# Dimension of measures : the probabilistic approach

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**Abstract**

Various tools can be used to calculate or estimate the dimension of measures. Using a probabilistic interpretation, we propose very simple proofs for the main inequalities related to this notion. We also discuss the case of quasi-Bernoulli measures and point out the deep link existing between the calculation of the dimension of auxiliary measures and the multifractal analysis.

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The notion of dimension is an important tool to classify the subsets in $\mathbb{R}^d$ and in particular to compare the size of small sets. There exist various definitions of dimension. The Hausdorff and the packing dimensions are probably the most famous one and can be considered as "extremal" notions of dimension. We refer to [Fal90] for precise definitions and we denote $\mathcal{H}^s$ (resp. $\hat{\mathcal{P}}^s$) the Hausdorff (resp. packing) measures. Finally, $\dim(E)$ and $\mathrm{Dim}(E)$ are respectively the Hausdorff and the packing dimension of a set $E$.

The computation of the dimension of a set $E$ is naturally connected to the analysis of auxilliary Borel measures. The first elementary result in this direction is the following.

**Proposition 0.1.** *Let $E$ be a Borel subset in $\mathbb{R}^d$ and $m$ be a Borel measure such that $m(E) > 0$. Suppose that there exist $s > 0$ and $C > 0$ such that*

$$\forall x \in E, \quad m(B(x,r)) \le Cr^s \quad \textit{if } r \textit{ is small enough .}$$

*Then, $\mathcal{H}^s(E) > 0$ and $\dim(E) \ge s$.*

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There is a converse to Proposition 0.1 known as Frostman's Lemma (see for example [Mat95]).

**Proposition 0.2.** *Suppose that $E$ is a Borel subset in $\mathbb{R}^d$ such that $\mathcal{H}^s(E) > 0$. There exists a Borel measure $m$ such that $m(E) > 0$ satisfying*

$$\forall x \in E,\ \forall r > 0, \quad m(B(x,r)) \le Cr^s \ .$$

*In particular, the result is true if* $\dim(E) > s$.

Similar results, involving the packing dimension of the set $E$ are also true (see [Fal97], Proposition 2.2, 2.3 and 2.4).

In vue of Propositions 0.1 and 0.2, it is natural to introduce the local dimensions (also called Hölder exponents) of the measure $m$ which are defined as

$$\begin{aligned} \underline{\dim}\, m(x) &= \liminf_{r\to 0} \frac{\log(m(B(x,r)))}{\log r} \\ \overline{\dim}\, m(x) &= \limsup_{r\to 0} \frac{\log(m(B(x,r)))}{\log r} \ . \end{aligned}$$

The quantities $\underline{\dim}$ and $\overline{\dim}$ are respectively called the lower and the upper local dimension of the measure $m$ at point $x$.

Finally, Propositions 0.1 and 0.2 can be reformulated as

**Proposition 0.3.** *Let $E$ be a Borel subset in $\mathbb{R}^d$.*

$$\dim(E) = \sup\left\{s,\ \exists m,\ m(E) > 0 \text{ and } \underline{\dim}\, m(x) \ge s \quad \forall x \in E\right\} \ .$$

We can also refer to Tricot ([Tri82]) and Cutler ([Cut95]) who studied the link between the Hausdorff dimension (or the packing dimension) of a set $E$ and the local exponents of auxiliary measures.

The deep relation between the value of the local exponent of auxiliary measures and the dimension of a given set $E$ is very useful in practice. In many situations, this is the natural way to compute the dimension of the set $E$.

It is for example the case for self-similar sets. Let $S_1, \cdots, S_k$ be similarities in $\mathbb{R}^d$ with ratio $0 < r_i < 1$ and $E$ be the unique nonempty compact set such that $E = \bigcup_i S_i(E)$ (see [Hut81]). For the sake of simplicity, suppose that the compact sets $S_i(E)$ are disjoint. Then $E$ is a Cantor set and the application

$$i = (i_1, \cdots, i_n, \cdots) \in \{1, \cdots, k\}^{\mathbb{N}^*} \longmapsto \bigcap_n S_{i_1} \circ \cdots \circ S_{i_n}(E) \tag{1}$$

is an homeomorphism. Let $s$ be the unique positive real number such that $\sum_{i=1}^k r_i^s = 1$ and $m$ be the unique probability measure such that

$$m\left(S_{i_1} \circ \cdots \circ S_{i_n}(E)\right) = r_{i_1}^s \cdots r_{i_n}^s \ .$$

The measure $m$ is nothing else but the image of a multinomial measure on the symbolic Cantor set $\{1, \cdots, k\}^{\mathbb{N}^*}$ through the application (1). Computing the local exponents of $m$, we find

$$\dim(E) = \mathrm{Dim}(E) = s \ .$$

This result remains true if the so called Open Set Condition is satisfied (see [Hut81, Fal97]). The case of self-affine sets is much more difficult ([McM84, Urb90, Ols98]).

The thermodynamic formalism is an interesting tool to give the value of the Hausdorff dimension of sets that are obtained in more general dynamical contexts. This is for example the case for cookie-cutter sets ([Bed86, Bed91]), graph-directed sets ([MW88]) and Julia sets ([Rue82, Zin97]). We can also refer to [Fal97].

Another famous result, due to Eggleston ([Egg49]) concerns the occurence of digits in the $\ell$-adic decomposition of real numbers. Let $\ell \geq 2$, $p = (p_0, \cdots, p_{\ell-1})$ a probability vector and $x = \sum_{k=1}^{+\infty} x_k \ell^{-k} \in [0,1)$ be the (proper) decomposition of the real number $x$ in base $\ell$. Finally let

$$f_n^i(x) = \frac{1}{n} \sharp \{k \in \{1, \cdots, n\} \; ; \; x_k = i\}$$

be the frequency of the digit $i$. If $E(p)$ is the set of real numbers $x \in (0,1)$ such that for all $i \in \{0, \cdots, \ell-1\}$, $\lim_{n\to+\infty} f_n^i(x) = p_i$, then

$$\dim(E(p)) = \mathrm{Dim}(E(p)) = -\sum_{i=0}^{\ell-1} p_i \log_\ell p_i \ . \tag{2}$$

The proof of this result is based on the analysis of an auxiliary measure $m$ defined by

$$m\left(\left[\sum_{i=1}^{n} \varepsilon_i \ell^{-i}, \sum_{i=1}^{n} \varepsilon_i \ell^{-i} + \ell^{-n}\right)\right) = \prod_{i=1}^{n} p_{\varepsilon_i} \ .$$

The strong law of large numbers easily ensures that the measure $m$ is carried by the set $E(p)$ and that

$$\underline{\dim}\, m(x) = \overline{\dim}\, m(x) = -\sum_{i=0}^{\ell-1} p_i \log_\ell p_i \quad \text{if } x \in E(p) \ .$$

Formula (2) follows (see Part 1 of the present paper for a detailed study of the case $\ell = 2$).

We can also reverse the point of view and try, for a given measure $m$ in $\mathbb{R}^d$, to compute or to estimate the dimension of sets that are naturally related to the measure $m$. In that way, we can in particular think to the negligible sets and the sets of full measure and define the quantities

$$\begin{aligned} \dim_*(m) &= \inf(\dim(E) \; ; \; m(E) > 0) \\ \dim^*(m) &= \inf(\dim(E) \; ; \; m(E) = 1) \ . \end{aligned} \tag{3}$$

Dimension $\dim^*(m)$ first appears in [You82]. These two dimensions are respectively called the lower and the upper dimension of the measure $m$ (see for example [Fal97]or [Edg98]). They precise how much the measure $m$ is a “singular measure” or a “regular measure” and they are important quantities for the understanding of $m$. Similar definitions involving the packing dimension can also be proposed :

$$\mathrm{Dim}_*(m) = \inf(\mathrm{Dim}(E) \; ; \; m(E) > 0)$$

$$\mathrm{Dim}^*(m) \quad = \quad \inf(\mathrm{Dim}(E)\ ;\ m(E)=1)\ . \tag{4}$$

There are numerous works in which estimates of the dimension of a given measure are obtained.

In particular, a lot of papers deal with the harmonic measure $\omega$ in a domain $\Omega \subset \mathbb{R}^d$. Let us recall some results in this direction. A famous result due to Makarov ([Mak85]) states that the harmonic measure in a simply connected domain of $\mathbb{R}^2$ is always supported by a set of Hausdorff dimension 1 while every set with dimension strictly less than 1 is negligible with respect to the harmonic measure. A few years later, Jones and Wolff ([JW88]) extended this result and proved that in a general domain in $\mathbb{R}^2$, the harmonic measure is always supported by a set of dimension one. When $\Omega$ is the complementary of a self-similar Cantor set, Carleson proved that the dimension of the harmonic measure $\omega$ satisfies $\dim_*(\omega) = \dim^*(\omega) < \dim(\partial\Omega)$. In that case, the harmonic measure can be seen as a Gibbs measure on a symbolic Cantor set and the properties of the harmonic measures are consequences of Ergodic theory (see also [MV86]). Such approach was also used in the more general situation of "conformal Cantor sets", generalized snowflakes and Julia sets of hyperbolic polynomials (see the survey paper [Mak98] on this subject). In a nondynamical context, Batakis proved in [Bat96] the relation $\dim^*(\omega) < \dim(\partial\Omega)$ for a large class of domains $\Omega$ for which $\Omega^c$ is a Cantor set. Let us finally recall Bourgain's result in higher dimension : the harmonic measure is always supported by a set of dimension $d-\varepsilon$ where $\varepsilon$ only depends on the dimension $d$ (see [Bou87]).

Explicit values of the dimension of measures can also often be obtained in dynamical contexts. This is for example the case for self-similar measures on self-similar Cantor sets. Let us briefly explain the calculus. Let $S_1, \cdots, S_k$ be similarities in $\mathbb{R}^d$ with ratio $0 < r_i < 1$ and $E$ be the unique nonempty compact set such that $E = \bigcup_i S_i(E)$ (see [Hut81]). Suppose that the compact sets $S_i(E)$ are disjoint. Let $p = (p_1, \cdots, p_k)$ be a probability vector and $m$ be the unique probability measure such that

$$m = \sum_{i=1}^{k} p_i\, m \circ S_i^{-1}\ . \tag{5}$$

The measure $m$ is nothing else but the image of a multinomial measure on the symbolic Cantor set $\{1, \cdots, k\}^{\mathbb{N}^*}$ through the homeomorphism

$$i = (i_1, \cdots, i_n, \cdots) \in \{1, \cdots, k\}^{\mathbb{N}^*} \longmapsto \bigcap_n S_{i_1} \circ \cdots \circ S_{i_n}(E)$$

Let

$$E_{i_1, \cdots, i_n} = S_{i_1} \circ \cdots \circ S_{i_n}(E)\ .$$

For every $x \in E$ there exists a unique sequence $i_1(x), \cdots, i_n(x), \cdots$ such that $x \in E_{i_1(x), \cdots, i_n(x)}$ for all $n$. Moreover, if $f_i^n(x)$ is the frequency of the digit $i$ in the sequence $i_1(x), \cdots, i_n(x)$, we have

$$\frac{\log m(E_{i_1(x), \cdots, i_n(x)})}{\log \mathrm{diam}\,(E_{i_1(x), \cdots, i_n(x)})} = \frac{\sum_{i=1}^k f_i^n(x) \log p_i}{\sum_{i=1}^k f_i^n(x) \log r_i + \frac{1}{n} \log \mathrm{diam}\,(E)}\ .$$

Using the strong law of large numbers we get

$$\lim_{n\to+\infty} \frac{\log m(E_{i_1(x),\cdots,i_n(x)})}{\log \operatorname{diam}(E_{i_1(x),\cdots,i_n(x)})} = \frac{\sum_{i=1}^k p_i \log p_i}{\sum_{i=1}^k p_i \log r_i} \quad dm - \text{almost surely} . \tag{6}$$

If we observe that $E_{i_1(x),\cdots,i_n(x)}$ is in some sense similar to the ball of center $x$ and radius $\operatorname{diam}(E_{i_1(x),\cdots,i_n(x)})$, we get

$$\underline{\dim}\, m(x) = \overline{\dim}\, m(x) = \frac{\sum_{i=1}^k p_i \log p_i}{\sum_{i=1}^k p_i \log r_i} \quad dm - \text{almost surely}$$

and we can conclude that

$$\dim_*(m) = \dim^*(m) = \frac{\sum_{i=1}^k p_i \log p_i}{\sum_{i=1}^k p_i \log r_i} . \tag{7}$$

This formula is always true when the Open Set Condition is satisfied (see Part 1 for an elementary example). The calculus is much more complicated (and often impossible) in "overlapping" situations (see for example [LN98, FL02, Fen03, Tes04, Tes06a]).

More generally, the thermodynamic formalism and the ergodic theory are in practice good tools to compute the dimension of measures. Let us for example mention the nice paper of L. S. Young in which a formula (involving the entropy and the Lyapunov exponents) is given for the upper dimension of invariant ergodic measures with respect to a $C^{1+\alpha}$ diffeomorphism of a compact surface ([You82]).

Multifractal analysis is the natural way to obtain a more precise analysis of the measure $m$. The object is to compute the spectrum, defined as the following function :

$$d(\alpha) = \dim\left(\left\{x \;;\; \underline{\dim}\, m(x) = \overline{\dim}\, m(x) = \alpha\right\}\right) .$$

In many situations, $d(\alpha)$ is nothing else but the Legendre transform $\tau^*(\alpha)$ of the $L^q$-spectrum

$$\tau(q) = \limsup_{n\to+\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n \log \ell} \log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right) \tag{8}$$

where $(\mathcal{F}_n)_{n\geq 0}$ are the natural partitions in dyadic (or $\ell$-adic) cubes in $\mathbb{R}^d$. When $d(\alpha) = \tau^*(\alpha)$, we say that the multifractal formalism is valid.

A heuristic justification of the multifractal formalism runs as follows : First, the contribution to $\tau_n(q)$ of the set of points where the local exponents takes a value $\alpha$ is estimated. If the dimension of this set is $d(\alpha)$, then there are about $\ell^{nd(\alpha)}$ cubes in $\mathcal{F}_n$ which cover this set and such a cube $I$ satisfies $m(I) \approx \ell^{-\alpha n}$. Therefore, the order of magnitude of the required contribution is $\ell^{-(\alpha q - d(\alpha))n}$. When $n$ goes to $+\infty$, the maximum contribution is clearly obtained for the value of $\alpha$ that minimizes the exponent $\alpha q - d(\alpha)$; thus $\tau(q) = \inf_\alpha(\alpha q - d(\alpha))$. If $d(\alpha)$ is a concave function, then this formula can be inverted and $d(\alpha)$ can be obtained from $\tau(q)$ by an inverse Legendre transform :

$$d(\alpha) = \inf_q(\alpha q + \tau(q)) . \tag{9}$$

There are many papers who support formula (9). Frisch and Parisi ([FP85]) were the first to introduce the Legendre transform in multifractal analysis. Rigorous approaches are given by Brown, Michon Peyrière ([BMP92]) and Olsen ([Ols95]). They enlighten the link between formula (9) and the existence of auxiliary measures $m_q$ satisfying

$$\frac{1}{C}\, m(I)^q |I|^{\tau(q)} \le m_q(I) \le C\, m(I)^q |I|^{\tau(q)} \ . \tag{10}$$

In fact, it is shown in [Ben94, BBH02] that the existence of a measure $m_q$ satisfying

$$m_q(I) \le C\, m(I)^q |I|^{\tau(q)} \tag{11}$$

is sufficient to obtain the nontrivial inequality

$$d(\alpha) \ge \inf_q (\alpha q + \tau(q)) \ .$$

Now again, the dynamical context is a paradigm for multifractal analysis. In many situations, the existence of measures $m_q$ satisfying (10) and the validity of (9) are proved. This is for example the case for quasi-Bernoulli measures ([BMP92, Heu98, Pey92]), self-similar measures ([CM92, Fen03, Fen05, FO03, HL01, LN98, LN00, Rie95, Tes06a, Ye05]), measures on cookie-cutters ([Ran89]), graph-directed constructions ([EM92]), invariant measures of rational maps on the complex plane ([Lop89]). The context of self-affine measures is much more complicated ([Kin95, Ols98]). The case of random self-similar measures was also studied ([Man74, KP76, Bar99, Bar00a, Bar00b]).

Let us briefly explain the ideas that are used to validate the multifractal formalism in the context of self-similar measures on a self-similar Cantor set. The notations are the same as before (see (5) and the notations below). The partitions given by the compact sets $E_{i_1,\cdots,i_n}$ are prefered to the $(\mathcal{F}_n)_{n\ge 0}$. In fact, it is easy to show that the measure $m$ is doubling and that the sequence $E_{i_1(x),\cdots,i_n(x)}$ of neighborhoods of $x$ calculates the local exponents at point $x$. Let $q \in \mathbb{R}$ and let $\tau = \tau(q)$ be the unique real number such that

$$\sum_{i=1}^{k} p_i^q r_i^\tau = 1 \ . \tag{12}$$

The function $\tau = \tau(q)$ is similar to the $L^q$-spectrum defined by (8). The function $\tau$ is convex and real analytic. Let $m_q$ be the self-similar measure such that for all $i$,

$$m_q(E_i) = p_i^q r_i^\tau \ .$$

The measure $m_q$ is such that for all $i_1, \cdots, i_n$,

$$m_q(E_{i_1,\cdots,i_n}) = (p_{i_1}\cdots p_{i_n})^q\, (r_{i_1}\cdots r_{i_n})^\tau \approx m\left(E_{i_1,\cdots,i_n}\right)^q \operatorname{diam}\left(E_{i_1,\cdots,i_n}\right)^\tau ,$$

which is similar to (10). Let

$$\alpha = -\tau'(q) = \frac{\sum_{i=1}^k p_i^q r_i^\tau \log p_i}{\sum_{i=1}^k p_i^q r_i^\tau \log r_i}$$

and

$$E_\alpha = \left\{ x \in E \ ; \ \lim_{n\to+\infty} \frac{\log m\left(E_{i_1(x),\cdots,i_n(x)}\right)}{\log \operatorname{diam}\left(E_{i_1(x),\cdots,i_n(x)}\right)} = \alpha \right\} \ .$$

We observe that $x \in E_\alpha$ if and only if

$$\lim_{n\to\infty} \frac{\log m\left(E_{i_1(x),\cdots,i_n(x)}\right)}{\log \operatorname{diam}\left(E_{i_1(x),\cdots,i_n(x)}\right)} = \alpha q + \tau(q) = \frac{\sum_{i=1}^k p_i^q r_i^\tau \log(p_i^q r_i^\tau)}{\sum_{i=1}^k p_i^q r_i^\tau \log r_i} \ .$$

Applying (6) and (7) to the measure $m_q$, we obtain

$$\dim(E_\alpha) = \dim(m_q) = -q\tau'(q) + \tau(q) = \inf_q (\alpha q + \tau(q))$$

which is the desired formula.

This example points out the importance of auxiliary measures in the multifractal analysis. In Part 5, we will apply the same technique to quasi-Bernoulli measures.

The purpose of this survey paper is to revisit the notion of dimension of a measure in a very simple way. We do not refer to any dynamical context and we try to obtain estimates of the lower and the upper dimension which are always true. The probabilistic interpretation of the notion of dimension will be useful to achieve our purpose.

As it is shown in Part 3, the lower and the upper dimension of a measure $m$ are related to the asymptotic behaviour of a sequence of random variables. More precisely, if $(\mathcal{F}_n)_{n\geq 0}$ are the natural partitions in dyadic (or $\ell$-adic) cubes in $\mathbb{R}^d$ and if $I_n(x)$ is the unique cube that contains $x$, we will see that the lower dimension (resp. upper dimension) of the measure $m$ coincides with the lower essential bound (resp. upper essential bound) of the random variable $\liminf_{n\to+\infty} S_n/n$, where

$$\frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n} \quad \text{and} \quad X_n(x) = -\log_\ell\left(\frac{m(I_n(x))}{m(I_{n-1}(x))}\right) \ .$$

Similar interpretation of $\operatorname{Dim}_*(m)$ and $\operatorname{Dim}^*(m)$ in terms of the essential bounds of $\limsup_{n\to+\infty} S_n/n$ is also possible. It is then not surprising that the lower and the upper dimension of the measure $m$ are related to the log-Laplace transform of the sequence $S_n$ :

$$L(q) = \limsup_{n\to+\infty} \frac{1}{n} \log_\ell \mathbb{E}\left[\ell^{qS_n}\right],$$

where the expectation is related to the probability $m$.

An easy calculation gives

$$L(1-q) = \limsup_{n\to+\infty} \frac{1}{n\,\log\ell}\ \log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right) \ := \tau(q)$$

where we recognize the classical $L^q$-spectrum $\tau$ used in multifractal analysis. The lower and the upper entropy of the measure $m$ can also be expressed in terms of the sequence of random variables $S_n$. More precisely, we have

$$\begin{aligned} h_*(m) &:= \liminf_{n\to+\infty} \frac{-1}{n} \sum_{I\in\mathcal{F}_n} m(I)\log_\ell m(I) = \liminf_{n\to+\infty} \mathbb{E}\left[\frac{S_n}{n}\right] \\ h^*(m) &:= \limsup_{n\to+\infty} \frac{-1}{n} \sum_{I\in\mathcal{F}_n} m(I)\log_\ell m(I) = \limsup_{n\to+\infty} \mathbb{E}\left[\frac{S_n}{n}\right] \end{aligned}$$

and these quantities are also related to the dimension of the measure $m$.

All those estimates are gathered in Theorem 3.1 which states that

$$-\tau'_+(1) \leq \dim_*(m) \leq h_*(m) \leq h^*(m) \leq \mathrm{Dim}^*(m) \leq -\tau'_-(1)\ . \tag{13}$$

A probabilistic interpretation of (13) is proposed in Theorem 3.2 and the equality cases are discussed in Part 4. Classical examples and concrete estimates are also developed to illustrate the purpose.

In the last part (Part 5), we revisit the notion of quasi-Bernoulli measures in order to explain the importance of the estimates that are developed in the previous sections. Ergodicity properties are explained, the existence of the derivative function $\tau'$ is shown and an elementary proof of the validity of the multifractal formalism is given. Such a proof points out the important role played by the dimension of auxiliary measures in multifractal analysis.

# 1 A classical example : the Bernoulli product

We begin this paper with the study of a classical example. It is a convenient way to introduce the notion of dimension of measures and to precise some notations. Moreover, generalizations of this example will be developed later (see Part 3.1).

Let $\mathcal{F}_n$ be the family of dyadic intervals of the $n^{\text{th}}$ generation on $[0,1)$, $0 < p < 1$ and let $m$ be the Bernoulli product of parameter $p$. It is defined as follows. If $\varepsilon_1\cdots\varepsilon_n$ are integers in $\{0,1\}$, and if

$$I_{\varepsilon_1\cdots\varepsilon_n} = \left[\sum_{i=1}^n \frac{\varepsilon_i}{2^i}, \sum_{i=1}^n \frac{\varepsilon_i}{2^i} + \frac{1}{2^n}\right) \in \mathcal{F}_n$$

then

$$m\left(I_{\varepsilon_1\cdots\varepsilon_n}\right) = p^{s_n}(1-p)^{n-s_n}, \quad \text{where} \quad s_n = \varepsilon_1 + \cdots + \varepsilon_n\ .$$

If $x \in [0,1)$, we can find $\varepsilon_1, \cdots, \varepsilon_n, \cdots \in \{0,1\}$ uniquely determined and such that for every $n$, $x \in I_{\varepsilon_1\cdots\varepsilon_n}$. Recall that $0, \varepsilon_1\cdots\varepsilon_n\cdots$ is the proper dyadic expansion of the real number $x$. In the space $[0,1)$ equipped with the probability $m$, it is easy to see that $(\varepsilon_n)_{n\geq 1}$ are independent Bernoulli random variables with parameter $p$. More precisely,

$$m(\{\varepsilon_i = 1\}) = p \quad \text{and} \quad m(\{\varepsilon_i = 0\}) = 1-p\ .$$

Using the strong law of large numbers, we know that $s_n/n$ converges $dm$-almost surely to $p$. If $I_n(x)$ is the unique interval in $\mathcal{F}_n$ which contains $x$, we deduce that for almost every $x \in [0,1)$,

$$\begin{aligned}\lim_{n\to+\infty} \frac{\ln(m(I_n(x)))}{\ln|I_n(x)|} &= \lim_{n\to+\infty} -\frac{s_n \ln p + (n-s_n)\ln(1-p)}{n\ln 2} \\ &= -(p\log_2(p) + (1-p)\log_2(1-p)) \ .\end{aligned}$$

Let $h(p) = -(p\log_2(p) + (1-p)\log_2(1-p))$. Using Billingsley's theorem (see for example Proposition 2.3 in [Fal97]), it is easy to conclude that

$$\dim_*(m) = \dim^*(m) = h(p)$$

where $\dim_*(m)$ and $\dim^*(m)$ are the lower and the upper dimension defined in (3). It means that the measure $m$ is supported by a set of Hausdorff dimension $h(p)$ and that every set of dimension less that $h(p)$ is negligible. We say that the measure $m$ is unidimensional with dimension $h(p)$.

We also have

$$\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = h(p)$$

where $\mathrm{Dim}_*(m)$ and $\mathrm{Dim}^*(m)$ are the lower and the upper packing dimension defined in (4). This example is well known. The measure $m$ allows to prove that the set $F_p$ of real numbers $x$ such that $s_n/n$ converges to $p$ has dimension $h(p)$ (see for example [Bes35], [Egg49] or [Fal90]) .

# 2 Dimensions of a measure

## 2.1 Lower and upper dimension of a measure

In general, a probability measure $m$ is not unidimensional (in the sense described in the previous example). Nevertheless, we can always define the so called lower and upper dimension in the following way.

**Definition 2.1.** *Let $m$ be a probability measure on $\mathbb{R}^d$. The quantities*

$$\dim_*(m) = \inf(\dim(E)\ ;\ m(E) > 0) \text{ and } \dim^*(m) = \inf(\dim(E)\ ;\ m(E) = 1)$$

*are respectively called the lower and the upper dimension of the measure $m$.*

The inequalities $0 \le \dim_*(m) \le \dim^*(m) \le d$ are always true. When the equality $\dim_*(m) = \dim^*(m)$ is satisfied, we say that the measure $m$ is unidimensional and we denote by $\dim(m)$ the common value.

Recall that $m_1 << m_2$ (resp. $m_1 \perp m_2$) says that the measure $m_1$ is absolutely continuous (resp. singular) with respect to $m_2$. Quantities $\dim_*(m)$ and $\dim^*(m)$ allow us to compare the measure $m$ with Hausdorff measures. More precisely, we have the following quick result :

**Proposition 2.2.** *Let $m$ be a probability measure on $\mathbb{R}^d$. Then*

$$\dim_*(m) = \sup(\alpha\ ;\ m << \mathcal{H}^\alpha) \quad \text{and} \quad \dim^*(m) = \inf(\alpha\ ;\ m \perp \mathcal{H}^\alpha)\ .$$

When the upper dimension of the measure is small, it means that the measure $m$ is "very singular". In the same way, when the lower dimension of the measure is large, then the measure $m$ is "quite regular".

Quantities $\dim_*(m)$ and $\dim^*(m)$ are also related to the asymptotic behavior of the functions $\Phi_r(x) = \frac{\ln m(B(x,r))}{\ln(r)}$. More precisely, we have

**Theorem 2.3.** ([Fan94, Fal97, Edg98, Heu98]) *Let $m$ be a probability measure on $\mathbb{R}^d$. Let*

$$\Phi_*(x) = \liminf_{r\to 0} \Phi_r(x) \quad \textit{where} \quad \Phi_r(x) = \frac{\ln m(B(x,r))}{\ln(r)} \ .$$

*We have*

$$\dim_*(m) = \operatorname{ess\,inf}(\Phi_*) \quad \textit{and} \quad \dim^*(m) = \operatorname{ess\,sup}(\Phi_*),$$

*the essential bounds being related to the measure $m$. In particular, the inequalities $0 \le \Phi_* \le d$ are true $dm$-almost surely.*

*Proof.* Let us prove the equality $\dim_*(m) = \operatorname{ess\,inf}(\Phi_*)$. The proof of the equality $\dim^*(m) = \operatorname{ess\,sup}(\Phi_*)$ is quite similar. Let $\alpha < \operatorname{ess\,inf} \Phi_*$. For $dm$-almost every $x$, there exists $r_0$ such that if $r < r_0$, $m(B(x,r)) < r^\alpha$. Let

$$E_n = \{x \ ; \ \forall r < 1/n, \ m(B(x,r)) < r^\alpha\} \ .$$

The measure $m$ is carried by $\bigcup_n E_n$. If $m(E) > 0$, we can then find an integer $n$ such that $m(E \cap E_n) > 0$. Using the definition of $E_n$, it follows that $\mathcal{H}^\alpha(E \cap E_n) > 0$ and that $\dim(E) \ge \alpha$. We have proved that $\operatorname{ess\,inf}(\Phi_*) \le \dim_*(m)$.

Conversely, if $\alpha > \operatorname{ess\,inf} \Phi_*$, we can find $E$ such that $m(E) > 0$ and such that for every $x \in E$, $\Phi_*(x) < \alpha$. If $x \in E$, and if $\delta > 0$, we can find $r_x < \delta$ such that $m(B(x, r_x)) > r_x^\alpha$. The balls $B(x, r_x)$ constitute a $2\delta$-covering of $E$. The problem is that these balls are not disjoint. Nevertheless, using Besicovich's covering lemma, we can find a constant $\xi$ which only depends on the dimension $d$ and we can choose $\xi$ sub families $B(x_{1,j}, r_{x_{1,j}})_j, \cdots, B(x_{\xi,j}, r_{x_{\xi,j}})_j$ of disjoint balls which always cover $E$. We then have

$$\begin{aligned}
\forall i, \qquad \sum_j (\text{diam } (B(x_{i,j}, r_{x_{i,j}})))^\alpha &= \sum_j (2r_{x_{i,j}})^\alpha \\
&\le 2^\alpha \sum_j m(B(x_{i,j}, r_{x_{i,j}})) \\
&\le 2^\alpha m(\mathbb{R}^d) \\
&< +\infty \ .
\end{aligned}$$

Finally, $\mathcal{H}^\alpha_{2\delta}(E) \le \xi 2^\alpha m(\mathbb{R}^d)$ and we can conclude that $\mathcal{H}^\alpha(E) < +\infty$ and that $\dim(E) \le \alpha$. When $\alpha \to \operatorname{ess\,inf} \Phi_*$, we obtain $\dim_*(m) \le \operatorname{ess\,inf}(\Phi_*)$. •

*Remark 1.* We can also use Proposition 2.2 in [Fal97] to give another proof of Theorem 2.3.

*Remark 2.* The measure $m$ is unidimensional (that is $\dim_*(m) = \dim^*(m)$) if and only if there exists $\alpha \ge 0$ such that $m$ is carried by a a set of dimension $\alpha$ while $m(E) = 0$ for every Borel set $E$ satisfying $\dim(E) < \alpha$. In that case, $\alpha = \dim_*(m) = \dim^*(m)$. This notion was first introduced by Rogers and Taylor ([RT59]) and revived by Cutler [[Cut86]).

## 2.2 And what about packing dimensions ?

It is then natural to ask about the interpretation of the essential bounds of the function $\Phi^* = \limsup_{r\to 0}\Phi_r$. Those are related to the packing dimension of the measure $m$ (for more details on packing dimension, see [Fal90] or the original paper of Tricot [Tri82]). Without any new idea, we can prove the twin results of Proposition 2.2 and Theorem 2.3.

**Proposition 2.4.** *Let $m$ be a probability measure on $\mathbb{R}^d$. Let us denote*

$$\mathrm{Dim}_*(m) = \inf(\mathrm{Dim}(E)\ ;\ m(E)>0) \textit{ and } \mathrm{Dim}^*(m) = \inf(\mathrm{Dim}(E)\ ;\ m(E)=1)\ .$$

*Then,*

$$\mathrm{Dim}_*(m) = \sup(\alpha\ ;\ m << \hat{\mathcal{P}}^\alpha) \quad \textit{and} \quad \mathrm{Dim}^*(m) = \inf(\alpha\ ;\ m\perp\hat{\mathcal{P}}^\alpha),$$

*where $(\hat{\mathcal{P}}^\alpha)_{\alpha>0}$ are the packing measures and* Dim *the packing dimension.*

**Theorem 2.5.** ([Fal97, Edg98, Heu98]) *Let $m$ be a probability measure on $\mathbb{R}^d$. Let*

$$\Phi^*(x) = \limsup_{r\to 0}\Phi_r(x) \quad \textit{where} \quad \Phi_r(x) = \frac{\ln m(B(x,r))}{\ln(r)}\ .$$

*We have*

$$\mathrm{Dim}_*(m) = \mathrm{ess\,inf}\,(\Phi^*) \quad \textit{and} \quad \mathrm{Dim}^*(m) = \mathrm{ess\,sup}\,(\Phi^*),$$

*the essential bounds being related to the measure $m$. In particular, the inequalities $0 \le \Phi^* \le d$ are true $dm$-almost surely.*

## 2.3 Unidimensionality and ergodicity

Let us come back to the Bernoulli product which is described in Section 1. This measure satisfies :

$$\begin{cases} \mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = \dim_*(m) = \dim^*(m) = h(p) \\ \Phi^*(x) = \Phi_*(x) = h(p) \quad dm - \text{almost surely}\ . \end{cases}$$

In particular, it is a unidimensional measure.

Moreover, the Bernoulli product has interesting properties with respect to the doubling operator

$$\sigma(x) = 2x - [2x]$$

where $[2x]$ is the integer part of $2x$.

Let us precise those properties. Denote by

$$IJ = I_{\varepsilon_1\cdots\varepsilon_{n+p}} \quad \text{if} \quad I = I_{\varepsilon_1\cdots\varepsilon_n} \text{ and } J = I_{\varepsilon_{n+1}\cdots\varepsilon_{n+p}}\ .$$

Independence properties of the random variables $\varepsilon_n$ easily ensure that

$$m(\sigma^{-1}(I)) = m(I), \qquad \forall I \in \bigcup_n \mathcal{F}_n \tag{14}$$

and

$$m(I \cap \sigma^{-n}(J)) = m(IJ) = m(I)m(J) \quad \text{if} \quad I \in \mathcal{F}_n \ . \tag{15}$$

Finally, using a monotone class argument, it is easy to deduce from (14) and (15) that the measure $m$ is $\sigma$-invariant and ergodic (see also Part 5).

This result is not surprising. More generally we can prove the following property which can be found in [Fal97].

**Proposition 2.6.** *Let $X$ be a closed subset of $\mathbb{R}^d$, $T : X \to X$ a lipschitz function and $m$ a $T$-invariant and ergodic probability measure on $X$. Then :*

$$\dim_*(m) = \dim^*(m) \quad \textit{and} \quad \mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) \ .$$

*Proof.* Let us give a proof of this proposition which is somewhat simpler to the one proposed by Falconer in [Fal97] and which does not need the use of the ergodic theorem. If $T$ is $C$-lipschitz, $T(B(x,r)) \subset B(T(x), Cr)$. We can deduce that

$$m(B(x,r)) \leq m(T^{-1}(T(B(x,r)))) \leq m(T^{-1}(B(T(x),Cr))) = m(B(T(x),Cr)) \ .$$

So, $\Phi_r(x) \geq \Phi_{Cr}(T(x))\frac{\ln(Cr)}{\ln(r)}$, which proves that $\Phi_*(x) \geq \Phi_*(T(x))$. The function $\Phi_*(x) - \Phi_*(T(x))$ is then positive and satisfies $\int(\Phi_*(x) - \Phi_*(T(x)))dm(x) = 0$. We can conclude that $\Phi_*(x) = \Phi_*(T(x))$ almost surely and that $\Phi_*$ is $T$-invariant. On the other hand, $\Phi_*$ is essentially bounded (see Theorem 2.3) and the measure $m$ is ergodic. It follows that $\Phi_*$ is almost surely constant, which says that $\dim_*(m) = \dim^*(m)$. The proof of $\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m)$ is similar. •

*Remark 1.* The function $\sigma(x) = 2x - [2x]$ is not lipschitz. Apparently, Proposition 2.6 is not relevant for this function. Nevertheless, if we identify the points 0 and 1, that is, if we imagine the measure $m$ defined on the circle $\mathbb{R}/\mathbb{Z} = S_1$, then, $m$ is invariant with respect to the doubling function which is a smooth function on $S_1$.

*Remark 2.* Another way to study $m$ is to consider that the Bernoulli product is defined on the Cantor set $\{0,1\}^{\mathbb{N}^*}$. Then, the intervals $I_{\varepsilon_1 \cdots \varepsilon_n}$ become the cylinders $\varepsilon_1 \cdots \varepsilon_n$ of the Cantor set and the function $\sigma$ is nothing else but the shift operator $(\varepsilon_n)_{n\geq 1} \mapsto (\varepsilon_n)_{n \geq 2}$ on the Cantor set.

*Remark 3.* Ergodic criteria for unidimensionality are also given by Cutler ([Cut90]) and Fan ([Fan94]).

# 3 The discrete point of view

The Hausdorff dimension may be calculated with the use of the $\ell$-adic cubes. Therefore we can obtain discrete versions of the previous results. Let $\ell \geq 2$ be an integer and $\mathcal{F}_n$ the dyadic cubes of the $n^{\text{th}}$ generation. Suppose that $m$ is a probability measure on $[0,1)^d$. If $I_n(x)$ is the unique cube in $\mathcal{F}_n$ which contains $x$ and if $\log_\ell$ is the logarithm in base $\ell$, we can introduce the sequence of random variables $X_n$ defined by

$$X_n(x) = -\log_\ell \left( \frac{m(I_n(x))}{m(I_{n-1}(x))} \right) \ . \tag{16}$$

If $|I_n(x)| = \ell^{-n}$ is the “length” of the cube $I_n(x)$, we have

$$\frac{S_n(x)}{n} = \frac{X_1(x) + \ldots + X_n(x)}{n} = \frac{\log m(I_n(x))}{\log |I_n(x)|}$$

and the quantities $\dim_*(m)$ and $\dim^*(m)$ are related to the asymptotic behavior of the sequence $\frac{S_n}{n}$. More precisely, we have the two following relations

$$\begin{aligned} \dim_*(m) &= \operatorname{ess\,inf} \left( \liminf_{n\to\infty} \frac{S_n}{n} \right) \\ \dim^*(m) &= \operatorname{ess\,sup} \left( \liminf_{n\to\infty} \frac{S_n}{n} \right) \ . \end{aligned} \tag{17}$$

In the same way, we can also prove that

$$\begin{aligned} \mathrm{Dim}_*(m) &= \operatorname{ess\,inf} \left( \limsup_{n\to\infty} \frac{S_n}{n} \right) \\ \mathrm{Dim}^*(m) &= \operatorname{ess\,sup} \left( \limsup_{n\to\infty} \frac{S_n}{n} \right) \ . \end{aligned} \tag{18}$$

## 3.1 An example

Let us describe a well known elementary example (see for example [BK90] or [Bis95]) which is more general than the Bernoulli product and indicates that the probabilistic point of view is useful. Let $d = 1$, $\ell = 2$ and let us consider a sequence $(p_n)_{n\geq 1}$ of real numbers satisfying $0 < p_n < 1$. With the notations of Section 1, let us construct the measure $m$ in the following way.

$$m\left(I_{\varepsilon_1\cdots\varepsilon_{n-1}1}\right) = p_n m\left(I_{\varepsilon_1\cdots\varepsilon_{n-1}}\right) \quad \text{and} \quad m\left(I_{\varepsilon_1\cdots\varepsilon_{n-1}0}\right) = (1-p_n) m\left(I_{\varepsilon_1\cdots\varepsilon_{n-1}}\right) \ .$$

The random variables $\varepsilon_i$ are independent and verify

$$m\left(\{\varepsilon_n = 1\}\right) = p_n \quad \text{and} \quad m\left(\{\varepsilon_n = 0\}\right) = 1 - p_n \ .$$

The random variables $X_n$, defined by (16) are independent and bounded in $L^2$. The strong law of large numbers ensures that the sequence

$$\frac{S_n - \mathbb{E}\left[S_n\right]}{n} \tag{19}$$

is almost surely converging to 0. We can easily conclude that for $dm$-almost every $x \in [0,1)$,

$$\begin{aligned} \liminf_{n\to\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} &= \liminf_{n\to\infty} \mathbb{E}\left[\frac{S_n}{n}\right] \\ &= \liminf_{n\to\infty} \frac{-1}{n} \sum_{k=1}^{n} p_k \log_2 p_k + (1-p_k)\log_2(1-p_k) \ . \end{aligned}$$

We write $h_*(m) = \liminf_{n\to\infty} \mathbb{E}\left[\frac{S_n}{n}\right]$. This quantity is called the lower entropy of the measure $m$ (see Section 3.2).

In this case, the measure $m$ is always a unidimensional measure with dimension $\dim(m) = h_*(m)$. More precisely, we can deduce from (19) the existence of a subsequence $n_k$ such that for almost every $x \in [0,1)$,

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{\log |I_{n_k}(x)|} = h_*(m) \ .$$

We will see in Section 4.3 that this kind of property characterizes measures for which the dimension can be calculated with an entropy formula.

Of course, a similar result can be written with packing dimensions. The measure $m$ is unidimensional and satisfies

$$\begin{aligned} \mathrm{Dim}(m) &= \limsup_{n\to\infty} \mathbb{E}\left[\frac{S_n}{n}\right] \\ &= \limsup_{n\to\infty} \frac{-1}{n} \sum_{k=1}^{n} p_k \log_2 p_k + (1-p_k)\log_2(1-p_k) \\ &:= h^*(m) \ . \end{aligned}$$

Note that we may have $\dim(m) \neq \mathrm{Dim}(m)$.

## 3.2 The function $\tau$, probabilistic interpretation and links with entropy

Relations (17) and (18) do not help to find the dimensions of the measure $m$. From now on we try to obtain estimates of the quantities $\dim_*(m)$, $\dim^*(m)$, $\mathrm{Dim}_*(m)$, $\mathrm{Dim}^*(m)$ and describe some equality cases.

Let us introduce the function $\tau$ which is well known in multifractal analysis. It is defined as

$$\tau(q) = \limsup_{n\to+\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n \log \ell} \log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right) \tag{20}$$

where $m$ is a probability measure on $[0,1)^d$. The function $\tau$ is finite on $[0,+\infty)$ and may be degenerated on the open interval $(-\infty,0)$. It is convex, non increasing on its definition domain. If we equip the set $[0,1)^d$ with the probability $m$, we can write :

$$\tau_n(1-q) = \frac{1}{n}\log_\ell \mathbb{E}\left[\ell^{qS_n}\right] \quad \text{and} \quad \tau(1-q) = \limsup_{n\to\infty} \frac{1}{n}\log_\ell \mathbb{E}\left[\ell^{qS_n}\right] \ . \tag{21}$$

Taking the derivative, we get

$$-\tau_n'(1) = \mathbb{E}\left[\frac{S_n}{n}\right] = \frac{-1}{n}\sum_{I\in\mathcal{F}_n} m(I)\log_\ell m(I) \ .$$

This quantity is nothing else but the entropy of the probability $m$ related to the partition $\mathcal{F}_n$. It will be denoted by $h_n(m)$. In a general setting the sequence $h_n(m)$ does not necessarily converge. Nevertheless, one can always define the lower and the upper entropy with the formula

$$h_*(m) = \liminf_{n\to\infty} h_n(m) \quad \text{and} \quad h^*(m) = \limsup_{n\to\infty} h_n(m) \ . \tag{22}$$

If $h_*(m) = h^*(m)$, the common value is denoted by $h(m)$. It is the entropy of the measure $m$.

Let us remark that convexity properties ensure that

$$-\tau'_+(1) \le h_*(m) \le h^*(m) \le -\tau'_-(1) \ , \tag{23}$$

where $\tau'_-$ et $\tau'_+$ are respectively the left and the right derivative of the convex function $\tau$.

Let us finish this section with the example described in Part 3.1. Easy calculations give

$$\begin{aligned}
\tau(q) &= \limsup_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \log_2 \left(p_k^q + (1-p_k)^q\right) \\
h_*(m) &= \liminf_{n\to\infty} \frac{-1}{n} \sum_{k=1}^{n} p_k \log_2 p_k + (1-p_k)\log_2(1-p_k) \\
h^*(m) &= \limsup_{n\to\infty} \frac{-1}{n} \sum_{k=1}^{n} p_k \log_2 p_k + (1-p_k)\log_2(1-p_k) \ .
\end{aligned}$$

In particular, if $m$ is a Bernoulli product with parameter $p$ (that is, if $p_k = p$ for all $k$), we get

$$\tau(q) = \log_2 \left(p^q + (1-p)^q\right) \quad \text{and} \quad h(m) = -\left(p\log_2(p) + (1-p)\log_2(1-p)\right) \ .$$

### 3.3 General estimates

There are deep links between the function $\tau$, entropy and the dimension of the measure $m$. These can be resumed in the following theorem.

**Theorem 3.1.** ([Heu98, BH02]) *Let $m$ be a probability measure on $[0,1)^d$. We have ;*

$$-\tau'_+(1) \le \dim_*(m) \le h_*(m) \le h^*(m) \le \mathrm{Dim}^*(m) \le -\tau'_-(1) \ . \tag{24}$$

*Remarks.* 1. In particular, (17) and (18) ensure that if $\dim_*(m) = \mathrm{Dim}^*(m)$, then the entropy $h(m)$ exists and

$$\lim_{n\to\infty} \frac{-\log_\ell m(I_n(x))}{n} = h(m), \qquad dm\text{-almost surely} \ .$$

We then obtain some kind of "Shannon-McMillan conclusion" in a non dynamical context. It is in particular the case if $\tau'(1)$ exists.

2. Conversely, if there exists a real number $h$ such that $\lim_{n\to\infty} \frac{-\log_\ell m(I_n(x))}{n} = h$ almost surely, we have

$$\dim_*(m) = \mathrm{Dim}^*(m) \quad \text{and} \quad h_*(m) = h^*(m) = h \ .$$

3. In [Nga97], S.M. Ngai proves inequalities like $-\tau'_+(1) \le \dim_*(m)$ and $\mathrm{Dim}^*(m) \le -\tau'_-(1)$. His purpose is then to consider the case where $\tau'(1)$ exists. Here we will first consider the non differentiable case (see Part 3.4 and 4.2) and then find conditions that ensure that $\tau'(1)$ exists (see Part 5).

Formulas (17) and (18) give links between the dimension of the measure $m$ and the asymptotic behavior of the sequence $S_n/n$. They allow us to propose a very simple proof of Theorem 3.1. This is not the way used in [Heu98] but we can isolate the following result which immediately gives Theorem 3.1.

**Theorem 3.2.** *Let $(S_n)_{n\geq 0}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that the function*

$$L(q) = \limsup_{n\to\infty} \frac{1}{n} \log_\ell \mathbb{E}\left[\ell^{qS_n}\right]$$

*is finite on a neighborhood $\mathcal{V}$ of 0. Then we have :*

$$L'_-(0) \leq \operatorname{ess\,inf}\left(\liminf_{n\to\infty} \frac{S_n}{n}\right) \quad \textit{and} \quad \operatorname{ess\,sup}\left(\limsup_{n\to\infty} \frac{S_n}{n}\right) \leq L'_+(0)\ .$$

*Moreover, the sequence $\frac{S_n}{n}$ is dominated in $L^1(\mathbb{P})$ and*

$$\operatorname{ess\,inf}\left(\liminf_{n\to+\infty} \frac{S_n}{n}\right) \leq \liminf_{n\to+\infty} \mathbb{E}\left[\frac{S_n}{n}\right] \leq \limsup_{n\to+\infty} \mathbb{E}\left[\frac{S_n}{n}\right] \leq \operatorname{ess\,sup}\left(\limsup_{n\to+\infty} \frac{S_n}{n}\right)\ .$$

*Proof of Theorem 3.2.* Let $\alpha > L'_+(0)$ and $q > 0$. Using Cramer-Chernov's idea, we have

$$\mathbb{P}\left(\frac{S_n}{n} \geq \alpha\right) \leq \frac{1}{\ell^{qn\alpha}} \mathbb{E}\left[\ell^{qS_n}\right]\ .$$

Taking the logarithm and the lim sup, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log_\ell\left(\mathbb{P}\left(\frac{S_n}{n} \geq \alpha\right)\right) \leq L(q) - q\alpha$$

and we can conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log_\ell\left(\mathbb{P}\left(\frac{S_n}{n} \geq \alpha\right)\right) \leq - \sup_{q>0\,,\, q\in\mathcal{V}} (q\alpha - L(q)) = -L^*(\alpha) < 0,$$

where $L^*$ is the Legendre transform of $L$. If $0 < \varepsilon < L^*(\alpha)$ and if $n$ is sufficiently large, we obtain

$$\mathbb{P}\left(\frac{S_n}{n} \geq \alpha\right) \leq e^{-n(L^*(\alpha)-\varepsilon)}\ .$$

Then, Borel-Cantelli's lemma gives

$$\mathbb{P}\left(\limsup_{n\to\infty}\left\{\frac{S_n}{n} \geq \alpha\right\}\right) = 0,$$

which clearly implies that $\limsup_{n\to+\infty} \frac{S_n}{n} \leq \alpha$ almost surely. The inequality

$$\operatorname{ess\,sup}\left(\limsup_{n\to\infty} \frac{S_n}{n}\right) \leq L'_+(0)$$

follows. With a similar argument, we can also prove the other inequality

$$L'_-(0) \leq \operatorname{ess\,inf}\left(\liminf_{n\to\infty} \frac{S_n}{n}\right)\ .$$

In order to obtain the second point of the theorem, we first observe that the sequence $\frac{S_n}{n}$ is dominated in $L^1(\mathbb{P})$. Indeed, let $X = \sup_n \left|\frac{S_n}{n}\right|$. We have :

$$\mathbb{P}\left(X > t\right) \leq \sum_{n\geq 1} \mathbb{P}\left(\left|\frac{S_n}{n}\right| > t\right) = \sum_{n\geq 1} \mathbb{P}\left(\frac{S_n}{n} > t\right) + \mathbb{P}\left(\frac{S_n}{n} < -t\right) .$$

On the other hand, if $q > 0$ is such that $L(q) < +\infty$ and if $\varepsilon > 0$, the preceding calculus allows us to find an integer $n_0$ such that for every $n \geq n_0$,

$$\frac{1}{n}\log_\ell\left(\mathbb{P}\left(\frac{S_n}{n} > t\right)\right) \leq L(q) + \varepsilon - qt .$$

If $t$ is large enough, we get

$$\sum_{n\geq n_0} \mathbb{P}\left(\frac{S_n}{n} > t\right) \leq \sum_{n\geq n_0} \ell^{n(L(q)+\varepsilon-qt)} \leq \frac{\ell^{L(q)+\varepsilon-qt}}{1-\ell^{L(q)+\varepsilon-qt}}$$

which proves that the function

$$t \mapsto \sum_{n\geq 1} \mathbb{P}\left(\frac{S_n}{n} > t\right)$$

is integrable with respect to the Lebesgue's measure. A similar result is true for the function $t \mapsto \sum_{n\geq 1} \mathbb{P}\left(\frac{S_n}{n} < -t\right)$. Finally,

$$\mathbb{E}\left[X\right] = \int_0^{+\infty} \mathbb{P}\left(X > t\right) dt < +\infty .$$

Having just proved that the sequence $\frac{S_n}{n}$ is dominated in $L^1(\mathbb{P})$ by the random variable $X$, Fatou's lemma applied to the positive sequence $X + \frac{S_n}{n}$ gives

$$\begin{aligned}
\mathbb{E}\left[X\right] + \operatorname{ess\,inf}\left(\liminf_{n\to+\infty}\frac{S_n}{n}\right) &= \mathbb{E}\left[X + \operatorname{ess\,inf}\left(\liminf_{n\to+\infty}\frac{S_n}{n}\right)\right] \\
&\leq \mathbb{E}\left[X + \left(\liminf_{n\to+\infty}\frac{S_n}{n}\right)\right] \\
&\leq \liminf_{n\to+\infty}\mathbb{E}\left[X + \frac{S_n}{n}\right] \\
&= \mathbb{E}\left[X\right] + \liminf_{n\to+\infty}\mathbb{E}\left[\frac{S_n}{n}\right] ,
\end{aligned}$$

and the first inequality follows. In order to prove the second inequality, it suffices to apply Fatou's lemma to the positive sequence $X - \frac{S_n}{n}$. •

## 3.4 How to use Theorem 3.1

In general it is awkward or even impossible to obtain exact values for the function $\tau$ and the numbers $\tau'_-(1)$ and $\tau'_+(1)$. Nevertheless, if we can estimate in a neighborhood of 1 the function $\tau$ by a function $\chi$ satisfying $\chi(1) = 0$, we obtain

$$\dim_*(m) \geq -\chi'_+(1) \quad \text{and} \quad \mathrm{Dim}^*(m) \leq -\chi'_-(1) .$$

In particular, this remark can be applied to $\chi = \log_\ell(\beta)$ where

$$\beta(q) = \limsup_{n\to+\infty} \beta_n(q) \quad \text{and} \quad \beta_n(q) = \sup_{I\in\mathcal{F}_n} \left( \sum_{J\subset I,\ J\in\mathcal{F}_{n+1}} \left(\frac{m(J)}{m(I)}\right)^q \right) .$$

This is a consequence of the inequalities

$$\begin{aligned} \tau(q) &\le \limsup_{n\to+\infty} \frac{\log\beta_{n-1}(q) + \cdots + \log\beta_0(q)}{n\log\ell} \\ &\le \limsup_{n\to+\infty} \frac{\log\beta_n(q)}{\log\ell} = \frac{\log\beta(q)}{\log\ell} . \end{aligned}$$

Finally, using $\beta(1) = 1$, we get the following corollary :

**Corollary 3.3.** ([Heu95, Heu98]) *Let $m$ be a probability measure on $[0,1)^d$ and $\beta$ defined as above. We have*

$$\dim_*(m) \ge -\frac{\beta'_+(1)}{\ln(\ell)} \quad \textit{and} \quad \mathrm{Dim}^*(m) \le -\frac{\beta'_-(1)}{\ln(\ell)} .$$

## 3.5 Contrasts and dimension's estimates

The function $\beta_n$ gives estimates of the contrasts between the mass of a cube $I$ and the mass of its sons. In numerous situations, those contrasts can be estimated and we can then deduce estimates of the dimension of the measure. In particular, this is what is done by Bourgain in [Bou87] and Batakis in [Bat96] when they give estimates of the dimension of the harmonic measure. Some elementary situations, which are particular cases of Proposition 3.4 and 3.5 are also proposed in [Heu95].

Let us describe a general way to obtain concrete estimates. Suppose that every cube $I \in \bigcup_n \mathcal{F}_n$ has a positive mass. Let $k \in \{1, \cdots, \ell^d - 1\}$ and if $I \in \mathcal{F}_n$, $n \ge 1$, let

$$\delta_k(I) = \max\left(\frac{m(I_1 \cup \cdots \cup I_k)}{m(I)}\ ,\ I_1, \cdots, I_k \text{ sons of } I\right) .$$

We first remark that if $J_1, \cdots, J_{\ell^d}$ are the sons of $I$ and satisfy $m(J_1) \ge \cdots \ge m(J_{\ell^d})$, we have

$$\delta_k(I) = \frac{m(J_1 \cup \cdots \cup J_k)}{m(I)} \quad \text{and} \quad \forall j > k,\ km(J_j) \le m(J_1 \cup \cdots \cup J_k) .$$

It follows that

$$1 = \delta_k(I) + \sum_{j>k} \frac{m(J_i)}{m(I)} \le \delta_k(I) + \frac{(\ell^d - k)}{k}\delta_k(I)$$

and we can claim that

$$\frac{k}{\ell^d} \le \delta_k(I) \le 1 . \tag{25}$$

If $\delta_k(I) \approx \frac{k}{\ell^d}$, the measure $m$ is quite homogenous in the cube $I$. If it is true in every cube, we can hope that the dimension of $m$ is big. On the other hand, if for every cube $I$, $\delta_k(I) \approx 1$, a small part of $I$ contains a large part of the mass and we can hope that the dimension of $m$ is small.

These remarks can be made precise in the following propositions.

**Proposition 3.4.** *Let $m$ be a probability measure on $[0,1)^d$, $1 \le k < \ell^d$ and $k\ell^{-d} < \delta < 1$ such that for every $I \in \bigcup_n \mathcal{F}_n$, $\delta_k(I) \ge \delta$. Then, the measure $m$ satisfies*

$$\mathrm{Dim}^*(m) \le -\delta \log_\ell\left(\frac{\delta}{k}\right) - (1-\delta)\log_\ell\left(\frac{1-\delta}{\ell^d - k}\right) .$$

**Proposition 3.5.** *Let $m$ be a probability measure on $[0,1)^d$, $1 \le k < \ell^d$ and $k\ell^{-d} < \delta < 1$ such that for every $I \in \bigcup_n \mathcal{F}_n$, $\delta_k(I) \le \delta$. Let $p = \left[\delta^{-1}\right]$. Then, the measure $m$ satisfies*

$$\dim_*(m) \ge -p\delta \log_\ell(\delta) - (1-p\delta)\log_\ell(1-p\delta) .$$

Proposition 3.5 is in fact an elementary consequence of the more general following result.

**Proposition 3.6.** *Let $m$ be a probability measure on $[0,1)^d$ and $0 < \delta \le 1$. Let $p = \left[\delta^{-1}\right]$ and suppose that for every cube $I \in \bigcup_n \mathcal{F}_n$, we can find a partition $A_1, \cdots, A_j$ of the set of sons of $I$ such that*

$$\forall i \in \{1, \cdots, j\}, \qquad \frac{m\left(\bigcup_{J\in A_i} J\right)}{m(I)} \le \delta .$$

*Then*

$$\dim_*(m) \ge -p\delta \log_\ell(\delta) - (1-p\delta)\log_\ell(1-p\delta) .$$

*Remark 1.* If $\delta > 1/2$, then $p = 1$. This is in particular the case when $\ell = 2$ and $d = 1$.

*Remark 2.* When $k = 1$ and $\ell = 2$, similar estimations are also obtained by Llorente and Nicolau in [LN04]. Logarithm corrections are also proposed.

*Proof of Proposition 3.4.* This proposition can be found in [Heu98]. Let us sketch the proof in order to be self contained. Let $I \in \mathcal{F}_n$ and $I_1, \cdots, I_k$ the sons of $I$ such that $\delta_k(I) = \frac{m(I_1\cup\cdots\cup I_k)}{m(I)}$. Denote $S = \{I_1, \cdots, I_k\}$. If $q < 1$, Hölder's inequality gives

$$\begin{aligned}\sum_{J\subset I,\ J\in\mathcal{F}_{n+1}} \left(\frac{m(J)}{m(I)}\right)^q &= \sum_{J\in S}\left(\frac{m(J)}{m(I)}\right)^q + \sum_{J\notin S}\left(\frac{m(J)}{m(I)}\right)^q \\ &\le k^{1-q}\left(\delta_k(I)\right)^q + (\ell^d - k)^{1-q}\left(1-\delta_k(I)\right)^q .\end{aligned}$$

Let us observe that the function $t \mapsto k^{1-q}t^q + (\ell^d - k)^{1-q}(1-t)^q$ is decreasing on the interval $[k\ell^{-d}, 1]$. Under the hypothesis of Proposition 3.4, we obtain

$$\forall q \in ]0,1[, \quad \beta_n(q) \le k^{1-q}(\delta)^q + (\ell^d - k)^{1-q}(1-\delta)^q ,$$

and the conclusion follows from Corollary 3.3.

*Proof of Proposition 3.6.* We begin with the following lemma.

**Lemma 3.7.** *Let $q > 1$, $j \geq 2$ and $\frac{1}{j} < \delta \leq 1$. Denote by $M(\delta, j)$ the maximum of the function $F(a_1, \cdots, a_j) = a_1^q + \cdots + a_j^q$ under the constraints $a_1 + \cdots + a_j = 1$ and $0 \leq a_i \leq \delta$ $\forall i$. Then*

$$M(\delta, j) = p\delta^q + (1 - p\delta)^q$$

*where $p = \left[\delta^{-1}\right]$.*

*Proof.* The function $F$ being symmetric, we can add the constraint $a_1 \geq \cdots \geq a_j$. Observe that we have $j \geq p + 1$.

If $0 < a_2 \leq a_1 < \delta$, the function $\varepsilon > 0 \mapsto (a_1 + \varepsilon)^q + (a_2 - \varepsilon)^q$ is increasing, so that the maximum is obtained when $a_1 = \delta$. We then prove the lemma by recurrence on the integer $p$.

Suppose first that $p = 1$, that is $\frac{1}{2} < \delta \leq 1$. We have

$$F(\delta, a_2, \cdots, a_j) \leq \delta^q + (a_2 + \cdots + a_j)^q = \delta^q + (1 - \delta)^q \ .$$

Moreover, under the hypothesis $p = 1$, we have $0 \leq 1 - \delta < \delta$, $F(\delta, 1 - \delta, 0, \cdots, 0) = \delta^q + (1 - \delta)^q$ and we can conclude that $M(\delta, j) = \delta^q + (1 - \delta)^q$.

Suppose now that the conclusion of the lemma is satisfied for every value of $\left[\delta^{-1}\right]$ between 1 and $p - 1$ and let $\delta$ such that $\left[\delta^{-1}\right] = p$. The real number $\delta$ satisfies the inequalities $\frac{1}{p+1} < \delta \leq \frac{1}{p}$ and we observe that

$$F(\delta, a_2, \cdots, a_j) = \delta^q + (1 - \delta)^q \left( \left( \frac{a_2}{1 - \delta} \right)^q + \cdots + \left( \frac{a_j}{1 - \delta} \right)^q \right) \ .$$

The real numbers $\frac{a_i}{1-\delta}$ satisfy the constraints

$$0 \leq \frac{a_i}{1 - \delta} \leq \frac{\delta}{1 - \delta} \ .$$

Moreover,

$$\left[ \frac{1 - \delta}{\delta} \right] = p - 1 \quad \text{and} \quad \frac{1}{j - 1} < \frac{\delta}{1 - \delta} \ .$$

We can then use the recurrence hypothesis and obtain

$$\begin{aligned} F(\delta, a_2, \cdots, a_j) &\leq \delta^q + M\left( \frac{\delta}{1 - \delta}, j - 1 \right) \\ &= \delta^q + (1 - \delta)^q \left( (p - 1) \left( \frac{\delta}{1 - \delta} \right)^q + \left( 1 - (p - 1) \frac{\delta}{1 - \delta} \right)^q \right) \\ &= p\delta^q + (1 - p\delta)^q \ . \end{aligned}$$

It follows that $M(\delta, j) \leq p\delta^q + (1 - p\delta)^q$. In fact, the last inequality is an equality if we remark that $1 - p\delta \leq \delta$ and

$$F(\delta, \cdots, \delta, (1 - p\delta), 0, \cdots, 0) = p\delta^q + (1 - p\delta)^q \ .$$

We can now finish the proof of Proposition 3.6. We want to estimate the function $\beta$ of Part 3.4. Let $I \in \mathcal{F}_n$. If $q > 1$, Lemma 3.7 ensures that

$$\sum_{J \subset I,\, J \in \mathcal{F}_{n+1}} \left( \frac{m(J)}{m(I)} \right)^q = \sum_{i=1}^{j} \sum_{J \in A_i} \left( \frac{m(J)}{m(I)} \right)^q$$

$$\begin{aligned} &\leq \sum_{i=1}^{j} \left( \frac{m\left(\bigcup_{J\in A_i} J\right)}{m(I)} \right)^q \\ &\leq p\delta^q + (1-p\delta)^q \ . \end{aligned}$$

We can deduce that

$$\beta(q) \leq p\delta^q + (1-p\delta)^q \qquad \text{if } q > 1$$

and conclude that

$$\dim_*(m) \geq -\frac{\beta'_+(1)}{\log \ell} \geq -p\delta \log_\ell(\delta) - (1-p\delta)\log_\ell(1-p\delta) \ .$$

# 4 Situations where it is possible to obtain an exact formula for the dimension

## 4.1 Equalities $-\tau'_-(1) = \mathrm{Dim}^*(m)$ and $-\tau'_+(1) = \dim_*(m)$ are often false

In general $-\tau'_+(1) \neq \dim_*(m)$ and $-\tau'_-(1) \neq \mathrm{Dim}^*(m)$. For example, Olsen in [Ols00] gives an example of a discrete measure such that $-\tau'_-(1) = 1$ and $-\tau'_+(1) = 0$. We give here a more convincing example.

**Proposition 4.1.** *Let $\mu$ be a continuous measure with support $[0,1]$. We can construct a measure $m$ which is equivalent to $\mu$ and for which the function $\tau$ satisfies*

$$\tau(q) = \sup(1-q, 0) \quad \textit{if } q > 0 \ .$$

*In particular, the measures $\mu$ and $m$ have the same dimensions but the function $\tau$ associated to $m$ is degenerated.*

Applying this proposition to a Bernoulli product for which the parameter $p$ satisfies

$$-(p \log_2(p) + (1-p) \log_2(1-p)) = h,$$

we obtain the following corollary.

**Corollary 4.2.** *Let $0 < h < 1$. There exists a probability measure $m$ on $[0,1)$ such that*

$$\tau(q) = \sup(1-q, 0) \ \textit{if } q > 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = h \ dm\textit{-almost surely} \ .$$

*Proof of Proposition 4.1.* Suppose that $\ell = 2$ (the construction is quite similar if $\ell > 2$). Let $\mu$ be a measure with support $[0,1]$ and for which the points have no mass. The construction of the measure $m$ needs two steps. If $I \in \mathcal{F}_n$, let $\mu_I = (\mu(I))^{-1} \mathbb{1}_I \mu$ be the "localized measure" on $I$. Define the measure $m_1$ with the formula

$$m_1 = \sum_{n=1}^{\infty} \sum_{I\in\mathcal{F}_n} cn^{-2}2^{-n}\mu_I,$$

where $c$ is chosen such that $c\sum_{n\geq 1} n^{-2} = 1$. The measure $m_1$ is clearly equivalent to the measure $\mu$. Moreover, if $I \in \mathcal{F}_n$, we remark that

$$m_1(I) \geq cn^{-2}2^{-n}$$

which implies that for every $0 < q < 1$,

$$\sum_{I\in\mathcal{F}_n} m_1(I)^q \geq 2^n \left[cn^{-2}2^{-n}\right]^q \ .$$

With obvious notations, we get $\tau_1(q) \geq 1-q$ if $0 < q < 1$. Moreover, the inequality $\tau_1(q) \leq 1-q$ is always true in dimension 1. So,

$$\tau_1(q) = 1-q \qquad \text{if } 0 < q < 1 \ .$$

In the second step, we denote by $J_n$ the interval $J_n = [2^{-n}, 2^{-n+1})$ and observe that the open interval $(0,1)$ his the union of all the $J_n$. Let

$$\alpha_n = \sup\left(\frac{1}{n^2 m_1(J_n)}, 1\right)$$

and

$$m = \sum_{n=1}^{+\infty} c\alpha_n \mathbb{1}_{J_n} m_1$$

where $c$ is chosen such that $m$ is a probability measure. Using that $m \geq c\,m_1$, we find (with obvious notations) $\tau(q) \geq \tau_1(q)$ if $q > 0$. In particular, the equality $\tau(q) = 1-q$ if $0 < q < 1$ is always true. On the other hand,

$$\sum_{I\in\mathcal{F}_n} m(I)^q \geq m(J_n)^q \geq \left[\frac{c}{n^2}\right]^q ,$$

which implies that $\tau(q) \geq 0$ if $q \geq 1$. The inequality $\tau(q) \leq 0$ being always true if $q \geq 1$ we finally get

$$\tau(q) = 0 \qquad \text{if } q > 1$$

and the proof is finished. •

## 4.2 A sufficient condition for the equalities $-\tau'_+(1) = \dim_*(m)$ and $-\tau'_-(1) = \mathrm{Dim}^*(m)$

Corollary 4.2 proves that homogeneity properties are necessary if we want to obtain the equalities

$$\tau'_+(1) = \dim_*(m) \quad \text{and} \quad \tau'_-(1) = \mathrm{Dim}^*(m) \ .$$

A possible way to obtain such equalities is the following. Suppose for simplicity that $d = 1$ and let us code the intervals of $\mathcal{F}_n$ with the words $\varepsilon_1 \cdots \varepsilon_n$ where $\varepsilon_i \in \{0, \cdots, \ell - 1\}$. More precisely, let

$$I_{\varepsilon_1\cdots\varepsilon_n} = \left[\sum_{i=1}^n \frac{\varepsilon_i}{2^i}, \sum_{i=1}^n \frac{\varepsilon_i}{2^i} + \frac{1}{\ell^n}\right) \ .$$

Let us introduce the following notation

$$IJ = I_{\varepsilon_1 \cdots \varepsilon_{n+p}} \quad \text{if} \quad I = I_{\varepsilon_1 \cdots \varepsilon_n} \text{ and } J = I_{\varepsilon_{n+1} \cdots \varepsilon_{n+p}} \ . \tag{26}$$

Suppose that there exists a constant $C \geq 1$ such that

$$\forall I, J \in \bigcup_n \mathcal{F}_n, \qquad m(IJ) \leq C\, m(I)\, m(J) \ . \tag{27}$$

We have the following result.

**Theorem 4.3.** ([Heu98]) *Under the hypothesis* (27)*,*

$$\dim_*(m) = -\tau'_+(1) \quad \textit{and} \quad \mathrm{Dim}^*(m) = -\tau'_-(1) \ .$$

*Remark.* Hypothesis (27) is in particular satisfied if $m$ is a Bernoulli product (in fact, the equality $m(IJ) = m(I)m(J)$ is true in this case). More generally, it is also satisfied if $m$ is a quasi-Bernoulli measure (see Part 5). Nevertheless, there are measures satisfying (27) which are not quasi-Bernoulli measures. In particular every barycenter of two quasi-Bernoulli measures satisfies inequality (27) but is in general not a quasi-Bernoulli measure (see the example developed page 333 in [Heu98]).

Suppose that (27) is satisfied and let $q > 0$. As a consequence of the sub-multiplicative property of the sequence $a_n = C^q \sum_{I \in \mathcal{F}_n} m(I)^q$, we know that $(a_n)^{1/n}$ converges to its lower bound. It follows that the sequence $\tau_n(q)$ converges and that

$$\sum_{I \in \mathcal{F}_n} m(I)^q \geq C^{-q} \ell^{n\tau(q)} \ . \tag{28}$$

In particular, near $q = 1$, we have the inequality

$$\tau_n(q) \geq \tau(q) - \frac{c}{n} \ . \tag{29}$$

In fact, inequality (29) is sufficient to obtain Theorem 4.3. This remark can also be found in Benoît Testud thesis ([Tes04]) and we have the general following result.

**Theorem 4.4.** *Let $m$ be a probability measure on $[0,1)^d$. Suppose that there exists a constant $c > 0$ and a neighborhood $\mathcal{V}$ of 1 such that*

$$\forall n \geq 1, \quad \forall q \in \mathcal{V}, \quad \tau_n(q) \geq \tau(q) - \frac{c}{n} \ .$$

*Then, the measure $m$ satisfies*

$$\dim_*(m) = -\tau'_+(1) \quad \textit{and} \quad \mathrm{Dim}^*(m) = -\tau'_-(1) \ .$$

As in Part 3.3, Theorem 4.4 is a consequence of a result which is true in a general probability context. More precisely, we have

**Theorem 4.5.** *Let $(S_n)_{n\geq 0}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let*

$$L_n(q) = \frac{1}{n}\log_\ell \mathbb{E}\left[\ell^{qS_n}\right] \quad \textit{and} \quad L(q) = \limsup_{n\to\infty} L_n(q)$$

*and suppose that $L(q)$ is finite on a neighborhood $\mathcal{V}$ of 0. Suppose moreover that there exists a constant $C > 0$ such that*

$$\forall q \in \mathcal{V}, \qquad L_n(q) \geq L(q) - \frac{C}{n} \tag{30}$$

*Then we have*

$$\operatorname{ess\,inf}\left(\liminf_{n\to\infty}\frac{S_n}{n}\right) = L'_-(0) \quad \textit{and} \quad \operatorname{ess\,sup}\left(\limsup_{n\to\infty}\frac{S_n}{n}\right) = L'_+(0)\ .$$

*Remark.* Inequality (30) ensures that $\lim\limits_{n\to+\infty} L_n(q)$ exists if $q \in \mathcal{V}$.

*Proof of Theorem 4.5.* We first prove the inequality $\operatorname{ess\,sup}\left(\limsup_{n\to\infty}\frac{S_n}{n}\right) \geq L'_+(0)$. Replacing $S_n$ by $S_n + nA$ where $A$ is a sufficiently large number, we can suppose that $L'_+(0) > 0$. Let $\alpha_0 = L'_+(0)$, $\alpha < \alpha_0$ and $q > 0$. The convexity of the function $L$ ensures that $L(q) \geq \alpha_0 q$. We get

$$\begin{array}{rcl} \ell^{-C}\ell^{\alpha_0 nq} & \leq & \mathbb{E}\left[\ell^{qS_n}\right] \\ & = & \mathbb{E}\left[\ell^{qS_n}1\!\!1_{S_n<n\alpha}\right] + \mathbb{E}\left[\ell^{qS_n}1\!\!1_{S_n\geq n\alpha}\right] \\ & \leq & \left[1-\mathbb{P}[S_n\geq n\alpha]\right]\ell^{qn\alpha} + \mathbb{P}[S_n\geq n\alpha]^{1/2}\mathbb{E}\left[\ell^{2qS_n}\right]^{1/2}\ . \end{array}$$

We claim that we can find $\alpha_1 > 0$ and $q_0 > 0$ such that if $0 \leq q \leq q_0$, $\mathbb{E}\left[\ell^{qS_n}\right] \leq \ell^{qn\alpha_1}$ for all $n$. More precisely, if $L_n(q_0) \leq \lambda$ for all $n$, convexity inequalities imply that $L_n(q) \leq \frac{\lambda}{q_0}q \equiv \alpha_1 q$.

If $q = \frac{\delta}{n} \leq \frac{q_0}{2}$, we get

$$\ell^{\delta\alpha}\mathbb{P}[S_n\geq n\alpha] - \ell^{\delta\alpha_1}\mathbb{P}[S_n\geq n\alpha]^{1/2} \leq \ell^{\delta\alpha} - \ell^{-C}\ell^{\delta\alpha_0}\ . \tag{31}$$

We can chose $\delta$ sufficiently large such that $\ell^{\delta\alpha} - \ell^{-C}\ell^{\delta\alpha_0} < 0$. The zeros of the polynome $\Phi(t) = \ell^{\delta\alpha}t^2 - \ell^{\delta\alpha_1}t$ are nonnegative and we can deduce from inequality (31) the existence of a positive real number $\gamma$ such that

$$\mathbb{P}[S_n \geq n\alpha] \geq \gamma^2$$

if $n$ is large enough. Finally

$$\mathbb{P}\left[\limsup_{n\to+\infty}\left\{\frac{S_n}{n}\geq\alpha\right\}\right] > 0\ .$$

In that set, $\frac{S_n}{n} \geq \alpha$ infinitely often and $\limsup_{n\to+\infty}\frac{S_n}{n} \geq \alpha$. We have proved that

$$\operatorname{ess\,sup}\left(\limsup_{n\to\infty}\frac{S_n}{n}\right) \geq \alpha$$

and the conclusion follows when $\alpha \to \alpha_0$.

In order to prove that $\operatorname{ess\,inf}\left(\liminf_{n\to\infty}\frac{S_n}{n}\right) = L'_-(0)$, it suffices to apply the previous result to the sequence $-S_n$. •

## 4.3 Measures whose dimensions can be calculated with an entropy formula

In this part, we are interested in probability measures such that

$$\dim_*(m) = h_*(m) \quad \text{or} \quad \mathrm{Dim}^*(m) = h^*(m) \ .$$

This kind of property is due to a very special behavior of the sequence $\frac{S_n}{n} = \frac{\log m(I_n(x)}{|I_n(x)|}$. This is the object of the following theorem.

**Theorem 4.6.** ([BH02]) *Let $m$ be a probability measure on $[0,1)^d$. The following are equivalent.*

(i) $\dim_*(m) = h_*(m)$

(ii) $\dim_*(m) = \dim^*(m) = h_*(m)$

(iii) *There exists a sub-sequence $(n_k)_{k\geq 1}$ such that*

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{\log |I_{n_k}(x)|} = \lim_{k\to+\infty} \frac{S_{n_k}(x)}{n_k} = \dim_*(m) \quad dm - \textit{almost surely} \ .$$

*Remarks.*

1. In particular, measures for which dimension can be calculated with an entropy formula are unidimensional. Nevertheless, the equality $\dim_*(m) = h_*(m)$ corresponds to a deeper homogeneity property : the measure $m$ is unidimensional if and only if for almost every $x$, there exists a subsequence $n_k$ such that $S_{n_k}/n_k$ converges to $\dim_*(m)$, but it satisfies $\dim_*(m) = h_*(m)$ if and only if there exists a sub-sequence $n_k$ such that for almost every $x$, $S_{n_k}/n_k$ converges to $\dim_*(m)$. In particular, we can construct unidimensional measures for which the dimension is not equal to the entropy (see [BH02]).

2. Conclusion (iii) is some kind of "Shannon-McMillan result" obtained in a non dynamical context.

3. We can of course also prove the equivalence between

(i) $\mathrm{Dim}^*(m) = h^*(m)$

(ii) $\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = h^*(m)$

(iii) There exists a sub-sequence $(n_k)_{k\geq 1}$ such that

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{\log |I_{n_k}|} = \lim_{k\to+\infty} \frac{S_{n_k}(x)}{n_k} = \mathrm{Dim}^*(m) \quad dm - \text{almost surely} \ .$$

Like in Section 3.3 and 4.2, Theorem 4.6 is a consequence of a result which is valid in a general probability context.

**Theorem 4.7.** *Let $(Z_n)_{n\geq 0}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that the sequence $(Z_n)_{n\geq 0}$ is dominated in $L^1(\mathbb{P})$. Let*

$$Z_* = \liminf_{n\to+\infty} Z_n \ .$$

*The following are equivalent :*

(i) $\operatorname{ess\,inf}\,(Z_*) = \liminf_{n\to+\infty} \mathbb{E}[Z_n]$

(ii) $Z_* = \liminf_{n\to+\infty} \mathbb{E}[Z_n]$ $\quad d\mathbb{P}-$*almost surely*

(iii) *There exists a sub-sequence* $(n_k)_{k\geq 1}$ *such that*

$$\lim_{k\to+\infty} Z_{n_k} = \operatorname{ess\,inf}\,(Z_*) \quad d\mathbb{P} - \textit{almost surely} .$$

*Remark.* To obtain Theorem 4.6, it suffices to apply Theorem 4.7 to the sequence $Z_n = \frac{S_n}{n}$ where $\frac{S_n(x)}{n} = \frac{\log m(I_n(x))}{\log |I_n(x)|}$.

*Proof of Theorem 4.7.* Let $X$ be a non negative random variable such that $\mathbb{E}[X] < +\infty$ and $|Z_n| \leq X$ for all $n$. Fatou's Lemma applied to the positive sequence $X + Z_n$ shows that

$$\mathbb{E}[X] + \operatorname{ess\,inf}(Z_*) \leq \mathbb{E}[X + Z_*] \leq \mathbb{E}[X] + \liminf_{n\to+\infty} \mathbb{E}[Z_n] \ . \tag{32}$$

*Proof of* (iii) $\Rightarrow$ (i)*.* The dominated convergence theorem applied to the sequence $Z_{n_k}$ gives

$$\operatorname{ess\,inf}\,(Z_*) = \mathbb{E}\left[\lim_{k\to+\infty} Z_{n_k}\right] = \lim_{k\to+\infty} \mathbb{E}[Z_{n_k}] \geq \liminf_{n\to+\infty} \mathbb{E}[Z_n] \ .$$

The reverse inequality follows from (32).

*Proof of* (i) $\Rightarrow$ (ii)*.* We are in the equality case in (32) so that $Z_* = \liminf_{n\to+\infty} \mathbb{E}[Z_n]$ $d\mathbb{P}-$almost surely.

*Proof of* (ii) $\Rightarrow$ (iii)*.* Replacing $Z_n$ by $Z_n + X$, we can suppose that $Z_n \geq 0$. Let $\delta = \liminf_{n\to+\infty} \mathbb{E}[Z_n]$. We begin with the following lemma.

**Lemma 4.8.** *Let* $0 < \eta < 1$ *and* $n_0 \geq 1$*. We can find* $n_1 \geq n_0$ *such that*

$$\mathbb{P}[Z_{n_1} > \delta + \eta] \leq (2 + \delta)\eta \ .$$

*Proof.* Hypothesis (ii) says that $Z_* = \delta$ almost surely. We can then find $n_0' \geq n_0$ such that

$$\mathbb{P}\left[\bigcap_{n\geq n_0'} \left\{Z_n > \delta - \eta^2\right\}\right] > 1 - \eta^2 \ .$$

Moreover, we can find $n_1 \geq n_0'$ such that

$$\mathbb{E}[Z_{n_1}] < \delta + \eta^2 \ .$$

Let

$$A = \left\{Z_{n_1} > \delta - \eta^2\right\} \quad \text{and} \quad B = \{Z_{n_1} > \delta + \eta\} \ .$$

Recalling that $Z_n \geq 0$, we get

$$\begin{aligned} \delta + \eta^2 &\geq \mathbb{E}[Z_{n_1}] \\ &\geq \int_{A\setminus B} Z_{n_1} d\mathbb{P} + \int_B Z_{n_1} d\mathbb{P} \end{aligned}$$

$$\geq \quad (\delta - \eta^2)(\mathbb{P}[A] - \mathbb{P}[B]) + (\delta + \eta)\mathbb{P}[B] \ .$$

Moreover, $\mathbb{P}[A] \geq 1 - \eta^2$, so that

$$\mathbb{P}[B] \leq \frac{2\eta^2 + \delta\eta^2}{\eta + \eta^2} \leq (2 + \delta)\eta \ .$$

In order to prove Theorem 4.7, we use Lemma 4.8 with $\eta = 2^{-k}$ and then construct a subsequence $n_k$ such that

$$\forall k, \qquad \mathbb{P}\left[Z_{n_k} > \delta + 2^{-k}\right] \leq (2 + \delta)2^{-k} \ .$$

Using Borel-Cantelli's lemma, we deduce that

$$\limsup_{k \to +\infty} Z_{n_k} \leq \delta \qquad d\mathbb{P} - \text{almost surely} \ .$$

Moreover

$$\delta = S_* \leq \liminf_{k \to +\infty} Z_{n_k} \qquad d\mathbb{P} - \text{almost surely}$$

and we can conclude that the subsequence $Z_{n_k}$ is almost surely converging to $\delta$. The proof is finished if we observe that under hypothesis (ii), $Z_* = \operatorname{ess\,inf}(Z_*) = \delta$ $d\mathbb{P}-$almost surely. •

## 4.4 Entropy is a bad notion of dimension

Entropy can not allow us to classify measures. For example, there exist equivalent probability measures with different entropies. Let us precise this phenomenon in the following example.

**Proposition 4.9.** *Let $m_0$ and $m_1$ be two probability measures on $[0,1)^d$ such that the entropies $h(m_0)$ et $h(m_1)$ exist and are different. If $0 < \alpha < 1$, let*

$$m_\alpha = \alpha m_1 + (1 - \alpha) m_0 \ .$$

*Then,*

$$h(m_\alpha) = \alpha h(m_1) + (1 - \alpha) h(m_0) \ .$$

*In particular, the family $(m_\alpha)_{0<\alpha<1}$ is constituted of equivalent measures for which entropy varies in a non trivial interval.*

*Proof.* The notations are the same as in Part 3.2. We remark that the function $x \mapsto -x \log_\ell(x)$ is concave. It follows that

$$h_n(m_\alpha) \geq \alpha h_n(m_1) + (1 - \alpha) h_n(m_0),$$

and

$$h_*(m_\alpha) \geq \alpha h(m_1) + (1 - \alpha) h(m_0) \ . \tag{33}$$

On the other hand, if $q < 1$ and if $x$ and $y$ are two positive numbers, it is well known that

$$(\alpha x + (1 - \alpha) y)^q \leq \alpha^q x^q + (1 - \alpha)^q y^q \ .$$

We can deduce that

$$\sum_{I\in\mathcal{F}_n} m(I)^q \le \alpha^q \sum_{I\in\mathcal{F}_n} m_1(I)^q + (1-\alpha)^q \sum_{I\in\mathcal{F}_n} m_0(I)^q \ .$$

These two quantities are equal to 1 if $q=1$. We can then take the derivative at $q=1$ and obtain

$$h_n(m_\alpha) \le \alpha h_n(m_1) - \frac{\alpha\log_\ell \alpha}{n} + (1-\alpha)h_n(m_0) - \frac{(1-\alpha)\log_\ell(1-\alpha)}{n},$$

Finally,

$$h^*(m_\alpha) \le \alpha h(m_1) + (1-\alpha)h(m_0) \ . \tag{34}$$

Inequalities (33) and (34) give the conclusion of Proposition 4.9.

# 5 Quasi-Bernoulli measures

In this section, we suppose for simplicity that $d=1$. The notations are the same as in Section 4.2. We say that the probability measure $m$ is a quasi-Bernoulli measure if we can find $C\ge 1$ such that

$$\forall I,J\in\bigcup_n \mathcal{F}_n,\quad \frac{1}{C}\, m(I)\, m(J) \le m(IJ) \le C\, m(I)\, m(J) \ . \tag{35}$$

Quasi-Bernoulli property does appear in many situations. In particular, this is the case for the harmonic measure in regular Cantor sets ([Car85, MV86]) and for the caloric measure in domains delimited by Weierstrass type graphs ([BH00]).

Let us introduce the natural applications between $[0,1)$ and the Cantor set $\mathcal{C}=\{0,\dots,\ell-1\}^{\mathbb{N}^*}$ :

$$J\ :\ [0,1)\longrightarrow \mathcal{C} \quad \text{and} \quad S\ :\ \mathcal{C}\longrightarrow [0,1]\ .$$

They are defined by :

$$J(x)=(\varepsilon_i)_{i\ge 1} \text{ if } \{x\}=\bigcap_n I_{\varepsilon_1\dots\varepsilon_n} \quad \text{and} \quad S((\varepsilon_i)_{i\ge 1})=\bigcap_n \bar{I}_{\varepsilon_1\dots\varepsilon_n}\ .$$

The application $J$ is a bijection between $[0,1)$ and the complement of a countable subset of $\mathcal{C}$. Observing that a quasi Bernoulli measure does not contain any Dirac mass, we can carry the measure $m$ through the application $J$ and work on the Cantor set $\mathcal{C}$. We always denote by $m$ this new measure and every property that is proved for this new measure can be pulled back.

Let $\mathcal{M}$ be the set of words written with the alphabet $\{0,\cdots,\ell-1\}$. There is a link between the words of $\mathcal{M}$ and the cylinders in the Cantor set $\mathcal{C}$, so that Property (35) can be rewritten

$$\forall a,b\in\mathcal{M},\quad \frac{1}{C}\, m(a)m(b) \le m(ab) \le C\, m(a)m(b)\ . \tag{36}$$

($ab$ is the concatenation of the words $a$ and $b$). We say that the measure $m$ is a quasi Bernoulli measure on the Cantor set $\mathcal{C}$.

Let $\mathcal{M}_n$ be the set of words of length $n$, and if $x = x_1 x_2 \cdots \in \mathcal{C}$, let $I_n(x) = x_1 \cdots x_n$ be the unique cylinder $\mathcal{M}_n$ that contains $x$.

In this new context, it is always possible to define $\tau_n$ and $\tau$. Sub-multiplicative properties like in Part 4.2 ensure that the sequence $\tau_n(q)$ is convergent when $m$ is a quasi-Bernoulli measure. We then have

$$\tau(q) = \lim_{n\to+\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n \log \ell} \log \left( \sum_{a\in\mathcal{M}_n} m(a)^q \right) . \tag{37}$$

and the following inequalities are true

$$C^{-|q|} \ell^{n\tau(q)} \le \sum_{a\in\mathcal{M}_n} m(a)^q = \ell^{n\tau_n(q)} \le C^{|q|} \ell^{n\tau(q)} . \tag{38}$$

Let us finally remark that we can suppose that for every $a \in \mathcal{M}$, $m(a) > 0$. Indeed, if it is not the case, quasi-Bernoulli property ensures that there exists a cylinder $a \in \mathcal{M}_1$ such that $m(a) = 0$. Finally, several letters are not useful in the alphabet and one can work in a smaller Cantor set.

## 5.1 0-1 law and mixing properties

The interest in working on the Cantor set $\mathcal{C}$ is the dynamical context related to the shift

$$\sigma \;:\; (\varepsilon_n)_{n\ge1} \in \mathcal{C} \longmapsto (\varepsilon_n)_{n\ge2} \in \mathcal{C} . \tag{39}$$

In particular, if $a \in \mathcal{M}_n$, then $ab = a \cap \sigma^{-n}(b)$.

We can isolate the following properties that precise some previous remarks due to Carleson and Makarov-Volberg ([Car85, MV86]).

**Proposition 5.1.** *Let $m$ be a quasi-Bernoulli measure on the Cantor set $\mathcal{C}$. Let $B_0$ be the $\sigma$-field of Borel sets , $B_n = \sigma^{-n}(B_0)$ and $B_\infty = \bigcap_n B_n$.*

(i) *For every $E \in B_\infty$, $\quad m(E) = 0$ or $m(E) = 1$. (0-1 law)*

(ii) *Moreover, if $m$ is $\sigma$-invariant, the strong mixing property is true. That is*

$$\forall A, B \in B_0, \quad \lim_{n\to\infty} m\left(A \cap \sigma^{-n}(B)\right) = m(A)\, m(B) .$$

*Remark.* In particular, every $\sigma$-invariant quasi-Bernoulli measure is ergodic.

*Proof.* Let $E \in B_\infty$ be such that $m(E) > 0$. For every $n \in \mathbb{N}$ we can find a Borel set $F$ such that $E = \sigma^{-n}(F)$. We can also find a cylinder $a_0 \in \mathcal{M}_n$ such that

$$\frac{m(a_0 \cap E)}{m(a_0)} \ge \frac{1}{2} m(E) .$$

Quasi-Bernoulli property ensures that

$$\forall a \in \mathcal{M}_n,\ \forall b \in \mathcal{M}, \quad \frac{m(a \cap \sigma^{-n}(b))}{m(a)} \ge \frac{1}{C^2} \frac{m(a_0 \cap \sigma^{-n}(b))}{m(a_0)} .$$

Observing that an open set is the union of a countable family of disjoint cylinders, the previous inequality is also true if $b$ is an open set. Finally, using the regularity properties of the measure $m$, it is true for every Borel set $b$. Replacing $b$ by $F$, we obtain

$$\forall a \in \mathcal{M}_n, \quad \frac{m(a \cap E)}{m(a)} \geq \frac{1}{C^2} \frac{m(a_0 \cap E)}{m(a_0)} \geq \frac{1}{2C^2} m(E) \ .$$

A similar argument proves that the inequality $m(a \cap E) \geq (2C^2)^{-1} m(E)\, m(a)$ is also true for every Borel set $a$. In particular, $m((\mathcal{C} \setminus E) \cap E) \geq (2C^2)^{-1} m(E)\, m(\mathcal{C} \setminus E)$, which says that $m(\mathcal{C} \setminus E) = 0$. That is what we wanted to prove.

The proof of (ii) is then classical. Let $Z_n = \mathbb{E}\left[\mathbb{1}_A | B_n\right]$. It is a martingale with respect to de decreasing sequence of $\sigma$-fields $B_n$. It is converging in the $L^2$ sense (and also almost-surely) to $Z_\infty = \mathbb{E}\left[\mathbb{1}_A | B_\infty\right]$. But $B_\infty$ is the trivial $\sigma$-field. Then $Z_\infty$ is a constant random variable. Moreover, $\mathbb{E}[Z_n] = m(A)$. Taking the limit, we get

$$Z_\infty = \mathbb{E}\left[Z_\infty\right] = m(A) \qquad dm - \text{almost-surely} \ .$$

Finally,

$$\begin{aligned} \left|m(A \cap \sigma^{-n}(B)) - m(A)m(B)\right| &= \left|\mathbb{E}\left[\mathbb{1}_A \mathbb{1}_{\sigma^{-n}(B)}\right] - \mathbb{E}\left[m(A)\mathbb{1}_{\sigma^{-n}(B)}\right]\right| \\ &= \left|\mathbb{E}\left[(Z_n - Z_\infty)\mathbb{1}_{\sigma^{-n}(B)}\right]\right| \\ &\leq \left(\mathbb{E}\left[\left|Z_n - Z_\infty\right|^2\right]\right)^{1/2} , \end{aligned}$$

and the strong mixing property is proved. •

Let us now introduce the following definition.

**Definition 5.2.** *Let $m_1$ and $m_2$ be two probability measures on $\mathcal{C}$. We say that $m_1$ and $m_2$ are strongly equivalent if we can find $c > 0$ such that :*

$$\frac{1}{c} m_1 \leq m_2 \leq c\, m_1 \ .$$

We then have the following corollary.

**Corollary 5.3.** *Let $m$ be a quasi-Bernoulli measure on $\mathcal{C}$. There exists a unique probability measure, which is quasi-Bernoulli, $\sigma$-invariant and strongly equivalent to $m$. Moreover, it is obtained as the weak limit of the sequence $m_n$ defined by*

$$m_n(E) = \frac{1}{n} \sum_{k=1}^{n} m\left(\sigma^{-k}(E)\right) \ .$$

*Proof.* Observe that every probability measure which is strongly equivalent to a quasi-Bernoulli measure is also a quasi-Bernoulli measure. Moreover, it is well known that two equivalent ergodic probabilities are equal. These two facts prove the uniqueness.

In order to prove the existence, we first compare the measures $m_n$ and $m$. If $a \in \mathcal{M}$, we have :

$$m(\sigma^{-k}(a)) = m\left(\bigcup_{b \in \mathcal{M}_k} ba\right) = \sum_{b \in \mathcal{M}_k} m(ba) \leq C \sum_{b \in \mathcal{M}_k} m(b)m(a) = Cm(a) \ .$$

It follows that $m_n \leq Cm$ with a constant $C$ that does not depend on $n$. The inequality $m_n \geq \frac{1}{C}m$ is also true. We can then deduce that the measures $m_n$ are quasi-Bernoulli with a constant that does not depend on $n$. It follows that every weak limit of a subsequence $m_{n_k}$ is quasi-Bernoulli and strongly equivalent to $m$.

Let us finally consider an adherent value $\mu$ of the sequence $m_n$ and a subsequence $m_{n_k}$ which is weakly convergent to $\mu$. If $f$ is a continuous function on $\mathcal{C}$, then

$$\begin{aligned}\int f\circ\sigma(x)dm_{n_k}(x) &= \frac{1}{n_k}\sum_{j=1}^{n_k}\int f\circ\sigma^{j+1}(x)dm(x)\\ &= \int f(x)dm_{n_k}(x) + \frac{1}{n_k}\left[\int f\circ\sigma^{n_k+1}(x)dm(x) - \int f\circ\sigma(x)dm(x)\right] .\end{aligned}$$

Taking the limit, we obtain $\int f\circ\sigma(x)d\mu(x) = \int f(x)d\mu(x)$, which says that $\mu$ is $\sigma$-invariant.

Finally, using the uniqueness, there is only one adherent value for the sequence $m_n$. Then, the sequence $m_n$ is converging.

## 5.2 Showing that $\tau$ is differentiable at point 1

Corollary 5.3, Theorem 4.3 and the Shannon-McMillan's theorem allow us to prove that $\tau'(1)$ exists. This was done in [Heu98].

**Theorem 5.4.** *Let $m$ be a quasi-Bernoulli measure on $\mathcal{C}$. Quantities $\tau'(1)$ and $h(m)$ exist and we have*

$$\lim_{n\to\infty}\frac{-\log_\ell m(I_n(x))}{n} = -\tau'(1) = h(m) \quad dm\text{-almost surely} .$$

*Remark.* If the Cantor set $\mathcal{C}$ is equipped with the natural ultra metric which gives the diameter $\ell^{-n}$ to each cylinder in $\mathcal{M}_n$, then $\frac{-\log_\ell m(I_n(x))}{n}$ is nothing else but the quotient of the logarithm of the mass of $I_n(x)$ and the logarithm of its diameter. So, the measure $m$ is unidimensional with dimension $\dim(m) = -\tau'(1) = h(m)$.

Let us now introduce the sets

$$E_\alpha = \left\{x\in\mathcal{C}\ ;\ \lim_{n\to\infty}\frac{-\log_\ell m(I_n(x))}{n} = \alpha\right\} . \tag{40}$$

Using Billingsley's theorem (see [Fal90]), Theorem 5.4 shows that

$$\dim(E_{-\tau'(1)}) = \dim(m) = -\tau'(1) . \tag{41}$$

This is the first step in the multifractal analysis of the measure $m$.

*Proof of Theorem 5.4.* Let $\mu$ be the unique quasi-Bernoulli probability which is strongly equivalent to $m$ and $\sigma$-invariant. The measures $m$ and $\mu$ have the same function $\tau$ and the same dimensions. Moreover, results of Part 4.2 can be applied to the measures $m$ and $\mu$. It follows that

$$\dim_*(m) = \dim_*(\mu) = -\tau'_+(1) \quad \text{and} \quad \mathrm{Dim}^*(m) = \mathrm{Dim}^*(\mu) = -\tau'_-(1) .$$

Let us apply Shannon-McMillan's theorem (see [Zin97]) to the measure $\mu$. It says that the entropy

$$h(\mu) = \lim_{n\to+\infty} \frac{-1}{n} \sum_{a\in\mathcal{M}_n} \mu(a)\log_\ell(\mu(a))$$

exists and that for $d\mu$ almost every $x = x_1x_2\cdots \in \mathcal{C}$,

$$\frac{-\log_\ell \mu(I_n(x))}{n} = \frac{-\log_\ell \mu(x_1\cdots x_n)}{n} \xrightarrow[n\to+\infty]{} h(\mu)\ . \tag{42}$$

So, the measure $\mu$ is unidimensional. Measures $m$ and $\mu$ being strongly equivalent, one can replace $\mu$ by $m$ in (42). Finally, we have

$$\dim_*(m) = -\tau'_+(1) = h(m) = -\tau'_-(1) = \mathrm{Dim}^*(m),$$

which proves that $\tau'(1)$ exists. •

Let us finally remark that Theorem 5.4 and Corollary 3.3 allow us to deduce the following corollary.

**Corollary 5.5.** *Let $m$ be a quasi-Bernoulli probability on $\mathcal{C}$. Let $m_0$ be the homogenous probability on $\mathcal{C}$ which gives the mass $\ell^{-n}$ to each cylinder in $\mathcal{M}_n$. We have :*

$$\dim(m) = 1 \iff \tau'(1) = -1 \iff m \text{ is strongly equivalent to } m_0\ .$$

*Proof.* Suppose that $m$ is not strongly equivalent to $m_0$. We can for example suppose that the inequality $m_0 \le cm$ is never satisfied. We can then find an integer $n_0$ and a cylinder $a_0 \in \mathcal{M}_{n_0}$ such that $m(a_0) < \frac{1}{\ell C} m_0(a_0)$ where $C$ is the constant which appears in the quasi-Bernoulli property. If $a \in \mathcal{M}$, we have

$$\frac{m(aa_0)}{m(a)} \le \frac{1}{\ell} m_0(a_0) = \ell^{-(n_0+1)}\ .$$

If $0 < q < 1$, then

$$\begin{aligned}
\sum_{b\in\mathcal{M}_{n_0}} m(ab)^q &\le m(aa_0)^q + (\ell^{n_0}-1)\left[\frac{m(a)-m(aa_0)}{\ell^{n_0}-1}\right]^q \\
&\le \left(\ell^{-(n_0+1)q} + (\ell^{n_0}-1)\left[\frac{1-\ell^{-(n_0+1)}}{\ell^{n_0}-1}\right]^q\right) m(a)^q \\
&:= \gamma(q) m(a)^q\ .
\end{aligned}$$

We can then sum this inequality on every cylinder of the same generation and then iterate the process. We get

$$\sum_{a\in\mathcal{M}_{pn_0}} m(a)^q \le (\gamma(q))^p, \qquad \forall p \ge 0,$$

which gives

$$\tau(q) \le \frac{1}{n_0}\log_\ell \gamma(q)\ .$$

Finally we have

$$\dim(m) = -\tau'(1) \le \frac{-\gamma'(1)}{n_0 \log \ell} < 1\ .$$

## 5.3 Multifractal analysis of quasi-Bernoulli measures

In [BMP92], Brown, Michon and Peyrière proved that the multifractal formalism is valid for quasi-Bernoulli measures at every point $\alpha$ which can be written $\alpha = -\tau'(q)$. This result was one of the first rigorous results on multifractal analysis of measures. Unfortunately, they could not prove that the function $\tau$ is of class $C^1$. This has been done a few years later in [Heu98] and we can resume these two results in the following theorem.

**Theorem 5.6.** ([BMP92, Heu98]) *Let $m$ be a quasi-Bernoulli measure on $\mathcal{C}$. The function $\tau$ is of class $C^1$. Moreover, for every $-\tau'(+\infty) < \alpha < -\tau'(-\infty)$,*

$$\dim(E_\alpha) = \tau^*(\alpha)$$

*where the level set $E_\alpha$ is defined like in formula* (40) *and $\tau^*(\alpha) = \inf_q(\alpha q + \tau(q))$ is the Legrendre transform of the function $\tau$.*

*Remark.* In [Tes06a], Testud introduces a weaker notion which is called weak quasi-Bernoulli property. In this more general context, he proves that the function $\tau$ is differentiable on $[0, +\infty)$ and satisfies $\dim(E_\alpha) = \tau^*(\alpha)$ for every $-\tau'(+\infty) < \alpha < -\tau'_+(0)$. Moreover, he also proves in [Tes06b] that the function $\tau$ is not necessary differentiable on $(-\infty, 0]$. His results can be applied to a large class of self-similar measures with overlaps.

## 5.4 An easy proof of Theorem 5.6

We can give a proof of Theorem 5.6 which is much simpler than the original one and which points out the important role of auxiliary measures in multifractal analysis of measures. This approach is quite different to the one used in [BMP92] and [Heu98]. It was already present in my "mémoire d'habilitation" [Heu99] but never published. It makes use of the relation between the real number $\tau'(1)$ (when it exists) and the asymptotic behavior of $m(I_n(x))$ (see Theorem 3.1 and the associated remarks).

We begin with the construction of auxiliary measures $m_q$, $q \in \mathbb{R}$ (so called Gibbs measures) which satisfy $m_q(a) \approx m(a)^q |a|^{\tau(q)}$ for every $a \in \mathcal{M}$ (here $|a| = \ell^{-n}$ if $a \in \mathcal{M}_n$).

**Lemma 5.7.** *Let $q \in \mathbb{R}$. There exists a probability measure $m_q$ and a constant $c \geq 1$ such that*

$$\forall a \in \mathcal{M} \qquad \frac{1}{c}\, m(a)^q |a|^{\tau(q)} \leq m_q(a) \leq c\, m(a)^q |a|^{\tau(q)} \ .$$

*The measure $m_q$ is called the Gibbs measure at state $q$.*

*Proof.* In [Mic83], Michon proposed a construction of such measures. Let us present a simpler proof.

Let us introduce some notation. If $F_1$ and $F_2$ are two functions which depend on $q$ and on cylinders in $\mathcal{M} = \bigcup_n \mathcal{M}_n$, we will write $F_1 \approx F_2$ if there exists a constant $C > 0$ which eventually depends on $q$ but which does not depend on the cylinders such that $\frac{1}{C} F_1 \leq F_2 \leq C F_1$. Let us first observe that

$$\ell^{(n+p)\tau_{n+p}(q)} = \sum_{a \in \mathcal{M}_n, b \in \mathcal{M}_p} m(ab)^q \approx \sum_{a \in \mathcal{M}_n} \sum_{b \in \mathcal{M}_p} m(a)^q m(b)^q = \ell^{n\tau_n(q)}\, \ell^{p\tau_p(q)} \ .$$

Let $\mu_n$ be the unique measure such that $\mu_n(a) = m(a)^q |a|^{\tau_n(q)} = m(a)^q \ell^{-n\tau_n(q)}$ if $a \in \mathcal{M}_n$ and which is homogenous on the cylinders of $\mathcal{M}_n$. The measure $\mu_n$ is a probability measure. If $a \in \mathcal{M}_n$ and if $p \geq 1$, we have

$$\begin{aligned}
\mu_{n+p}(a) &= \sum_{b\in\mathcal{M}_p} \mu_{n+p}(ab) \\
&= \sum_{b\in\mathcal{M}_p} m(ab)^q \ell^{-(n+p)\tau_{n+p}(q)} \\
&\approx m(a)^q \ell^{-n\tau_n(q)} \sum_{b\in\mathcal{M}_p} m(b)^q \ell^{-p\tau_p(q)} \\
&= m(a)^q \ell^{-n\tau_n(q)} \ .
\end{aligned}$$

Moreover, we saw in (38) that $\ell^{n\tau_n(q)} \approx \ell^{n\tau(q)}$. Finally,

$$\forall a \in \mathcal{M}_n, \quad \forall k > n, \qquad \mu_k(a) \approx m(a)^q \ell^{-n\tau(q)} = m(a)^q |a|^{\tau(q)} \ .$$

Let $m_q$ be an adherent value of the sequence $(\mu_k)_{k\geq 1}$. The function $\mathbb{1}_a$ being continuous on the Cantor set $\mathcal{C}$, we can take the limit and obtain

$$\forall a \in \mathcal{M}, \quad \frac{1}{c}\, m(a)^q |a|^{\tau(q)} \leq m_q(a) \leq c\, m(a)^q |a|^{\tau(q)}, \tag{43}$$

which finishes the proof of Lemma 5.7. •

We can now prove Theorem 5.6. An elementary computation shows that the function $\tau$ associated with the measure $m_q$ (which is denoted by $\tau_q$) satisfies :

$$\tau_q(t) = \tau(qt) - t\tau(q) \ .$$

Moreover,

$$m_q(ab) \approx m(ab)^q |ab|^{\tau(q)} \approx [m(a)m(b)]^q (|a||b|)^{\tau(q)} \approx m_q(a) m_q(b),$$

which says that $m_q$ is a quasi-Bernoulli measure. The existence of $\tau_q'(1)$ proves the existence of $\tau'(q)$ and the relation

$$-\tau_q'(1) = -q\tau'(q) + \tau(q) = \tau^*(-\tau'(q)) \ .$$

Let $\alpha = -\tau'(q)$. Inequalities (43) ensure that

$$E_\alpha = \left\{ x \in \mathcal{C} \ ; \ \lim_{n\to\infty} \frac{-\log_\ell m_q(I_n(x))}{n} = -\tau_q'(1) \right\} \ .$$

Finally, Relation (41) written for the measure $m_q$ gives

$$\dim(E_\alpha) = \dim(m_q) = -\tau_q'(1) = \tau^*(\alpha) \ .$$

Of course, we need another argument to prove the existence of $\tau'(0)$. Taking the logarithm in (38), we have

$$|\tau_n(q) - \tau(q)| \leq \frac{|q| \log_\ell C}{n} \ .$$

In particular, $\tau_n(0) = \tau(0)$ and we deduce that

$$\left| \frac{\tau_n(q) - \tau_n(0)}{q} - \frac{\tau(q) - \tau(0)}{q} \right| \leq \frac{\log_\ell C}{n} \ .$$

If $q \to 0^+$ and $q \to 0^-$, we get

$$\begin{cases} \left|\tau_n'(0) - \tau_+'(0)\right| \leq \dfrac{\log_\ell C}{n} \\ \left|\tau_n'(0) - \tau_-'(0)\right| \leq \dfrac{\log_\ell C}{n} \end{cases}$$

and we can conclude that $\tau_+'(0) = \tau_-'(0)$. •

## 5.5 Coming back to the case of Bernoulli products

Let us finish this paper by applying the previous results to the Bernoulli products which are the simplest cases of quasi-Bernoulli measures. The notations are the same as in Part 1 and $m$ is a Bernoulli product with parameter $p$. Let

$$E_\alpha = \left\{ x \ ; \ \lim_{n\to\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = \alpha \right\} \quad \text{and} \quad F_\beta = \left\{ x \ ; \ \lim_{n\to\infty} \frac{s_n}{n} = \beta \right\} \ .$$

Let us remember that the quantities $m(I_n(x))$ and $s_n$ satisfy the relation

$$m(I_n(x)) = p^{s_n}(1-p)^{n-s_n} \ .$$

So, if $0 \leq \beta \leq 1$ and if $\alpha = -\beta \log_2 p - (1-\beta)\log_2(1-p)$, we have $E_\alpha = F_\beta$. Moreover, let us remark that the sets $F_\beta$ are empty if $\beta \notin [0,1]$. It follows that the sets $E_\alpha$ are empty if $\alpha \notin [-\log_2 p, -\log_2(1-p)]$.

Let $\mu_\beta$ be Bernoulli product with parameter $\beta$. The results of Part 1 say that

$$\dim(\mu_\beta) = \dim(F_\beta) = h(\beta) = -(\beta \log_2(\beta) + (1-\beta)\log_2(1-\beta))$$

and we can write

$$\dim(E_\alpha) = -(\beta \log_2(\beta) + (1-\beta)\log_2(1-\beta))$$

where

$$\alpha = -(\beta \log_2 p + (1-\beta)\log_2(1-p)) \ .$$

In other words,

$$\dim(E_\alpha) = h\left( \frac{\alpha + \log_2(1-p)}{\log_2(1-p) - \log_2(p)} \right) \tag{44}$$

where $h(t) = -t\log_2 t - (1-t)\log_2(1-t)$.

*Remark 1.* We know that $\tau(q) = \log_2(p^q + (1-p)^q)$. Another way to obtain (44) is to calculate the Legendre transform $\tau^*$ and to use Theorem 5.6.

*Remark 2.* If $\alpha = -(\beta \log_2 p + (1-\beta)\log_2(1-p))$ and if $q$ is such that $\alpha = -\tau'(q)$, it is easy to show that $\mu_\beta$ is nothing else but the Gibbs measure at state $q$ for the measure $m$ (see Lemma 5.7).

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