathbf{b}}$. In the following lemma we assume that $\varphi_\lambda = \rho_\lambda^{\mathbf{a}}$, but a similar statement holds true for $\rho_{\lambda^{-1}}^{\mathbf{b}}$.

5.8 Lemma *Let $\varphi_\lambda = \rho_\lambda^{\mathbf{a}}$ be a non-abelian representation as in Theorem 1.1. Let $b_1, b_2, c_1, c_2 \in \mathbb{C}[R(M)]$ be the algebraic functions defined by*

$$\rho(S_1) = \begin{pmatrix} a_1(\rho) & b_1(\rho) \\ c_1(\rho) & d_1(\rho) \end{pmatrix} \quad \text{and} \quad \rho(S_2) = \begin{pmatrix} a_2(\rho) & b_2(\rho) \\ c_2(\rho) & d_2(\rho) \end{pmatrix} \qquad \forall \rho \in R(M).$$

We can choose the generators $S_1, \dots, S_n$ so that the map

$$F := (b_1, b_2, c_1, c_2)\colon\ R(M) \to \mathbb{C}^4$$

is locally invertible at φ_λ.

Proof of Lemma 5.7. We construct the map $\tilde{f}\colon\ \Delta \to R(M)$ by using Lemma 5.8. After conjugation we may assume that

$$\varphi_\lambda(S_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \text{and} \qquad \varphi_\lambda(S_2) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Since $F(\varphi_\lambda) = (0, 1, 0, 0)$, we define $\tilde{f}(z) := F^{-1}(0, 1, 0, z)$, for every z in a small disk $\Delta \subset \mathbb{C}$ around the origin. In particular, if $\tilde{f}(z) = \rho_z$, then

$$\rho_z(S_1) = \begin{pmatrix} a_1(z) & 0 \\ 0 & d_1(z) \end{pmatrix} \quad \text{and} \quad \rho_z(S_2) = \begin{pmatrix} a_2(z) & 1 \\ z & d_2(z) \end{pmatrix}$$

with $a_1 d_1 = 1$ and $a_2 d_2 - z = 1$. In order to construct g, we observe that if $\tilde{f}(z) = \rho_z$, then

$$(4 - \operatorname{tr}^2(\rho_z(S_1)))\, z = \operatorname{tr}(\rho_z([S_1, S_2])) - 2, \tag{16}$$

where $[S_1, S_2] = S_1 S_2 S_1^{-1} S_2^{-1}$ denotes the commutator. Thus we define

$$g = \frac{I_{[S_1,S_2]} - 2}{4 - I_{S_1}^2},$$

where $I_\gamma(\chi_\rho) = \chi_\rho(\gamma) = \operatorname{tr}(\rho(\gamma))$. It is clear from equation (16) that $g \circ f = \mathrm{Id}_\Delta$ where $f := \pi \circ \tilde{f}$. □

The following two results conclude the proof of Theorem 1.2:

5.9 Proposition *The character χ_λ is a smooth point of $X_\lambda(M)$.*

Proof. We already know by Corollary 5.5 that the analytic germ of $X_\lambda(M)$ at χ_λ is irreducible. If this analytic germ was singular, then the composition $g \circ f$ would be a map of degree $r > 1$, where f and g are the maps of Lemma 5.7. Therefore this germ is non-singular, because $g \circ f = \mathrm{Id}_\Delta$. □

5.10 Remark It follows from this proof that g defines a local parameter of $X_\lambda(M)$ at χ_λ.

5.11 Corollary $T^{\mathrm{Zar}}_{\chi_\lambda}(X_\lambda(M)) \cap T^{\mathrm{Zar}}_{\chi_\lambda}(Y(M)) = \{0\}$.

Proof. The corollary follows from the following properties of the rational function $g \in \mathbb{C}(X(M))$ of Lemma 5.7:

(i) $g(Y(M)) = \{0\}$, because $g = (I_{[S_1,S_2]} - 2)/(4 - I_{S_1}^2)$ and every character χ of an abelian representation satisfies $I_{[S_1,S_2]}(\chi) = 2$;

(ii) $d_{\chi_\lambda} g(T^{\mathrm{Zar}}_{\chi_\lambda}(X_\lambda(M))) \cong \mathbb{C}$, because $g \circ f = \mathrm{Id}_\Delta$.

This finishes the proof of the corollary and also of Theorem 1.2. □

5.3 The local geometry of $R_\lambda(M)$ at the abelian representation ρ_λ

In this section we prove Corollary 1.3 and then we use it to construct a slice. In order to prove it, we need the following:

5.12 Lemma *If ρ_λ denotes the abelian representation as in Corollary 1.3, then* $\dim T^{\mathrm{Zar}}_{\rho_\lambda}(R(M)) = 5$ *and there is a vector $v \in T^{\mathrm{Zar}}_{\rho_\lambda}(R(M))$ such that $d_{\rho_\lambda}\pi(v)$ generates $T^{\mathrm{Zar}}_{\chi_\lambda}(Y(M))$.*

Proof. As in the proof of Theorem 2.7, to compute $Z^1(\Gamma, \mathfrak{sl}_2^{\rho_\lambda}) \cong T^{\mathrm{Zar}}_{\rho_\lambda}(R(M))$ we use the decomposition:

$$Z^1(\Gamma, \mathfrak{sl}_2^{\rho_\lambda}) = Z^1(\Gamma, \mathbb{C}_{\lambda^2}) \oplus Z^1(\Gamma, \mathbb{C}) \oplus Z^1(\Gamma, \mathbb{C}_{\lambda^{-2}})$$

where $\mathbb{C}_\alpha$, $\alpha \in \mathbb{C}^*$, denotes the Γ–module $\mathbb{C}$ (the action is given by $S_i \circ z = \alpha z$). As in Theorem 2.7, $Z^1(\Gamma, \mathbb{C}_{\lambda^2}) \cong \operatorname{Ker} J(\lambda^2)$ and $\dim(\operatorname{Ker} J(\lambda^2)) = 2$, because λ^2 is a simple root of the Alexander polynomial. In addition

$$Z^1(\Gamma, \mathbb{C}) \cong \operatorname{Hom}(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \operatorname{Ker} J(1) \cong \mathbb{C},$$

which completes the computation of the dimension.

Finally, for any non-vanishing $v \in Z^1(\Gamma, \mathbb{C})$, we prove that $d_{\rho_\lambda}\pi(v)$ generates $T^{\mathrm{Zar}}_{\chi_\lambda}(Y(M))$. Up to multiplication by a constant, the cocycle v satisfies $v(S_i) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. As a vector, v is tangent to a path of representations ρ_s at $s = 0$, where

$$\rho_s(\gamma) = \exp(s v(\gamma))\, \rho_\lambda(\gamma) \qquad \forall \gamma \in \Gamma.$$

Since $I_{S_i}(\pi(\rho_s)) = \operatorname{tr}(\rho_s(S_i)) = \lambda + \frac{1}{\lambda} + s(\lambda - \frac{1}{\lambda}) + O(s^2)$, it follows that $d_{\chi_\lambda} I_{S_i}(d_{\rho_\lambda}\pi(v)) = \lambda - \frac{1}{\lambda} \neq 0$. Thus $d_{\rho_\lambda}\pi(v)$ is a basis for $T^{\mathrm{Zar}}_{\chi_\lambda}(Y(M)) \cong \mathbb{C}$. □

Proof of Corollary 1.3. By Proposition 5.3 we know that the analytic germ of $R(M)$ at ρ_λ has precisely two irreducible components, which are the germ of $R_\lambda(M)$ and the germ of $S(M)$. In addition, Proposition 5.3 says that ρ_λ is a smooth point of $S(M)$.

Let $v \in T^{\mathrm{Zar}}_{\rho_\lambda}(R(M))$ be a vector such that $d_{\rho_\lambda}\pi(v)$ generates $T^{\mathrm{Zar}}_{\chi_\lambda}(Y(M))$. Since $T^{\mathrm{Zar}}_{\chi_\lambda}(X_\lambda(M)) \cap T^{\mathrm{Zar}}_{\chi_\lambda}(Y(M)) = \{0\}$, we have that $v \notin T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M))$. Thus

$$\dim(T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M))) \leq \dim(T^{\mathrm{Zar}}_{\rho_\lambda}(R(M))) - 1 = 4$$

and, since by Theorem 1.1 $\dim(R_\lambda(M)) = 4$, it follows that ρ_λ is a smooth point of $R_\lambda(M)$.

In addition, since $v \in T^{\mathrm{Zar}}_{\rho_\lambda}(S(M))$ and $v \notin T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M))$, we have:

$$\dim(T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M)) \cap T^{\mathrm{Zar}}_{\rho_\lambda}(S(M))) \leq \dim(T^{\mathrm{Zar}}_{\rho_\lambda}(S(M))) - 1 = 2.$$

As the orbit $\mathcal{O}(\rho_\lambda)$ is a two-dimensional subvariety of the intersection $R_\lambda(M) \cap S(M)$, it follows that $\mathcal{O}(\rho_\lambda)$ is a proper component of $R_\lambda(M) \cap S(M)$ and that

$$T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M)) \cap T^{\mathrm{Zar}}_{\rho_\lambda}(S(M)) = T^{\mathrm{Zar}}_{\rho_\lambda}(\mathcal{O}(\rho_\lambda)).$$

This concludes the proof of the corollary □

Next we construct a *local parametrization* of a neighborhood of ρ_λ in $R_\lambda(M)$. Let S_1 and S_2 be a system of generators as in Lemma 5.8. We consider again the algebraic functions $b_1, b_2, c_1, c_2 \in \mathbb{C}[R_\lambda(M)]$ defined by

$$\rho(S_1) = \begin{pmatrix} a_1(\rho) & b_1(\rho) \\ c_1(\rho) & d_1(\rho) \end{pmatrix} \quad \text{and} \quad \rho(S_2) = \begin{pmatrix} a_2(\rho) & b_2(\rho) \\ c_2(\rho) & d_2(\rho) \end{pmatrix} \qquad \forall \rho \in R(M).$$

5.13 Lemma *The map $F|_{R_\lambda(M)} = (b_1, b_2, c_1, c_2) : R_\lambda(M) \to \mathbb{C}^4$ is locally invertible at ρ_λ.*

We postpone the proof of this lemma to the last subsection and we use it to construct a slice. Following [BA98b], we define:

5.14 Definition The *slice* $\mathcal{S}_\lambda$ is the following analytic germ at ρ_λ:

$$\mathcal{S}_\lambda = \{\rho \in R_\lambda(M) \mid \rho(S_1) \text{ is diagonal and } \rho \text{ is in a neighborhood of } \rho_\lambda\}.$$

Of course $\rho_\lambda(S_1)$ is diagonal and the definition makes sense.

By using lemma 5.13, $F(\mathcal{S}_\lambda)$ is a neighborhood of the origin in the two dimensional subspace of $\mathbb{C}^4$ defined by $b_1 = c_1 = 0$. In particular $\mathcal{S}_\lambda$ is smooth, two dimensional and locally parametrized by $(b_2, c_2) : \mathcal{S}_\lambda \to \mathbb{C}^2$.

5.15 Lemma *Two representations $\rho, \rho' \in \mathcal{S}_\lambda$ are conjugate if and only if*

$$(b_2(\rho), c_2(\rho)) = (e^t b_2(\rho'), e^{-t} c_2(\rho')) \qquad \textit{for some } t \in \mathbb{C}.$$

Proof. When two representations $\rho, \rho' \in \mathcal{S}_\lambda$ are conjugate, the conjugation matrix is of the form $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, because $\rho(S_1)$ and $\rho'(S_1)$ are both diagonal and close to $\rho_\lambda(S_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Therefore the lemma follows from the fact that (b_2, c_2) are local parameters. $\square$

5.16 Remark (i) It follows from this lemma that the quotient of $\mathcal{S}_\lambda$ by conjugation is not Hausdorff.

(ii) Let g be the rational function on $R(M)$ defined in Lemma 5.7. A straightforward computation shows that:

$$g \circ \pi|_{\mathcal{S}_\lambda} = b_2\, c_2$$

(iii) It follows from the previous point and from Remark 5.10 that the restriction $\pi|_{\mathcal{S}_\lambda}$ of the projection map $\pi : R(M) \to X(M)$ is open and surjective in a neighborhood of $\chi_\lambda \in X_\lambda(M)$. This is, $\pi(\mathcal{S}_\lambda)$ is a neighborhood of χ_λ in $X_\lambda(M)$.

(iv) However the restriction $\pi|_{\mathcal{S}_\lambda}$ is singular at ρ_λ, i.e. $d_{\rho_\lambda}\pi(T^{\text{Zar}}_{\rho_\lambda}(\mathcal{S}_\lambda)) = 0$. In fact, we have that $d_{\rho_\lambda}\pi(T^{\text{Zar}}_{\rho_\lambda}(R_\lambda(M))) = 0$.

(v) Since $d_{\rho_\lambda}\pi(T^{\text{Zar}}_{\rho_\lambda}(S(M))) \neq 0$ (see Lemma 5.12) we obtain that

$$\operatorname{Ker}(d_{\rho_\lambda}\pi\colon\, T^{\text{Zar}}_{\rho_\lambda}(R(M)) \to T^{\text{Zar}}_{\chi_\lambda}(X(M))) = T^{\text{Zar}}_{\rho_\lambda}(R_\lambda(M)).$$

5.4 Real characters

In this subsection we prove Corollary 1.4 about the deformations of real characters. Along all the subsection we will assume that χ_λ satisfies the hypothesis of Corollary 1.4.

We recall that a character χ is said to be *real* if $\chi(\gamma) \in \mathbb{R}$ for every $\gamma \in \Gamma$. We also recall that the character χ_λ is real-valued iff λ is real or lies in the complex unit circle.

5.17 Lemma *Assume that χ_λ is real. Let $\mathcal{S}_\lambda$, b_2 and c_2 as in previous subsection. For every $\rho \in \mathcal{S}_\lambda$, χ_ρ is real if and only if $g(\chi_\rho) = b_2(\rho)\, c_2(\rho) \in \mathbb{R}$.*

Proof. When χ_ρ is real, then $g(\chi_\rho) \in \mathbb{R}$, because $g = (I_{[S_1,S_2]} - 2)/(4 - I_{S_1}^2)$.

Assuming that $|\lambda| = 1$, we want to prove that if $g(\chi_\rho) \in \mathbb{R}$ then χ_ρ is real. To prove it, we construct an involution $\iota : R(M) \to R(M)$ that preserves $\mathcal{S}_\lambda$ and fixes ρ_λ. We take ι to be the composition of complex conjugation of coefficients with conjugation with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The involution ι fixes ρ_λ and, by taking an invariant neighborhood of ρ_λ, ι preserves $\mathcal{S}_\lambda$.

Let χ_ρ be a character such that $g(\chi_\rho) = b_2(\rho)\, c_2(\rho) \in \mathbb{R}$. If $g(\chi_\rho) = 0$ then $\chi_\rho = \chi_\lambda$ is real. If $g(\chi_\rho) \neq 0$ then there is a $\alpha \in \mathbb{R}^*$ such that $c_2(\rho) = \alpha \overline{b_2(\rho)}$ and we obtain

$$(b_2(\iota(\rho)), c_2(\iota(\rho))) = (-\overline{c_2(\rho)}, -\overline{b_2(\rho)}) = (-\alpha\, b_2(\rho), -\frac{1}{\alpha}\, c_2(\rho)).$$

Therefore ρ and $\iota(\rho)$ are conjugate by Lemma 5.15 and $\chi_\rho = \chi_{\iota(\rho)} = \overline{\chi_\rho}$.

When $\lambda \in \mathbb{R}$ the same argument applies, by taking the involution $\iota : R(M) \to R(M)$ which is just complex conjugation of the coefficients. □

5.18 Lemma *Let $\rho \in \mathcal{S}_\lambda$.*

(i) If $\lambda \in \mathbb{R}$ and $b_2(\rho), c_2(\rho) \in \mathbb{R}$, then $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{R}))$.

(ii) If $|\lambda| = 1$ and $b_2(\rho) = \overline{c_2(\rho)}$, then $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SU}(1,1))$.

(iii) If $|\lambda| = 1$ and $b_2(\rho) = -\overline{c_2(\rho)}$, then $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SU}(2))$.

Proof. We prove only (i), the proof of the other points being similar and easier. We distinguish several cases. If $b_2(\rho) = c_2(\rho) = 0$, then $\rho = \rho_\lambda$ and there is nothing to prove. If $c_2(\rho) = 0$ but $b_2(\rho) \neq 0$, then ρ is metabelian and conjugate to φ_λ. In this case, by Corollary 4.3, the coefficients of ρ are real. When $b_2(\rho) = 0$ but $c_2(\rho) \neq 0$ the same argument applies.

Finally, we consider the case where $b_2(\rho)\, c_2(\rho) \neq 0$. In this case we observe first that $\rho(S_1), \rho(S_2) \in \mathrm{SL}_2(\mathbb{R})$ because $\chi_\rho(S_1), \chi_\rho(S_2), \chi_\rho(S_1 S_2) \in \mathbb{R}$ and $\lambda \neq \pm 1$. Given an element $\gamma \in \Gamma$, since the character χ_ρ evaluated at γ, γS_1, γS_2 and $\gamma S_1 S_2$ is real, it follows easily that $\rho(\gamma)$ has real coefficients, by using $b_2(\rho)\, c_2(\rho) \neq 0$ and $\lambda \neq \pm 1$. □

Proof of Corollary 1.4. The fact that the set of real points of $X_\lambda(M)$ is a smooth real curve in a neighborhood of χ_λ follows Remarks 5.16(iii) and 5.10.

To prove the second half of the corollary when $\lambda \in \mathbb{R}$, we consider two paths ρ_s and ρ'_s, with $s \in (0, \varepsilon)$, which are paths of representations in the slice such that

$$\begin{cases} b_2(\rho_s) = c_2(\rho_s) = s \\ b_2(\rho'_s) = -c_2(\rho'_s) = s \end{cases}$$

By Lemma 5.18, ρ_s and ρ'_s are paths of representations into $\mathrm{SL}_2(\mathbb{R})$. Since $g(\chi_{\rho_s}) = s^2$ and $g(\chi_{\rho'_s}) = -s^2$, it suffices to take $\chi_{s^2} = \chi_{\rho_s}$ and $\chi_{-s^2} = \chi_{\rho'_s}$ so that χ_t parametrizes $X_\lambda(M) \cap X(M)^{\mathbb{R}}$.

When $|\lambda| = 1$, the same construction applies, the only difference is that Lemma 5.18 says that ρ is a path of representations into $\mathrm{SU}(1,1)$ and ρ' is a path of representations into $\mathrm{SL}_2(\mathbb{R})$. □

5.19 Remark The path of characters χ_t, with $t \in (-\varepsilon^2, \varepsilon^2)$, constructed in this proof does not lift to a smooth path of representations because the projection $\pi : R_\lambda(M) \to X_\lambda(M)$ is singular at ρ_λ.

5.5 Proof of Lemmas 5.4, 5.8 and 5.13

Proof of Lemma 5.4. We want to prove that the projection $\pi : R(M) \to X(M)$ is open at the abelian representation ρ_λ.

As affine subset of $\mathbb{C}^N$, $R(M)$ is equipped with a distance that we denote by d. Since π is invariant on orbits, Lemma 5.4 will follow for the following limit:

$$\lim_{\pi(\rho)\to\pi(\rho_\lambda)} d(\mathcal{O}(\rho), \rho_\lambda) = 0$$

that we prove next.

Let $S_1, \ldots, S_n$ be a system of generators of $\Gamma = \pi_1(M)$. Given a representation $\rho \in R(M)$ with $\pi(\rho) = \chi_\rho$ close to $\pi(\rho_\lambda) = \chi_\lambda$, we have $\chi_\rho(S_1) \neq \pm 2$ and we can conjugate ρ so that

$$\rho(S_1) = \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}.$$

Since $\chi_\rho(S_1) = x + 1/x$ and $\chi_\lambda(S_1) = \lambda + 1/\lambda$, we have that $x \to \lambda^{\pm 1}$ as $\chi_\rho \to \chi_\lambda$. After conjugating ρ, we may assume that $x \to \lambda$.

For any $\gamma \in \pi_1(M)$, $\rho_\lambda(\gamma) = \left(\begin{smallmatrix} \lambda^r & 0 \\ 0 & \lambda^{-r} \end{smallmatrix}\right)$, where $r = \phi(\gamma) \in \mathbb{Z}$. Therefore, if $\rho(\gamma) = \left(\begin{smallmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{smallmatrix}\right)$, then the equations

$$\begin{aligned} a + d &= \chi_\rho(\gamma) \\ xa + d/x &= \chi_\rho(\gamma S_1) \end{aligned} \tag{17}$$

imply that $a \to \lambda^r$ and $d \to \lambda^{-r}$. In particular, if we set $\rho(S_i) = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$, then $a_i \to \lambda$ and $d_i \to \lambda^{-1}$.

After permuting the elements $S_2, \ldots, S_n$, we may assume that

$$|b_2| \geq \max\{|b_2|, \ldots, |b_n|, |c_2|, \ldots, |c_n|\}$$

(the following argument works by permuting b_i and c_i if necessary). We distinguish two cases:

Case 1. There is an entry c_j, with $j \in \{2, \ldots, n\}$, such that

$$|c_j| \geq \frac{|b_2|}{16n}.$$

In this case, since $\rho(S_2 S_j) = \left(\begin{smallmatrix} a_j a_2 + b_2 c_j & * \\ * & * \end{smallmatrix}\right)$ formula (17) above implies that $a_j a_2 + b_2 c_j$ converges to λ^2. Since both a_2 and a_j converge to λ, it follows that b_2 converges to 0. In particular, all coefficients b_i and c_i converge to zero. Therefore $\rho(S_i) \to \rho_\lambda(S_i)$, which means that $\rho \to \rho_\lambda$.

Case 2. $\max\{|c_2|,\dots,|c_n|\} < \frac{|b_2|}{16n}$. In this case, we conjugate ρ by $\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$, to obtain a representation ρ_1. By construction, this representation ρ_1 is contained in the orbit of ρ. If

$$\rho_1(S_i) = \begin{pmatrix} a'_i & b'_i \\ c'_i & d'_i \end{pmatrix}$$

then $a'_i = a_i$, $b'_i = b_i/4$, $c'_i = 4c_i$ and $d'_i = d_i$. Thus

$$\begin{aligned} |b'_2| + \cdots + |b'_n| + |c'_2| + \cdots + |c'_n| &= \frac{|b_2| + \cdots + |b_n|}{4} + 4(|c_2| + \cdots + |c_n|) \\ &< \frac{|b_2| + \cdots + |b_n|}{4} + \frac{|b_2|}{4} \\ &\leq \frac{1}{2}(|b_2| + \cdots + |b_n| + |c_2| + \cdots + |c_n|) \end{aligned} \tag{18}$$

And we distinguish again two cases for ρ_1. In this way, either we obtain a representation in the orbit of ρ such that Case 1 applies, or we obtain a sequence of representations $(\rho_k)_{k\in\mathbb{N}}$ such that, for each k, ρ_k is in Case 2 and ρ_{k+1} is obtained by conjugating ρ_k by $\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$. By inequality (18), ρ_k converges to ρ_λ and therefore $d(\rho_\lambda, \mathcal{O}(\rho)) = 0$. □

Proof of Lemma 5.8. The proof will follow directly from the description of a basis for $Z^1(\Gamma, \mathfrak{sl}_2^{\varphi_\lambda}) = T^{\mathrm{Zar}}_{\varphi_\lambda}(R(M))$.

Let $S_1,\dots,S_n$ denote the usual system of generators of $\Gamma = \pi_1(M)$ and let $J(t)$ denote the corresponding Alexander matrix. We recall that $\ker(J(\lambda^2))$ is a two dimensional subspace of $\mathbb{C}^n$ with basis $\{\mathbf{e},\mathbf{a}\}$, and $\ker(J(\lambda^{-2}))$ is a two dimensional subspace of $\mathbb{C}^n$ with basis $\{\mathbf{e},\mathbf{b}\}$, where:

$$\mathbf{e} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{a} := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} := \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

We will assume that $a_1 = b_1 = 0$.

We claim that we can also assume that $a_2 = b_2 = 1$. To prove this claim, we remark that if this is not possible to achieve by permuting $S_2,\dots,S_n$, then we can always assume that $a_2 = b_3 = 1$ and $a_3 = b_2 = 0$. If this was the case, then it would be sufficient to replace the generator S_2 by $S_1^{-1}S_2S_3$ to have $a_2 = b_2 = 1$.

According to the normalization $a_1 = 0$ and $a_2 = 1$, the metabelian representation satisfies:

$$\varphi_\lambda(S_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \varphi_\lambda(S_2) = \begin{pmatrix} \lambda & \lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix}.$$

By the computations of Section 3 and the normalization $b_1 = 0$ and $b_2 = 1$, there is a basis $\{v_1, v_2, v_3, v_4\}$ for $Z^1(\Gamma, \mathfrak{sl}_2^{\varphi_\lambda}) = T^{\mathrm{Zar}}_{\varphi_\lambda}(R(M))$ such that $\{v_1, v_2, v_3\}$ is a basis for the coboundary space and v_4 is a cocycle that satisfies:

$$v_4(S_1) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad \text{and} \quad v_4(S_2) = \begin{pmatrix} * & * \\ 1 & * \end{pmatrix}.$$

In addition, an elementary computation (see Equation (13)) shows that the basis for the coboundary space may be chosen satisfying:

$$v_1(S_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v_1(S_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$
$$v_2(S_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad v_2(S_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$
$$v_3(S_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad v_3(S_2) = \begin{pmatrix} * & * \\ 1 & * \end{pmatrix}.$$

The lemma follows straightforward from this description of this basis for the tangent space $Z^1(\Gamma, \mathfrak{sl}_2^{\varphi_\lambda}) = T^{\mathrm{Zar}}_{\varphi_\lambda}(R(M))$. $\square$

Proof of Lemma 5.13. By Corollary 1.3 we know that $R_\lambda(M)$ is smooth at ρ_λ and it suffices to show that there is a basis $\{\nu_1, \nu_2, \nu_3, \nu_4\}$ for $T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M))$ such that

$$\nu_1(S_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \nu_1(S_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$
$$\nu_2(S_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \nu_2(S_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$
$$\nu_3(S_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \nu_3(S_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$
$$\nu_4(S_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \nu_4(S_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since $T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M)) \cong \mathbb{C}^4$ and $\nu_1, \nu_2, \nu_3, \nu_4$ are linearly independent, we only need to prove that $\nu_1, \nu_2, \nu_3, \nu_4$ belong to $T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M))$. The cocycles ν_1 and ν_2 belong to $T^{\mathrm{Zar}}_{\rho_\lambda}(R_\lambda(M))$ because they are coboundaries, and therefore they are tangent to some conjugation orbit. In addition ν_3 and ν_4 are tangent to spaces of metabelian representations provided by Corollary 4.3. $\square$

6 Examples

Let $k \subset S^3$ be a tame knot. The exterior of k, i.e. the complement of a open tubular neighborhood of k, is denoted by $M(k)$. We shall write $\Gamma(k)$ for the *knot group* i.e. $\Gamma(k) = \pi_1(M(k))$.

The representation spaces for knot groups have been studied by different authors and some historic remarks can be found in Section 5 of [HLMA95].

6.1 The complement of the knot $\mathfrak{b}(49, 17)$

Let k be the 2-bridge knot $\mathfrak{b}(49, 17)$ (see [BZ85, Kaw96, Sie75, Sch56]). This is an alternating knot with 12 crossings in a minimal projection. The fundamental group of a 2–bridge knot is generated by two elements which are conjugated. More precisely, $\Gamma(k) = \langle S_1, S_2 \mid L_{S_1} S_1 = S_2 L_{S_1} \rangle$ where $L_{S_1} := L_{S_1}(S_1, S_2) \in F_2(S_1, S_2)$. We have $\phi(S_1) = \phi(S_2) = t$ and $L_{S_1} \in \mathrm{Ker}(\phi)$ (see [BZ85]). The character variety $X(k)$ is hence algebraic subset of $\mathbb{C}^2$ (see [HLMA95, Ril84] for the details). The Alexander polynomial $\Delta_{\mathfrak{b}(49,17)}(t) = (2t^2 - 3t + 2)^2$ has a double zero on the complex unit circle and we denote by ζ a complex number such that $\Delta_k(\zeta^2) = 0$.

For ζ and $\mathbf{a} = \binom{0}{1}$ we obtain a reducible non abelian representations $\rho_\zeta^{\mathbf{a}}\colon \Gamma \to \mathrm{SL}_2$ and computer supported calculations give

$$\mathcal{A}(\zeta, \mathbf{a}) = \begin{pmatrix} 0 & 0 & 1-4\zeta & 4\zeta-1 & 7/8 & 49/8-6\zeta \\ 0 & 0 & 1 & -1 & 0 & 2\zeta-1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\mathcal{A}(\zeta, \mathbf{a})$ is the coefficient-matrix in Equation 12. It is clear that $\mathrm{rk}(\mathcal{A}(\zeta, \mathbf{a})) = 2$ and $\mathrm{rk}(\tilde{\mathcal{A}}(\zeta, \mathbf{a})) = 1$ where $\tilde{\mathcal{A}}(\zeta, \mathbf{a})$ is defined as in proof of Lemma 4.6. We obtain therefore $\dim Z^1(\Gamma(k), \mathfrak{sl}_2^{\rho_\zeta^{\mathbf{a}}}) = 4$ and hence $\rho_\zeta^{\mathbf{a}} \in R(M(k))$ is a smooth point, contained in a unique component of the representation variety of dimension four. The transversality statement is not valid. The component $X_\zeta(M)$ and $Y(M)$ do not have a transversal intersection at χ_ζ.

In figure 1 we can see how the real branch of $X_\zeta(M)$ and $Y(M)$ intersect each other. The characters of the abelian representations are parametrized by the line $\tau = 1$ (see [Bur90] for the details).

$\leftarrow \chi_{\rho_\zeta}$

PSfrag replacements

$X(\mathfrak{b}(49,17)) \cap \mathbb{R}^2$

τ

γ

1

-1

2

-2

$\leftarrow \chi_{\rho_\zeta}$

Figure 1: The real branch of the representation variety.

6.2 The complement of the knot 8_{20}

Let $k \subset S^3$ be the knot 8_{20}. The group $\Gamma(k)$ is has the following presentation: $\langle S_1, S_2, S_3 \mid R_1, R_2 \rangle$ where

$$\begin{aligned} R_1 &:= S_1^{-1} S_3^{-1} S_1 S_2^{-1} S_1^{-1} S_2 S_1^{-1} S_3 S_1 S_3, \\ R_2 &:= S_1 S_3^{-1} S_2 S_1^{-1} S_3 S_1 S_2^{-1} S_1^{-1} S_3^{-1} S_1 S_2^{-1} S_3. \end{aligned} \tag{19}$$

The Alexander module of k is cyclic and the Alexander polynomial is given by $\Delta_k(t) = (t^2 - t + 1)^2$. It follows that $\mathrm{rk}\, J(\xi^2) = 1$ and hence $\dim Z^1(\Gamma(k), \mathfrak{sl}_2^{\rho_\xi}) = 5$ where $\xi := \exp(i\pi/6)$.

The following lemma shows that the representation ρ_ξ can not be contained in a five dimensional component of the representation variety.

6.1 Lemma *Let $\lambda \in \mathbb{C}^*$ be given and assume that $\dim Z^1(\Gamma, \mathfrak{sl}_2^{\rho_\lambda}) = 2n+1$. For every irreducible component V of the representation variety $R(\Gamma)$ such that $\rho_\lambda \in V$ we have $\dim V \leq 2n$.*

Proof. Since $\dim Z^1(\Gamma, \mathfrak{sl}_2^{\rho_\lambda}) \geq \dim T^{\mathrm{Zar}}_{\rho_\lambda}(V) \geq \dim V$ we have $\dim V \leq 2n+1$. If $\dim V = 2n+1$ then ρ_λ is a simple point of $R(\Gamma)$ and V is the unique component through ρ_λ (see Lemma 2.6). This is a contradiction since $\rho_\lambda \in S(M)$ and $\dim S(M) = 3$. □

In order to find components of the representation variety which contains ρ_ξ we consider the following surjection:

$$\varphi\colon\ \Gamma(k) \to \Gamma(k') \text{ given by } \varphi\colon\ S_1 \mapsto S,\ S_2 \mapsto S, S_3 \mapsto T$$

where $\Gamma(k') = \langle S, T \mid STS = TST\rangle$ is the group of the trefoil $k' \subset S^3$. This surjection induces a proper embedding $\varphi^*\colon\ X(k') \to X(k)$.

The representation space of the trefoil knot is well known: since ξ is a simple root of $\Delta_{k'}(t) = t^2 - t + 1$ it follows from Theorem 1.2 that χ_{ρ_ξ} is a proper component of the intersection $X'_\xi \cap Y(k')$. It is clear that $\varphi^*(Y(k')) = Y(k)$ and we denote $X_\xi := \varphi^*(X'_\xi)$. It is now clear that $\chi_\xi \in X_\xi$ and it follows from Lemma 6.1 that X_ξ is a one dimensional component of $X(k)$.

Note that the component X_ξ has a real branch which corresponds to path of irreducible representations $\rho_t\colon\ \Gamma(k) \to \mathrm{SU}(2)$. It follows that there must be a second real branch of the character variety $X(k)$ which contains χ_ξ. For if not it would follow from Section 4 of [Heu98a] and from the Theorem 1.2 of [HK98] that the absolute value of the signature $|\sigma(k)| = 2$ (see also [Heu98b]). This gives a contradiction since k is a slice knot and $\sigma(k) = 0$.

This shows that the character χ_ξ is not a smooth point of $\overline{X^{irr}(k)}$. More precisely, the analytic germ of $\overline{X^{irr}(k)}$ at χ_ξ is not irreducible. Since X_ξ is a component with irreducible analytic germ at χ_ξ it follows that there are at least two irreducible components of $\overline{X^{irr}(k)}$ passing through χ_ξ.

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