On multifractal phenomena

Yanick Heurteaux

Université Clermont-Auvergne – Clermont-Ferrand

ANR Front, October 2019

Summary

1- Multifractal formalism : the beginning of the story

2- Two notions of genericity

3- Generic multifractal phenomena (with Frédéric Bayart)

Summary

1- Multifractal formalism : the beginning of the story

2- Two notions of genericity

3- Generic multifractal phenomena (with Frédéric Bayart)

Fully developed turbulence

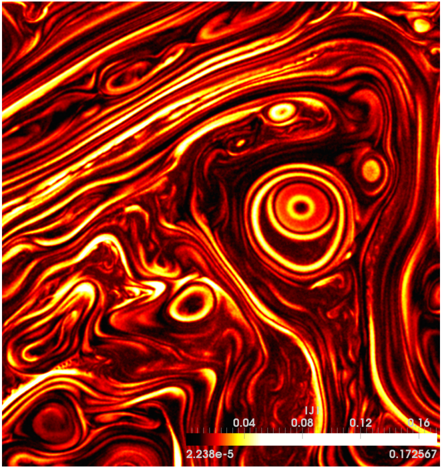


Figure – a velocity vector field

Fully developed turbulence

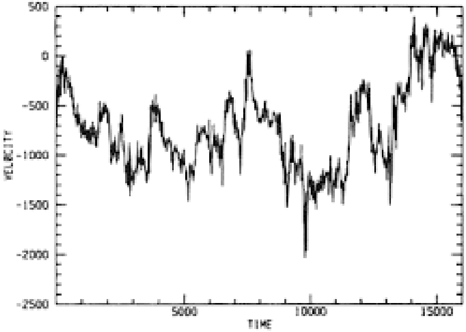


Figure – a velocity time series

The local Hölder exponent

Definition

Let $\alpha > 0$, $x_0 \in \mathbb{R}^d$ end $f \ : \ R^d \to \mathbb{R}$. We say that f is $C^\alpha(x_0)$ if there exists a polynomial P_{x_0} such that

$$|f(x) - P_{x_0}(x)| = O(|x - x_0|^\alpha) \ .$$

The local Hölder exponent

Definition

Let $\alpha > 0$, $x_0 \in \mathbb{R}^d$ end $f\ :\ R^d \to \mathbb{R}$. We say that f is $C^\alpha(x_0)$ if there exists a polynomial P_{x_0} such that

$$|f(x) - P_{x_0}(x)| = O(|x - x_0|^\alpha)\ .$$

The Hölder exponent of f at x_0 is then defined by

$$h_f(x_0) = \sup\{\alpha;\ f \text{ is } C^\alpha(x_0)\}\ .$$

The local Hölder exponent

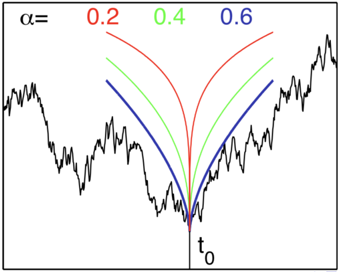
Definition

Let $\alpha > 0$, $x_0 \in \mathbb{R}^d$ end $f \ : \ R^d \to \mathbb{R}$. We say that f is $C^\alpha(x_0)$ if there exists a polynomial P_{x_0} such that

$$|f(x) - P_{x_0}(x)| = O(|x - x_0|^\alpha) \ .$$

The Hölder exponent of f at x_0 is then defined by

$$h_f(x_0) = \sup\{\alpha; \ f \text{ is } C^\alpha(x_0)\} \ .$$

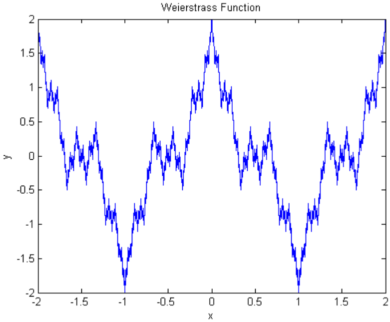


Example 1 : the Weierstrass series

$$f(x) = \sum_{n=0}^{+\infty} b^{-\alpha n} \cos(b^n x)$$

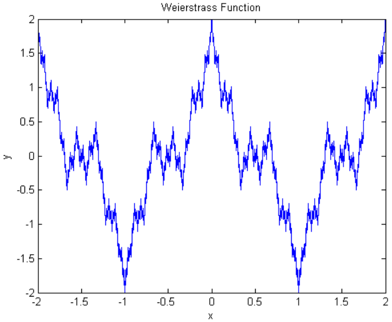
Example 1 : the Weierstrass series

$$f(x) = \sum_{n=0}^{+\infty} b^{-\alpha n} \cos(b^n x)$$



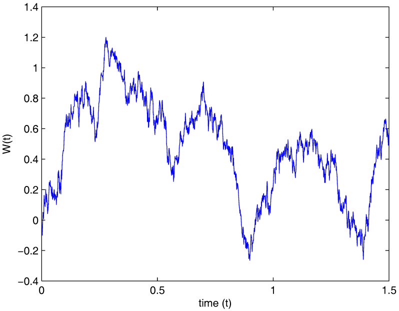
Example 1 : the Weierstrass series

$$f(x) = \sum_{n=0}^{+\infty} b^{-\alpha n} \cos(b^n x)$$

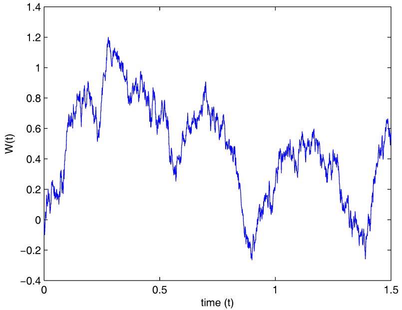


$$\text{for any } x_0, \quad h_f(x_0) = \alpha$$

Example 2 : the Brownian motion

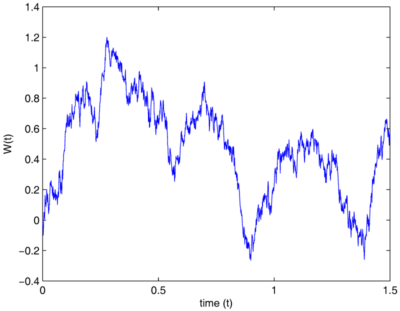


Example 2 : the Brownian motion



$$\text{a.s.} \qquad \limsup_{t \to 0} \frac{B_t}{\sqrt{2t \log \log \frac{1}{t}}} = 1 \ .$$

Example 2 : the Brownian motion



$$\text{a.s.} \quad \limsup_{t \to 0} \frac{B_t}{\sqrt{2t \log\log \frac{1}{t}}} = 1 \ .$$

$$\text{a.s., for any } t_0, \quad h_B(t_0) = \frac{1}{2}$$

What is a multifractal function ?

What is a multifractal function ?

- Iso Hölder sets :

$$E_\alpha = \{x \ ; \ h_f(x) = \alpha\}$$

What is a multifractal function ?

- Iso Hölder sets :

$$E_\alpha = \{x \; ; \; h_f(x) = \alpha\}$$

- A multifractal function is a function f for which $E_\alpha \neq \emptyset$ for many values of α

What is a multifractal function ?

- Iso Hölder sets :

$$E_\alpha = \{x \ ; \ h_f(x) = \alpha\}$$

- A multifractal function is a function f for which $E_\alpha \neq \emptyset$ for many values of α
- The multifractal spectrum :

$$\mathcal{D} \ : \ \alpha \mapsto \dim_H(E_\alpha)$$

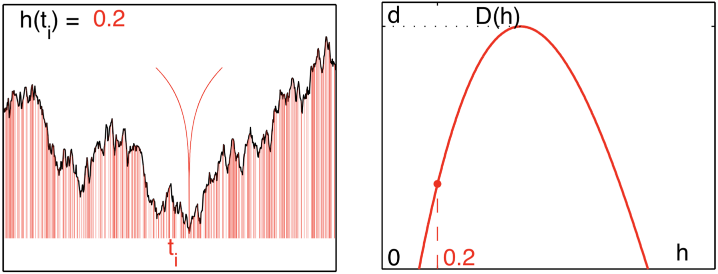
What is a multifractal function ?

- Iso Hölder sets :

$$E_\alpha = \{x \; ; \; h_f(x) = \alpha\}$$

- A multifractal function is a function f for which $E_\alpha \neq \emptyset$ for many values of α
- The multifractal spectrum :

$$\mathcal{D} \; : \; \alpha \mapsto \dim_H(E_\alpha)$$



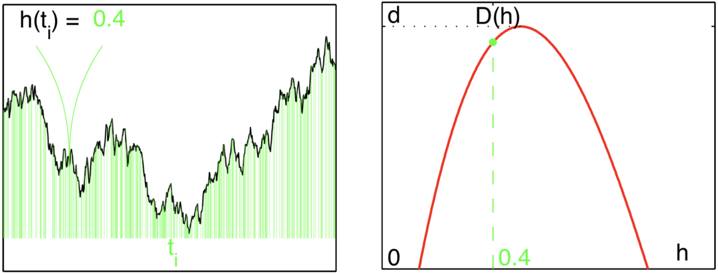
What is a multifractal function ?

- Iso Hölder sets :

$$E_\alpha = \{x \ ; \ h_f(x) = \alpha\}$$

- A multifractal function is a function f for which $E_\alpha \neq \emptyset$ for many values of α
- The multifractal spectrum :

$$\mathcal{D} \ : \ \alpha \mapsto \dim_H(E_\alpha)$$



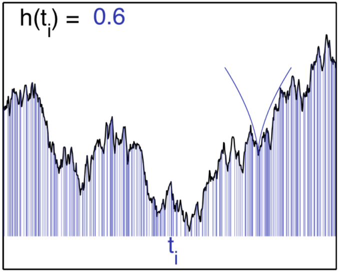
What is a multifractal function ?

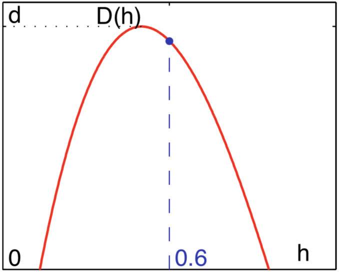
- Iso Hölder sets :

$$E_\alpha = \{x \; ; \; h_f(x) = \alpha\}$$

- A multifractal function is a function f for which $E_\alpha \neq \emptyset$ for many values of α
- The multifractal spectrum :

$$\mathcal{D} \; : \; \alpha \mapsto \dim_H(E_\alpha)$$





How to compute the multifractal spectrum ?

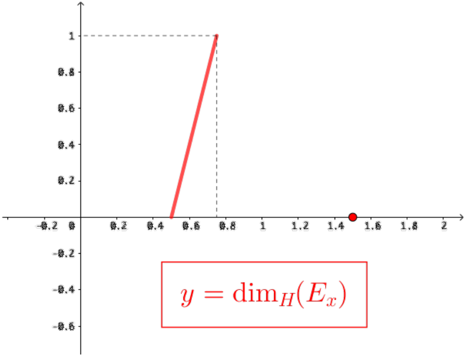
An ad hoc example : the Riemann function (S. Jaffard - 1996)

$$f(x) = \sum_{n=1}^{+\infty} \frac{1}{n^2} \sin(\pi n^2 x)$$

How to compute the multifractal spectrum ?

An ad hoc example : the Riemann function (S. Jaffard - 1996)

$$f(x) = \sum_{n=1}^{+\infty} \frac{1}{n^2} \sin(\pi n^2 x)$$



What about measures ?

The analogue of the Hölder index :

$$h_m(x_0) = \sup \{\alpha \; ; \; m(B(x_0, r)) = O(r^\alpha)\}$$
$$= \liminf_{r \to 0} \frac{\log(m(B(x_0, r))}{\log r}$$

$$\boxed{E_\alpha = \{x \; ; \; h_m(x) = \alpha\}}$$

What about measures ?

The analogue of the Hölder index :

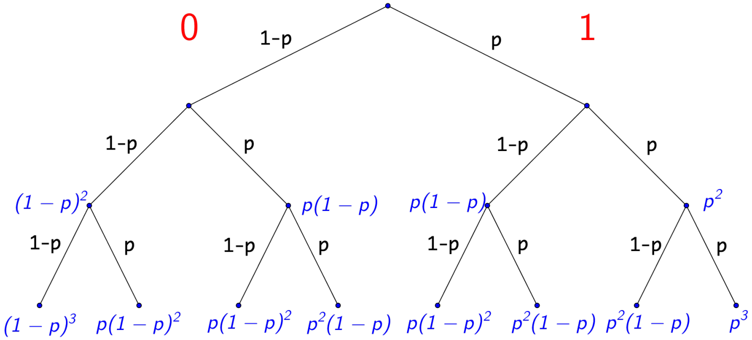
$$h_m(x_0) = \sup\{\alpha\ ;\ m(B(x_0,r)) = O(r^\alpha)\}$$
$$= \liminf_{r\to 0} \frac{\log(m(B(x_0,r))}{\log r}$$

$$\boxed{E_\alpha = \{x\ ;\ h_m(x) = \alpha\}}$$

An alternative definition of E_α :

$$\boxed{E_\alpha = \left\{x\ ;\ \lim_{r\to 0} \frac{\log(m(B(x_0,r))}{\log r} = \alpha\right\}}$$

A toy example : the Bernoulli measure



$$\boxed{m\left(I_{\varepsilon_1 \cdots \varepsilon_n}\right) = p^{s_n}(1-p)^{n-s_n} \quad \text{where} \quad s_n = \varepsilon_1 + \cdots + \varepsilon_n}$$

$$E_\alpha = \left\{ x \ ; \ \lim \frac{\log(m(I_n(x))}{\log(|I_n(x)|)} = \alpha \right\} = \{ x \ ; \ m(I_n(x)) \text{"} \approx \text{"} |I_n(x)|^\alpha \}$$

Bernoulli measure : the almost sure behavior

$$\boxed{m\left(I_{\varepsilon_1\cdots\varepsilon_n}\right) = p^{s_n}(1-p)^{n-s_n}}$$

where $s_n = \varepsilon_1 + \cdots + \varepsilon_n$

Bernoulli measure : the almost sure behavior

$$\boxed{m\left(I_{\varepsilon_1\cdots\varepsilon_n}\right) = p^{s_n}(1-p)^{n-s_n}}$$

where $s_n = \varepsilon_1 + \cdots + \varepsilon_n$

$$\frac{\log m(I_n(x))}{\log|I_n(x)|} = -\frac{s_n}{n}\log_2(p) - \left(1-\frac{s_n}{n}\right)\log_2(1-p)$$

Bernoulli measure : the almost sure behavior

$$\boxed{m\left(I_{\varepsilon_1\cdots\varepsilon_n}\right) = p^{s_n}(1-p)^{n-s_n}}$$

where $s_n = \varepsilon_1 + \cdots + \varepsilon_n$

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\frac{s_n}{n}\log_2(p) - \left(1 - \frac{s_n}{n}\right)\log_2(1-p)$$

Strong law of large numbers : $\dfrac{s_n}{n} \to p$ dm-almost surely.

Bernoulli measure : the almost sure behavior

$$\boxed{m\left(I_{\varepsilon_1\cdots\varepsilon_n}\right) = p^{s_n}(1-p)^{n-s_n}}$$

where $s_n = \varepsilon_1 + \cdots + \varepsilon_n$

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\frac{s_n}{n}\log_2(p) - \left(1 - \frac{s_n}{n}\right)\log_2(1-p)$$

Strong law of large numbers : $\dfrac{s_n}{n} \to p$ dm-almost surely.

$$\lim_{n\to\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = -p\log_2 p - (1-p)\log_2(1-p)$$

$$:= h(p) \quad dm - \text{a.s.}$$

Bernoulli measure : the almost sure behavior

$$\boxed{m\left(I_{\varepsilon_1\cdots\varepsilon_n}\right) = p^{s_n}(1-p)^{n-s_n}}$$

where $s_n = \varepsilon_1 + \cdots + \varepsilon_n$

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\frac{s_n}{n}\log_2(p) - \left(1 - \frac{s_n}{n}\right)\log_2(1-p)$$

Strong law of large numbers : $\dfrac{s_n}{n} \to p$ dm-almost surely.

$$\lim_{n\to\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = -p\log_2 p - (1-p)\log_2(1-p)$$
$$:= h(p) \quad dm - \text{a.s.}$$

$$\boxed{E_{h(p)} = \left\{\frac{s_n}{n} \to p\right\} \quad \text{and} \quad \dim_H(E_{h(p)}) = h(p)}$$

The other level sets

The other level sets

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\left(\frac{s_n}{n}\log_2 p + \left(1 - \frac{s_n}{n}\right)\log_2(1-p)\right)$$

The other level sets

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\left(\frac{s_n}{n}\log_2 p + \left(1 - \frac{s_n}{n}\right)\log_2(1-p)\right)$$

Let $\alpha = -(\theta \log_2 p + (1-\theta)\log_2(1-p))$

The other level sets

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\left(\frac{s_n}{n}\log_2 p + \left(1 - \frac{s_n}{n}\right)\log_2(1-p)\right)$$

Let $\alpha = -(\theta \log_2 p + (1-\theta)\log_2(1-p))$

$$\boxed{E_\alpha = \left\{\frac{s_n}{n} \to \theta\right\}}$$

The other level sets

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\left(\frac{s_n}{n}\log_2 p + \left(1 - \frac{s_n}{n}\right)\log_2(1-p)\right)$$

Let $\alpha = -(\theta \log_2 p + (1-\theta)\log_2(1-p))$

$$\boxed{E_\alpha = \left\{\frac{s_n}{n} \to \theta\right\}}$$

$$\dim_H(E_\alpha) = -(\theta \log_2 \theta + (1-\theta)\log_2(1-\theta)) = h(\theta)$$

The other level sets

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\left(\frac{s_n}{n}\log_2 p + \left(1 - \frac{s_n}{n}\right)\log_2(1-p)\right)$$

Let $\alpha = -(\theta \log_2 p + (1-\theta)\log_2(1-p))$

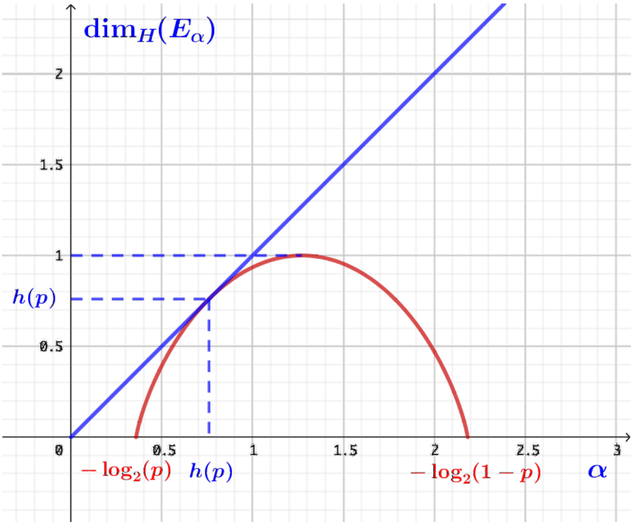
$$\boxed{E_\alpha = \left\{\frac{s_n}{n} \to \theta\right\}}$$

$$\dim_H(E_\alpha) = -(\theta \log_2 \theta + (1-\theta)\log_2(1-\theta)) = h(\theta) := F(\alpha)$$

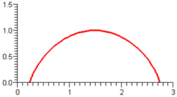
where

$$F(\alpha) = h\left(\frac{\alpha + \log_2(1-p)}{\log_2(1-p) - \log_2 p}\right)$$

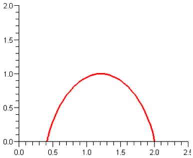
The spectrum of the Bernoulli measure m



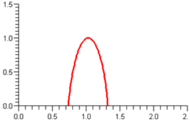
Different values of p



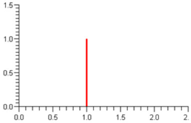
p = 0,85



p = 0,75



p = 0,60



p = 0,50

A natural way to compute the spectrum

Define the structure function τ :

$$\tau(x) = \limsup_{n \to +\infty} \frac{1}{n} \log_2 \left(\sum_{I \in \mathcal{F}_n} m(I)^x \right) = \log_2 \left(p^x + (1-p)^x \right)$$

A natural way to compute the spectrum

Define the structure function τ :

$$\tau(x) = \limsup_{n\to+\infty} \frac{1}{n} \log_2 \left(\sum_{I\in\mathcal{F}_n} m(I)^x \right) = \log_2 \left(p^x + (1-p)^x\right)$$

$$-\tau'(x) = -(\theta \log_2 p + (1-\theta)\log_2(1-p))$$

with $\theta = \frac{p^x}{p^x+(1-p)^x}$

A natural way to compute the spectrum

Define the structure function τ :

$$\boxed{\tau(x) = \limsup_{n \to +\infty} \frac{1}{n} \log_2 \left(\sum_{I \in \mathcal{F}_n} m(I)^x \right) = \log_2 \left(p^x + (1-p)^x \right)}$$

$$-\tau'(x) = -(\theta \log_2 p + (1-\theta) \log_2 (1-p)) = \alpha$$

with $\theta = \frac{p^x}{p^x + (1-p)^x}$

A natural way to compute the spectrum

Define the structure function τ :

$$\boxed{\tau(x) = \limsup_{n\to+\infty} \frac{1}{n} \log_2 \left(\sum_{I\in\mathcal{F}_n} m(I)^x \right) = \log_2 \left(p^x + (1-p)^x\right)}$$

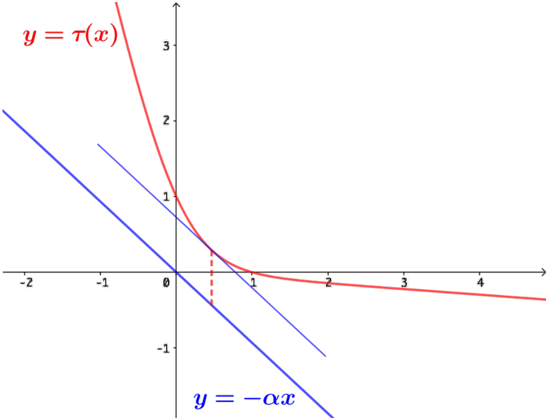
$$-\tau'(x) = -(\theta \log_2 p + (1-\theta)\log_2(1-p)) = \alpha$$

with $\theta = \frac{p^x}{p^x+(1-p)^x}$

$$\begin{aligned}\dim_H(E_\alpha) &= -(\theta \log_2 \theta + (1-\theta)\log_2(1-\theta)) \\ &= -x\tau'(x) + \tau(x) \\ &= \tau^*(-\tau'(x)) \\ &= \tau^*(\alpha)\end{aligned}$$

where $\tau^*(\alpha) = \inf_t(t\alpha + \tau(t))$ is the Legendre transform of τ.

The Legendre transform



The multifractal formalism scheme

Compute the structure function τ

$$\Downarrow$$

Compute the Legendre transform τ^*

$$\Downarrow$$

The expected value is $\quad \dim_H(E_\alpha) = \tau^*(\alpha)$

Summary

1- Multifractal formalism : the beginning of the story

2- Two notions of genericity

3- Generic multifractal phenomena (with Frédéric Bayart)

What is genericity ?

To define a notion of genericity in E, we need a family $\mathcal{G}$ of "big subsets" of E such that :

- Any $A \in \mathcal{G}$ is dense in E
- If $B \supset A$ and $A \in \mathcal{G}$, then $B \in \mathcal{G}$
- (A_n) is a sequence of sets in $\mathcal{G}$, then $\bigcap_{n \geq 0} A_n \in \mathcal{G}$

What is genericity ?

To define a notion of genericity in E, we need a family $\mathcal{G}$ of "big subsets" of E such that :

- Any $A \in \mathcal{G}$ is dense in E
- If $B \supset A$ and $A \in \mathcal{G}$, then $B \in \mathcal{G}$
- (A_n) is a sequence of sets in $\mathcal{G}$, then $\bigcap_{n \geq 0} A_n \in \mathcal{G}$

We say that a property $\mathcal{P}$ is generic if the set of points where it is satisfied is in $\mathcal{G}$

The Baire property

The Baire property

- In a complete metric space E, the intersection of a sequence of dense open sets is a dense $\mathcal{G}_\delta$ set

The Baire property

- In a complete metric space E, the intersection of a sequence of dense open sets is a dense $\mathcal{G}_\delta$ set
- A set A is called *residual* if it contains a dense $\mathcal{G}_\delta$ set

The Baire property

- In a complete metric space E, the intersection of a sequence of dense open sets is a dense $\mathcal{G}_\delta$ set
- A set A is called *residual* if it contains a dense $\mathcal{G}_\delta$ set
- A set A is called *meager* if its complement is residual

The Baire property

- In a complete metric space E, the intersection of a sequence of dense open sets is a dense $\mathcal{G}_\delta$ set
- A set A is called *residual* if it contains a dense $\mathcal{G}_\delta$ set
- A set A is called *meager* if its complement is residual

Definition

If a property $\mathcal{P}$ is true in a residual subset of E, we say that $\mathcal{P}$ is true for *quasi all* element of E.

The Baire property

- In a complete metric space E, the intersection of a sequence of dense open sets is a dense $\mathcal{G}_\delta$ set
- A set A is called *residual* if it contains a dense $\mathcal{G}_\delta$ set
- A set A is called *meager* if its complement is residual

Definition

If a property $\mathcal{P}$ is true in a residual subset of E, we say that $\mathcal{P}$ is true for *quasi all* element of E.

This is a notion of genericity !

Three remarks to introduce prevalence

Three remarks to introduce prevalence

Goal. We want to extend the notion "almost every where" in an infinite dimensional vector space E.

Three remarks to introduce prevalence

Goal. We want to extend the notion "almost every where" in an infinite dimensional vector space E.

- The Lebesgue measure m doesn't exist when $\dim(E) = +\infty$

Three remarks to introduce prevalence

Goal. We want to extend the notion "almost every where" in an infinite dimensional vector space E.

- The Lebesgue measure m doesn't exist when $\dim(E) = +\infty$
- Let m be the Lebesgue measure in $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$. Suppose that there exists a compactly supported measure μ such that for any x, $\mu(x + A) = 0$. By Fubini's theorem :

$$\int \mu(x + A)\, dm(x) = \int m(y - A)\, d\mu(y) = m(A) = 0$$

Three remarks to introduce prevalence

Goal. We want to extend the notion "almost every where" in an infinite dimensional vector space E.

- The Lebesgue measure m doesn't exist when $\dim(E) = +\infty$
- Let m be the Lebesgue measure in $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$. Suppose that there exists a compactly supported measure μ such that for any x, $\mu(x + A) = 0$. By Fubini's theorem :

$$\int \mu(x + A)\, dm(x) = \int m(y - A)\, d\mu(y) = m(A) = 0$$

- The reverse is true

Three remarks to introduce prevalence

Goal. We want to extend the notion "almost every where" in an infinite dimensional vector space E.

- The Lebesgue measure m doesn't exist when $\dim(E) = +\infty$
- Let m be the Lebesgue measure in $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$. Suppose that there exists a compactly supported measure μ such that for any x, $\mu(x + A) = 0$. By Fubini's theorem :

$$\int \mu(x + A)\, dm(x) = \int m(y - A)\, d\mu(y) = m(A) = 0$$

- The reverse is true (μ = restriction of the Lebesgue measure)

Prevalence

Definition

Let E be a (infinite dimensional) complete metric vector space

- A Borel set $A \subset E$ is called *Haar-null* if there exists a compactly supported probability measure μ such that for any $x \in E$, $\mu(x + A) = 0$.

Prevalence

Definition

Let E be a (infinite dimensional) complete metric vector space

- A Borel set $A \subset E$ is called *Haar-null* if there exists a compactly supported probability measure μ such that for any $x \in E$, $\mu(x + A) = 0$.
- A subset of E is called *Haar-null* if it is included in a Haar-null Borel set

Prevalence

Definition

Let E be a (infinite dimensional) complete metric vector space

- A Borel set $A \subset E$ is called *Haar-null* if there exists a compactly supported probability measure μ such that for any $x \in E$, $\mu(x + A) = 0$.
- A subset of E is called *Haar-null* if it is included in a Haar-null Borel set
- A subset of E is called *prevalent* if its complement is Haar-null

Prevalence

Definition

Let E be a (infinite dimensional) complete metric vector space

- A Borel set $A \subset E$ is called *Haar-null* if there exists a compactly supported probability measure μ such that for any $x \in E$, $\mu(x + A) = 0$.
- A subset of E is called *Haar-null* if it is included in a Haar-null Borel set
- A subset of E is called *prevalent* if its complement is Haar-null

Proposition

- *Prevalence is a notion of genericity*

Prevalence

Definition

Let E be a (infinite dimensional) complete metric vector space

- A Borel set $A \subset E$ is called *Haar-null* if there exists a compactly supported probability measure μ such that for any $x \in E$, $\mu(x + A) = 0$.
- A subset of E is called *Haar-null* if it is included in a Haar-null Borel set
- A subset of E is called *prevalent* if its complement is Haar-null

Proposition

- *Prevalence is a notion of genericity*
- *If A is Haar-null, then $x + A$ is Haar-null*

Prevalence

Definition

Let E be a (infinite dimensional) complete metric vector space

- A Borel set $A \subset E$ is called *Haar-null* if there exists a compactly supported probability measure μ such that for any $x \in E$, $\mu(x + A) = 0$.
- A subset of E is called *Haar-null* if it is included in a Haar-null Borel set
- A subset of E is called *prevalent* if its complement is Haar-null

Proposition

- *Prevalence is a notion of genericity*
- *If A is Haar-null, then $x + A$ is Haar-null*
- *Compact subsets of E are Haar-null*

Pointwise regularity : generic results (1)

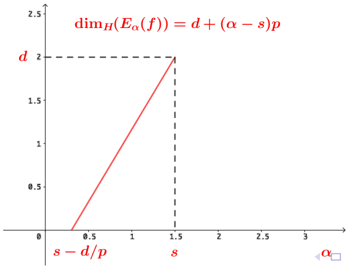
Theorem (Jaffard, 2000)

(i) *For any function* $f \in B^s_{p,q}([0,1]^d)$ *and any* $\alpha \in [s-d/p, s]$,

$$\dim_H\left(E_\alpha(f)\right) \leq d + (\alpha - s)p.$$

(ii) *For quasi all function* $f \in B^s_{p,q}([0,1]^d)$, *for any* $\alpha \in [s-d/p, s]$,

$$\dim_H\left(E_\alpha(f)\right) = d + (\alpha - s)p.$$



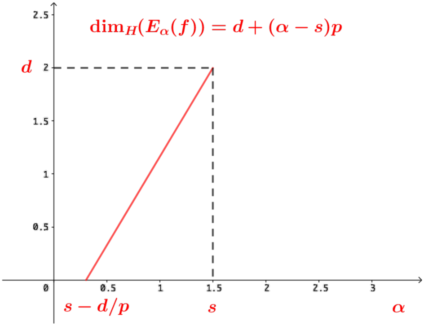
Pointwise regularity : generic results (2)

Theorem (Fraysse-Jaffard, 2006)

The set of functions $f \in B^s_{p,q}([0,1]^d)$*, such that for any* $\alpha \in [s - d/p, s]$*,*

$$\dim_H \left(E_\alpha(f)\right) = d + (\alpha - s)p.$$

is prevalent in $B^s_{p,q}([0,1]^d)$*.*



Summary

1- Multifractal formalism : the beginning of the story

2- Two notions of genericity

3- Generic multifractal phenomena (with Frédéric Bayart)

General context for multifractal analysis

General context for multifractal analysis

Any situation where we observe a collection of possible asymptotic behavior that change from point to point

General context for multifractal analysis

Any situation where we observe a collection of possible asymptotic behavior that change from point to point

$\longrightarrow$ Define the level sets

General context for multifractal analysis

Any situation where we observe a collection of possible asymptotic behavior that change from point to point

$\longrightarrow$ Define the level sets

$\longrightarrow$ Compute the expected multifractal spectrum
(i.e. give a natural upper bound for $\dim_H(E_\alpha)$)

General context for multifractal analysis

Any situation where we observe a collection of possible asymptotic behavior that change from point to point

$\longrightarrow$ Define the level sets

$\longrightarrow$ Compute the expected multifractal spectrum (i.e. give a natural upper bound for $\dim_H(E_\alpha)$)

$\longrightarrow$ Generic behavior ?

Some examples

- Pointwise regularity of functions (Fraysse & Jaffard)
- Local regularity of measures (Buczolich & Seuret, Bayart)
- Fourier series : asymptotic behavior of $S_n f(x)$ (Bayart & H.)
- Harmonic functions : radial behavior of harmonic functions near the boundary (Bayart & H.)
- Dirichlet series : behavior of $\sum_{k=1}^{n} a_k k^{-1/2+it}$ (Bayart & H.)
- Wavelet series : behavior of the partial sums $\langle f, \varphi \rangle \varphi + \sum_{j=0}^{n-1} \sum_{\mu \in \Lambda_j} 2^{j/2} \langle f, \psi_\mu \rangle \psi_\mu$ (Esser & Jaffard, Bayart & H.)
- Fourier integral : behavior of $\int_{-R}^{R} f(t) e^{-it\xi} dt$
- ...

Some examples

- Pointwise regularity of functions (Fraysse & Jaffard)
- Local regularity of measures (Buczolich & Seuret, Bayart)
- Fourier series : asymptotic behavior of $S_n f(x)$ (Bayart & H.)
- Harmonic functions : radial behavior of harmonic functions near the boundary (Bayart & H.)
- Dirichlet series : behavior of $\sum_{k=1}^{n} a_k k^{-1/2+it}$ (Bayart & H.)
- Wavelet series : behavior of the partial sums $\langle f,\varphi\rangle\varphi + \sum_{j=0}^{n-1} \sum_{\mu\in\Lambda_j} 2^{j/2}\langle f,\psi_\mu\rangle\psi_\mu$ (Esser & Jaffard, Bayart & H.)
- Fourier integral : behavior of $\int_{-R}^{R} f(t)e^{-it\xi}dt$
- ...

Divergence of Fourier series : historic results

Divergence of Fourier series : historic results

- Du Bois Raymond (1872) $\exists f \in \mathcal{C}(\mathbb{T})$; $S_n f(0)$ diverges

Divergence of Fourier series : historic results

- Du Bois Raymond (1872) $\exists f \in \mathcal{C}(\mathbb{T})$; $S_n f(0)$ diverges
- Baire : $S_n f(0)$ diverges for quasi all function in $\mathcal{C}(\mathbb{T})$

Divergence of Fourier series : historic results

- Du Bois Raymond (1872) $\exists f \in \mathcal{C}(\mathbb{T})$; $S_n f(0)$ diverges
- Baire : $S_n f(0)$ diverges for quasi all function in $\mathcal{C}(\mathbb{T})$
- Baire : $S_n f(0)$ diverges for quasi all function in $L^p(\mathbb{T})$

Divergence of Fourier series : historic results

- Du Bois Raymond (1872) $\exists f \in \mathcal{C}(\mathbb{T})$; $S_n f(0)$ diverges
- Baire : $S_n f(0)$ diverges for quasi all function in $\mathcal{C}(\mathbb{T})$
- Baire : $S_n f(0)$ diverges for quasi all function in $L^p(\mathbb{T})$
- Kolmogorov (1923) $\exists f \in L^1(\mathbb{T})$; $S_n f(x)$ diverges a.s.

Divergence of Fourier series : historic results

- Du Bois Raymond (1872) $\exists f \in \mathcal{C}(\mathbb{T})$; $S_n f(0)$ diverges
- Baire : $S_n f(0)$ diverges for quasi all function in $\mathcal{C}(\mathbb{T})$
- Baire : $S_n f(0)$ diverges for quasi all function in $L^p(\mathbb{T})$
- Kolmogorov (1923) $\exists f \in L^1(\mathbb{T})$; $S_n f(x)$ diverges a.s.
- Kolmogorov (1926) $\exists f \in L^1(\mathbb{T})$; $S_n f(x)$ diverges surely

Divergence of Fourier series : historic results

- Du Bois Raymond (1872) $\exists f \in \mathcal{C}(\mathbb{T})$; $S_n f(0)$ diverges
- Baire : $S_n f(0)$ diverges for quasi all function in $\mathcal{C}(\mathbb{T})$
- Baire : $S_n f(0)$ diverges for quasi all function in $L^p(\mathbb{T})$
- Kolmogorov (1923) $\exists f \in L^1(\mathbb{T})$; $S_n f(x)$ diverges a.s.
- Kolmogorov (1926) $\exists f \in L^1(\mathbb{T})$; $S_n f(x)$ diverges surely
- Carleson (1966) If $f \in L^2(\mathbb{T})$, $S_n f(x)$ converges a.s.

Divergence of Fourier series : historic results

- Du Bois Raymond (1872) $\exists f \in \mathcal{C}(\mathbb{T})$; $S_n f(0)$ diverges
- Baire : $S_n f(0)$ diverges for quasi all function in $\mathcal{C}(\mathbb{T})$
- Baire : $S_n f(0)$ diverges for quasi all function in $L^p(\mathbb{T})$
- Kolmogorov (1923) $\exists f \in L^1(\mathbb{T})$; $S_n f(x)$ diverges a.s.
- Kolmogorov (1926) $\exists f \in L^1(\mathbb{T})$; $S_n f(x)$ diverges surely
- Carleson (1966) If $f \in L^2(\mathbb{T})$, $S_n f(x)$ converges a.s.
- Carleson Hunt (1967) Always true if $f \in L^p(\mathbb{T})$, $p > 1$

Natural questions

Question

Let x be a divergence point for $S_n f$. What is the size of $S_n f(x)$?

Natural questions

Question

Let x be a divergence point for $S_n f$. What is the size of $S_n f(x)$?

Nikolsky Inequality : If $f \in L^p(\mathbb{T})$,

$$\|S_n f\|_\infty \leq C n^{1/p} \|S_n f\|_p \leq C n^{1/p} \|f\|_p \ .$$

Natural questions

Question

Let x be a divergence point for $S_n f$. What is the size of $S_n f(x)$?

Nikolsky Inequality : If $f \in L^p(\mathbb{T})$,

$$\|S_n f\|_\infty \leq C n^{1/p} \|S_n f\|_p \leq C n^{1/p} \|f\|_p \ .$$

Question

Let $\beta \in [0, 1/p]$ and $f \in L^p(\mathbb{T})$. What is the size of the set of points x such that $|S_n f(x)| \approx n^\beta$ when $n \to +\infty$?

The divergence index

The divergence index

Let $f \in L^p(\mathbb{T})$ and $x_0 \in \mathbb{T}$.

$$\beta(x_0) \quad = \quad \inf \Big(\beta \ ; \ |S_n f(x_0)| = O(n^\beta)\Big)$$

The divergence index

Let $f \in L^p(\mathbb{T})$ and $x_0 \in \mathbb{T}$.

$$\begin{aligned} \beta(x_0) &= \inf\left(\beta \; ; \; |S_n f(x_0)| = O(n^\beta)\right) \\ &= \limsup_{n \to +\infty} \frac{\log |S_n f(x_0)|}{\log n} \; . \end{aligned}$$

The divergence index

Let $f \in L^p(\mathbb{T})$ and $x_0 \in \mathbb{T}$.

$$\begin{aligned} \beta(x_0) &= \inf\left(\beta \; ; \; |S_n f(x_0)| = O(n^\beta)\right) \\ &= \limsup_{n\to+\infty} \frac{\log|S_n f(x_0)|}{\log n} \; . \end{aligned}$$

Level sets :

$$E(\beta, f) = \{x \in \mathbb{T}; \; \beta(x) = \beta\} \; .$$

The divergence index

Let $f \in L^p(\mathbb{T})$ and $x_0 \in \mathbb{T}$.

$$\begin{aligned} \beta(x_0) &= \inf\left(\beta\ ;\ |S_n f(x_0)| = O(n^\beta)\right) \\ &= \limsup_{n\to+\infty} \frac{\log|S_n f(x_0)|}{\log n}\ . \end{aligned}$$

Level sets :

$$E(\beta, f) = \{x \in \mathbb{T};\ \beta(x) = \beta\}\ .$$

Multifractal spectrum :

$$\beta \mapsto \dim_H \left(E(\beta, f)\right)\ .$$

A generic result

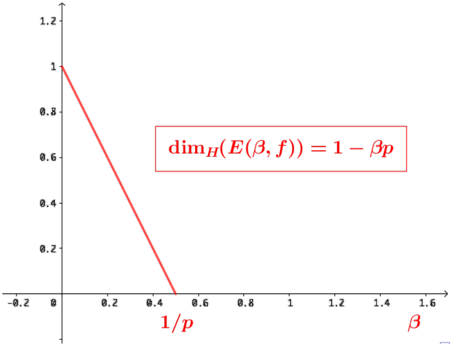
Theorem (Bayart, H.)

Let $p \geq 1$

- *For quasi all function $f \in L^p(\mathbb{T})$,*

$$\forall \beta \in [0, 1/p], \quad \dim_H \big(E(\beta, f)\big) = 1 - \beta p \ . \qquad (3.1)$$

- *Property (3.1) is also prevalent in $L^p(\mathbb{T})$*



First step : the upper bound

Define

$$F(\beta, f) = \left\{ x \in \mathbb{T};\ \limsup_{n\to+\infty} n^{-\beta} |S_n f(x)| > 0 \right\}.$$

First step : the upper bound

Define

$$F(\beta, f) = \left\{ x \in \mathbb{T};\ \limsup_{n \to +\infty} n^{-\beta} |S_n f(x)| > 0 \right\}.$$

$(E(\beta, f) \subset F(\beta - \varepsilon, f))$

First step : the upper bound

Define

$$F(\beta, f) = \left\{ x \in \mathbb{T};\ \limsup_{n \to +\infty} n^{-\beta} |S_n f(x)| > 0 \right\}.$$

$(E(\beta, f) \subset F(\beta - \varepsilon, f))$

Theorem (J.M. Aubry, 2006)

Suppose $p > 1$ and $f \in L^p(\mathbb{T})$. Then

$$\dim_H \big(F(\beta, f)\big) \leq 1 - \beta p\ .$$

First step : the upper bound

Define

$$F(\beta,f)=\left\{x\in\mathbb{T};\ \limsup_{n\to+\infty} n^{-\beta}|S_nf(x)|>0\right\}.$$

$(E(\beta,f)\subset F(\beta-\varepsilon,f))$

Theorem (J.M. Aubry, 2006)

Suppose $p>1$ and $f\in L^p(\mathbb{T})$. Then

$$\dim_H\left(F(\beta,f)\right)\leq 1-\beta p\ .$$

Tool : Carleson Hunt maximal inequality

$$\|S^*f\|_p\leq C\|f\|_p$$

where $S^*f(x)=\sup_n|S_nf(x)|$.

And if $p = 1$?

Carleson Hunt maximal inequality is false when $p = 1$

And if $p = 1$?

Carleson Hunt maximal inequality is false when $p = 1$

Lemma (Bayart, H.)

There exists a constant $C > O$ such that for any $f \in L^1(\mathbb{T})$,

$$\int_{\mathbb{T}} \sup_n \left| \frac{S_n f(x)}{(\log n)^{1+\varepsilon}} \right| dx \leq C \|f\|_1 \ .$$

And if $p = 1$?

Carleson Hunt maximal inequality is false when $p = 1$

Lemma (Bayart, H.)

There exists a constant $C > O$ such that for any $f \in L^1(\mathbb{T})$,

$$\int_{\mathbb{T}} \sup_n \left| \frac{S_n f(x)}{(\log n)^{1+\varepsilon}} \right| dx \leq C \|f\|_1 \ .$$

Remark

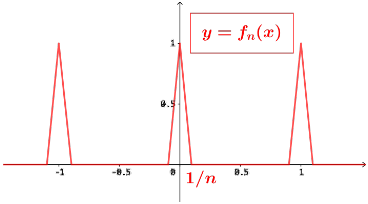
In the $L^p(\mathbb{T})$ context, we don't really need the Carleson Hunt maximal inequality. The inequality

$$\int_{\mathbb{T}} \sup_n \frac{|S_n f(x)|^p}{(\log n)^{p+\varepsilon}} \, dx \leq C \|f\|_p^p$$

is sufficient.

A function $f \in L^p(\mathbb{T})$ such that $\beta(0) = 1/p$

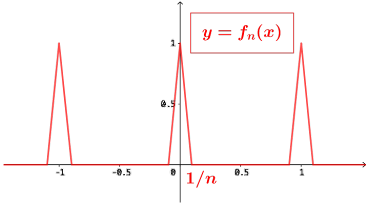
Lemma



Let $g_n = \sigma_n(f_n)$.
Then, $g_n(0) \geq \frac{1}{4}$.

A function $f \in L^p(\mathbb{T})$ such that $\beta(0) = 1/p$

Lemma

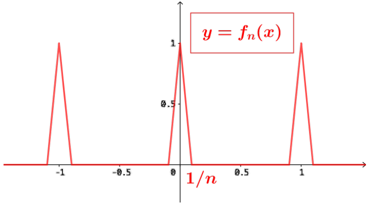


Let $g_n = \sigma_n(f_n)$.
Then, $g_n(0) \geq \frac{1}{4}$.

$$f = \sum_{k=1}^{+\infty} \frac{1}{k^2} 2^{k/p} e_{k2^k} g_{2^k} := \sum_{k=1}^{+\infty} \frac{1}{k^2} \varphi_k$$

A function $f \in L^p(\mathbb{T})$ such that $\beta(0) = 1/p$

Lemma



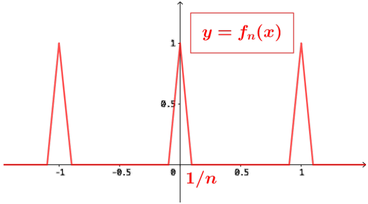
Let $g_n = \sigma_n(f_n)$.
Then, $g_n(0) \geq \frac{1}{4}$.

$$f = \sum_{k=1}^{+\infty} \frac{1}{k^2} 2^{k/p} e_{k2^k} g_{2^k} := \sum_{k=1}^{+\infty} \frac{1}{k^2} \varphi_k$$

- $\|\varphi_k\|_p^p \leq 2$

A function $f \in L^p(\mathbb{T})$ such that $\beta(0) = 1/p$

Lemma



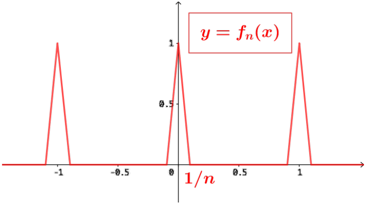
Let $g_n = \sigma_n(f_n)$*.*
Then, $g_n(0) \geq \frac{1}{4}$*.*

$$f = \sum_{k=1}^{+\infty} \frac{1}{k^2} 2^{k/p} e_{k2^k} g_{2^k} := \sum_{k=1}^{+\infty} \frac{1}{k^2} \varphi_k$$

- $\|\varphi_k\|_p^p \leq 2$
- $|S_{(n+1)2^n} f(0) - S_{(n-1)2^n - 1} f(0)| = \frac{1}{n^2} 2^{n/p} |g_{2^n}(0)| \geq \frac{2^{n/p}}{4n^2}$

A function $f \in L^p(\mathbb{T})$ such that $\beta(0) = 1/p$

Lemma



Let $g_n = \sigma_n(f_n)$*.*
Then, $g_n(0) \geq \frac{1}{4}$*.*

$$f = \sum_{k=1}^{+\infty} \frac{1}{k^2} 2^{k/p} e_{k2^k} g_{2^k} := \sum_{k=1}^{+\infty} \frac{1}{k^2} \varphi_k$$

- $\|\varphi_k\|_p^p \leq 2$
- $|S_{(n+1)2^n} f(0) - S_{(n-1)2^n - 1} f(0)| = \frac{1}{n^2} 2^{n/p} |g_{2^n}(0)| \geq \frac{2^{n/p}}{4n^2}$
- There exists $N \sim n2^n$ with $|S_N f(0)| \geq C \frac{N^{1/p}}{\log^{2+1/p}(N)}$

To go further : dyadic approximation

The real number x is said to be α-approximable by dyadics if $\left|x - \frac{k}{2^j}\right| \leq \frac{1}{2^{\alpha j}}$ for infinitely many j.

To go further : dyadic approximation

The real number x is said to be α-approximable by dyadics if $\left|x - \frac{k}{2^j}\right| \leq \frac{1}{2^{\alpha j}}$ for infinitely many j.

$$\begin{aligned} D_\alpha &= \{x \in [0,1] \ ; \ x \text{ is } \alpha\text{-approximable}\} \\ &= \limsup_{j \to +\infty} \bigcup_{k=0}^{2^j - 1} \left[\frac{k}{2^j} - \frac{1}{2^{\alpha j}}, \frac{k}{2^j} + \frac{1}{2^{\alpha j}}\right] \ . \end{aligned}$$

To go further : dyadic approximation

The real number x is said to be α-approximable by dyadics if $\left|x - \frac{k}{2^j}\right| \leq \frac{1}{2^{\alpha j}}$ for infinitely many j.

$$\begin{aligned} D_\alpha &= \{x \in [0,1] \ ; \ x \text{ is } \alpha\text{-approximable}\} \\ &= \limsup_{j \to +\infty} \bigcup_{k=0}^{2^j-1} \left[\frac{k}{2^j} - \frac{1}{2^{\alpha j}}, \frac{k}{2^j} + \frac{1}{2^{\alpha j}}\right] . \end{aligned}$$

Well known : $\dim_H(D_\alpha) = 1/\alpha$

To go further : dyadic approximation

The real number x is said to be α-approximable by dyadics if $\left|x - \frac{k}{2^j}\right| \leq \frac{1}{2^{\alpha j}}$ for infinitely many j.

$$\begin{aligned} D_\alpha &= \{x \in [0,1] \; ; \; x \text{ is } \alpha\text{-approximable}\} \\ &= \limsup_{j \to +\infty} \bigcup_{k=0}^{2^j - 1} \left[\frac{k}{2^j} - \frac{1}{2^{\alpha j}}, \frac{k}{2^j} + \frac{1}{2^{\alpha j}} \right] . \end{aligned}$$

Well known : $\dim_H(D_\alpha) = 1/\alpha$ (in fact $\mathcal{H}^{1/\alpha}(D_\alpha) > 0$).

To go further : a saturating function

We can construct a function $f \in L^p(\mathbb{T})$ such that

$$x \in D_\alpha \Rightarrow \beta(x) = \limsup \frac{\log S_n f(x)}{\log n} \geq \frac{1}{p}\left(1 - \frac{1}{\alpha}\right)$$

To go further : a saturating function

We can construct a function $f \in L^p(\mathbb{T})$ such that

$$x \in D_\alpha \Rightarrow \beta(x) = \limsup \frac{\log S_n f(x)}{\log n} \geq \frac{1}{p}\left(1 - \frac{1}{\alpha}\right)$$

$$\beta = \frac{1}{p}\left(1 - \frac{1}{\alpha}\right) \Rightarrow \frac{1}{\alpha} = 1 - \beta p$$

To go further : a saturating function

We can construct a function $f \in L^p(\mathbb{T})$ such that

$$x \in D_\alpha \Rightarrow \beta(x) = \limsup \frac{\log S_n f(x)}{\log n} \geq \frac{1}{p}\left(1 - \frac{1}{\alpha}\right)$$

$$\beta = \frac{1}{p}\left(1 - \frac{1}{\alpha}\right) \Rightarrow \frac{1}{\alpha} = 1 - \beta p$$

$$\dim_H(E(\beta, f)) \geq \dim_H(D_\alpha) = \frac{1}{\alpha} = 1 - \beta p$$

Au patrimoine mondial de l'Unesco