Measures and the law of the iterated logarithm

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Dimension of measures

m : probability measure in $\mathbb{R}^p$.

$$\begin{cases} \dim_*(m) = \inf\{\dim(E); \quad m(E) > 0\} \\ \\ \dim^*(m) = \inf\{\dim(E); \quad m(E) = 1\} \end{cases}$$

In other words

$$\begin{cases} \dim_*(m) = \sup\{s \geq 0; \quad m \ll \mathcal{H}^s\} \\ \\ \dim^*(m) = \inf\{s \geq 0; \quad m \perp \mathcal{H}^s\} \end{cases}$$

Unidimensional measures

$$\boxed{\dim_*(m) = \dim^*(m) = d}$$

In other words :

- There exists E_0 such that $\dim(E_0) = d$ and $m(E_0) = 1$
- $\dim(E) < d \Rightarrow m(E) = 0$

Example : the natural measure on the triadic Cantor set K.

$$\dim_*(m) = \dim^*(m) = \dim(K) = \frac{\ln 2}{\ln 3} .$$

$\dim(m)$ in terms of local regularity

Hölder index at point x :

$$\underline{\dim}\, m(x) = \liminf_{r\to 0} \frac{\log m\left(B(x,r)\right)}{\log r} \ .$$

$\underline{\dim}\, m(x)$ is the best of the α such that $m\left(B(x,r)\right) \leq C\, r^{\alpha}$ for small r.

Proposition (Falconer, Fan, H.)

$$\begin{cases} \dim_*(m) = \operatorname{ess\,inf}\left(\underline{\dim}\, m(x)\right) \\ \\ \dim^*(m) = \operatorname{ess\,sup}\left(\underline{\dim}\, m(x)\right) \end{cases}$$

In particular, $\dim_*(m) = \dim^*(m) = d$ if and only if $\underline{\dim}\, m(x) = d$ dm-almost surely.

Natural questions

Let m be a unidimensional measure with dimension d.

Question

Does there exist a set $E_0 \subset \mathrm{supp}\,(m)$ such that

$$\forall A, \quad m(A) = C\,\mathcal{H}^d(A \cap E_0)\ ?$$

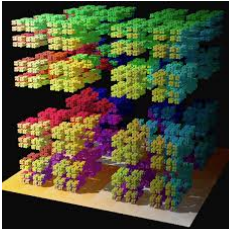
Or in a weaker form :

Question

Does there exist a set $E_0 \subset \mathrm{supp}\,(m)$ such that

$$\forall A, \quad \frac{1}{C}\,\mathcal{H}^d(A \cap E_0) \leq m(A) \leq C\,\mathcal{H}^d(A \cap E_0)\ ?$$

Sometimes YES !



Selfsimilar Cantor set

$$K = \bigcup_{i=1}^{8} S_i(K) \quad m = \sum_{i=1}^{8} \frac{1}{8} m \circ S_i^{-1}$$

$$m(A) \approx \mathcal{H}^d(A \cap K) \quad \text{where } d = \dim(K)$$

Bernoulli products on $[0, 1]$

$$m([0,1/2]) = (1-p) \quad \text{and} \quad m([1/2,1]) = p$$

- $\mathcal{F}_n$: dyadic intervals of the n^{th} generation
- $I_{\varepsilon_1 \cdots \varepsilon_n} = \left[\sum_{i=1}^{n} \frac{\varepsilon_i}{2^i}, \sum_{i=1}^{n} \frac{\varepsilon_i}{2^i} + \frac{1}{2^n}\right) \in \mathcal{F}_n$

$$m\left(I_{\varepsilon_1 \cdots \varepsilon_n}\right) = p^{s_n}(1-p)^{n-s_n}, \quad \text{where} \quad s_n = \varepsilon_1 + \cdots + \varepsilon_n \ .$$

$$X_n(x) = -\log_2\left(p^{\varepsilon_n}(1-p)^{1-\varepsilon_n}\right) = -\log_2\left(\frac{m(I_n(x))}{m(I_{n-1}(x))}\right)$$

$$\boxed{\frac{S_n}{n} = \frac{\log m(I_n(x))}{\log |I_n(x)|}}$$

Consequence of the law of large numbers

$$\lim_{n\to+\infty} \frac{S_n}{n} = \lim_{n\to+\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|}$$

$$\lim_{n\to+\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = -p\log_2 p - (1-p)\log_2(1-p) \quad a.s.$$

The measure m is unidimensional with dimension

$$\dim_*(m) = \dim^*(m) = -p\log_2 p - (1-p)\log_2(1-p) := d$$

More precise estimations

Law of the Iterated Logarithm : $\liminf_{n\to+\infty} \frac{S_n - nd}{\sqrt{2n \ln \ln n}} = -\sigma$ a.s.

Proposition

Take $\Theta(t) = 2^{\sqrt{2\log_2(1/t)\ln\ln\log_2(1/t)}}$. *Let*

$$\sigma^2 = p(1-p)\left(\log_2(\frac{p}{1-p})\right)^2$$

Then

1. $m \ll \mathcal{H}^{\Psi}$, *where* $\Psi(t) = t^d\Theta(t)^{\sigma+\varepsilon}$
2. $m \perp \mathcal{H}^{\Psi}$, *where* $\Psi(t) = t^d\Theta(t)^{\sigma-\varepsilon}$.

In particular, $m \perp \mathcal{H}^d$.

And what about $\limsup_{n\to+\infty} \frac{S_n - nd}{\sqrt{2n \ln \ln n}} = \sigma$ a.s ?

Related to the packing dimension and the packing measures.

$$\begin{cases} \mathrm{Dim}_*(m) = \inf\{\mathrm{Dim}(E);\quad m(E) > 0\} = \sup\{s \geq 0;\quad m \ll \widehat{\mathcal{P}}^s\} \\ \\ \mathrm{Dim}^*(m) = \inf\{\mathrm{Dim}(E);\quad m(E) = 1\} = \inf\{s \geq 0;\quad m \perp \widehat{\mathcal{P}}^s\} \ . \end{cases}$$

Proposition

1. $m \ll \widehat{\mathcal{P}}^\Psi$, *where* $\Psi(t) = t^d \Theta(t)^{-(\sigma-\varepsilon)}$
2. $m \perp \widehat{\mathcal{P}}^\Psi$, *where* $\Psi(t) = t^d \Theta(t)^{-(\sigma+\varepsilon)}$.

In particular, $\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = d$ *and* $m \ll \widehat{\mathcal{P}}^d$.

Bernoulli products are self-similar measures

$$S_1(x) = x/2 \quad \text{and} \quad S_2(x) = (1+x)/2$$

$$[0,1] = S_1([0,1]) \cup S_2([0,1])$$

$$m = (1-p)\, m \circ S_1^{-1} + p\, m \circ S_2^{-1}$$

Self-similar measures : the general case

$$K = \bigcup_{i=1}^{k} S_i(K)$$

where S_i are similarity transformations with ratio $0 < r_i < 1$.

$$m = \sum_{i=1}^{k} p_i \, m \circ S_i^{-1} \ .$$

OSC : $U \neq \emptyset$, $\bigcup_{i=1}^{k} S_i(U) \subset U$ and $\forall i \neq j,\ S_i(U) \cap S_j(U) = \emptyset$

- $\dim(K) = \delta$ such that $\sum_{i=1}^{k} r_i^{\delta} = 1$
- $dim(m) = d = \frac{\sum_{i=1}^{k} p_i \ln p_i}{\sum_{i=1}^{k} p_i \ln r_i}$

Log-Log corrections

Theorem (Bhouri, H. 09)

There are two possible cases :

- $d = \delta$ *and* $m \approx \mathcal{H}^d$
- $d \neq \delta$ *and* $m \perp \mathcal{H}^d$. *More precisely,*

$$m \ll \mathcal{H}^{\psi_\varepsilon} \quad \textit{but} \quad m \perp \mathcal{H}^{\psi_{-\varepsilon}}$$

where

$$\psi_\varepsilon(t) = t^d \theta(t)^{\sigma+\varepsilon} \quad \textit{and} \quad \sigma^2 = \frac{\sum_{i=1}^k p_i (\ln p_i - d \ \ln r_i)^2}{-\sum_{i=1}^k p_i \ln r_i}$$

Makarov Theorem

- Ω : Jordan domain in $\mathbb{R}^2$
- ω : harmonic measure on Ω

Theorem (Makarov, 1985)

$\dim_*(\omega) = \dim^*(\omega) = 1$. *More precisely*

- *if* $\psi(t) = o(t)$, *then* $\omega \perp \mathcal{H}^\psi$
- $\exists c > 0$ *such that* $\omega \ll \mathcal{H}^\psi$ *where* $\psi(t) = t\theta(t)^c$

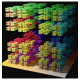
Theorem (Makarov, 1985)

There exists a Jordan domain Ω *such that*

$$\omega \perp \mathcal{H}^\psi \quad \textit{where} \quad \psi(t) = t\theta(t)^c$$

for some $c > 0$.

Harmonic measure on self-similar Cantor sets



Theorem (Makarov-Volberg, Carleson, 1986)

- $\dim_*(\omega) = \dim^*(\omega) = d < \dim(K)$
- *There exists* $\sigma > 0$ *such that*

$$\begin{cases} \omega \ll \mathcal{H}^\psi \quad \textit{where} \quad \psi(t) = t^d \theta(t)^{\sigma+\varepsilon} \\ \\ \omega \perp \mathcal{H}^\psi \quad \textit{where} \quad \psi(t) = t^d \theta(t)^{\sigma-\varepsilon} \end{cases}$$

Tool : asymptotic independance of the random variables :

$$\left| \log \frac{\omega(XYZ)}{\omega(XY)} - \log \frac{\omega(YZ)}{\omega(Y)} \right| \leq C q^{|Y|} \quad \text{where } q < 1 \ .$$

A tool : the structure function τ

- m : probability measure on $[0,1)^p$
- $\ell \geq 2$ and $\mathcal{F}_n$: ℓ-adic cubes of the n^{th} generation
- $X_n(x) = -\log_\ell \frac{m(I_n(x))}{m(I_{n-1}(x))}$ and $\frac{S_n}{n} = \frac{\log m(I_n(x))}{\log |I_n(x)|}$

$$\tau(q) = \limsup_{n\to+\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n\log \ell} \log\Big(\sum_{I\in\mathcal{F}_n} m(I)^q\Big) .$$

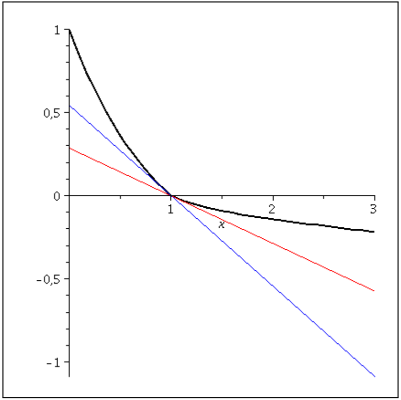
Remark : $\tau_n(1-q) = \frac{1}{n}\log_\ell \mathbb{E}[\ell^{qS_n}]$

Theorem (H. 98)

$$-\tau'_+(1) \leq \dim_*(m) \leq \mathrm{Dim}^*(m) \leq -\tau'_-(1)$$

In particular, if $\tau'(1)$ exists,
$\dim_*(m) = \dim^*(m) = \mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = -\tau'(1).$

The structure function τ



τ is convex, decreasing, satisfies $\tau(1) = 0$ and $\tau(0) = \dim(\text{supp}\,(m))$

Quasi-Bernoulli measures

Definition

A measure m is called a quasi-Bernoulli measure on $[0,1)$ if

$$\boxed{\frac{1}{C}\, m(I)m(J) \le m(IJ) \le C\, m(I)m(J)}$$

where $I = I_{\varepsilon_1 \cdots \varepsilon_n}$, $J = I_{\varepsilon_1 \cdots \varepsilon_p}$ and $IJ = I_{\varepsilon_1 \cdots \varepsilon_{n+p}}$.

In other words : $\dfrac{m(IJ)}{m(I)} \approx m(J)$.

Quasi-Bernoulli measures : ergodic properties

The left shift :

$\sigma(x) = \ell x \bmod 1$ or equivalently $\sigma(0, \varepsilon_1 \cdots \varepsilon_n \cdots) = 0, \varepsilon_2 \cdots \varepsilon_n \cdots$.

Proposition (Carleson, H.)

- *0-1 law :* $B_\infty = \bigcap_n \sigma^{-n}(Bor([0,1)))$.

$$\forall E \in B_\infty,\ m(E) = 0 \text{ or } m(E) = 1 \ .$$

- *Strong mixing property : (m is suppposed σ-invariant)*

$$\lim_{n \to +\infty} m(A \cap \sigma^{-n}(B)) = m(A)\, m(B) \ .$$

- *General case : let* $\tilde{m} = \lim_n \frac{1}{n}(m + \cdots + m \circ \sigma^{-(n-1)})$.
  $\tilde{m}$ *is σ-invariant, ergodic, quasi-Bernoulli and equivalent to m.*

Sub and super multiplicative properties

$$\frac{1}{C}\left(m(I)m(J)\right)^q \le m(IJ)^q \le C\left(m(I)m(J)\right)^q$$

where C does not depend on q when q is bounded. It follows

$$\sum_{K\in\mathcal{F}_{n+p}} m(K)^{1-q} \approx \left[\sum_{I\in\mathcal{F}_n} m(I)^{1-q}\right]\left[\sum_{J\in\mathcal{F}_p} m(J)^{1-q}\right]$$

$$\boxed{\mathbb{E}\left[\ell^{qS_{n+p}}\right] \approx \mathbb{E}\left[\ell^{qS_n}\right]\mathbb{E}\left[\ell^{qS_p}\right]}$$

So that

$$\boxed{\mathbb{E}\left[\ell^{qS_n}\right] \approx \ell^{n\tau(1-q)} \quad \text{near } 0}$$

Quasi-Bernoulli measure are unidimenional measures

Theorem (H. 98)

Let m be a quasi-Bernoulli measure. Then τ is of class C^1 on $\mathbb{R}$. In particular, m is unidimensional and

$$\lim_{n\to+\infty} \frac{\log(m(I_n(x)))}{\log|I_n(x)|} = \lim_{n\to+\infty} \frac{S_n}{n} = -\tau'(1) = d \quad a.s. \; .$$

A log-log upper bound

Theorem (Bhouri, H. 09)

Let m be a quasi-Bernoulli measure with dimension $d = -\tau'(1)$. Suppose that

$$\tau(1-q) = qd + \frac{\sigma^2}{2} q^2 + o(q^2) \quad \textit{near } 0\ .$$

Then,

$$\begin{cases} \limsup_{n\to+\infty} \dfrac{S_n - nd}{\sqrt{2n \log_\ell \log_\ell n}} \le \sigma \\ \\ \liminf_{n\to+\infty} \dfrac{S_n - nd}{\sqrt{2n \log_\ell \log_\ell n}} \ge -\sigma \end{cases}$$

dm-almost surely.

The main lemma

Using sub and super multiplicative properties, the following maximal lemma holds :

Lemma

Let $\varepsilon > 0$, $a > 0$ *and* $n_0 < n_1$. *Then,*

$$m\left(\exists\ k \in \{n_0, \cdots, n_1\}\ ;\ S_k \geq a + kd\right) \leq C\,\ell^{\frac{-a^2}{2n_1(\sigma+\varepsilon)^2}}$$

for $\dfrac{a}{n_1(\sigma+\varepsilon)^2}$ *small enough.*

Consequences for the measure m

Corollary

Let $\Theta(t) = \ell^{\sqrt{2\log_\ell 1/t \log_\ell \log_\ell \log_\ell 1/t}}$. *Then, for all* $\varepsilon > 0$,

1. $m \ll \mathcal{H}^{\Psi}$, *where* $\Psi(t) = t^d \Theta(t)^{\sigma+\varepsilon}$
2. $m \perp \widehat{\mathcal{P}}^{\Psi}$, *where* $\Psi(t) = t^d \Theta(t)^{-(\sigma+\varepsilon)}$.

Sketch of Proof of 1 : a. s., for n large enough,

$$\log_\ell(m(I_n(x))) = S_n \geq nd - (\sigma+\varepsilon)\sqrt{2n \ln\ln n}$$

We get

$$\text{a.s,} \quad m(I_n(x)) \leq |I_n(x)|^d \, \Theta(|I_n(x)|)^{\sigma+\varepsilon} \quad \text{for } n \text{ large enough}$$

There exists a full measure set E_0 such that

$$\forall E, \quad m(E \cap E_0) \leq \mathcal{H}^{\Psi}(E)$$

Estimations in the reverse sense

Theorem (Bhouri, H. 09)

Let m be a quasi-Bernoulli measure with dimension $d = -\tau'(1)$. Suppose that for some $\sigma > 0$

$$\tau(1-q) = qd + \frac{\sigma^2}{2} q^2 + o(q^2) \quad \textit{near } 0 \ .$$

Then,

$$\begin{cases} \limsup\limits_{n \to +\infty} \dfrac{S_n - nd}{\sqrt{n}} = +\infty \\ \\ \liminf\limits_{n \to +\infty} \dfrac{S_n - nd}{\sqrt{n}} = -\infty \end{cases}$$

dm-almost surely.

Consequences for the measure m

Corollary

- $\exists E \subset [0,1)$ *such that* $m(E) = 1$ *and* $\mathcal{H}^d(E) = 0$ *(*$m \perp \mathcal{H}^d$*)*
- *If* $\widehat{\mathcal{P}}^d(E) < +\infty$ *then* $m(E) = 0$ *(in particular* $m \ll \widehat{\mathcal{P}}^d$*)*

More precisely,

Corollary

If $\psi_a(t) = t^d \ell^{a\sqrt{\log_\ell 1/t}}$, $a \in \mathbb{R}$,

$$\begin{cases} m \perp \mathcal{H}^{\psi_a} \\ \\ m \ll \widehat{\mathcal{P}}^{\psi_a} \end{cases}$$

More generaly

Theorem (Bhouri, H. 09)

Let m be a quasi-Bernoulli measure with dimension $d = -\tau'(1)$. Suppose that for some $\alpha > 0$ and $\beta > 1$

$$\tau(1-q) = qd + \alpha|q|^{\beta} + o(q^{\beta}) \quad \textit{near } 0 \ .$$

Then,

$$\begin{cases} \limsup_{n\to+\infty} \dfrac{S_n - nd}{n^{1/\beta}} = +\infty \\ \\ \liminf_{n\to+\infty} \dfrac{S_n - nd}{n^{1/\beta}} = -\infty \end{cases}$$

dm-almost surely.
In particular, $m \perp \mathcal{H}^d$ *but* $m \ll \widehat{\mathcal{P}}^d$.

The case where τ is analytic

Corollary (Bhouri, H. 09)

Suppose that τ is analytic. Let

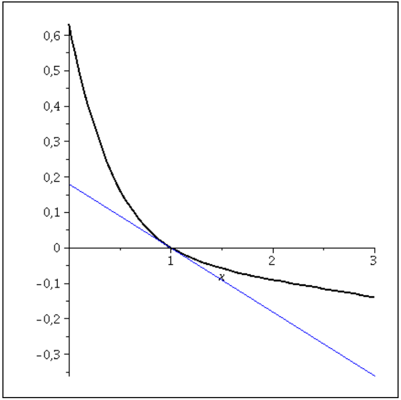
$$\delta = \dim(\mathrm{supp}\,(m)) \quad \textit{and} \quad d = \dim(m)\ .$$

There are only two possible cases :

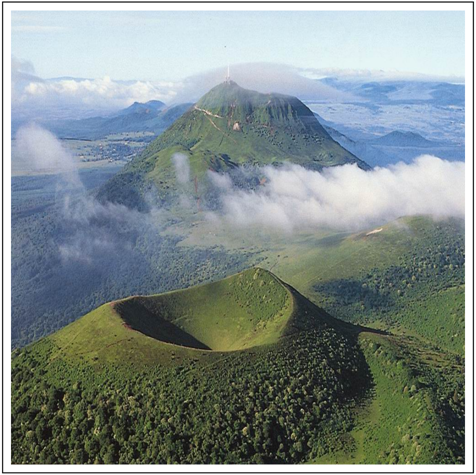
- $d = \delta$ *and* $\quad m \approx \mathcal{H}^\delta \approx \widehat{\mathcal{P}}^\delta$ *on* $\mathrm{supp}\,(m)$
- $d < \delta$ *and* $\quad m \perp \mathcal{H}^d$ *but* $m \ll \widehat{\mathcal{P}}^d$

Example : this is the case when the measure m is a Gibbs measure related to an Hölder potential.

Proof



If $\dim(m) < \dim(\text{supp}\,(m))$, then $\quad \tau(1-q) = dq + \alpha q^{2k} + o(q^{2k})$



Merci !