MEASURES AND THE LAW OF THE ITERATED LOGARITHM

IMEN BHOURI AND YANICK HEURTEAUX

Abstract. Let m be a unidimensional measure with dimension d. A natural question is to ask if the measure m is comparable with the Hausdorff measure (or the packing measure) in dimension d. We give an answer (which is in general negative) to this question in several situations (self-similar measures, quasi-Bernoulli measures). More precisely we obtain fine comparisons between the mesure m and generalized Hausdorff type (or packing type) measures. The Law of the Iterated Logarithm or estimations of the L^q-spectrum in a neighborhood of $q = 1$ are the tools to obtain such results.

1. Introduction

For a given probability measure m in $\mathbb{R}^D$ we define as usual

$$\begin{cases} \dim_*(m) = \inf\{\dim(E);\quad m(E) > 0\} = \sup\{s \geq 0;\quad m \ll \mathcal{H}^s\} \\ \\ \dim^*(m) = \inf\{\dim(E);\quad m(E) = 1\} = \inf\{s \geq 0;\quad m \perp \mathcal{H}^s\} \end{cases} \tag{1.1}$$

respectively the lower and upper dimension of the measure m, where $\mathcal{H}^s$ define the Hausdorff measure and $\dim(E)$ is the Hausdorff dimension of a set E.

When the equality $\dim_*(m) = \dim^*(m)$ is satisfied, we say that the measure m is unidimensional and we denote by $\dim(m)$ the common value. In this situation, the measure m is carried by a set of dimension $d = \dim(m)$ while $m(E) = 0$ for every Borel set E satisfying $\dim(E) < d$.

For such a unidimensional measure, it is natural to try to compare the measure m with the Hausdorff measure $\mathcal{H}^d$ and to ask the following question :

Question 1.1. Does there exist a set $E_0 \subset \operatorname{supp}(m)$ and a constant $C > 0$ such that for every Borel set A, $m(A) = C\,\mathcal{H}^d(A \cap E_0)$?

Or in a weaker form :

Question 1.2. Does there exist a set $E_0 \subset \operatorname{supp}(m)$ and a constant $C > 0$ such that for every Borel set A, $\frac{1}{C}\,\mathcal{H}^d(A \cap E_0) \leq m(A) \leq C\,\mathcal{H}^d(A \cap E_0)$?

The answer to Question 1.2 is sometimes positive. Let us describe a classical example. Let K be a self-similar compact set in $\mathbb{R}^D$, that is

$$K = \bigcup_{i=1}^{k} S_i(K) \tag{1.2}$$

where $S_1, \cdots, S_k$ are similarities in $\mathbb{R}^D$ with ratio $0 < r_i < 1$. In the case where the Open Set Condition is satisfied (see for example [8] or Section 3 for a precise

---

2000 *Mathematics Subject Classification.* Primary:28A12,28A78,28D20; Secondary:60F10.

Key words and phrases. Iterated Logarithm Law, quasi-Bernoulli measure, Hausdorff dimension, Packing dimension.

definition) it is well known that the Hausdorff dimension of the compact set K is the unique positive real number δ such that $\sum_{i=1}^k r_i^\delta = 1$. A way to obtain this result is the following. Let $\mathbf{p} = (p_1, \cdots, p_k)$ be a probability vector and m be the unique probability measure such that

$$m = \sum_{i=1}^k p_i \, m \circ S_i^{-1} \ . \tag{1.3}$$

The measure m is unidimensional with dimension

$$\dim(m) = \frac{\sum_{i=1}^k p_i \log p_i}{\sum_{i=1}^k p_i \log r_i} \ . \tag{1.4}$$

We can refer to [9] or [12] for a proof of this formula. The maximal value of $\dim(m)$ in (1.4) is obtained when $p_i = r_i^\delta$ for all i. In that case, $\dim(m) = \delta$ and it is possible to show that $m(A) \approx \mathcal{H}^d(A \cap K)$ for all A. That is the reason why in particular $\dim(K) = \delta$. As we will see in Section 3, this situation is exceptional. For any other choice of the probability vector $\mathbf{p}$, the measure m is singular with respect to the Hausdorff measure $\mathcal{H}^{\dim(m)}$. More precisely, using the Law of the Iterated Logarithm, we will obtain in Theorem 3.1 precise logarithm corrections in the comparison between the measure m and Hausdorff type measures.

Similar quantities involving the packing measure $\widehat{\mathcal{P}}^s$ and packing dimension Dim can be defined

$$\begin{cases} \mathrm{Dim}_*(m) = \inf\{\mathrm{Dim}(E); \quad m(E) > 0\} = \sup\{s \geq 0; \quad m \ll \widehat{\mathcal{P}}^s\} \\ \\ \mathrm{Dim}^*(m) = \inf\{\mathrm{Dim}(E); \quad m(E) = 1\} = \inf\{s \geq 0; \quad m \perp \widehat{\mathcal{P}}^s\} \ . \end{cases} \tag{1.5}$$

They are respectively called the lower and upper packing dimension of the measure m. For more details on packing dimension and packing measures, we can refer to [8] or to the original paper of Tricot [22].

There are two fondamental ways to compute or to estimate the dimension of a measure.

On one hand, the calculation of the dimension of measures may be a consequence of some independance properties and the use of the strong law of large numbers. This is in particular the case for self-similar measures under some separation condition.

On the other hand, in a more abstract and general context, it is well known that the quantities defined in (1.1) and (1.5) are strongly related to the derivatives $\tau'_-(1)$ and $\tau'_+(1)$ of the L^q-spectrum at point 1 (see [11] or [20] for example). In particular, if $\tau'(1)$ exists, the measure is unidimensional and we have

$$-\tau'(1) = \dim_*(m) = \dim^*(m) = \mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) \ . \tag{1.6}$$

This is in particular the case for the so called quasi-Bernoulli measures (see Section 4 or [6] for a precise definition and [11] where it is proved that the L^q-spectrum is differentiable in this situation).

There are numerous works dealing with the comparison between measures and Hausdorff measures or packing measures. This can be done in an abstract context

([6], [11], [3], [14]), in a dynamical context ([16], [23]) or for concrete measures as harmonic measures ([2], [4], [5], [17], [18]). Logarithmic corrections are also proposed in several situations. In particular, Makarov and Makarov-Volberg obtained such corrections for the harmonic measure, respectively in Jordan domains ([17]) and in self-similar Cantor sets ([18]). Using a dyadic martingale that approximates the logarithm of the densities of the measure at different scales, Llorente and Nicolau [15] also obtained some abstract (and in general non explicit) logarithmic corrections in the case of doubling measure.

In this work we pursue such studies and we try to obtain logarithmic corrections in several situations, using the following two ideas. On one hand, the Law of the Iterated Logarithm, which is more precise than the Strong Law of Large Numbers, can be used in the case where some independance properties are satisfied. On the other hand, estimations of $\tau(1+q)-q\tau'(1)$ near $q=0$ where τ is the L^q-spectrum of the measure m allow us to derive more precise comparisons beetwen m and generalized Hausdorff or packing measures. In particular, logarithm or iterated logarithm corrections may be obtained in some situations.

The paper is organised as follows. In Section 2, we study the case of Bernoulli products and give fine comparisons with Hausdorff type or packing type measures. In particular such a Bernoulli product m is singular with respect the the Hausdorff measure $\mathcal{H}^d$ but absolutely continuous with respect to the packing measure $\widehat{\mathcal{P}}^d$ where d is the dimension of the measure m. This elementary fact, which is a consequence of the Law of the Iterated Logarithm seems not to be very present in the litterature. The motivation in studying such a toy example is that it contains the fundamental ideas that will be developped in the next sections.

In Section 3 we generalize the results of Section 2 to the case of self-similar measures like (1.3), when the Open Set Condition is satisfied.

Finally, the last section is devoted to the study of quasi-Bernoulli measures. Of course, in such a general situation we do not have independance properties which allow to use the Law of the Iterated Logarithm. Nevertheless, in a large situation we obtain upper bounds of type LIL (Section 4.1) and we prove that fine comparisons between the measure m and Hausdorff (or packing) type measures in the reverse sens are strongly related to the behaviour of the L^q-spactrum near the point $q=1$ (Sections 4.2 and 4.3). In particular, in much situations, a quasi-Bernoulli measure is already singular with respect to the Hausdorff measure $\mathcal{H}^{\dim(m)}$ but absolutely continuous with respect to the packing measure $\widehat{\mathcal{P}}^{\dim(m)}$.

2. Bernoulli products

We begin by the study of a classical example. Let $(\mathcal{F}_n)_{n\geq 0}$ be the family of ℓ-adic cubes of the n^{th} generation on $[0,1)^D$, in other words :

$$\mathcal{F}_n = \left\{ I = \prod_{i=1}^{D} [k_i/\ell^n, (k_i+1)/\ell^n)\,;\quad 0 \leq k_i < \ell^n \right\}. \tag{2.1}$$

Suppose for simplicity $D=1$ and $\ell=2$. Let m be the Bernoulli product on $[0,1)$ with parameter $0<p<1$.

It is defined as follows. If $\varepsilon_1 \cdots \varepsilon_n$ are integers in $\{0,1\}$, and if

$$I_{\varepsilon_1\cdots\varepsilon_n} = \left[\sum_{i=1}^n \frac{\varepsilon_i}{2^i}, \sum_{i=1}^n \frac{\varepsilon_i}{2^i} + \frac{1}{2^n}\right) \in \mathcal{F}_n$$

then

$$m\left(I_{\varepsilon_1\cdots\varepsilon_n}\right) = p^{s_n}(1-p)^{n-s_n}, \quad \text{where} \quad s_n = \varepsilon_1 + \cdots + \varepsilon_n \ .$$

It is well known that the measure m is unidimensional with dimension

$$d = -p\log_2 p - (1-p)\log_2(1-p) \ .$$

This is an easy consequence of the strong law of large numbers applied to the sequence of independent Bernoulli random variables $(\varepsilon_n)_{n\geq 1}$. Here, the space $[0,1)$ is equipped with the probability measure m (see for example [9] or [12]).

It is then natural to think that the Law of the Iterated Logarithm gives a more precise result. Curiously this elementary fact is not very present in the litterature. Let us only mention [21] in which the law of the iterated logarithm is used in a weak form in order to prove that the measure m is singular with respect to the Hausdorff measure $\mathcal{H}^d$.

Proposition 2.1. *Let m be a Bernoulli product with parameter $0 < p < 1$ satisfying $p \neq 1/2$. Take*

$$d = -p\log_2 p - (1-p)\log_2(1-p) \quad \textit{and} \quad \sigma^2 = p(1-p)\left(\log_2(\frac{p}{1-p})\right)^2$$

and denote

$$\Theta(t) = 2^{\sqrt{2\log_2(1/t)\ln\ln\log_2(1/t)}} \ .$$

For all $\varepsilon > 0$ we have :

1. $m \ll \mathcal{H}^\Psi$, *where* $\Psi(t) = t^d\Theta(t)^{\sigma+\varepsilon}$
2. $m \perp \mathcal{H}^\Psi$, *where* $\Psi(t) = t^d\Theta(t)^{\sigma-\varepsilon}$.

In particular, $\dim_*(m) = \dim^*(m) = d$ *but* $m \perp \mathcal{H}^d$.

Remark 2.2. Bernoulli products are particular cases of the self-similar measures described in (1.3). We have $k = 2$, $S_1(x) = x/2$, $S_2(x) = (1+x)/2$, and $\mathbf{p} = (1-p,p)$. The self-similar compact set K is the unit interval $[0,1]$ with dimension 1. The case $p = 1/2$ corresponds to the Lebesgue measure : it is the natural self-similar measure on the compact set K. In the other cases, the measure m is singular with respect to the Hausdorff measure $\mathcal{H}^{\dim(m)}$.

Remark 2.3. In a famous paper ([17]), Makorov proved that the harmonic measure of a Jordan domain is unidimensional with dimension 1 and obtained similar iterated logarithm corrections.

Proof of Proposition 2.1. We first prove (1). Let

$$X_n(x) = -\log_2\left(p^{\varepsilon_n}(1-p)^{1-\varepsilon_n}\right) = -\log_2\left(\frac{m(I_n(x))}{m(I_{n-1}(x))}\right)$$

where $I_n(x)$ denotes the unique interval in $\mathcal{F}_n$ containing x and $I_n(x) = I_{\varepsilon_1\cdots\varepsilon_n}$. The random variables X_n are independant and identiquely distributed. An easy calculation gives

$$\mathbb{E}[X_n] = d \quad \text{and} \quad \mathbb{V}[X_n] = \sigma^2$$

where d and σ are the quantities introduced in the proposition. Derive from the Law of the Iterated Logarithm that, dm-almost surely,

$$\liminf_{n\to+\infty} \frac{S_n - nd}{\sqrt{2n\ln\ln n}} = -\sigma \tag{2.2}$$

where $S_n = X_1 + \cdots + X_n$.

Set $\varepsilon > 0$. Then, dm-almost surely,

$$\exists n_0 \in \mathbb{N}^* \; ; \quad \forall n \geq n_0, \quad S_n \geq nd - (\sigma + \varepsilon)\sqrt{2n\ln\ln n} \; .$$

Furthermore, $S_n = -\log_2(m(I_n(x)))$. Therefore, we get

$$\text{a.s,} \qquad \exists n_0 \in \mathbb{N}^* \; ; \quad \forall n \geq n_0, \quad m(I_n(x)) \leq |I_n(x)|^d \,\Theta(|I_n(x)|)^{\sigma+\varepsilon} \; .$$

where $|I_n(x)| = 2^{-n}$ is the length of the interval $I_n(x)$. It is classical to deduce (see for example [19]) that for every set E,

$$m(E \cap \liminf_n B_n) \leq \mathcal{H}^{\Psi}(E)$$

where $B_n = \{x \; ; \; m(I_n(x)) \leq \Psi(|I_n(x)|)\}$ and $\Psi(t) = t^d\Theta(t)^{\sigma+\varepsilon}$. Moreover, the measure m is carried by the set $\liminf_n B_n$ and the result yields.

We now prove (2). Let $\varepsilon > 0$. A consequence of (2.2) is also that dm-almost surely,

$$\forall n_0 \in \mathbb{N}^*, \; \exists n \geq n_0 \; ; \quad S_n \leq nd + (-\sigma + \varepsilon)\sqrt{2n\ln\ln n}$$

and we get,

$$\text{a.s,} \quad \text{i.o,} \quad m(I_n(x)) \geq |I_n(x)|^d \,\Theta(|I_n(x)|)^{\sigma-\varepsilon} \; .$$

The full measure set E_0 which is just described satisfies $\mathcal{H}^{\Psi}(E_0) < +\infty$ where $\Psi(t) = t^d\Theta(t)^{\sigma-\varepsilon}$. Using that ε is arbitrary small, we can deduce that m is singular with respect to $\mathcal{H}^{\Psi}$ for every $\varepsilon > 0$.

In particular, $t^d \ll \Psi(t)$ when t goes to 0, so that $m \perp \mathcal{H}^d$ and the proof of Proposition 2.1 is complete. □

The Law of the Iterated Logarithm also says that dm-almost surely,

$$\limsup_{n\to+\infty} \frac{S_n - nd}{\sqrt{2n\ln\ln n}} = \sigma \; . \tag{2.3}$$

This asymptotic behavior is deeply related to comparisons between the measure m and packing measures. That is what is shown in the following twin proposition.

Proposition 2.4. *The notations are the same as in Proposition 2.1. For all $\varepsilon > 0$ we have :*

1. $m \ll \widehat{\mathcal{P}}^{\Psi}$*, where* $\Psi(t) = t^d\Theta(t)^{-(\sigma-\varepsilon)}$
2. $m \perp \widehat{\mathcal{P}}^{\Psi}$*, where* $\Psi(t) = t^d\Theta(t)^{-(\sigma+\varepsilon)}$.

In particular, $\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = d$ *and* $m \ll \widehat{\mathcal{P}}^d$.
More precisely, $\widehat{\mathcal{P}}^d(E) < +\infty \Rightarrow m(E) = 0$.

Proof. The relation (2.3) implies that dm-almost surely

$$\forall n_0 \in \mathbb{N}^*, \; \exists n \geq n_0 \; ; \quad S_n \geq nd + (\sigma - \varepsilon)\sqrt{2n\ln\ln n} \; .$$

So,

$$\text{a.s,} \quad \text{i.o.,} \quad m(I_n(x)) \leq |I_n(x)|^d\Theta(|I_n(x)|)^{-(\sigma-\varepsilon)}$$

and assumption (1) holds with similar arguments as in Proposition 2.1 (using packing instead of coverings). More precisely, for every set E, one has

$$m(E \cap \limsup_n B_n) \leq \widehat{\mathcal{P}}^{\Psi}(E)$$

where $B_n = \{x\ ;\ m(I_n(x)) \leq \Psi(|I_n(x)|)\}$, $\Psi(t) = t^d \Theta(t)^{-(\sigma-\varepsilon)}$ and the sets B_n are such that $m\left(\limsup_n B_n\right) = 1$.

In particular, $t^d \Theta(t)^{-(\sigma-\varepsilon)} \leq t^d$ so that $m \ll \widehat{\mathcal{P}}^d$. More precisely, $t^d\Theta(t)^{-(\sigma-\varepsilon)} \ll t^d$ when t goes to 0. If E is a set such that $\widehat{\mathcal{P}}^d(E) < +\infty$, we have successively $\widehat{\mathcal{P}}^{\Psi}(E) = 0$ and $m(E) = 0$.

On the other hand, we have dm-almost surely

$$\exists n_0 \in \mathbb{N}\ ; \quad \forall n \geq n_0, \quad S_n \leq nd + (\sigma+\varepsilon)\sqrt{2n \ln\ln n}$$

which says that

$$\text{a.s,} \qquad \exists n_0 \in \mathbb{N}\ ; \quad \forall n \geq n_0, \quad m(I_n(x)) \geq |I_n(x)|^d \Theta(|I_n(x)|)^{-(\sigma+\varepsilon)}\ .$$

Let

$$E_{n_0} = \left\{ x \in [0,1)\ ; \quad \forall n \geq n_0, \quad m(I_n(x)) \geq |I_n(x)|^d \Theta(|I_n(x)|)^{-(\sigma+\varepsilon)} \right\}\ .$$

It is clear that $\mathcal{P}^{\psi_\varepsilon}(E_{n_0}) \leq 1$ where $\psi_\varepsilon(t) = t^d\theta(t)^{-(\sigma+\varepsilon)}$ and $\mathcal{P}^{\psi_\varepsilon}$ is the pre-measure related to the packing measure $\widehat{\mathcal{P}}^{\psi_\varepsilon}$ (see [8] for the link between $\mathcal{P}^{\psi_\varepsilon}$ and $\widehat{\mathcal{P}}^{\psi_\varepsilon}$). It follows that $\mathcal{P}^{\psi_{2\varepsilon}}(E_{n_0}) = 0$ and $\widehat{\mathcal{P}}^{\psi_{2\varepsilon}}\left(\bigcup_{n_0} E_{n_0}\right) \leq \sum_{n_0} \mathcal{P}^{\psi_{2\varepsilon}}(E_{n_0}) = 0$. Moreover, $m\left(\bigcup_{n_0} E_{n_0}\right) = 1$. This implies (2) ($\varepsilon$ is arbitrary small). □

3. A more general situation : self-similar measures

The results established in the previous section are particular cases of the more general situation of self-similar measures. We can prove the following general theorem.

Theorem 3.1. *Let K be the attractor of a family of similarity transformations $S_1, \cdots, S_k$ in $\mathbb{R}^D$ where S_i has similarity ratio $0 < r_i < 1$. Suppose that the Open Set Condition is satisfied and let $\delta = \dim(K)$ be the Hausdorff dimension of K. Recall that δ is the unique positive solution of the equation $\sum_{i=1}^k r_i^\delta = 1$.*

Let $\mathbf{p} = (p_1, \cdots, p_k)$ be a probability vector and m be the self-similar probability measure such that

$$m = \sum_{i=1}^k p_i\, m \circ S_i^{-1}\ .$$

Set

$$d = \frac{\sum_{i=1}^k p_i \ln p_i}{\sum_{i=1}^k p_i \ln r_i} \quad \text{and} \quad \sigma^2 = \frac{\sum_{i=1}^k p_i(\ln p_i - d\ln r_i)^2}{-\sum_{i=1}^k p_i \ln r_i}\ .$$

Suppose that $d \neq \delta$ (which is equivalent to $\sigma > 0$) and let

$$\theta(t) = e^{\sqrt{2\ln(1/t)\ln\ln\ln(1/t)}}\ .$$

For all $\varepsilon > 0$ we have :

(1) *$m \ll \mathcal{H}^{\Psi}$, where $\Psi(t) = t^d\Theta(t)^{\sigma+\varepsilon}$*
(2) *$m \perp \mathcal{H}^{\Psi}$, where $\Psi(t) = t^d\Theta(t)^{\sigma-\varepsilon}$.*

In particular, $m \perp \mathcal{H}^d$.

Recall that the Open Set Condition states that there exists a non-empty and bounded open set U in $\mathbb{R}^D$ with $\bigcup_{i=1}^k S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all i, j with $i \neq j$.

Proof of Theorem 3.1. We give the complete proof in the particular case where the strong separation condition is satisfied (i.e. in the case where the $S_1(K), \cdots, S_k(K)$ are disjoint compact sets) and then say a few words in the general case.

In the case where the strong separation condition is satisfied, the application

$$\pi \ : \quad i = (i_1, \cdots, i_n, \cdots) \in \{1, \cdots, k\}^{\mathbb{N}^*} \longmapsto \bigcap_n S_{i_1} \circ \cdots \circ S_{i_n}(K) \tag{3.1}$$

is an homeomorphism between the symbolic Cantor set $\{1, \cdots, k\}^{\mathbb{N}^*}$ and the self-similar set K. Moreover, the measure m is nothing else but the image of a multinomial measure on $\{1, \cdots, k\}^{\mathbb{N}^*}$ through this homeomorphism. Let

$$K_{i_1 \cdots i_n} = S_{i_1} \circ \cdots \circ S_{i_n}(K) \ .$$

For every $x \in K$ there exists a unique sequence $i_1(x), \cdots, i_n(x), \cdots$ such that $x \in K_{i_1(x) \cdots i_n(x)}$ for all n. Moreover, the random variables $i_1, \cdots, i_n, \cdots$ are independant and uniformly distributed with distribution

$$m(\{i_n = i\}) = p_i \quad \forall i \in \{1, \cdots k\} \ .$$

Set

$$K_n(x) = K_{i_1(x) \cdots i_n(x)} \quad \text{and} \quad R_n(x) = |K_n(x)|$$

where $|A|$ denotes the diameter of the set A.

We may suppose without lost of generality that $|K| = 1$ and we define for every $n \geq 1$ the random variable

$$S_n(x) = -\ln(m(K_n(x))) + d \ln(R_n(x)) \quad \text{and} \quad X_n = S_n - S_{n-1}$$

with the convention $S_0 = 0$.

The random variables X_n are independant, uniformly distributed and take the value $-\ln p_i + d \ln r_i$ with probability p_i. An easy calculation gives

$$\mathbb{E}[X_n] = 0 \quad \text{and} \quad \mathbb{V}[X_n] = \sum_{i=1}^k p_i \left(\ln p_i - d \ln r_i\right)^2 = \left(-\sum_{i=1}^k p_i \ln r_i\right) \sigma^2 \ .$$

The Law of the Iterated Logarithm states that almost surely,

$$\liminf_{n \to +\infty} \frac{S_n}{\sqrt{2n \ln \ln n}} = -\left(-\sum_{i=1}^k p_i \ln r_i\right)^{1/2} \sigma \ . \tag{3.2}$$

On the other hand, $\ln R_n = \rho_1 + \cdots + \rho_n$ where the ρ_j are independant, uniformly distributed and such that for all $i \in \{1, \cdots, k\}$, $m(\{\rho_n = \ln r_i\}) = p_i$. The strong law of large numbers says that almost surely,

$$\lim_{n \to +\infty} \frac{\ln R_n}{n} = \sum_{i=1}^k p_i \ln r_i \ . \tag{3.3}$$

Combining (3.2) and (3.3), we deduce that dm-almost surely,

$$\liminf_{n \to +\infty} \frac{-\ln(m(K_n(x))) + d \ln(|K_n(x)|)}{\sqrt{2 \ln\left(|K_n(x)|^{-1}\right) \ln \ln \ln\left(|K_n(x)|^{-1}\right)}} = -\sigma \ . \tag{3.4}$$

Let $\varepsilon > 0$. Using the notation introduced in the theorem, we conclude that almost surely

$$\begin{cases} \exists n_0\ ;\ \forall n \geq n_0, \quad m(K_n(x)) \leq |K_n(x)|^d \theta(|K_n(x)|)^{\sigma+\varepsilon} \\ \\ \forall n_0\ ;\ \exists n \geq n_0, \quad m(K_n(x)) \geq |K_n(x)|^d \theta(|K_n(x)|)^{\sigma-\varepsilon} \end{cases}$$

The size of the $K_n(x)$ are exponentialy decreasing in the sense that

$$\min_{1\leq i\leq k}(r_i)\,|K_n(x)| \leq |K_{n+1}(x)| \leq \max_{1\leq i\leq k}(r_i)\,|K_n(x)|\ .$$

It is then well known that Hausdorff measures of subsets of K computed with coverings using the $K_n(x)$ are comparable to the genuine ones. In the same way as in Section 2, we can then conclude that for all $\varepsilon > 0$,

$$\begin{cases} m \ll \mathcal{H}^\Psi, \quad \text{where } \Psi(t) = t^d\Theta(t)^{\sigma+\varepsilon} \\ \\ m \perp \mathcal{H}^\Psi, \quad \text{where } \Psi(t) = t^d\Theta(t)^{\sigma-\varepsilon} \end{cases}$$

and the proof is finished in the case where the strong separation condition is satisfied.

In the general case we have to adapt the argument. The difficulty is that the function π defined in (3.1) is always surjective but not one to one. We will use the following lemma which was proved by Graf in [10].

Lemma 3.2. ([10]) *The notations are the same as in Theorem 3.1. Under the Open Set Condition we have :*

$$\forall (i_1,\cdots,i_n) \in \{1,\cdots,k\}^n, \quad m(K_{i_1\cdots i_n}) = p_{i_1}\cdots p_{i_n}$$

and

$$\textit{if} \quad (i_1,\cdots,i_n) \neq (j_1,\cdots,j_n) \quad \textit{then} \quad m\left(K_{i_1\cdots i_n} \cap K_{j_1\cdots j_n}\right) = 0\ .$$

We define the following families of subsets of K. If $n \geq 1$ and $(i_1,\cdots,i_n) \in \{1,\cdots,k\}^n$,

$$K^0_{i_1\cdots i_n} = \{x \in K_{i_1\cdots i_n}\ ;\quad \forall (j_1,\cdots,j_n) \neq (i_1,\cdots,i_n), \quad x \notin K_{j_1\cdots j_n}\}\ .$$

It follows from Lemma 3.2 that for every integer $n \geq 1$ and for every $(i_1,\cdots,i_n) \in \{1,\cdots,k\}^n$,

$$m(K^0_{i_1\cdots i_n}) = p_{i_1}\cdots p_{i_n}\ .$$

Moreover, the family $K^0_{i_1\cdots i_n}$, where $(i_1,\cdots,i_n) \in \{1,\cdots,k\}^n$, is constituted of k^n disjoint $\mathcal{G}_\delta$ subsets of $\mathbb{R}^D$ and satisfies :

$$K^0_{i_1\cdots i_n j} \subset K^0_{i_1\cdots i_n} \quad \forall j \in \{1,\cdots,k\}\ .$$

Let

$$K^0 = \bigcap_{n\in\mathbb{N}^*} \bigcup_{(i_1,\cdots,i_n)} K^0_{i_1\cdots i_n}\ .$$

The set K^0 is a $\mathcal{G}_\delta$ subset of $\mathbb{R}^D$ such that $K^0 \subset K$ and $m(K^0) = 1$. Moreover for every $x \in K^0$, there exists a unique sequence $(i_1(x),\cdots,i_n(x),\cdots)$ such that for every integer $n \geq 1$, $x \in K^0_{i_1(x)\cdots i_n(x)}$. We can extend the applications $i_1,\cdots,i_n,\cdots$ in a mesurable way and define for every $x \in K$

$$K_n(x) = K_{i_1(x)\cdots i_n(x)}$$

such that $x \in K_n(x)$. Moreover the random variables $i_1, \cdots, i_n, \cdots$ are independant, uniformly distributed and such that $m(\{i_n = i\}) = p_i$. We can already use the Law of the Iterated Logarithm and obtain (3.4) which is the key to prove Theorem 3.1. □

Remark 3.3. In the case of Bernoulli products described in Section 2, the sets $K^0_{i_1\cdots i_n}$ are nothing else but the open dyadic intervals of the n^{th} generation and K^0 is the set of points $x \in [0,1]$ that are not dyadic numbers.

Remark 3.4. It is classical to establish that the L^q-spectrum of the measure m is given by the implicit equation

$$\sum_{i=1}^{k} p_i^q r_i^{\tau(q)} = 1 \ .$$

We can refer to [1] or [9] where this formula is obtained and where the link with multifractal formalism is shown. The function τ is analytic and an easy calculation gives $\tau'(1) = -d$ and $\tau''(1) = \sigma^2$. In other words, $\tau(1-q) = dq + \frac{\sigma^2}{2}q^2 + o(q^2)$ near $q = 0$. We will see in Section 4 that such an estimate is the key to obtain quite similar results for quasi-Bernoulli measures.

Of course a similar result involving packing measures is also true.

Theorem 3.5. *The hypothesis and the notations are the same as in Theorem 3.1. For all* $\varepsilon > 0$ *we have :*

(1) $m \ll \widehat{\mathcal{P}}^\Psi$, *where* $\Psi(t) = t^d \Theta(t)^{-(\sigma-\varepsilon)}$
(2) $m \perp \widehat{\mathcal{P}}^\Psi$, *where* $\Psi(t) = t^d \Theta(t)^{-(\sigma+\varepsilon)}$.

In particular, $\widehat{\mathcal{P}}^d(E) < +\infty \Rightarrow m(E) = 0$.

4. Quasi-Bernoulli measures

Natural generalisations of Bernoulli products or self-similar measures are the so called quasi-Bernoulli measures.

The notations are the same as in Section 2. Suppose that the ℓ-adic cubes in $\mathcal{F}_n$ are coded $I_{\varepsilon_1\cdots\varepsilon_n}$, $0 \le \varepsilon_i < \ell^D$ in such a way that

$$I_{\varepsilon_1\cdots\varepsilon_{n+1}} \subset I_{\varepsilon_1\cdots\varepsilon_n}, \qquad \forall \varepsilon_1, \cdots, \varepsilon_{n+1} \in \{0, \cdots, \ell^D - 1\} \ .$$

If $I = I_{\varepsilon_1\cdots\varepsilon_n} \in \mathcal{F}_n$ and $J = I_{\varepsilon_{n+1}\cdots\varepsilon_{n+p}} \in \mathcal{F}_p$, we note IJ the ℓ-adic cube

$$IJ = I_{\varepsilon_1\cdots\varepsilon_{n+p}} \in \mathcal{F}_{n+p}$$

obtained by the concatenation of the words $\varepsilon_1 \cdots \varepsilon_n$ and $\varepsilon_{n+1} \cdots \varepsilon_{n+p}$.

We say that the probability measure m is a quasi-Bernoulli measure on $[0,1)^D$, if we can find a constant $C \ge 1$ such that

$$\forall \ I, J \in \bigcup_n \mathcal{F}_n, \ \ \frac{1}{C} m(I)m(J) \le m(IJ) \le C m(I)m(J) \ . \tag{4.1}$$

Quasi-Bernoulli property appears in many situations. In particular, this is the case for the harmonic measure in regular Cantor sets ([7], [18]) and for the caloric measure in domains delimited by Weirstrass type graphs ([5]). The L^q-spectrum τ is defined as usual by

$$\tau(q) = \limsup_{n\to+\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n\log\ell} \log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right) .$$

In the case of quasi-Bernoulli measures, sub and super multiplicative properties of the sequences

$$C^{|q|}\sum_{I\in\mathcal{F}_n} m(I)^q \quad \text{and} \quad C^{-|q|}\sum_{I\in\mathcal{F}_n} m(I)^q$$

ensure that the sequence $\tau_n(q)$ converges and satisfies

$$C^{-|q|}\ell^{n\tau(q)} \le \sum_{I\in\mathcal{F}_n} m(I)^q \le C^{|q|}\ell^{n\tau(q)} . \tag{4.2}$$

We can see [6], [11] or [12] for more details.

It is well known that quasi Bernoulli measures satisfy the multifractal formalism (see [6]) and it is proved in [11] that the L^q-spectrum is of class C^1 on $\mathbb{R}$. In particular, according to (1.6), quasi-Bernoulli measures are unidimensional measures with dimension

$$d = -\tau'(1) .$$

The L^q-spectrum τ and the dimension d of the measure m have the following probabilistic interpretations which are detailed in [12]. If $I_n(x)$ is the unique cube in $\mathcal{F}_n$ containing x, let

$$\frac{S_n}{n} = \frac{X_1+\cdots+X_n}{n} \qquad \text{and} \qquad X_n(x) = -\log_\ell\left(\frac{m(I_n(x))}{m(I_{n-1}(x))}\right) .$$

In other words,

$$\frac{S_n}{n} = \frac{\log\left(m(I_n(x))\right)}{\log\left(|I_n(x)|\right)}$$

where $|I_n(x)| = \ell^{-n}$ is the "length" of the cube $I_n(x)$. The asymptotic behavior of the sequence of random variables S_n/n is then deeply related to the local behavior of the measure m and the dimension d of the measure m is the almost sure limit of the sequence of random variables S_n/n. Moreover,

$$\tau_n(1-q) = \frac{1}{n}\log_\ell \mathbb{E}[\ell^{qS_n}] \quad \text{and} \quad \tau(1-q) = \lim_{n\to+\infty}\frac{1}{n}\log_\ell \mathbb{E}[\ell^{qS_n}]$$

are related to the log-Laplace transform of the sequence S_n. Finally, (4.2) can be rewritten

$$C^{-1}\,\ell^{n\tau(1-q)} \le \mathbb{E}[\ell^{qS_n}] \le C\,\ell^{n\tau(1-q)} \tag{4.3}$$

where the constant C is independant of n and independant of q, provided q stays in a bounded set. Inequalities (4.3) will be usefull in the following sections.

There exists a symbolic counterpart μ to the quasi-bernoulli measure m which is defined on the symbolic Cantor space $\{0,\cdots,\ell^D-1\}^{\mathbb{N}^*}$ as the image of m through the application

$$J(x) = (\varepsilon_i)_{i\ge 1} \quad \text{if} \quad \{x\} = \bigcap_{n\ge 1} I_{\varepsilon_1\cdots\varepsilon_n} .$$

Carleson observed in [7] that such a quasi-Bernoulli measure μ on the Cantor set $\{0,\cdots,\ell^D-1\}^{\mathbb{N}^*}$ is strongly equivalent to a mesure $\tilde\mu$ (that is $\frac{1}{C}\mu \le \tilde\mu \le C\mu$ for some constant $C \geq 1$) which is shift-invariant and ergodic, where the shift operator S is defined by

$$S \;:\; (\varepsilon_i)_{i\geq 1} \in \{0,\cdots,\ell^D-1\}^{\mathbb{N}^*} \longmapsto (\varepsilon_i)_{i\geq 2} \in \{0,\cdots,\ell^D-1\}^{\mathbb{N}^*} \ .$$

Coming back to m, it follows that m is strongly equivalent to a quasi-Bernoulli measure $\tilde{m}$ which is T-invariant and ergodic where T is the "shift" operator on $[0,1)^D$ defined by

$$T \;:\; x = \bigcap_{n\geq 1} I_{\varepsilon_1\cdots\varepsilon_n} \longmapsto Tx = \bigcap_{n\geq 2} I_{\varepsilon_2\cdots\varepsilon_n} \ . \tag{4.4}$$

This will be a key in Section 4.2 and 4.3.

Let us finally describe the closed support of the quasi-Bernoulli measure m. If $\mathcal{G}_n$ is the set of ℓ-adic cubes $I \in \mathcal{F}_n$ such that $m(I) > 0$, it is clear that

$$\operatorname{supp}(m) = \bigcap_{n\geq 1} \bigcup_{I\in\mathcal{G}_n} \bar{I} \ .$$

More precisely, let

$$G = \left\{\varepsilon \in \{0,\cdots,\ell^D-1\} \ ; \quad m(I_\varepsilon) > 0\right\} \ .$$

Quasi-Bernoulli property ensures that

$$\mathcal{G}_n = \{I_{\varepsilon_1\cdots\varepsilon_n} \ ; \quad \forall i \in \{1,\cdots,n\},\ \varepsilon_i \in G\} \ .$$

In other words, in the symbolic counterpart, the associated measure μ is constructed on the smaller Cantor set $G^{\mathbb{N}^*}$.

Let $g = \sharp(G)$ be the cardinal of the set G. Define the homogeneous probability measure m_0 on $\operatorname{supp}(m)$ by the formula :

$$m_0(I) = g^{-n}, \quad \forall I \in \mathcal{G}_n \ .$$

Elementary properties of the measure m_0 allow us to conclude that the dimension δ of the compact set $\operatorname{supp}(m)$ satisfies $\delta = \log_\ell g$ and that there exists a constant $C > 0$ such that for every set Borel set A,

$$\frac{1}{C}\,\mathcal{H}^\delta(A \cap \operatorname{supp}(m)) \leq m_0(A) \leq C\,\mathcal{H}^\delta(A\cap\operatorname{supp}(m)) \ .$$

4.1. **A bound of type LIL.** According to Remark 3.4, it is natural to think that the quadratic term in the development of $\tau(1-q)$ near $q=0$ gives logarithmic corrections in the comparison between m and Hausdorff types measures. We are able to establish such estimations in the case of quasi-Bernoulli measures.

Theorem 4.1. *Let m be a quasi-Bernoulli measure with dimension $d = -\tau'(1)$. Suppose that there exists a real $\sigma \geq 0$ such that $\tau(1-q) = qd + \dfrac{\sigma^2}{2}q^2 + o(q^2)$ in a neighborhood of 0. Then, dm-almost surely,*

$$\begin{cases} \displaystyle\limsup_{n\to+\infty} \frac{S_n - nd}{\sqrt{2n\log_\ell\log_\ell n}} \leq \sigma \\[2ex] \displaystyle\liminf_{n\to+\infty} \frac{S_n - nd}{\sqrt{2n\log_\ell\log_\ell n}} \geq -\sigma \ . \end{cases}$$

Remark 4.2. Theorem 4.1 remains true when $\sigma = 0$. In that case, the conclusion is $\lim_{n\to+\infty} \frac{S_n - nd}{\sqrt{n \log_\ell \log_\ell n}} = 0$ dm-almost surely.

Remark 4.3. In general, we do not know if $\tau''(1)$ exists. Nevertheless, an important class of quasi-Bernoulli measures is constituted of Gibbs measures associated to an Hölder potential. In such a case, the L^q-spectrum is known to be analytic (see for example [24]) and the hypothesis in Theorem 4.1 are satisfied.

Remark 4.4. Return to the case of Bernoulli products described in Section 2. An easy calculation gives

$$\tau(q) = \log_2 \left(p^q + (1-p)^p\right) \quad \text{and} \quad \tau''(1) = \frac{p(1-p)}{\ln 2}\left(\ln(\frac{p}{1-p})\right)^2 = (\ln 2)\,\mathbb{V}[X_n]\ .$$

Here, $\ell = 2$ and the coefficient $\ln 2$ is due to the fact that functions Θ are not similarly normalised in Section 2 and in Theorem 4.1.

In order to prove Theorem 4.1, we need the following lemma, which is some kind of maximal lemma adapted to the situation .

Lemma 4.5. *Let $\varepsilon > 0$, $a > 0$ and $n_0 < n_1$ be two integers. Then, for $\frac{a}{n_1(\sigma+\varepsilon)^2}$ small enough, one has*

$$m\left\{\sup_{k\in\{n_0,\cdots,n_1\}} (S_k - kd) \ge a\right\} \le C\,\ell^{\frac{-a^2}{2n_1(\sigma+\varepsilon)^2}} \tag{4.5}$$

where C is a constant independent of all parameters.

Proof. For $n_0 \le k \le n_1$, let

$$A_k = \{x\ ;\quad (S_j - jd < a \ \text{ if } n_0 \le j < k) \quad \text{and} \quad S_k - kd \ge a\}\,.$$

We have to estimate $m\left(\bigcup_{k=n_0}^{n_1} A_k\right)$. Observe that A_k is the union of some cubes in $\mathcal{F}_k$ and denote by

$$\mathcal{A}_k = \{I \in \mathcal{F}_k\ ;\ I \subset A_k\}\ .$$

Let $0 < q < 1$. According to (4.3), we have

$$\begin{aligned}
\mathbb{E}[\ell^{qS_{n_1}}\mathbb{1}_{A_k}] &= \sum_{K\in\mathcal{F}_{n_1}\ K\subset A_k} m(K)^{1-q} \\
&= \sum_{I\in\mathcal{A}_k, J\in\mathcal{F}_{n_1-k}} m(IJ)^{1-q} \\
&\ge C\sum_{I\in\mathcal{A}_k} m(I)^{1-q} \sum_{J\in\mathcal{F}_{n_1-k}} m(J)^{1-q} \\
&= C\sum_{I\in\mathcal{A}_k} m(I)^{1-q}\mathbb{E}[\ell^{qS_{n_1-k}}] \\
&\ge C\,\mathbb{E}[\ell^{qS_k}\mathbb{1}_{\mathcal{A}_k}]\ell^{(n_1-k)\tau(1-q)} \\
&\ge C\,m(A_k)\ell^{q(kd+a)}\ell^{(n_1-k)\tau(1-q)}\ .
\end{aligned}$$

The constant C can change from line to line but is independant of k, n_1 and $q \in [0,1]$.

Remember that τ is convex and $d = -\tau'(1)$. We can finally find a constant $C > 0$, such that

$$\begin{aligned} m(A_k) &\leq C\,\mathbb{E}[\ell^{qS_{n_1}}\mathbb{1}_{A_k}]\ell^{-q(kd+a)}\ell^{-(n_1-k)\tau(1-q)} \\ &\leq C\,\mathbb{E}[\ell^{qS_{n_1}}\mathbb{1}_{A_k}]\ell^{-q(kd+a)}\ell^{-(n_1-k)dq} \\ &= C\,\mathbb{E}[\ell^{qS_{n_1}}\mathbb{1}_{A_k}]\ell^{-qa-qn_1d}\ . \end{aligned}$$

Let $\varepsilon > 0$. If q is small enough, we get

$$\begin{aligned} m(\bigcup_{k=n_0}^{n_1} A_k) &\leq C\,\mathbb{E}[\ell^{qS_{n_1}}]\ell^{-qa-qn_1d} \\ &\leq C\,\ell^{n_1(\tau(1-q)-qd)}\ell^{-qa} \\ &\leq C\,\ell^{n_1(\sigma+\varepsilon)^2\frac{q^2}{2}-qa}\ . \end{aligned}$$

This estimate is optimal for $q = \frac{a}{n_1(\sigma+\varepsilon)^2}$. Finally, if $\frac{a}{n_1(\sigma+\varepsilon)^2}$ is small enough, we obtain

$$m(\bigcup_{k=n_0}^{n_1} A_k) \leq C\,\ell^{-\frac{a^2}{2n_1(\sigma+\varepsilon)^2}}$$

which concludes the proof of Lemma 4.5. □

Proof of Theorem 4.1. The proof of Theorem 4.1 is quite standard. Fix $\varepsilon > 0$ and choose $\alpha > 1$ such that $\frac{(\sigma+2\varepsilon)^2}{\alpha(\sigma+\varepsilon)^2} > 1$. For $k \in \mathbb{N}$, let $n_k = [\alpha^k]$, (the integrand part of α^k) and

$$B_k = \left\{\exists\ n \in \{n_k, \cdots, n_{k+1}\} \quad : \quad S_n - nd \geq (\sigma + 2\varepsilon)\sqrt{2n_k \log_\ell \log_\ell n_k}\right\}$$

An easy calculation proves that

$$\frac{(\sigma + 2\varepsilon)\sqrt{2n_k \log_\ell \log_\ell n_k}}{n_{k+1}(\sigma+\varepsilon)^2}$$

goes to 0 when $k \to +\infty$, so that we can apply Lemma 4.5 when k is large enough. We get

$$m(B_k) \leq C\ [\log_\ell n_k]^{-\frac{(\sigma+2\varepsilon)^2 n_k}{(\sigma+\varepsilon)^2 n_{k+1}}}\ .$$

We claim that

$$[\log_\ell n_k]^{-\frac{(\sigma+2\varepsilon)^2 n_k}{(\sigma+\varepsilon)^2 n_{k+1}}} \sim [k \log_\ell \alpha]^{-\frac{(\sigma+2\varepsilon)^2}{\alpha(\sigma+\varepsilon)^2}}$$

so that $\sum_k m(B_k)$ converges. Hence, by the Borel-Cantelli lemma, almost surely, only finitely many of these events occur, which achieves the proof of the first estimation of Theorem 4.1. The second part of Theorem 4.1 can be proved in the same way. □

Theorem 4.1 allows us to compare the measure m with Hausdorff and packing measures . That is what is done in the following corollary.

Corollary 4.6. *Let* $\Theta(t) = \ell^{\sqrt{2\log_\ell 1/t \log_\ell \log_\ell \log_\ell 1/t}}$. *Then, for all* $\varepsilon > 0$,
1. $m \ll \mathcal{H}^\Psi$, *where* $\Psi(t) = t^h\Theta(t)^{\sigma+\varepsilon}$
2. $m \perp \widehat{\mathcal{P}}^\Psi$, *where* $\Psi(t) = t^h\Theta(t)^{-(\sigma+\varepsilon)}$.

Proof. The arguments are the same as in Section 2. □

4.2. **Estimations in the reverse sense.** The lack of independance does not allow us to have so precise estimations in the reverse sens. Nevertheless, we have the following general result.

Theorem 4.7. *Let m be a quasi-Bernoulli measure with dimension $d = -\tau'(1)$. Suppose that there exists a real $\sigma > 0$ such that $\tau(1-q) = qd + \frac{\sigma^2}{2}q^2 + o(q^2)$ in a neighborhood of* 0*. Then, almost surely,*

$$\begin{cases} \limsup_{n\to+\infty} \dfrac{S_n - nd}{\sqrt{n}} = +\infty \\ \\ \liminf_{n\to+\infty} \dfrac{S_n - nd}{\sqrt{n}} = -\infty \end{cases}$$

Proof. Remember that the mesure m is strongly equivalent to a quasi-Bernoulli measure $\tilde{m}$ which is "shift" invariant and ergodic (see the introduction of the section). The L^q-spectrum is the same for the two measures. Moreover, with obvious notations, there exists a constant $C > 0$ independant of n and x such that $|S_n - \tilde{S}_n| \le C$. It follows that the asymptotic behavior of the quantity $\frac{S_n - nd}{\sqrt{n}}$ is the same for the measure m and the measure $\tilde{m}$. We can then assume, without lost of generality, that the measure m is "shift" invariant and ergodic.

Let $A > 0$ and $0 < q < 1$,

$$\begin{aligned} \mathbb{E}[\ell^{qS_n}] &= \mathbb{E}[\ell^{qS_n}\mathbb{1}_{S_n \le nd + A\sqrt{n}}] + \mathbb{E}[\ell^{qS_n}\mathbb{1}_{S_n > nd + A\sqrt{n}}] \\ &\le \ell^{q(nd + A\sqrt{n})} + m(\{S_n > nd + A\sqrt{n}\})^{1/2}\mathbb{E}[\ell^{2qS_n}]^{1/2} . \end{aligned}$$

According to (4.3), we have

$$c_1\ell^{n\tau(1-q)} \le \ell^{q(nd+A\sqrt{n})} + c_2 m(\{S_n > nd + A\sqrt{n}\})^{1/2}\ell^{(n/2)\tau(1-2q)} .$$

So, if $\varepsilon > 0$ and q is small enough,

$$\begin{aligned} m(\{S_n > nd + A\sqrt{n}\})^{1/2} &\ge \frac{c_1\ell^{n(\tau(1-q)-qd)-qA\sqrt{n}} - 1}{c_2\ell^{n(\frac{1}{2}\tau(1-2q)-qd)-qA\sqrt{n}}} \\ &\ge \frac{c_1\ell^{n(\sigma-\varepsilon)^2\frac{q^2}{2}-qA\sqrt{n}} - 1}{c_2\ell^{n(\sigma+\varepsilon)^2q^2-qA\sqrt{n}}} . \end{aligned}$$

Take $q = \frac{\lambda}{\sqrt{n}}$. We get

$$m(\{S_n > nd + A\sqrt{n}\})^{1/2} \ge \frac{c_1\ell^{(\sigma-\varepsilon)^2\frac{\lambda^2}{2}-\lambda A} - 1}{c_2\ell^{(\sigma+\varepsilon)^2\lambda^2-\lambda A}} .$$

We can choose λ large enough such that $c_1\ell^{(\sigma-\varepsilon)^2\frac{\lambda^2}{2}-\lambda A} - 1 > 0$. It follows that there exists a constant $c > 0$ such that for sufficiently large n,

$$m(\{S_n > nd + A\sqrt{n}\}) \ge c .$$

Finally

$$m(\{S_n > nd + A\sqrt{n} \;\; i.o\}) \ge c .$$

Recall that $S_n = -\log_\ell(m(I_n(x)))$. Quasi-Bernoulli property implies that for dm-almost all $x \in [0,1)^D$,

$$|\log_\ell(m(I_n(x))) - \log_\ell(m(I_n(Tx)))| \le C,$$

where T is the "shift" operator described in (4.4). Finally, the set

$$\{S_n > nd + A\sqrt{n} \ \ i.o\}$$

is shift invariant and we can conclude that

$$m(\{S_n > nd + A\sqrt{n} \ \ i.o\}) = 1 \ .$$

The real A being arbitrary large, we obtain the first part of Theorem 4.7. We can prove the second part of Theorem 4.7 is a similar way, using estimations of $\mathbb{E}[\ell^{qS_n}]$ with $q < 0$. □

Corollary 4.8. *The hypothesis are the same as in Theorem 4.7. Let $a \in \mathbb{R}$ and $\psi_a(t) = t^d \ell^{a\sqrt{\log_\ell 1/t}}$. We have*

$$\forall a > 0,\ m \perp \mathcal{H}^{\psi_a} \qquad \textit{and} \qquad \forall a < 0,\ m \ll \widehat{\mathcal{P}}^{\psi_a} \ .$$

Proof. Let $a \in \mathbb{R}$. Theorem 4.7 naturally implies that dm-almost surely, infinitely often,

$$\begin{cases} m(I_n(x)) \leq |I_n(x)|^d \ell^{a\sqrt{-\log_\ell(|I_n(x)|)}} \\ \\ m(I_n(x)) \geq |I_n(x)|^d \ell^{a\sqrt{-\log_\ell(|I_n(x)|)}} \end{cases}$$

which gives the conclusion with similar arguments as in Section 2. □

In particular we can deduce :

Corollary 4.9. *The measure m satisfies the following properties :*

1. *There exists $E \subset \mathbb{R}^D$ such that $m(E) = 1$ and $\mathcal{H}^d(E) = 0$. In particular $m \perp \mathcal{H}^d$.*
2. *If $\widehat{\mathcal{P}}^d(E) < +\infty$ then $m(E) = 0$. In particular $m \ll \widehat{\mathcal{P}}^d$.*

4.3. **More general estimations.** As remarked in the previous section, in the general case, we do not know if $\tau''(1)$ exists and is strictly positive. Nevertheless, in the general case we can obtain the less precise following result.

Theorem 4.10. *Let m be a quasi-Bernoulli measure with dimension $d = -\tau'(1)$. Let $\chi(q) = \tau(1-q) - qd$. Suppose that there exists a constant $C > 0$ such that*

$$\forall q \in (0,1], \quad 0 < \chi(q) \leq C\,\chi(q/2) \ . \tag{4.6}$$

Then, dm-almost surely,

$$\limsup_{n \to +\infty} \frac{S_n - nd}{\theta(n)} \geq 1,$$

where $\theta(t) = \frac{1}{\chi^{-1}(1/t)}$ and χ^{-1} is the inverse fonction of χ on $[0, \chi(1)]$.
As a consequence, for all $0 < a < 1$ we have

$$m \ll \widehat{\mathcal{P}}^{\psi_a} \quad \textit{where} \quad \psi_a(t) = t^d \ell^{-a\theta(\log_\ell 1/t)} \ .$$

In particular, m is absolutely continuous with respect to $\widehat{\mathcal{P}}^d$.

Remark 4.11. Hypothesis (4.6) states that function χ is not flat when $q \to 0$, $q > 0$. We know that χ is a continuous convex function such that $\chi(0) = 0$. It follows that $\chi(q) \geq 2\,\chi(q/2)$. It is then easy to check that under hypothesis (4.6), there exists $\alpha > 1$ and $C > 0$ such that when $q > 0$ is small enough

$$\chi(q) \geq C\,q^\alpha \ .$$

Whe can then deduce that

$$\theta(t) \le C\, t^{1/\alpha}$$

for some $C > 0$ and we can replace $\theta(t)$ by t^β with $\beta = 1/\alpha$ in Theorem 4.10.

Of course, we can obtain a similar type result if we have some information on $\chi(q)$ when $q \to 0$ and $q < 0$.

Theorem 4.12. *The notations are the same as in Theorem 4.10. Suppose that there exists a constant $C > 0$ such that*

$$\forall q \in [-1, 0), \quad 0 < \chi(q) \le C\, \chi(q/2)\ . \tag{4.7}$$

Then, dm-almost surely,

$$\liminf_{n\to+\infty} \frac{S_n - nd}{\theta(n)} \le -1,$$

where $\theta(t) = \frac{1}{\chi^{-1}(-1/t)}$ and χ^{-1} is the inverse fonction of χ on $[0, \chi(-1)]$.
As a consequence, we have, for all $0 < a < 1$,

$$m \perp \mathcal{H}^{\psi_a} \quad \text{where} \quad \psi_a(t) = t^d \ell^{a\theta(\log_\ell 1/t)}\ .$$

In particular, m is singular with respect to $\mathcal{H}^d$.

Remark 4.13. As observed in Remark 4.11 we can replace $\theta(t)$ by t^β for some $\beta < 1$ in the conclusions of Theorem 4.12.

We now give the proof of Theorem 4.10. The proof of Theorem 4.12 is similar.

Proof of Theorem 4.10. The function χ is a continuous convex function on $[0, 1]$ such that $\chi(0) = 0$ and $\chi(q) > 0$ if $q > 0$. It follows that χ is increasing and we can define the inverse χ^{-1} on $[0, \chi(1)]$.

Let $0 < a < 1$ and $q > 0$ sufficiently small. Using the same argument as in Theorem 4.7, we have

$$\begin{aligned} m(\{S_n \ge nd + a\theta(n)\})^{1/2} &\ge \frac{c_1 \ell^{n\chi(q) - aq\theta(n)} - 1}{c_2 \ell^{\frac{n}{2}\chi(2q) - aq\theta(n)}} \\ &\ge \frac{c_1 \ell^{n\chi(q) - aq\theta(n)} - 1}{c_2 \ell^{\frac{nC}{2}\chi(q) - aq\theta(n)}}\ . \end{aligned}$$

If $\lambda > 0$ and $q = \chi^{-1}(\lambda/n)$, we get

$$m(\{S_n \ge nd + a\theta(n)\})^{1/2} \ge \frac{c_1 \ell^{\lambda - a\chi^{-1}(\lambda/n)\theta(n)} - 1}{c_2 \ell^{\frac{C\lambda}{2} - a\chi^{-1}(\lambda/n)\theta(n)}}\ .$$

Recall that χ^{-1} is a concave function on $[0, \chi(1)]$ such that $\chi^{-1}(0) = 0$. It follows that $\chi^{-1}(\lambda t) \le \lambda \chi^{-1}(t)$ if $\lambda \ge 1$ and $0 \le \lambda t \le \chi(1)$. Finally,

$$\lambda - a\chi^{-1}(\lambda/n)\theta(n) \ge \lambda(1 - a)$$

if $\lambda \ge 1$ and $n \ge \lambda/\chi(1)$.

Choose λ such that $c_1 \ell^{\lambda(1-a)} - 1 > 0$. We get

$$m(\{S_n \ge nd + a\theta(n)\})^{1/2} \ge \frac{c_1 \ell^{\lambda(1-a)} - 1}{c_2 \ell^{\frac{C\lambda}{2} - a\chi^{-1}(\lambda/n)\theta(n)}} \ge \frac{c_1 \ell^{\lambda(1-a)} - 1}{c_2 \ell^{\frac{C\lambda}{2}}} = c > 0$$

if n is sufficiently large. The end of the proof is the same as in Theorem 4.7. □

Theorem 4.10 and Theorem 4.12 can be applied in the important case where the function τ is analytic. This is in particular the case when the measure m is a Gibbs measure associated to an Hölder potential (see [24]).

Corollary 4.14. *Let m be a quasi-Bernoulli measure in $[0,1)^D$. Let*

$$\delta = \dim(\mathrm{supp}\,(m)) \quad \textit{and} \quad d = \dim(m)\ .$$

Suppose that τ is analytic. There are only two possible cases :

(i) *$d = \delta$ and the measure m is strongly equivalent to the Hausdorff measure $\mathcal{H}^\delta$ on* supp (m)*.*

or

(ii) *$d < \delta$ of the measure m is singular with respect to $\mathcal{H}^d$ but absolutely continuous with respect to $\widehat{\mathcal{P}}^d$.*

Proof. As in the introduction of the section, denote by m_0 the homogeneous measure on supp (m). The measure m_0 is strongly equivalent to the Hausdorff measure $\mathcal{H}^\delta$ on supp (m). Using a similar argument as in [12] Corollary 5.5, it is classical to prove that, if the quasi-Bernoulli measure m is not strongly equivalent to the measure m_0, its dimension d satisfies $d < \delta$. On the other hand, we know that $\tau(0) = \dim(\mathrm{supp}(m)) = \delta$. In the case where $d < \delta$, we can then conclude that $\tau(1-q) \not\equiv dq$. If τ is analytic, we obtain that there exists a smallest integer $n \geq 2$ such that $\tau^{(n)}(1) \neq 0$. Moreover τ is convex. It follows that $n = 2k$ is even and $\tau^{(n)}(1) = \lambda$ is a strictly positive real number. We can then write

$$\tau(1-q) = dq + \lambda q^{2k} + o(q^{2k})$$

in a neighborhood of $q = 0$. Finally, the hypothesis of Therorem 4.10 and Theorem 4.12 are satisfied. □

References

[1] M. Arbeiter and N. Patzschke. Random self-similar multifractals. (1996) 542. *Math. Nachr.*, **181** : 5–42, 1996.

[2] A. Batakis. Harmonic measure of some Cantor type sets. *Ann. Acad. Sci. Fenn.*, **21** : 255–270, 1996.

[3] F. BenNasr, I. Bhouri and Y. Heurteaux. The validity of the multifractal formalism : results and examples. *Adv. Math.*, **165** : 264–284, 2002.

[4] J. Bourgain. On the Hausdorff dimension of harmonic measure in higher dimension. *Invent. Math.*, **87** : 477–483, 1987.

[5] T. Bousch and Y. Heurteaux. Caloric measure on domains bounded by Weierstrass-type graphs. *Ann. Acad. Sci. Fenn.*, **25** : 501–522, 2000.

[6] G. Brown, G. Michon and J. Peyrière. On the Multifractal Analysis of Measures. *J. Stat. Phys.*, **66** : 775–790, 1992.

[7] L. Carleson. On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn.*, **10** : 113–123, 1985.

[8] K. Falconer. *Fractal Geometry : Mathematical Foundations and Applications.* John Wiley & Sons Ltd., New-York, 1990.

[9] K. Falconer. *Techniques in Fractal Geometry.* John Wiley & Sons Ltd., New-York, 1997.

[10] S. Graf. On Bandt's tangential distribution for self-similar measures. *Monatsh.Math.*, **120** : 223–246, 1995.

[11] Y. Heurteaux. Estimations de la dimension inférieure et de la dimension supérieure des mesures. *Ann. Inst. H. Poincaré Probab. Statist.*, **34** : 309–338, 1998.

[12] Y. Heurteaux. Dimension of measures: the probabilistic approach. *Publ. Mat.*, **51** : 243–290, 2007.

[13] J.E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, **30** : 713–747, 1981.

[14] K.S. Lau and S.M. Ngai. Multifractal measures and a weak separation condition. *Adv. Math.*, **141** : 45–96, 1999.
[15] J.G. Llorente and A. Nicolau. Regularity properties of measures, entropy and the law of the iterated logarithm. *Proc. London Math. Soc. (3)*, **89** : 485–524, 2004.
[16] A.O. Lopes. The dimension spectrum of the maximal measure. *SIAM J. Math. Anal.*, **20** : 1243–1254, 1989.
[17] N.G. Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc. (3)*, **51** : 369–384, 1985.
[18] N.G. Makarov and A. Volberg. On the harmonic measure of discontinous fractals. preprint LOMI E-6-86, Lenningrad, 1986.
[19] P. Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995.
[20] S.M. Ngai. A dimension result arising from the L^q spectrum of a measure. *Proc. Amer. Math. Soc.*, **125** : 2943–2951, 1997.
[21] M. Smorodinski. Singular measures and Hausdorff measures. *Israel J. Math.*, **7** : 203–206, 1969.
[22] C. Tricot. Two definitions of fractional dimension. *Math. Proc. Cambridge Philos. Soc.*, **91** : 57–74, 1982.
[23] Y.L. Ye. Multifractal of self-conformal measures. *Non Linearity*, **18** : 2111–2133, 2005.
[24] M. Zinsmeister. *Formalisme thermodynamique et systèmes dynamiques holomorphes*, volume **4** of *Panoramas et synthèses*. Société Mathématique de France, 1997.

Faculté des Sciences de Monastir, 5000 Monastir, Tunisia
E-mail address: Imen.Bhouri@fsm.rnu.tn

Clermont Université, Université Blaise Pascal, Laboratoire de Mathématiques, bp 10448, F-63000 Clermont-Ferrand, France

CNRS, UMR 6620, Laboratoire de Mathématiques, F-63177 Aubière, France
E-mail address: Yanick.Heurteaux@math.univ-bpclermont.fr