An introduction to Mandelbrot cascades

Yanick Heurteaux

Abstract In this course, we propose an elementary and self-contained introduction to canonical Mandelbrot random cascades. The multiplicative construction is explained and the necessary and sufficient condition of non-degeneracy is proved. Then, we discuss the problem of the existence of moments and the link with non-degeneracy. We also calculate the almost sure dimension of the measures. Finally, we give an outline on multifractal analysis of Mandelbrot cascades. This course was delivered in september 2013 during a meeting of the "Multifractal Analysis GDR" (GDR n$^{\text{o}}$ 3475 of the french CNRS).

1 Introduction

At the beginning of the seventies, Mandelbrot proposed a model of random measures based on an elementary multiplicative construction. This model, known as canonical Mandelbrot cascades, was introduced to simulate the energy dissipation in intermittent turbulence ([20]). It was probably inspired by previous heuristics described by Richardson in [23]. In two notes ([21] and [22]) published in '74, Mandelbrot described the fractal nature of the sets in which the energy is concentrated and proved or conjectured the main properties of this model. Two years later, in the fundamental paper [16], Kahane and Peyrière proposed a complete proof of the results announced by Mandelbrot. In particular, the questions of non-degeneracy, existence of moments and dimension of the measures were rigorously solved.

Mandelbrot also observed that in a multiplicative cascade, the energy is distributed along a large deviations principle: this was the beginning of the multifractal analysis.

Yanick Heurteaux
Clermont Université, Université Blaise Pascal, Laboratoire de Mathématiques, BP 10448, F-63000 CLERMONT-FERRAND - CNRS, UMR 6620, Laboratoire de Mathématiques, F-63177 AUBIERE e-mail: Yanick.Heurteaux@math.univ-bpclermont.fr

Multifractal analysis has been developed a lot in the 80's. Frisch an Parisi observed that in the context of the fully developed turbulence, the pointwise Hölder exponent of the dissipation of energy varies widely from point to point. They proposed in [13] a heuristic argument, showing that the Hausdorff dimension of the level sets of a measure or a function can be obtained as the Legendre transform of a free energy function (which will be called in this text the structure function). This principle is known as *Multifractal Formalism*. Such a formalism was then rigorously proved by Brown Michon and Peyrière for the so called quasi-Bernoulli measures ([10]). In particular, they highlighted the link between the multifractal formalism and the existence of auxiliary measures.

The problem of the multifractal analysis of Mandelbrot cascades appeared as a natural question at the end of the 80's. Holley and Waymire were the first to obtain results in this direction. Under restrictive hypotheses, they proved in [15] that for any value of the Hölder exponent, the multifractal formalism is almost surely satisfied. The expected stronger result which says that, almost surely, for any value of the Hölder exponent, the multifractal formalism is satisfied was finally proved by Barral at the end of the 20th century ([2]).

Let us finish this overview by saying that there exist now many generalizations of the Mandelbrot cascades (see for example [8] for the description of the principal ones).

In the following pages, we want to relate the beginning of the story of canonical Mandelbrot cascades. As a preliminary, we explain the well known determinist case of binomial cascades. It allows us to describe the multiplicative principle, to introduce the most important notations and definitions, and to show the way to calculate the dimension and to perform the multifractal analysis. Then, we introduce the canonical random Mandelbrot cascades (Theorem 1), solve the problem of non-degeneracy (Theorem 2) and its link with the existence of moments for the total mass of the cascade (Theorem 3). In Section 5, we prove that the Mandelbrot cascades are almost surely unidimensional and give the value of the dimension (Theorem 4). Finally, in a last section, we deal with the problem of multifractal analysis, and prove that for any value of the parameter β the Hausdorff dimension of the level set of points with Hölder exponent β is almost surely given by the multifractal formalism (Theorem 8). To obtain such a result, we use auxiliary cascades and we need to describe the simultaneous behavior of two cascades (Theorem 6) and to prove the existence of negative moments for the total mass (Proposition 5).

Part of this text is inspired by the founding article [16] by Kahane and Peyrière.

2 Binomial cascades

In order to understand the multiplicative construction principle, we begin with a very simple and classical example, known as Bernoulli product, which can be regarded as an introduction to the following.

Let $\mathscr{F}_n$ be the family of dyadic intervals of the n^{th} generation on $[0,1)$, $0<p<1$ and define the measure m as follows. If $\varepsilon_1\dots\varepsilon_n$ are integers in $\{0,1\}$, and if

$$I_{\varepsilon_1\cdots\varepsilon_n}=\left[\sum_{i=1}^n\frac{\varepsilon_i}{2^i},\sum_{i=1}^n\frac{\varepsilon_i}{2^i}+\frac{1}{2^n}\right)\in\mathscr{F}_n$$

then

$$m\left(I_{\varepsilon_1\cdots\varepsilon_n}\right)=p^{S_n}(1-p)^{n-S_n},\quad\text{where}\quad S_n=\varepsilon_1+\cdots+\varepsilon_n\,. \tag{1}$$

The measure m is constructed using a multiplicative principle: if $I=I_{\varepsilon_1\cdots\varepsilon_n}\in\mathscr{F}_n$ and in $I'=I_{\varepsilon_1\cdots\varepsilon_n 0}$ and $I''=I_{\varepsilon_1\cdots\varepsilon_n 1}$ are the two children of I in $\mathscr{F}_{n+1}$, then

$$m(I')=pm(I)\quad\text{and}\quad m(I'')=(1-p)m(I).$$

If $x\in[0,1)$, we can find $\varepsilon_1,\dots,\varepsilon_n,\dots\in\{0,1\}$ uniquely determined and such that for any $n\geq 1$, $x\in I_{\varepsilon_1\cdots\varepsilon_n}$. We also denote $I_{\varepsilon_1\cdots\varepsilon_n}=I_n(x)$ and we observe that

$$\frac{\log m(I_n(x))}{\log|I_n(x)|}=-\left(\frac{S_n}{n}\log_2 p+\left(1-\frac{S_n}{n}\right)\log_2(1-p)\right)$$

where $|I|$ is the length of the interval I. By the strong law of large numbers applied to the sequence (ε_n), we can then conclude that

$$\lim_{n\to\infty}\frac{\log m(I_n(x))}{\log|I_n(x)|}=h(p)\quad dm-\text{almost surely}$$

where $h(p)=-(p\log_2 p+(1-p)\log_2(1-p))$.

Using Billingsley's Theorem (see for example [11]), it is then easy to conclude that

$$\dim_*(m)=\dim^*(m)=h(p)$$

where $\dim_*(m)$ and $\dim^*(m)$ are the lower and the upper dimension defined by

$$\begin{cases}\dim_*(m)=\inf(\dim(E)\ ;\ m(E)>0)\\[2ex] \dim^*(m)=\inf(\dim(E)\ ;\ m([0,1]\setminus E)=0)\end{cases} \tag{2}$$

It means that the measure m is supported by a set of Hausdorff dimension $h(p)$ and that every set of dimension less that $h(p)$ is negligible. We say that the measure m is unidimensional with dimension $h(p)$.

If $\mathrm{Dim}(E)$ is the packing dimension of a set E and if

$$\begin{cases}\mathrm{Dim}_*(m)=\inf(\mathrm{Dim}(E)\ ;\ m(E)>0)\\[2ex] \mathrm{Dim}^*(m)=\inf(\mathrm{Dim}(E)\ ;\ m([0,1]\setminus E)=0)\end{cases} \tag{3}$$

we can also conclude that

$$\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = h(p).$$

2.1 Multifractal analysis of binomial cascades

Binomial cascades are also known to be multifractal measures and it is easy to compute their multifractal spectrum. Let

$$E_\beta = \left\{ x\ ;\ \lim_{n\to\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = \beta \right\}.$$

The multifractal spectrum of the measure m is the function $\beta \mapsto \dim(E_\beta)$.

Recall that

$$\frac{\log m(I_n(x))}{\log |I_n(x)|} = -\left(\frac{S_n}{n} \log_2 p + \left(1 - \frac{S_n}{n}\right) \log_2(1-p) \right).$$

Without lost of generality, we can suppose that $1/2 \le p < 1$, so that $-\log_2 p \le -\log_2(1-p)$.

Suppose that $\beta \in [-\log_2 p, -\log_2(1-p)]$. We can find $\theta \in [0,1]$ such that

$$\beta = -(\theta \log_2 p + (1-\theta)\log_2(1-p)).$$

It follows that $E_\beta = \left\{ \frac{S_n}{n} \to \theta \right\}$ and we can conclude that

$$\dim(E_\beta) = -(\theta \log_2 \theta + (1-\theta)\log_2(1-\theta)) = h(\theta) := F(\beta) \tag{4}$$

where $F(\beta) = h\left(\frac{\beta + \log_2(1-p)}{\log_2(1-p) - \log_2 p} \right)$.

Suppose now that $\beta \notin [-\log_2 p, -\log_2(1-p)]$ and observe that

$$-\log_2(p) \le \frac{\log m(I_n(x))}{\log |I_n(x)|} \le -\log_2(1-p).$$

It follows that $E_\beta = \emptyset$ and $\dim(E_\beta) = -\infty$.

We can finally give the graph of the multifractal spectrum of the measure m.

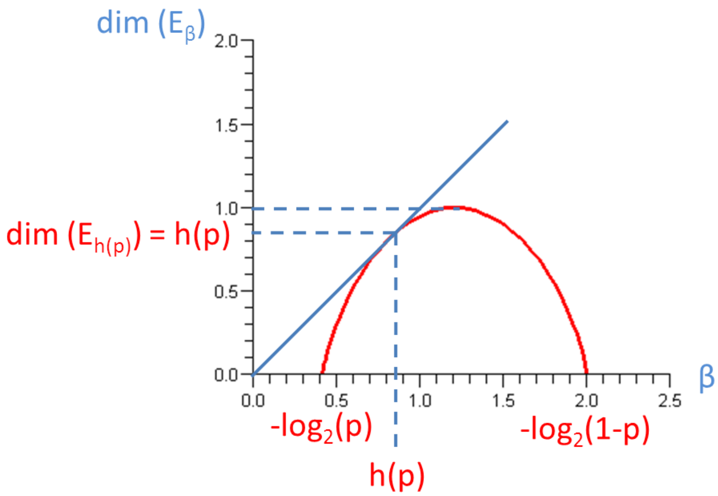

Fig. 1: The graph of the multifractal spectrum of the measure m. The dimension of the measure m is the abscissa of the point where the curve intersects the first bisector.

2.2 Binomial cascades satisfy the multifractal formalism

We can also rewrite formula (4) in the following way. If m_θ be the binomial cascade with parameter θ, the measure m_θ is supported by E_β and we have

$$\dim(E_\beta) = \dim(m_\theta) = h(\theta).$$

Moreover, if $q \in \mathbb{R}$ is such that

$$\theta = \frac{p^q}{p^q + (1-p)^q},$$

and if $I \in \mathscr{F}_n$, we have

$$\begin{aligned} m_\theta(I) &= \theta^{S_n}(1-\theta)^{n-S_n} \\ &= \frac{p^{qS_n}(1-p)^{q(n-S_n)}}{(p^q+(1-p)^q)^n} \\ &= m(I)^q |I|^{\tau(q)} \end{aligned}$$

where $\tau(q) = \log_2(p^q + (1-p)^q)$ is the structure function of the measure m at state q.

Finally, if we observe that $\beta = -(\theta \log_2 p + (1-\theta)\log_2(1-p)) = -\tau'(q)$, we can conclude that

$$\begin{aligned}\dim(E_\beta) &= -(\theta \log_2 \theta + (1-\theta)\log_2(1-\theta)) \\ &= -q\tau'(q) + \tau(q) \\ &= \tau^*(-\tau'(q)) \\ &= \tau^*(\beta)\end{aligned}$$

where $\tau^*(\beta) = \inf_t (t\beta + \tau(t))$ is the Legendre transform of τ.

We say that the measure m satisfies the multifractal formalism and that m_θ is a Gibbs measure at state q. Such a construction of an auxiliary cascade will be used in Section 7.

Remark 1. The new measure m_θ is obtained from m by changing the parameters $(p, 1-p)$ in $\left(\frac{p^q}{p^q+(1-p)^q}, \frac{(1-p)^q}{p^q+(1-p)^q}\right)$. The quantity $\frac{1}{p^q+(1-p)^q}$ is just the renormalization needed to ensure that the sum of the two parameters is equal to 1. A similar idea will be used to construct auxiliary Mandelbrot cascades (see the beginning of Section 7).

Remark 2. If m is a binomial cascade, we have

$$\sum_{I\in\mathscr{F}_{n+1}} m(I)^q = \sum_{I\in\mathscr{F}_n} p^q m(I)^q + (1-p)^q m(I)^q = (p^q + (1-p)^q) \sum_{I\in\mathscr{F}_n} m(i)^q.$$

Finally,

$$\log_2 (p^q + (1-p)^q) = \limsup_{n\to+\infty} \frac{1}{n} \log_2 \left(\sum_{I\in\mathscr{F}_n} m(I)^q \right)$$

which is the classical definition of the structure function τ (see Section 6).

2.3 Back to the existence of binomial cascades

We want to finish this section with an elementary proposition which gives a rigorous proof of the existence of a measure m satisfying (1). Denote by λ the Lebesgue measure on $[0,1)$ and let

$$m_n = f_n d\lambda \quad \text{where} \quad f_n = 2^n \sum_{\varepsilon_1\cdots\varepsilon_n} p^{S_n}(1-p)^{n-S_n} \mathbb{1}_{I_{\varepsilon_1\cdots\varepsilon_n}}.$$

If $I = I_{\varepsilon_1\cdots\varepsilon_j} \in \mathscr{F}_j$, we have

$$m_j(I) = p^{S_j}(1-p)^{j-S_j} = m_{j+1}(I) = \cdots = m_{j+k}(I) = \cdots$$

and the sequence $(m_n(I))_{n\geq 1}$ is convergent.

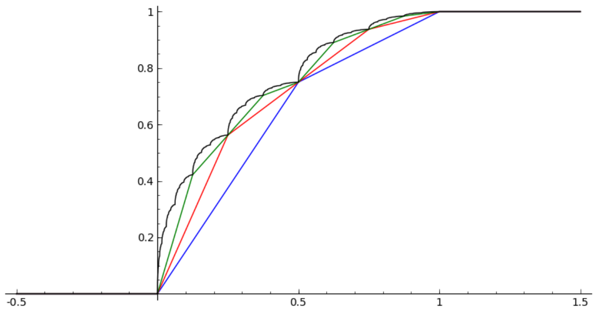

Fig. 2: The repartition function of the measures m_1, m_2, m_3 and m.

We can then use the following elementary proposition.

Proposition 1. *Let $(m_n)_{n\geq 1}$ be a sequence of finite Borel measures on $[0,1)$. Suppose that for any dyadic interval $I \in \bigcup_{j\geq 0} \mathscr{F}_j$, the sequence $(m_n(I))_{n\geq 1}$ is convergent. Then, the sequence $(m_n)_{n\geq 1}$ is weakly convergent to a finite Borel measure m.*

Remark 3. In Proposition 1, we can of course replace the family of dyadic intervals by the family of ℓ-adic intervals ($\ell \geq 2$). Proposition 1 will be used in Section 3 to prove the existence of Mandelbrot caseades.

Proof (Proof of Proposition 1). Observe that if f is a continuous function on $[0,1]$ and $\varepsilon > 0$, we can find a function φ which is a linear combinaison of functions $\mathbb{1}_I$ with $I \in \bigcup_{j\geq 0} \mathscr{F}_j$ and such that $\|f-\varphi\|_\infty \leq \varepsilon$. By the hypothesis, the sequence $\int \varphi(x)\,dm_n(x)$ is convergent and we have

$$\begin{aligned}\left|\int f\,dm_n - \int f\,dm_p\right| &\leq \left|\int \varphi\,dm_n - \int \varphi\,dm_p\right| + \|f-\varphi\|_\infty (m_n([0,1]) + m_p([0,1])) \\ &\leq \left|\int \varphi\,dm_n - \int \varphi\,dm_p\right| + C\varepsilon.\end{aligned}$$

It follows that the sequence $\int f(x)\,dm_n(x)$ is convergent. The conclusion is then a consequence of the Banach-Steinhaus theorem and of the Riesz representation theorem.

3 Canonical Mandelbrot cascades : construction and non-degeneracy conditions

3.1 Construction

In all the sequel, $\ell \geq 2$ is an integer and $\mathscr{F}_n$ is the set of ℓ-adic intervals of the n^{th} generation on $[0,1)$. We denote by $\mathscr{M}_n$ the set of words of length n written with the letters $0,\dots,\ell-1$ and $\mathscr{M}=\bigcup_n \mathscr{M}_n$. If $\varepsilon=\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n$, let

$$I_{\varepsilon_1\cdots\varepsilon_n}=\left[\sum_{k=1}^n \frac{\varepsilon_k}{\ell^k},\sum_{k=1}^n \frac{\varepsilon_k}{\ell^k}+\frac{1}{\ell^n}\right)\in\mathscr{F}_n.$$

Let W be a non-negative random variable such that $E[W]=1$ and $(W_\varepsilon)_{\varepsilon\in\mathscr{M}}$ be a family of independent copies of W.

If λ is the Lebesgue on $[0,1]$, we can define the sequence of random measures by

$$m_n=f_n\lambda \quad \text{where} \quad f_n=\sum_{\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}} W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n}\mathbb{1}_{I_{\varepsilon_1\cdots\varepsilon_n}}.$$

The construction of the measure m_n uses a multiplicative principle and

$$m(I_{\varepsilon_1\cdots\varepsilon_n})=\ell^{-n}W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n}.$$

We have the following existence theorem :

Theorem 1 (existence of m**).** *Almost surely, the sequence* $(m_n)_{n\geq 1}$ *is weakly convergent to a (random) measure m. The measure m is called the Mandelbrot cascade associated to the weight* W.

Remark 4. The condition $E[W]=1$ is a natural condition. Indeed if

$$Y_n:=m_n([0,1])=\int_0^1 f_n(t)\,d\lambda(t)=\ell^{-n}\sum_{\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n} W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n}$$

then,

$$\begin{aligned} E[Y_n]&=\ell^{-n}\sum_{\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n} E\left[W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_{n-1}}\right]E\left[W_{\varepsilon_1\cdots\varepsilon_n}\right]\\ &=E[Y_{n-1}]\times E[W] \end{aligned}$$

and the condition $E[W]=1$ ensures that the expectation of the total mass does not go to 0 or to $+\infty$.

Proof. Let $\mathscr{A}_n$ be the σ-algebra generated by the W_ε, $\varepsilon\in\mathscr{M}_1\cup\cdots\cup\mathscr{M}_n$. Define $Y_n:=m_n([0,1])$. An easy calculation says

$$\begin{aligned} E[Y_{n+1}|\mathscr{A}_n]&=\ell^{-(n+1)}\sum_{\varepsilon_1\cdots\varepsilon_{n+1}\in\mathscr{M}_{n+1}} E[W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n}W_{\varepsilon_1\cdots\varepsilon_{n+1}}|\mathscr{A}_n]\\ &=\ell^{-(n+1)}\sum_{\varepsilon_1\cdots\varepsilon_{n+1}\in\mathscr{M}_{n+1}} W_{\varepsilon_1}W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_n}E[W_{\varepsilon_1\cdots\varepsilon_{n+1}}]\\ &=Y_n \end{aligned}$$

and the sequence Y_n is a non negative martingale. So it is almost surely convergent.

More generally, if $I=I_{\alpha_1\cdots\alpha_k}\in\mathscr{F}_k$,

$$m_{k+n}(I) = \ell^{-(k+n)} \sum_{\varepsilon_{k+1}\cdots\varepsilon_{k+n} \in \mathscr{M}_n} W_{\alpha_1} \cdots W_{\alpha_1\cdots\alpha_k} W_{\alpha_1\cdots\alpha_k\varepsilon_{k+1}} \cdots W_{\alpha_1\cdots\alpha_k\varepsilon_{k+1}\cdots\varepsilon_{k+n}}$$

and a similar calculation says that $m_{k+n}(I)$ is a non-negative martingale. Finally, for any $I \in \bigcup_{k\geq 0} \mathscr{F}_k$ the random quantity $m_n(I)$ is almost surely convergent.

Observing that the set $\bigcup_{k\geq 0} \mathscr{F}_k$ is countable, we can also say that almost surely, for any $I \in \bigcup_{k\geq 0} \mathscr{F}_k$, $m_n(I)$ is convergent and the conclusion is a consequence of Proposition 1.

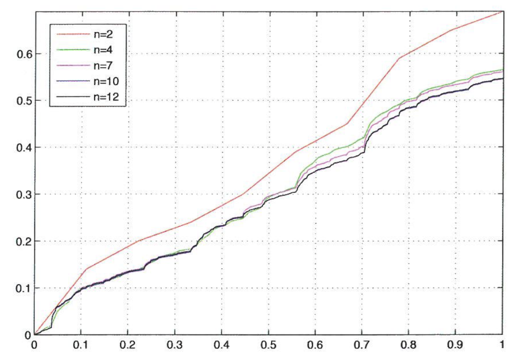

Fig. 3: The repartition function of the random measures m_2, m_4, m_7, m_{10} and m_{12} (from [9]).The total mass is not equal to 1.

3.2 Examples

3.2.1 Birth and death processes

We suppose in this example that the random variable W only takes the value 0 and another positive value. Let $p = 1 - P[W = 0]$. To ensure that $E[W] = 1$ we need to take $P\left[W = \frac{1}{p}\right] = p$. When $m \neq 0$, its support is a random Cantor set.

3.2.2 Log-normal cascades

This is the case where W is a log-normal random variable, that is $W = e^X$ where X follows a normal distribution with expectation m and variance σ^2. An easy calculation says that

$$\begin{aligned} E\left[e^X\right] &= \int e^x\, e^{-(x-m)^2/2\sigma^2}\, \frac{dx}{\sigma\sqrt{2\pi}} \\ &= \int e^{(m+\sigma u)}\, e^{-u^2/2}\, \frac{du}{\sqrt{2\pi}} \\ &= \int e^{-(u-\sigma)^2/2}\, e^{m+\sigma^2/2}\, \frac{du}{\sqrt{2\pi}} \\ &= e^{m+\sigma^2/2} \end{aligned}$$

In order to have $E[W]=1$ we need to choose $m=-\sigma^2/2$. In other words,

$$W = e^{\sigma N - \sigma^2/2}$$

where N follows a standard normal distribution.

3.3 The fundamental equations

Define $Y_n = m_n([0,1])$ as above. Then,

$$\begin{aligned} Y_{n+1} &= \ell^{-(n+1)} \sum_{\varepsilon_1\cdots\varepsilon_{n+1}} W_{\varepsilon_1} W_{\varepsilon_1\varepsilon_2}\cdots W_{\varepsilon_1\cdots\varepsilon_{n+1}} \\ &= \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j \left[\ell^{-n} \sum_{\varepsilon_2\cdots\varepsilon_{n+1}} W_{j\varepsilon_2}\cdots W_{j\cdots\varepsilon_{n+1}}\right] \end{aligned} \tag{5}$$

and the sequence (Y_n) is a solution in law of the equation

$$Y_{n+1} = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Y_n(j). \tag{6}$$

where the $Y_n(0),\ldots,Y_n(\ell-1)$ are independent copies of Y_n, and are independent to $W_0,\ldots,W_{\ell-1}$.

Taking the limit in the equality (5), the total mass $Y_\infty = m([0,1]$ is also a solution in law of the equation

$$Y_\infty = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Y_\infty(j) \tag{7}$$

where $Y_\infty(0),\ldots,Y_\infty(\ell-1)$ are independent copies of Y_∞, and are independent to $W_0,\ldots,W_{\ell-1}$.

Equations (6) and (7) are called the fundamental equations and will be very useful in the following.

3.4 Non-degeneracy

As proved in Theorem 1, the sequence $Y_n = m_n([0,1])$ is a non-negative martingale and we only know in the general case that $E[Y_\infty] \le 1$. In particular, the situation where $E[Y_\infty] = 0$ is possible and is called the degenerate case. The first natural problem related to the random measure m is then to find conditions that ensure that m is not almost surely equal to 0 (i.e. $E[Y_\infty] \neq 0$). An abstract answer is given by an equi-integrability property. We will see further a more concrete necessary and sufficient condition (Theorem 2) and more concrete sufficient conditions (Proposition 4 and Theorem 3).

Proposition 2. *Let m be a Mandelbrot cascade associated to a weight W. Denote as before $Y_n = m_n([0,1])$ and $Y_\infty = m([0,1])$. The following are equivalent*

1. $E[Y_\infty] = 1$
2. $E[Y_\infty] > 0$ *(i.e. $P[m([0,1]) \neq 0] > 0$)*
3. *The martingale (Y_n) is equi-integrable*

In that case, we say that the Mandelbrot cascade m is non-degenerate.

Proof. Suppose that 2. is true. Considering $Z = \frac{Y_\infty}{E[Y_\infty]}$, it follows that the fundamental equation

$$Z = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Z(j)$$

has a solution satisfying $E[Z] = 1$.

Iterating the fundamental equation, we get

$$Z = \frac{1}{\ell^n}\sum_{\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n} W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n} Z(\varepsilon_1\cdots\varepsilon_n)$$

in which the $Z(\varepsilon_1\cdots\varepsilon_n)$ are independant copies of Z, independent to the W_ε. Let $\mathscr{A}_n$ be again the σ-algebra generated by the W_ε, $\varepsilon \in \mathscr{M}_1 \cup \cdots \cup \mathscr{M}_n$. We get

$$E[Z|\mathscr{A}_n] = \frac{1}{\ell^n}\sum_{\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n} W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n} E[Z(\varepsilon_1\cdots\varepsilon_n)] = Y_n$$

and the martingale (Y_n) is equi-integrable.

The proof of $3 \Rightarrow 1$ is elementary. Indeed, if we suppose that the martingale (Y_n) is equi-integrable, it converges almost-surely and in L^1 to its limit Y_∞. In particular, $E[Y_\infty] = \lim_{n\to\infty} E[Y_n] = 1$.

Remark 5. In fact, the proof of Proposition 2 says that the condition of non degeneracy of the cascade m is equivalent to the existence of a non negative solution Z satisfying $E[Z] = 1$ for the fundamental equation

$$Z = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Z(j). \tag{8}$$

Remark 6. Equation (8) may have non-integrable solutions. For example, if $\ell = 2$ and $W = 1$, equation (8) becomes

$$Z = \frac{1}{2}(Z(1) + Z(2)).$$

If $Z(1)$ et $Z(2)$ are two independent Cauchy variables (with density $\frac{dz}{\pi(1+z^2)}$), then Z is also a Cauchy variable.

In the non-degenerate case, we only know that $P[m \neq 0] > 0$ almost surely. A natural question is then to ask if $P[m \neq 0] = 1$ almost surely. The answer to this question is easy.

Proposition 3. *Suppose that the Mandelbrot cascade m is non-degenerate. Then,*

$$P[m \neq 0] = 1 \quad \textit{if and only if} \quad P[W = 0] = 0.$$

Proof. Suppose that (Y_n) is equi-integrable. Let us write again the fundamental equation

$$Y_\infty = \frac{1}{\ell} \sum_{j=0}^{\ell-1} W_j Y_\infty(j).$$

Then

$$\begin{aligned} P[Y_\infty = 0] &= P\left[W_0 Y_\infty(0) = 0 \text{ and } \cdots \text{ and } W_{\ell-1} Y_\infty(\ell-1) = 0\right] \\ &= P[W Y_\infty = 0]^\ell \\ &= (1 - P[W \neq 0 \text{ and } Y_\infty \neq 0])^\ell \end{aligned}$$

If $r = P[W = 0]$, it follows that $P[Y_\infty = 0]$ is a fixed point of the function

$$f(x) = (r + (1-r)x)^\ell.$$

We know that $P[Y_\infty = 0] < 1$. The second fixed point of the function f is equal to 0 if and only if $r = 0$. The conclusion follows.

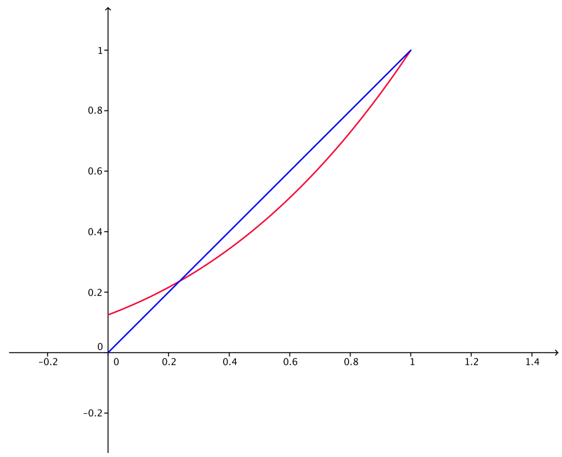

Fig. 4: The function f has two fixed points.

In the L^2 case it is easy to obtain a condition on the second order moment which gives non-degeneracy.

Proposition 4. *Suppose that $E[W^2] < +\infty$. The following are equivalent*

1. $E[W^2] < \ell$
2. *The sequence (Y_n) is bounded in L^2*
3. $0 < E[Y_\infty^2] < +\infty$

In particular, if 1. is true, the sequence (Y_n) is equi-integrable and the cascade m is non-degenerate.

Proof. Let us write the fundamental equation

$$Y_{n+1} = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Y_n(j).$$

We get

$$\begin{aligned} E[Y_{n+1}^2] &= \frac{1}{\ell^2}\left(\sum_{j=0}^{\ell-1} E[(W_j^2 Y_n(j))^2] + \sum_{i\neq j} E[W_i Y_n(i) W_j Y_n(j)]\right) \\ &= \frac{1}{\ell}E[W^2]E[Y_n^2] + \frac{1}{\ell^2}\times\ell(\ell-1) \end{aligned}$$

It follows that the sequence $(E[Y_n^2])$ is bounded if and only if the common ratio $\frac{1}{\ell}E[W^2]$ is lower than 1. So 1. is equivalent to 2.

2. $\Rightarrow$ 3. Suppose that the sequence (Y_n) is bounded in L^2. We know that the martingale (Y_n) converges in L^2. In particular

$$E[Y_\infty^2] = \lim_{n\to+\infty} E[Y_n^2] < +\infty.$$

Moreover the sequence (Y_n^2) is a submartingale and the sequence $(E[Y_n^2])$ is non-decreasing. It follows that $E[Y_\infty^2] > 0$. which gives 3.

3. $\Rightarrow$ 1. Suppose that $0 < E[Y_\infty^2] < +\infty$. According to Proposition 2, the martingale (Y_n) is non-degenerate. In particular, $E[Y_\infty] = 1$. The fundamental equation says that

$$Y_\infty = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Y_\infty(j).$$

It follows that

$$\begin{aligned} E[Y_\infty^2] &= \frac{1}{\ell^2}\left(\sum_{j=0}^{\ell-1} E[(W_j^2 Y_\infty(j))^2] + \sum_{i\neq j} E[W_i Y_\infty(i) W_j Y_\infty(j)]\right) \\ &= \frac{1}{\ell}E[W^2]E[Y_\infty^2] + \frac{1}{\ell^2}\times \ell(\ell-1) \end{aligned}$$

so that

$$\left(\ell - E[W^2]\right)E[Y_\infty^2] = \ell - 1.$$

In particular, $E[W^2] < \ell$.

A generalization of Proposition 4 in the case where the weight W admits an L^q moment is possible. This is the object of Section 4. Nevertheless, we can also give a characterization on the non-degeneracy of the cascade m. It is given in terms of the $L\log L$ moment of the weight W.

Theorem 2 (Kahane, 1976, [16]). *Let m be a Mandelbrot cascade associated to a weight W. The following are equivalent*

1. *The cascade m is non-degenerate*
2. *The martingale (Y_n) is equi-integrable*
3. *$E[W\log W] < \log \ell$*

We begin with a geometric interpretation of the condition $E[W\log W] < \log\ell$. Let us introduce the structure function τ, which is defined by

$$\tau(q) = \log_\ell E\left[\sum_{j=0}^{\ell-1}\left[\frac{1}{\ell}W_j\right]^q\right] = \log_\ell\left(E[W^q]\right) - (q-1). \tag{9}$$

Such a formula makes sense when $0 \le q \le 1$ (and perhaps for other values of q) and we always use the convention $0^q = 0$. In particular,

$$\tau(0) = 1 + \log_\ell(P[W\neq 0])$$

which will be seen as the almost sure Hausdorff dimension of the closed support of the measure m.

The function τ is continuous and convex on $[0,1]$ and we will show that

$$\tau'(1^-) = E[W\log_\ell W] - 1 \le +\infty.$$

It follows that Condition 3 in Theorem 2 is equivalent to $\tau'(1^-) < 0$.

Set $\phi(q) = E[W^q]$. In order to prove that $\tau'(1^-) = E[W\log_\ell W] - 1$, we have to understand why we can write $\phi'(1^-) = E[W\log W]$, with a possible value equal to $+\infty$. Indeed, using the dominated convergence theorem, we have $\phi'(q) = E[W^q \log W]$ when $0 \le q < 1$. On one hand, the convexity of the function ϕ allows us to write

$$\lim_{q\to 1^-} \phi'(q) = \phi'(1^-) \le +\infty.$$

On the other hand,

$$\phi'(q) = E[W^q\log W] = E[W^q \log W\, \mathbb{1}_{\{W<1\}}] + E[W^q\log W\, \mathbb{1}_{\{W\ge 1\}}].$$

The non-negative quantity $E[W^q\log W\, \mathbb{1}_{\{W\ge 1\}}]$ increases to $E[W\log W\, \mathbb{1}_{\{W\ge 1\}}]$ and by the dominated convergence theorem, the quantity $E[W^q\log W\, \mathbb{1}_{\{W<1\}}]$ goes to $E[W\log W\, \mathbb{1}_{\{W<1\}}]$. The formula $\phi'(1^-) = E[W\log W]$ follows.

Proof (Proof of Theorem 2). According to Proposition 2, we just have to prove that Conditions 2 and 3 are equivalent.

Step 1. $\tau'(1^-) \le 0$ *is a necessary condition.*

Suppose that the sequence (Y_n) is equi-integrable. Then, the fundamental equation

$$Z = \frac{1}{\ell}\sum_{j=0}^{\ell-1} W_j Z(j)$$

has a non-negative solution with expectation equal to 1. If $0 < q \le 1$, the function $x \mapsto x^q$ is subadditive (that is satisfies $(a+b)^q \le a^q + b^q$). We get

$$E\left[\ell^q Z^q\right] \le \sum_{j=0}^{\ell-1} E[W_j^q Z(j)^q] = \ell E[W^q]E[Z^q].$$

Observe that $E[Z^q] > 0$, so that

$$\ell^q \le \ell E[W^q].$$

Finally, $\tau(q) \ge 0$ if $q \le 1$ and $\tau'(1^-) \le 0$.

Step 2. *More precisely,* $\tau'(1^-) < 0$ *is a necessary condition.*

We have to improve the previous result. We need a lemma which gives a more precise estimate than the subadditivity of the function $x \mapsto x^q$.

Lemma 1. *If* $0 < q < 1$ *and if* $0 < y \le x$*, then* $(x+y)^q \le x^q + qy^q$.

Proof. Using homogeneity, we may assume that $y = 1$ and $x \geq 1$. The inequality $(x+1)^q - x^q \leq q$ is then an easy consequence of the mean value theorem.

We also need the following elementary lemma on random variables.

Lemma 2. *Let X and X' be two non-negative i.i.d. random variables such that $E[X] > 0$. There exists $\delta > 0$ such that for any $q \in [0,1]$, $E[X^q 1\!\!1_{X' \geq X}] \geq \delta E[X^q]$.*

Proof. We claim that for any $q \in [0,1]$, $E[X^q 1\!\!1_{X' \geq X}] > 0$. Indeed, if $E[X^q 1\!\!1_{X' \geq X}] = 0$ for some q, then X is almost surely equal to 0 on the set $\{X' \geq X\}$. By symmetry, X' is almost surely equal to 0 on the set $\{X \geq X'\}$. Then $XX' = 0$ almost surely, which is in contradiction with $E[XX'] = E[X]E[X'] > 0$. Moreover, the functions $q \mapsto E[X^q 1\!\!1_{X' \geq X}]$ and $q \mapsto E[X^q]$ are continuous on $[0,1]$ and the conclusion follows.

We can now prove that $\tau'(1^-) < 0$ is a necessary condition. Let

$$A = \{W_1 Z(1) \geq W_0 Z(0)\}.$$

Using subadditivity of $x \mapsto x^q$ and Lemma 1, we have :

$$\begin{cases} (\ell Z)^q \leq \sum_{j=0}^{\ell-1} (W_j Z(j))^q \\ (\ell Z)^q \leq q(W_0 Z(0))^q + \sum_{j=1}^{\ell-1} (W_j Z(j))^q \qquad \text{on } A. \end{cases}$$

Then,

$$\begin{aligned} E\left[(\ell Z)^q\right] &= E\left[(\ell Z)^q 1\!\!1_A\right] + E\left[(\ell Z)^q 1\!\!1_{A^c}\right] \\ &\leq qE\left[(W_0 Z(0))^q 1\!\!1_A\right] + \sum_{j=1}^{\ell-1} E\left[(W_j Z(j))^q 1\!\!1_A\right] + \sum_{j=0}^{\ell-1} E\left[(W_j Z(j))^q 1\!\!1_{A^c}\right] \\ &= (q-1)E\left[(W_0 Z(0))^q 1\!\!1_A\right] + \ell E[W^q]E[Z^q] \\ &\leq (q-1)\delta E[W^q]E[Z^q] + \ell E[W^q]E[Z^q]. \end{aligned}$$

We get

$$\ell^{1-q} E[W^q] \geq \frac{1}{1 + (q-1)\frac{\delta}{\ell}}$$

so that

$$\tau(q) \geq -\log_\ell \left(1 + (q-1)\frac{\delta}{\ell}\right).$$

Finally,

$$\tau'(1^-) \leq -\frac{\delta}{\ell \log \ell} < 0.$$

Step 3. *$\tau'(1^-) < 0$ is a sufficient condition.*

We suppose that $E[W\log W] < \log \ell$ (i.e. $\tau'(1^-) < 0$) and, according to Proposition 2, we want to prove that $E[Y_\infty] > 0$. Now, we need a precise lower bound of quantities such as $\left(\sum_{j=1}^{\ell} x_j\right)^q$. We will use the following lemma.

Lemma 3. *If* $x_1, \ldots, x_\ell \geq 0$*, and if* $0 < q \leq 1$*, then*

$$\left(\sum_{j=1}^{\ell} x_j\right)^q \geq \sum_{j=1}^{\ell} x_j^q - 2(1-q)\sum_{i<j}(x_i x_j)^{q/2}. \tag{10}$$

Suppose first that the lemma is true and let us write again the fundamental equation

$$\ell Y_n = \sum_{j=0}^{\ell-1} W_j Y_{n-1}(j).$$

Lemma 3 ensures that

$$(\ell Y_n)^q \geq \sum_{j=0}^{\ell-1}(W_j Y_{n-1}(j))^q - 2(1-q)\sum_{i<j}(W_i Y_{n-1}(i) W_j Y_{n-1}(j))^{q/2}.$$

Taking the expectation and using that Y_n^q is a supermartingale, we get

$$\begin{aligned}\ell^q E[Y_n^q] &\geq \ell E[W^q] E\left[Y_{n-1}^q\right] - \ell(\ell-1)(1-q)E[W^{q/2}]^2 \times E\left[Y_{n-1}^{q/2}\right]^2 \\ &\geq \ell E[W^q] E\left[Y_n^q\right] - \ell(\ell-1)(1-q)E[W^{q/2}]^2 \times E\left[Y_{n-1}^{q/2}\right]^2\end{aligned}$$

Finally,

$$\begin{aligned}E\left[Y_n^q\right](\ell^{\tau(q)}-1) &= E\left[Y_n^q\right](\ell^{1-q}E[W^q]-1) \\ &\leq \ell^{1-q}(\ell-1)(1-q)E\left[Y_{n-1}^{q/2}\right]^2 \times E\left[W^{q/2}\right]^2 \\ &\leq \ell^{1-q}(\ell-1)(1-q)E\left[Y_{n-1}^{q/2}\right]^2 \times E\left[W^q\right].\end{aligned}$$

Dividing by $1-q$ and taking the limit when q goes to 1^-, we get

$$1 \times (-\tau'(1^-) \times \log \ell) \leq (\ell-1)E\left[Y_{n-1}^{1/2}\right]^2 \times 1$$

which gives that $E\left[Y_{n-1}^{1/2}\right] \geq C > 0$. Observing that the supermartingale $\left(Y_n^{1/2}\right)$ converges almost surely to $Y_\infty^{1/2}$ and is bounded in L^2, we conclude that $\left(Y_n^{1/2}\right)$ is equi-integrable and converges in L^1. In particular,

$$E[Y_\infty^{1/2}] = \lim_{n \to +\infty} E[Y_n^{1/2}] \geq C.$$

So $E[Y_\infty] > 0$ and the cascade m is non-degenerate.

Let us now finish this part with the proof of Lemma 3. Suppose first that $\ell = 2$. By homogeneity the inequality is equivalent to

$$\left(x + x^{-1}\right)^q \geq x^q + x^{-q} - 2(1-q)$$

for any $x > 0$. Let

$$\varphi(x) = x^q + x^{-q} - \left(x + x^{-1}\right)^q.$$

If $0 < x \leq 1$, we have

$$\begin{aligned}
\varphi'(x) &= qx^{-(q+1)}\left[x^{2q} - 1 + \left(1 - x^2\right)\left(1 + x^2\right)^{q-1}\right] \\
&\geq qx^{-(q+1)}\left[x^{2q} - 1 + \left(1 - x^2\right)\left(1 + (q-1)x^2\right)\right] \\
&= qx^{-(q+1)}\left[x^{2q} + (q-2)x^2 - (q-1)x^4\right].
\end{aligned}$$

By studying the function $\psi(y) = y^q + (q-2)y - (q-1)y^2$, it is then easy to see that $\psi(y) \geq 0$ for any $y \in [0,1]$.

Finally, for any $x > 0$,

$$\varphi(x) = \varphi(x^{-1}) \leq \varphi(1) = 2 - 2^q \leq 2\ln 2(1-q) \leq 2(1-q).$$

and the proof is done in the case $\ell = 2$.

The general case is easily obtained by induction on ℓ, using once again that the function $x \mapsto x^{q/2}$ is subadditive if $0 < q < 1$.

Remark 7. In fact, the proof of Lemma 3 says that the constant $-2(1-q)$ in (10) can replaced by $-2\ln 2(1-q)$ which is the optimal one.

Example 1 (Birth and death processes). Suppose that the law of the random wariable W is given by $dP_W = (1-p)\delta_0 + p\delta_{\frac{1}{p}}$. Then

$$E\left[W^q\right] = 0P[W=0] + \left(\frac{1}{p}\right)^q P\left[W = \frac{1}{p}\right] = p^{1-q}$$

and

$$\tau(q) = \log_\ell(E[W^q]) - (q-1) = (1-q) \times (1 + \log_\ell p).$$

The cascade is non-degenerate if and only if $p > 1/\ell$, that is if and only if $P[W = 0] < 1 - \frac{1}{\ell}$. In that case, the box dimension of the closed support of the measure m is almost surely $d = \tau(0) = 1 + \log_\ell p$ on the set $\{m \neq 0\}$.

Example 2 (Log-normal cascades). Suppose that

$$W = e^{\sigma N - \sigma^2/2}$$

where N follows a standard normal distribution.

$$\begin{aligned} E[W^q] &= \int e^{q(\sigma x - \sigma^2/2)} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \\ &= \int e^{-(x-q\sigma)^2/2} e^{q^2\sigma^2/2} e^{-q\sigma^2/2} \frac{dx}{\sqrt{2\pi}} \\ &= e^{q^2\sigma^2/2} e^{-q\sigma^2/2} \end{aligned}$$

and

$$\tau(q) = \log_\ell(E[W^q]) - (q-1) = \frac{\sigma^2}{2\ln\ell}(q^2 - q) - (q-1).$$

The cascade is non-degenerate if and only if $\sigma^2 < 2\log\ell$.

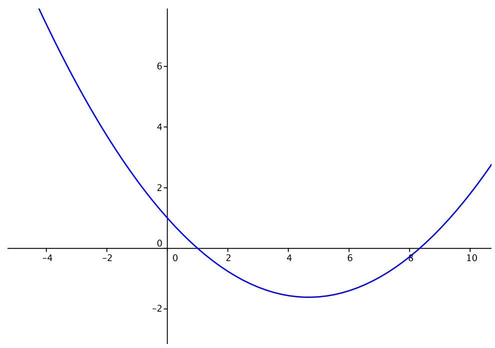

Fig. 5: The graph of the structure function τ for a non-degenerate log-normal cascade.

4 On the existence of moments for the random variable Y_∞

In Proposition 4, we obtained a necessary and sufficient condition for the martingale (Y_n) to be bounded in L^2. This condition can be generalized in the following way.

Theorem 3 (Kahane, 1976, [16]). *Let $q > 1$. Suppose that $E[W^q] < +\infty$. The following are equivalent*

1. $E[W^q] < \ell^{q-1}$ *(i.e. $\tau(q) < 0$)*
2. *The sequence (Y_n) is bounded in L^q*
3. $0 < E[Y_\infty^q] < +\infty$

In particular, if 1. is true, the sequence (Y_n) is equi-integrable and the cascade m is non-degenerate.

Remark 8. The condition $E[W^q] < \ell^{q-1}$ is equivalent to $\tau(q) < 0$. The graph of the function τ allows us to determine the set of values of $q > 1$ such that (Y_n) is bounded in L^q (see Figure 5 for the case of log-normal cascades).

Proof (Proof of Theorem 3).

2. $\Rightarrow$ 3. If (Y_n) is bounded in L^q, the martingale (Y_n) converges in L^q. In particular,

$$E[Y_\infty^q] = \lim_{n\to+\infty} E[Y_n^q] < +\infty.$$

Moreover the sequence (Y_n^q) is a submartingale and the sequence $(E[Y_n^q])$ is non-decreasing. It follows that $E[Y_\infty^q] > 0$. which gives 3.

3. $\Rightarrow$ 1. Suppose that $0 < E[Y_\infty^q] < +\infty$ and write the fundamental equation

$$(\ell Y_\infty)^q = \left(\sum_{j=0}^{\ell-1} W_j Y_\infty(j)\right)^q. \tag{11}$$

Recall that the function $x \mapsto x^q$ is super additive. More precisely, $(a+b)^q \geq a^q + b^q$ with equality if and only if $ab = 0$.

Taking the expectation in (11) we get

$$E[\ell^q Y_\infty^q] \geq \sum_{j=0}^{\ell-1} E\left[(W_j Y_\infty(j))^q\right] = \ell E[W^q] E[Y_\infty^q]$$

and the equality case would imply that $\prod_{j=0}^{\ell-1}(W_j Y_\infty(j)) = 0$ almost surely, which is impossible by independence. In particular, $E[W^q] < \ell^{q-1}$.

1. $\Rightarrow$ 2. This is the difficult part of the theorem. Let us begin with the easier case $1 < q \leq 2$. Recall once again the fundamental equation

$$\ell Y_{n+1} = \sum_{j=0}^{\ell-1} W_j Y_n(j).$$

The function $x \mapsto x^{q/2}$ is sub-additive so that

$$\begin{aligned}(\ell Y_{n+1})^q &\leq \left(\sum_{j=0}^{\ell-1} (W_j Y_n(j))^{q/2}\right)^2 \\ &= \sum_{j=0}^{\ell-1} (W_j Y_n(j))^q + \sum_{i\neq j} (W_i Y_n(i))^{q/2} (W_j Y_n(j))^{q/2}\end{aligned}$$

Taking the expectation, and using that (Y_n^q) is a submartingale, we get

$$\begin{aligned}\ell^q E\left[Y_{n+1}^q\right] &\leq \ell E\left[Y_n^q\right] E\left[W^q\right] + \ell(\ell-1) E[W^{q/2}]^2 E\left[Y_n^{q/2}\right]^2 \\ &\leq \ell E\left[Y_n^q\right] E\left[W^q\right] + \ell(\ell-1) E[W]^q E[Y_n]^q \\ &= \ell E\left[Y_{n+1}^q\right] E\left[W^q\right] + \ell(\ell-1)\end{aligned}$$

Finally,

$$E[Y_{n+1}^q] \le \frac{\ell - 1}{\ell^{q-1} - E[W^q]}. \tag{12}$$

Suppose now that $k < q \le k+1$ where $k \ge 2$ is an integer and write

$$(\ell Y_{n+1})^q \le \left(\sum_{j=0}^{\ell-1} (W_j Y_n(j))^{q/(k+1)} \right)^{k+1} = \sum_{j=0}^{\ell-1} (W_j Y_n(j))^q + T$$

where the quantity T is a sum of $\ell^{k+1} - \ell$ terms of the form

$$(W_{j_1} Y_n(j_1))^{\alpha_1 q/(k+1)} \times \cdots \times (W_{j_p} Y_n(j_p))^{\alpha_p q/(k+1)}$$

with $p \ge 2$ and $\alpha_1 + \cdots + \alpha_p = k+1$. The expectation of such a term satisfies

$$\begin{aligned} & E\left[(W_{j_1} Y_n(j_1))^{\alpha_1 q/(k+1)} \times \cdots \times (W_{j_p} Y_n(j_p))^{\alpha_p q/(k+1)} \right] \\ & \le E\left[(W_{j_1} Y_n(j_1))^k \right]^{\alpha_1 q/k(k+1)} \times \cdots \times E\left[(W_{j_p} Y_n(j_p))^k \right]^{\alpha_p q/k(k+1)} \\ & = \left(E\left[W^k\right] E\left[Y_n^k\right] \right)^{q/k} \end{aligned}$$

so that

$$\ell^q E\left[Y_{n+1}^q\right] \le \ell E\left[Y_n^q\right] E\left[W^q\right] + (\ell^{k+1} - \ell) \left(E\left[W^k\right] E\left[Y_n^k\right] \right)^{q/k}.$$

Using that (Y_n^q) is a submartingale, we get

$$E\left[Y_{n+1}^q\right] (\ell^{q-1} - E[W^q]) \le (\ell^k - 1) \left(E[W^k] E\left[Y_n^k\right] \right)^{q/k} \tag{13}$$

which is the generalization of (12). It follows that (Y_n) is bounded in L^q as soon as (Y_n) is bounded in L^k.

Let us finally observe that the hypothesis $E[W^q] < \ell^{q-1}$ (i.e.$\tau(q) < 0$) implies that $E[W^t] < \ell^{t-1}$ (i.e.$\tau(t) < 0$) for any t such that $1 < t < q$. Replacing q by $j+1$ in (13), we also have

$$E\left[Y_{n+1}^{j+1}\right] (\ell^j - E[W^{j+1}]) \le (\ell^j - 1) \left(E[W^j] E\left[Y_n^j\right] \right)^{q/j}$$

for any integer j such that $2 \le j < k$. Step by step we get that (Y_n) is bounded in L^2, L^3,..., L^k, L^q.

5 On the dimension of non-degenerate cascades

The Mandelbrot cascade is almost-surely a unidimensional measure as was proved by Peyrière in [16].

Theorem 4 (Peyrière, 1976, [16]). *Suppose that* $0 < E[Y_\infty \log Y_\infty] < +\infty$*. Then, almost surely,*

$$\lim_{n\to+\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = 1 - E[W \log_\ell W] \quad dm - \textit{almost everywhere}$$

Let us recall that it is possible that $m = 0$ with positive probability. So, the good way to rewrite Theorem 4 is

Corollary 1. *Suppose that* $0 < E[Y_\infty \log Y_\infty] < +\infty$*. Almost surely on* $\{m \neq 0\}$ *we have :*

1. *There exists a Borel set E such that*

$$\dim(E) = 1 - E[W \log_\ell W] \quad \textit{and} \quad m([0,1] \setminus E) = 0$$

2. *If* $\dim(F) < 1 - E[W \log_\ell W]$*, then* $m(F) = 0$*.*

It follows that

$$\dim_*(m) = \dim^*(m) = 1 - E[W \log_\ell W]$$

where $\dim_*(m)$ *and* $\dim^*(m)$ *are respectively the lower and the upper dimension of the measure m as defined on* (2)*.*
The measure m is unidimensional with dimension

$$\dim(m) = 1 - E[W \log_\ell W].$$

Remark 9. The condition $0 < E[Y_\infty \log Y_\infty] < +\infty$ is stronger than $E[W \log W] < \log \ell$ which ensures the non-degeneracy of the cascade m. Indeed, suppose that $0 < E[Y_\infty \log Y_\infty] < +\infty$ and observe that the function $t \mapsto t \log t$ is superadditive. More precisely, $(a+b)\log(a+b) \geq a \log a + b \log b$ with equality if and only if $ab = 0$. The fundamental equation implies that

$$\ell Y_\infty \log(\ell Y_\infty) \geq \sum_{j=0}^{\ell-1} (W_j Y_\infty(j)) \log(W_j Y_\infty(j)).$$

Taking the expectation,

$$E[\ell Y_\infty \log(\ell Y_\infty)] \geq \ell E[(W Y_\infty) \log(W Y_\infty)]$$

and the equality case would imply that $\prod_{j=0}^{\ell-1}(W_j Y_\infty(j)) = 0$ almost surely, which is impossible by independence.

Finaly,

$$E[Y_\infty \log(\ell Y_\infty)] > E[W \log W] E[Y_\infty] + E[Y_\infty \log Y_\infty] E[W]$$

so that

$$E[Y_\infty] \log \ell > E[W \log W] E[Y_\infty].$$

It follows that $E[W \log W] < \log \ell$ and the cascade is non-degenerate.

Remark 10. Under the hypothesis of Theorem 4 and Corollary 1, we have the following relation :

$$\dim(m) = 1 - E[W \log_\ell W] = -\tau'(1^-).$$

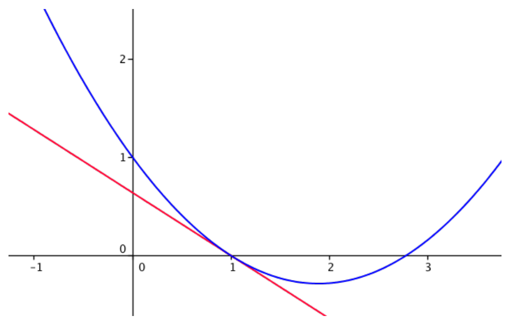

Fig. 6: The graph of the structure function τ and its tangente at point 1. The quantity $\dim(m)$ is the intercept of the tangente.

In particular, $0 < \dim(m) \leq 1$ almost surely on the event $\{m \neq 0\}$.

Example 3 (Birth and death processes). Suppose that $dP_W = (1-p)\delta_0 + p\delta_{\frac{1}{p}}$ with $p > 1/\ell$. Then,

$$\tau(q) = (1-q) \times (1 + \log_\ell p) \qquad \text{and} \qquad \dim(m) = 1 + \log_\ell p$$

almost surely on the event $\{m \neq 0\}$.

Let $\mathscr{M}_n^*$ be the set of words $\varepsilon \in \mathscr{M}_n$ such that $m(I_\varepsilon) > 0$. The closed support of m is nothing else but the Cantor set

$$\operatorname{supp}(m) = K = \bigcap_{n \geq 1} \bigcup_{\varepsilon \in \mathscr{M}_n^*} \overline{I_\varepsilon}.$$

Theorem 4 and Corollary 1 ensure that almost surely on the event $\{m \neq 0\}$, the Hausdorff dimension of K satisfies $\dim(K) \geq 1 + \log_\ell p$ which is also known as the box dimension of K. finally,

$$\dim(K) = 1 + \log_\ell p$$

almost surely on the event $\{m \neq 0\} = \{K \neq \emptyset\}$.

Example 4 (Log-normal cascades). Suppose that N follows a standard normal distribution and $W = e^{\sigma N - \sigma^2/2}$ with $\sigma^2 < 2 \log \ell$. We know that

$$\tau(q) = \frac{\sigma^2}{2\log \ell}(q^2 - q) + 1 - q$$

and we find

$$\dim(m) = 1 - \frac{\sigma^2}{2\log \ell} \quad \text{almost surely.}$$

Proof (Proof of Theorem 4). As observed before, under the hypothesis

$$0 < E[Y_\infty \log Y_\infty] < +\infty,$$

the cascade m is non-degenerate. In particular, $E[Y_\infty] = 1$.

We first need to precisely define the sentence "almost surely dm-almost everywhere". Let $\tilde{\Omega} = \Omega \times [0,1]$ endowed with the product σ-algebra. Define the measure Q by

$$Q[A] = E\left[\int \mathbb{1}_A \, dm\right].$$

Observe that the measure m depends on $\omega \in \Omega$ so that Q is not a product measure. Nevertheless,

$$Q[\tilde{\Omega}] = E\left[\int dm\right] = E[Y_\infty] = 1$$

so that Q is a probability measure. If a property is true on a set $A \subset \tilde{\Omega}$ satisfying $Q[A] = 1$, then, almost surely, the properly is true dm-almost everywhere. The measure Q is now very often referred as the Peyrière measure.

Recall that the measure m is constructed as the weak limit of the sequence $m_n = f_n \lambda$ where

$$f_n = \sum_{\varepsilon_1 \cdots \varepsilon_n \in \mathscr{M}_n} W_{\varepsilon_1} W_{\varepsilon_1 \varepsilon_2} \cdots W_{\varepsilon_1 \cdots \varepsilon_n} \mathbb{1}_{I_{\varepsilon_1 \cdots \varepsilon_n}}.$$

The proof of Theorem 4 is an easy consequence of the two following lemmas.

Lemma 4. *Suppose that $E[W \log W] < \log \ell$. Then, almost surely,*

$$\lim_{n \to +\infty} \frac{\log f_n(x)}{n} = E[W \log W] \qquad \text{for } dm-\text{almost every } x.$$

Lemma 5. *Suppose that $E[Y_\infty \log Y_\infty] < +\infty$. Let $\mu_n = \frac{1}{f_n} m$. Then, almost surely,*

$$\lim_{n \to +\infty} \frac{\log \mu_n(I_n(x))}{n} = -\log \ell \qquad \text{for } dm-\text{almost every } x.$$

Suppose first that Lemma 4 and Lemma 5 are true and recall that $dm = f_n d\mu_n$. The density f_n is constant on any interval of the n^{th} generation, so that

$$m(I_n(x)) = \int_{I_n(x)} f_n(y) d\mu_n(y) = f_n(x)\mu_n(I_n(x)).$$

It follows that

$$\begin{aligned}\frac{\log(m(I_n(x))}{\log|I_n(x)|} &= \frac{\log f_n(x)+\log\mu_n(I_n(x))}{-n\log\ell}\\ &\to -E[W\log_\ell W]+1\end{aligned}$$

almost surely dm-almost everywhere.

Proof (Proof of Lemma 4). Let us write

$$f_n = g_1\times\cdots\times g_n \qquad \text{where} \qquad g_n = \sum_{\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n} W_{\varepsilon_1\cdots\varepsilon_n}1\!\!1_{I_{\varepsilon_1\cdots\varepsilon_n}}.$$

We get

$$\frac{\log f_n}{n} = \frac{1}{n}\sum_{k=1}^{n}\log g_k$$

and Lemma 4 will be a consequence of the strong law of large numbers in the space $\tilde{\Omega}$ associated to the probability Q.

Let us calculate the law of the random variable $g_n = \sum_{\varepsilon\in\mathscr{M}_n} W_\varepsilon 1\!\!1_{I_\varepsilon}$. If $\mathbb{E}$ is the expectation related to the probability Q and if ϕ is bounded and measurable,

$$\begin{aligned}\mathbb{E}[\phi(g_n)] &= E\left[\int\sum_{\varepsilon\in\mathscr{M}_n}\phi(W_\varepsilon)1\!\!1_{I_\varepsilon}\,dm\right]\\ &= \sum_{\varepsilon\in\mathscr{M}_n}E\left[\phi(W_\varepsilon)m(I_\varepsilon)\right]\end{aligned}$$

Moreover, if $k\geq 0$, using the independence properties,

$$\begin{aligned}E\left[\phi(W_\varepsilon)m_{n+k}(I_\varepsilon)\right] &= \sum_{\alpha_1\cdots\alpha_k\in\mathscr{M}_k}E\left[\phi(W_\varepsilon)\ell^{-(n+k)}W_{\varepsilon_1}\cdots W_\varepsilon W_{\varepsilon\alpha_1}\cdots W_{\varepsilon\alpha_1\cdots\alpha_k}\right]\\ &= \ell^{-n}E\left[\phi(W)W\right].\end{aligned}$$

Taking the limit, we get

$$\mathbb{E}[\phi(g_n)] = E[\phi(W)W]. \tag{14}$$

Equation (14) remains true if ϕ is such that $E\left[|\phi(W)W|\right]<+\infty$. In particular, the random variables g_n have the same law and $\log(g_n)$ are integrable with respect to Q.

The independence of the sequence (g_n) is obtained in a similar way. If $\phi_1,\ldots,\phi_n$ are bounded and measurable, we can also write

$$\begin{aligned}\mathbb{E}\left[\phi_1(g_1)\cdots\phi_n(g_n)\right] &= \sum_{\varepsilon\in\mathscr{M}_n}E\left[\phi_1(W_{\varepsilon_1})\cdots\phi_n(W_{\varepsilon_1\cdots\varepsilon_n})m(I_\varepsilon)\right]\\ &= \cdots\\ &= E[\phi_1(W)W]\times\cdots\times E[\phi_n(W)W]\\ &= \mathbb{E}\left[\phi_1(g_1)\right]\times\cdots\times\mathbb{E}\left[\phi_n(g_n)\right]\end{aligned}$$

and the independence follows. Finally, the strong law of large numbers gives

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\log g_k=\mathbb{E}[\log g_1]=E[W\log W]\quad dQ-\text{almost surely,}$$

which says that almost surely,

$$\lim_{n\to+\infty}\frac{f_n(x)}{n}=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\log g_k(x)=E[W\log W]\quad \text{for } dm-\text{almost every } x.$$

Remark 11 (On the importance of the order of the quantifiers). Let $x\in[0,1]$ and $\varepsilon_1\dots\varepsilon_n\dots$ such that $x\in I_{\varepsilon_1\cdots\varepsilon_n}$ for any n. We have

$$\frac{\log(f_n(x))}{n}=\frac{1}{n}(\log W_{\varepsilon_1}+\cdots+\log W_{\varepsilon_1\cdots\varepsilon_n}).$$

Using the strong law of large numbers, we get:

$$\text{For any } x\in[0,1],\ \text{almost surely,}\quad \lim_{n\to+\infty}\frac{\log(f_n(x))}{n}=E[\log W]$$

which is different of the conclusion of Lemma 4 !

Proof (Proof of Lemma 5). Let us begin with a comment on the definition of the measure μ_n. The function f_n is constant on any interval of the n^{th} generation. Moreover, if f_n is equal to zero on some interval I of the n^{th} generation, then $m(I)=0$. Finally, $f_n\neq 0$ dm-almost surely and $\mu_n=\frac{1}{f_n}m$ is well defined. We can also write $m=f_n\mu_n$ and if $\varepsilon=\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n$,

$$m(I_\varepsilon)=W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n}\,\mu_n(I_\varepsilon).$$

We claim that $\mu_n(I_\varepsilon)$ is independent to $W_{\varepsilon_1},\dots,W_{\varepsilon_1\cdots\varepsilon_n}$ and has the same distribution as $\ell^{-n}Y_\infty$. Indeed,

$$m_{n+k}=f_n(g_{n+1}\cdots g_{n+k})\,d\lambda$$

and

$$\mu_n=\lim_{k\to+\infty}(g_{n+1}\cdots g_{n+k})\,d\lambda.$$

In particular,

$$\mu_n(I_\varepsilon)=\lim_{k\to+\infty}\int_{I_\varepsilon}g_{n+1}(x)\cdots g_{n+k}(x)\,d\lambda(x)$$

is clearly independent to $W_{\varepsilon_1},\dots,W_{\varepsilon_1\cdots\varepsilon_n}$ and an easy calculation gives that it has the same distribution as $\ell^{-n}Y_\infty$.

Using the previous remark, we get

$$\begin{aligned}\mathbb{E}\left[(\ell^n\mu_n(I_n(x)))^{-1/2}\right] &= E\left[\ell^{-n/2}\int \mu_n(I_n(x))^{-1/2}dm(x)\right]\\ &= \ell^{-n/2}\sum_{\varepsilon\in\mathscr{M}_n} E\left[\mu_n(I_\varepsilon)^{-1/2}m(I_\varepsilon)\right]\\ &= \ell^{-n/2}\sum_{\varepsilon\in\mathscr{M}_n} E\left[W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n}\right]E\left[\mu_n(I_\varepsilon)^{1/2}\right]\\ &= E\left[Y_\infty^{1/2}\right].\end{aligned}$$

It follows that

$$\mathbb{E}\left[\sum_{n\geq 1}\frac{1}{n^2}\left(\ell^n\mu_n(I_n(x))\right)^{-1/2}\right] < +\infty.$$

In particular, dQ-almost surely, $\ell^n\mu_n(I_n(x)) \geq 1/n^4$ if n is large enough and we can conclude that almost surely,

$$\liminf_{n\to+\infty}\frac{\log\left(\ell^n\mu_n(I_n(x))\right)}{n} \geq 0 \quad dm-\text{almost everywhere.}$$

In other words, almost surely,

$$\liminf_{n\to+\infty}\left(\frac{\log\left(\mu_n(I_n(x))\right)}{n}\right) \geq -\log\ell \quad dm-\text{almost everywhere.}$$

We have now to prove that almost surely,

$$\limsup_{n\to+\infty}\left(\frac{\log\left(\mu_n(I_n(x))\right)}{n}\right) \leq -\log\ell \quad dm-\text{almost everywhere.}$$

Recall that $m(I_\varepsilon) = W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n}\mu_n(I_\varepsilon)$ with independence properties. If $\alpha > 0$,

$$\begin{aligned}Q[\ell^n\mu_n(I_n(x)) > \alpha^n] &= E\left[\int 1\!\!1_{\{\ell^n\mu_n(I_n(x))>\alpha^n\}}(x)\,dm(x)\right]\\ &= \sum_{\varepsilon\in\mathscr{M}_n} E\left[\int_{I_\varepsilon} 1\!\!1_{\{\ell^n\mu_n(I_\varepsilon)>\alpha^n\}}(x)\,dm(x)\right]\\ &= \sum_{\varepsilon\in\mathscr{M}_n} E\left[m(I_\varepsilon)\,1\!\!1_{\{\ell^n\mu_n(I_\varepsilon)>\alpha^n\}}\right]\\ &= \sum_{\varepsilon\in\mathscr{M}_n} E[W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n}]E\left[\mu_n(I_\varepsilon)\,1\!\!1_{\{\ell^n\mu_n(I_\varepsilon)>\alpha^n\}}\right]\\ &= E\left[Y_\infty\,1\!\!1_{\{Y_\infty>\alpha^n\}}\right]\end{aligned}$$

In particular,

$$\sum_{n\geq 1} Q[\ell^n \mu_n(I_n(x)) > \alpha^n] = E\left[\sum_{n\geq 1} Y_\infty 1\!\!1_{\{Y_\infty > \alpha^n\}}\right] \\ \leq E\left[Y_\infty \log_\alpha^+(Y_\infty)\right] \\ < +\infty$$

Using Borel Cantelli's lemma, we get

$$dQ-\text{almost surely,}\quad \ell^n \mu_n(I_n(x)) \leq \alpha^n \quad \text{if } n \text{ is large enough.}$$

In particular, almost surely,

$$\limsup_{n\to+\infty} \frac{\log\left(\ell^n \mu_n(I_n(x))\right)}{n} \leq \alpha \quad dm-\text{almost everywhere}$$

and the conclusion is a consequence of the arbitrary value of α.

Remark 12. In the eighties, Kahane proved that the condition $0 < E[Y_\infty \log Y_\infty] < +\infty$ is not necessary.

6 A digression on multifractal analysis of measures

In order to understand the approach developed in Section 7, let us recall some basic facts on multifractal analysis of measures. In this part, m is a deterministic measure on $[0,1]$ with finite total mass. As usual, we define the structure function as

$$\tau(q) = \limsup_{n\to+\infty} \frac{1}{n} \log_\ell \left(\sum_{I\in\mathscr{F}_n} m(I)^q\right)$$

and we want to briefly recall the way to improve the formula

$$\dim(E_\beta) = \tau^*(\beta)$$

where

$$E_\beta = \left\{x\ ;\ \lim_{n\to\infty} \frac{\log m(I_n(x))}{\log|I_n(x)|} = \beta\right\}$$

and

$$\tau^*(\beta) = \inf_{q\in\mathbb{R}} (q\beta + \tau(q))$$

is the Legendre transform of τ.

The function τ is known to be a non-increasing convex function on $\mathbb{R}$ such that $\tau(1) = 0$. Moreover, the right and the left derivative $-\tau'(1^+)$ and $-\tau'(1^-)$ are related to the dimensions of the measure m which are defined in formula (2) and (3). In the general case, as we can see for example in [14], we have

Theorem 5.

$$-\tau'(1^+) \leq \dim_*(m) \leq \mathrm{Dim}^*(m) \leq -\tau'(1^-).$$

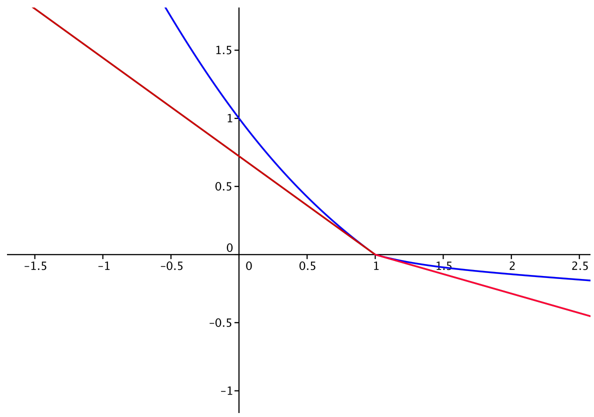

Fig. 7: A structure function τ such that $\tau'(1)$ does not exist.

We can not ensure in general that $\tau'(1^+) = \tau'(1^-)$. Nevertheless, if $\tau'(1)$ exists, the measure m is uni-dimensional and the following are true.

Corollary 2. *Suppose that $\tau'(1)$ exists. Then*

1. *dm-almost-surely,* $\lim_{n\to+\infty} \dfrac{\log(m(I_n(x)))}{\log|I_n(x)|} = -\tau'(1)$
2. $\dim\left(E_{-\tau'(1)}\right) = -\tau'(1)$
3. $\dim_*(m) = \dim^*(m) = \mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = -\tau'(1)$.

The equality $\dim\left(E_{-\tau'(1)}\right) = -\tau'(1)$ can be rewritten in terms of the Legendre transform of the function τ. More precisely, if $\beta = -\tau'(1)$, then $\tau^*(\beta) = \beta$ and $\dim(E_\beta) = \tau^*(\beta)$. This is the first step in mutlifractal formalism.

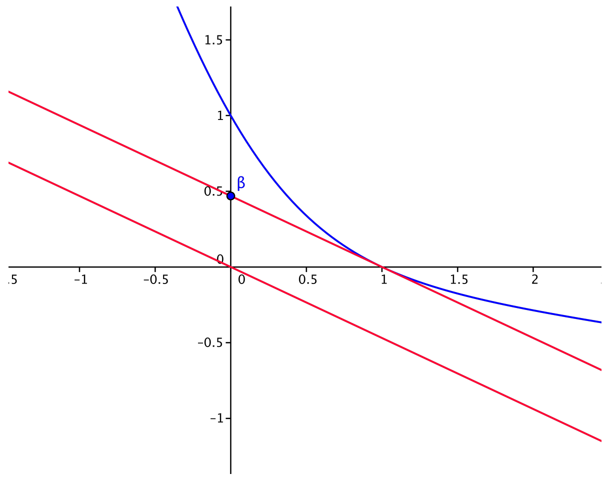

Fig. 8: If $\beta=-\tau'(1)$, then $\tau^*(\beta)=\beta$

In order to obtain the formula $\dim(E_\beta)=\tau^*(\beta)$ for another value of β, the usual way is to write $\beta=-\tau'(q)$ and to construct an auxiliary measure m_q satisfying for any ℓ-adic interval

$$\frac{1}{C}\,m(I)^q|I|^{\tau(q)}\le m_q(I)\le C\,m(I)^q|I|^{\tau(q)}.$$

This is the way used in [10]. If such a measure m_q exists, its structure function τ_{m_q} is such that

$$\tau_{m_q}(t)=\tau(qt)-t\tau(q).$$

In particular,

$$-\tau'_{m_q}(1)=-q\tau'(q)+\tau(q)=\tau^*(\beta).$$

If we observe that

$$\frac{\log(m_q(I_n(x)))}{\log|I_n(x)|}=q\,\frac{\log(m(I_n(x)))}{\log|I_n(x)|}+\tau(q)+o(1),$$

we can conclude that

$$\dim(E_\beta)=\dim(m_q)=-\tau'_{m_q}(1)=\tau^*(\beta).$$

7 Multifractal analysis of Mandelbrot cascades: an outline

In this section, we make the following additional assumptions:

$$\begin{cases} P[W=0]=0 \\ \\ \text{For any real } q, \quad E\left[W^q\right]<+\infty. \end{cases} \tag{15}$$

In particular, assumption (15) is satisfied when m is a log-normal cascade or when $\frac{1}{C} \le W \le C$ almost surely.

We can then list some easy consequences.

- The function $\tau(q) = \log_\ell(E[W^q]) - (q-1)$ is defined on $\mathbb{R}$, convex and of class C^∞
- There exists $r > 1$ such that $\tau(r) < 0$ (and so $E[Y_\infty^r] < +\infty$)
- The cascade is non-degenerate
- $P[m=0] = P[Y_\infty = 0] = 0$.

In order to perform the multifractal analysis of the Mandelbrot cascades, we want to mimic the proof developed for the binomial cascades. It is then natural to introduce the auxiliary cascade m' associated to the weight $W' = \frac{W^q}{E[W^q]}$ (the renormalization ensures that $E[W'] = 1$). The structure function $\tau_{m'}$ of the cascade m' is

$$\begin{aligned} \tau_{m'}(t) &= \log_\ell(E[W'^t]) - (t-1) \\ &= \log_\ell\left(E\left[\frac{W^{qt}}{E[W^q]^t}\right]\right) - (t-1) \\ &= \log_\ell\left(E\left[W^{qt}\right]\right) - t\log_\ell(E[W^q]) - (t-1) \\ &= \tau(tq) - t\tau(q) \end{aligned}$$

In particular,

$$-\tau'_{m'}(1^-) = -q\tau'(q) + \tau(q) = \tau^*(-\tau'(q))$$

and the cascade m' is non-degenerate if and only if $\tau^*(-\tau'(q)) > 0$. This suggests to consider the interval

$$(q_{min}, q_{max}) = \{q \in \mathbb{R}\ ;\ \tau^*(-\tau'(q)) > 0\}.$$

Example 5 (The interval (q_{min}, q_{max}) in the case of log-normal cascades). If m is a log-normal cascade, the function τ is given by

$$\tau(q) = \log_\ell(E[W^q]) - (q-1) = \frac{\sigma^2}{2\ln\ell}(q^2 - q) - (q-1)$$

and the numbers q_{min} and q_{max} are the solutions of the equation

$$\tau(q) = q\tau'(q).$$

We find

$$q_{min} = -\frac{\sqrt{2\ln\ell}}{\sigma} \qquad \text{and} \qquad q_{max} = \frac{\sqrt{2\ln\ell}}{\sigma}.$$

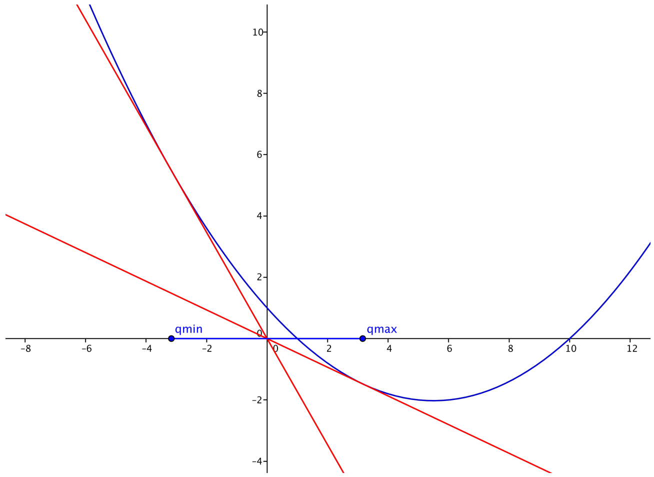

Fig. 9: The structure function τ and the interval (q_{min}, q_{max}) in the case of a log-normal cascade.

When $q \in (q_{min}, q_{max})$, we would like to compare the behavior of the cascades m and m'. In the following subsection, we give a general result which can be applied to the present situation.

7.1 Simultaneous behavior of two Mandelbrot cascades

Theorem 6. *Let (W, W') be a random vector (with coordinates not necessarily independent) such that the Mandelbrot cascades m and m' associated to the weight W et W' are non-degenerate. Suppose that :*

- *There exists $r > 1$ such that $E[Y_\infty^r] < +\infty$ and $E[Y_\infty'^r] < +\infty$*
- *There exists $\alpha > 0$ such that $E[Y_\infty^{-\alpha}] < +\infty$*

Then, almost surely,

$$\lim_{n \to +\infty} \frac{\log m(I_n(x))}{\log |I_n(x)|} = 1 - E[W' \log_\ell W] \quad dm' - \text{almost everywhere.}$$

Proof. The ideas are quite similar to those developed in the proof of Theorem 4. The probability measure on the product space $\tilde{\Omega} = \Omega \times [0,1]$ is now

$$Q'(A) = E\left[\int 1\!\!1_A \, dm'\right]$$

and the related expectation is denoted by $\mathbb{E}'$. The notations are the same as in Theorem 4. In particular

$$dm = f_n\,d\mu_n, \qquad f_n = g_1 \times \cdots \times g_n, \qquad \frac{\log(m(I_n(x)))}{\log|I_n(x)|} = \frac{\log f_n(x) + \log \mu_n(I_n(x))}{-n\log \ell}$$

and we have to prove :

1. $\frac{1}{n}\sum_{j=1}^n \log g_j(x)$ converges to $E[W'\log W] \quad dQ'$ almost surely
2. $\frac{1}{n}\log \mu_n(I_n(x))$ converges to $-\log \ell \quad dQ'$ almost surely.

Step 1 : *behavior of* $\frac{1}{n}\sum_{j=1}^n \log g_j$.
In the same way as in Lemma 4, we have :

$$\mathbb{E}'[\phi(g_n)] = \sum_{\varepsilon\in\mathscr{M} n} E\left[\phi(W_\varepsilon) m'(I_\varepsilon)\right] = E[\phi(W)W']$$

when ϕ is a bounded measurable function and the g_n are identically distributed. On the other hand,

$$\begin{aligned}\mathbb{E}'\left[\phi_1(g_1)\cdots\phi_n(g_n)\right] &= E[\phi_1(W)W'] \times \cdots \times E[\phi_n(W)W'] \\ &= \mathbb{E}'\left[\phi_1(g_1)\right] \times \cdots \times \mathbb{E}'\left[\phi_n(g_n)\right]\end{aligned}$$

which proves the independance of the random variables (g_n) with respect to Q'. Observing that $\mathbb{E}'[|\log g_n|] = E[W'|\log W|] < +\infty$, the strong law of large numbers says that

$$\frac{1}{n}\sum_{k=1}^n \log g_k \xrightarrow[n\to+\infty]{} E[W'\log W] \quad dQ' - \text{almost surely.}$$

Step 2 : *behavior of* $\frac{1}{n}\log \mu_n(I_n(x))$.
Let $\varepsilon = \varepsilon_1\cdots\varepsilon_n \in \mathscr{M}_n$ and recall that $m(I_\varepsilon) = W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n}\,\mu_n(I_\varepsilon)$. It is easy to see that $\mu_n(I_\varepsilon)$ is independent to $W_{\varepsilon_1},\dots,W_{\varepsilon_1\cdots\varepsilon_n}$ and has the same law as $\ell^{-n}Y_\infty$. If we write $m'(I_\varepsilon) = W'_{\varepsilon_1}\cdots W'_{\varepsilon_1\cdots\varepsilon_n}\,\mu'_n(I_\varepsilon)$, we can more precisely say that the vector $(m(I_\varepsilon), m'(I_\varepsilon))$ is identically distributed to $(\ell^{-n}Y_\infty, \ell^{-n}Y'_\infty)$ and independent to $W_{\varepsilon_1},\dots,W_{\varepsilon_1\dots\varepsilon_n}, W'_{\varepsilon_1},\dots,W'_{\varepsilon_1\dots\varepsilon_n}$. It follows that

$$\begin{aligned}\mathbb{E}'\left[(\ell^n\mu_n(I_n(x)))^{-\eta}\right] &= E\left[\ell^{-n\eta}\int \mu_n(I_n(x))^{-\eta}\,dm'(x)\right] \\ &= \ell^{-n\eta}\sum_{\varepsilon\in\mathscr{M}_n} E\left[\mu_n(I_\varepsilon)^{-\eta} m'(I_\varepsilon)\right] \\ &= \ell^{-n\eta}\sum_{\varepsilon\in\mathscr{M}_n} E\left[W'_{\varepsilon_1}\cdots W'_{\varepsilon_1\cdots\varepsilon_n}\right] E\left[\mu_n(I_\varepsilon)^{-\eta}\mu'_n(I_\varepsilon)\right] \\ &= E[Y_\infty^{-\eta}Y'_\infty] \\ &\le E[Y_\infty^{-\eta r'}]^{1/r'} E\left[Y'^{r}_\infty\right]^{1/r}\end{aligned}$$

where r' is such that $\frac{1}{r} + \frac{1}{r'} = 1$. If we choose η such that $\eta r' = \alpha$, we get

$$\mathbb{E}'\left[\sum_{n=1}^{+\infty}\frac{1}{n^2}\left(\ell^n\mu_n(I_n(x))\right)^{-\eta}\right]<+\infty$$

and we can conclude as in Lemma 4 that

$$\text{almost surely,}\quad \liminf_{n\to+\infty}\frac{\log\left(\ell^n\mu_n(I_n(x))\right)}{n}\geq 0\quad dm'-\text{almost everywhere.}$$

In the same way,

$$\begin{aligned}\mathbb{E}'\left[\left(\ell^n\mu_n(I_n(x))\right)^{\eta}\right] &= E[Y_\infty^{\eta}Y'_\infty]\\ &\leq E[Y_\infty^{\eta r'}]^{1/r'}E\left[Y'^{r}_\infty\right]^{1/r}\end{aligned}$$

which is finite and independent of n if we choose η such that $\eta r' = r$. We can then conclude that

$$\text{almost surely,}\quad \limsup_{n\to+\infty}\frac{\log\left(\ell^n\mu_n(I_n(x))\right)}{n}\leq 0\quad dm'-\text{almost everywhere.}$$

7.2 Application to the multifractal analysis of Mandelbrot cascades

If we apply Theorem 6 to the case where $W'=\frac{W^q}{E[W^q]}$, we obtain the following result on multifractal analysis of Mandelbrot cascades.

Theorem 7. *Let m be a Mandelbrot cascade associated to a weight W. Suppose that* (15) is satisfied and define q_{min} and q_{max} as above. Let $\beta=-\tau'(q)$ with $q\in(q_{min},q_{max})$. Then

$$\dim(E_\beta)\geq\tau^*(\beta)$$

where

$$E_\beta=\left\{x\,;\ \lim_{n\to\infty}\frac{\log m(I_n(x))}{\log|I_n(x)|}=\beta\right\}.$$

Proof. As suggested at the beginning of Section 7, let $W'=\frac{W^q}{E[W^q]}$. The condition $q\in(q_{min},q_{max})$ ensures that the associated cascade m' is non-degenerate. More precisely, observing that $\tau'(1)<0$ and $\tau'_q(1)=-\tau^*(-\tau'(q))=-\tau^*(\beta)<0$, we can find $r>1$ such that $E[Y_\infty^r]<+\infty$ and $E[Y'^{r}_\infty]<+\infty$. Finally, all the hypotheses of Theorem 6 are satisfied. Observe that

$$1-E[W'\log_\ell W]=1-E\left[\frac{W^q}{E[W^q]}\log_\ell W\right]=-\tau'(q)=\beta.$$

The conclusion of Theorem 6 says that almost surely, the set E_β is of full measure m'. It follows that

$$\dim(E_\beta)\geq\dim(m')=-\tau'_q(1)=-q\tau'(q)+\tau(q)=\tau^*(-\tau'(q))=\tau^*(\beta).$$

Remark 13. We can observe some analogies between the construction proposed in Theorem 7 and similar ones developed in the context of the thermodynamic formalism (see for example [24]) or in the context of quasi-Bernoulli measures (see [10] or [14]).

7.3 To go further

It is natural to ask if the inequality proved in Theorem 7 is an equality. Indeed we know that the inequality

$$\dim(E_\beta) \le \tilde{\tau}^*(\beta) \quad \text{where} \quad \tilde{\tau}(q) = \limsup_{n\to+\infty} \frac{1}{n} \log_\ell \left(\sum_{I\in\mathscr{F}_n} m(I)^q \right)$$

is always true (see for example [10]).

Our goal is then to compare the convex functions τ and $\tilde{\tau}$. Such a comparison can be deduced from the existence of negative moments for the random variable Y_∞.

Proposition 5 (Existence of negative moments). *Suppose that* (15) is satisfied. Then, for any $\alpha > 0$, $E\left[Y_\infty^{-\alpha}\right] < +\infty$.

Proof. The argument is developed for example in [1] or [19]. Let

$$F(t) = E\left[e^{-tY_\infty}\right]$$

be the generating function of Y_∞. The fundamental equation $\ell Y_\infty = \sum_{j=0}^{\ell-1} W_j Y_\infty(j)$ gives the following duplication formula :

$$F(\ell t) = \left(\int_0^{+\infty} F(tw)\, dP_W(w) \right)^\ell . \tag{16}$$

We claim that it is sufficient to prove that for any $\alpha > 0$, $F(t) = O(t^{-\alpha})$ when $t \to +\infty$. Indeed, if it is the case,

$$P\left[Y_\infty \le t^{-1}\right] = P\left[e^{-tY_\infty} \ge e^{-1}\right] \le eF(t) = O(t^{-\alpha})$$

and we can conclude that

$$E\left[Y_\infty^{-\alpha'}\right] = \int_0^{+\infty} P\left[Y_\infty^{-\alpha'} \ge t\right] dt = \int_0^{+\infty} P\left[Y_\infty \le t^{-1/\alpha'}\right] dt < +\infty$$

for any $\alpha' < \alpha$.

Let us now observe that (16) gives for any $t > 0$ and any $u \in (0,1]$,

$$\begin{aligned}F(t) &\leq \left(\int_0^{+\infty} F\left((t\ell^{-1})w\right)\, dP_W(w)\right)^2\\ &\leq \left(P[W \leq \ell u] + F(tu)\right)^2\\ &\leq 2P[W \leq \ell u]^2 + 2F(tu)^2.\end{aligned}$$

Moreover, for any $\beta > 0$, assumption (15) ensures that

$$P\left[W \leq \ell u\right] = P\left[W^{-\beta} \geq (\ell u)^{-\beta}\right] \leq (\ell u)^{\beta} E\left[W^{-\beta}\right] = Cu^{\beta}.$$

Proposition 5 is then a consequence of the following elementary lemma.

Lemma 6. *Let* $\beta > 0$ *and* $\psi\ :\ [0,+\infty) \to [0,+\infty)$ *a continuous function such that*

$$\lim_{t\to+\infty} \psi(t) = 0.$$

Suppose that there exists $K > 0$ *such that for any* $t > 0$ *and any* $u \in (0,1]$*,*

$$\psi(t) \leq Ku^{2\beta} + 2\psi(tu)^2. \tag{17}$$

Then, for any $\alpha < \beta$*,* $\psi(t) = O(t^{-\alpha})$ *when* $t \to +\infty$*.*

Proof. Let $\alpha < \beta$ and $t_0 > 1$ such that $4Kt_0^{2(\alpha-\beta)} + \frac{1}{2} \leq 1$. Let $\lambda > 1$ such that

$$\text{for any } t \in \left[t_0, t_0^2\right], \quad \psi(\lambda t) \leq \frac{1}{4t^{\alpha}}.$$

Define $\psi_\lambda(t) = \psi(\lambda t)$. Equation (17) remains true if we replace ψ by ψ_λ. Moreover, if $u = \frac{1}{t}$ we get

$$\psi_\lambda(t^2) \leq Kt^{-2\beta} + 2\psi_\lambda(t)^2.$$

If $t \in \left[t_0, t_0^2\right]$, we obtain

$$\begin{aligned}\psi_\lambda(t^2) &\leq Kt^{-2\beta} + 2\left(\frac{1}{4t^{\alpha}}\right)^2\\ &= \frac{1}{4t^{2\alpha}}\left[4Kt^{2(\alpha-\beta)} + \frac{1}{2}\right]\\ &\leq \frac{1}{4(t^2)^{\alpha}}.\end{aligned}$$

Define the sequence (t_n) by $t_{n+1} = t_n^2$. Using the same argument, we obtain step by step

$$\text{for any } n \geq 0, \quad \text{for any } t \in [t_n, t_{n+1}], \qquad \psi_\lambda(t) \leq \frac{1}{4t^{\alpha}}$$

and the conclusion follows.

Corollary 3. *Suppose that* (15) is satisfied. Then,

$$\text{almost surely,}\quad \text{for any } q\in\mathbb{R},\qquad \tilde{\tau}(q)\le\tau(q).$$

Proof. Using the continuity of the convex functions $\tilde{\tau}$ ans τ, it is sufficient to prove that for any $q\in\mathbb{R}$, almost surely, $\tilde{\tau}(q)\le\tau(q)$. Let

$$q_0=\sup\{q>1\ ;\ \tau(q)<0\}.$$

It is possible that $q_0=+\infty$. Nevertheless, if $q_0<+\infty$ and if $q\ge q_0$, we obviously have $\tilde{\tau}(q)\le 0\le\tau(q)$.

We can now suppose that $q<q_0$ and we claim that

$$E\left[Y_\infty^q\right]<+\infty. \tag{18}$$

Indeed, the case $q<0$ is due to Proposition 5, the case $0\le q\le 1$ is obvious and the case $1<q<q_0$ is due to Theorem 3.

Let $\varepsilon=\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}_n$. As observed in Lemma 5, we have

$$m(I_\varepsilon)=W_{\varepsilon_1}\cdots W_{\varepsilon_1\cdots\varepsilon_n}\,\mu_n(I_\varepsilon).$$

where $\mu_n(I_\varepsilon)$ is independent to $W_{\varepsilon_1},\cdots,W_{\varepsilon_1\cdots\varepsilon_n}$ and has the same distribution as $\ell^{-n}Y_\infty$. It follows that

$$\begin{aligned}E\left[\sum_{\varepsilon\in\mathscr{M}_n}m(I_\varepsilon)^q\right]&=\sum_{\varepsilon\in\mathscr{M}_n}E\left[W_{\varepsilon_1}^q\cdots W_{\varepsilon_1\cdots\varepsilon_n}^q\,\mu_n(I_\varepsilon)^q\right]\\&=\ell^nE\left[W^q\right]^n\ell^{-nq}E\left[Y_\infty^q\right]\\&=\ell^{n\tau(q)}E\left[Y_\infty^q\right].\end{aligned}$$

Let $t>\tau(q)$. In view of (18) we get

$$E\left[\sum_{n\ge1}\ell^{-nt}\sum_{\varepsilon\in\mathscr{M}_n}m(I_\varepsilon)^q\right]=\sum_{n\ge1}\ell^{-nt}\ell^{n\tau(q)}E\left[Y_\infty^q\right]<+\infty.$$

It follows that almost surely, $\sum_{\varepsilon\in\mathscr{M}_n}m(I_\varepsilon)^q\le\ell^{nt}$ if n is large enough and we can conclude that $\tilde{\tau}(q)\le t$ almost surely. This gives the conclusion.

We can now prove the following result.

Theorem 8. *Suppose that* (15) is satisfied. Then, for any $\beta\in(-\tau'(q_{max}),-\tau'(q_{min}))$,

$$\dim(E_\beta)=\tau^*(\beta)\quad\text{almost surely.}$$

Indeed Theorem 7 and Corollary 3 ensure that for any $\beta\in(-\tau'(q_{max}),-\tau'(q_{min}))$,

$$\tau^*(\beta)\le\dim(E_\beta)\le\tilde{\tau}^*(\beta)\le\tau^*(\beta)$$

which gives the conclusion of Theorem 8.

Remark 14. The proof of Theorem 8 shows that

$$\tau^*(\beta) = \tilde{\tau}^*(\beta) \quad \text{for any } \beta \in (-\tau'(q_{max}), -\tau'(q_{min})).$$

It follows that $\tau(q) = \tilde{\tau}(q)$ for any $q \in (q_{min}, q_{max})$. When q_{min} and q_{max} are finite, it is possible to prove that

$$\tilde{\tau}(q) = \tau'(q_{min})q \quad \text{if } q \leq q_{min} \qquad \text{and} \qquad \tilde{\tau}(q) = \tau'(q_{max})q \quad \text{if } q \geq q_{max}$$

(see for example [8]).

Let us finish this text by recalling that Barral proved in [2] the much more difficult result:

$$\text{almost surely,} \quad \begin{cases} \text{for any } \beta \in (-\tau'(q_{max}), -\tau'(q_{min})), & \dim(E_\beta) = \tau^*(\beta) \\ \\ \text{for any } \beta \notin [-\tau'(q_{max}), -\tau'(q_{min})], & E_\beta = \emptyset. \end{cases}$$

This text about Mandelbrot cascades can be viewed as the begining of the story. Since then, important generalizations have been proposed. In 1985 Kahane introduced the so-called T-martingales (see [17, 18]). In particular, Log-infinitely divisible multifractal processes ([4]) and compound Poisson cascades ([3]) are models for which similar fine results on multifractal analysis have been proved (see [5, 6]).

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