Generic boundary behaviour for harmonic functions in the ball

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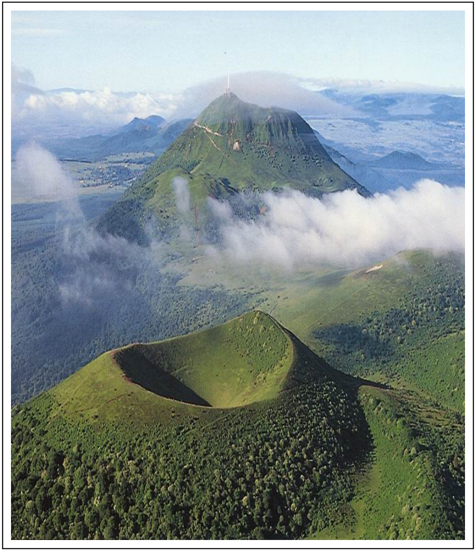
Joint work with Frédéric Bayart



The beginning of the story

- If $f \in L^p(\mathbb{T})$, $p > 1$, $S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{inx}$ is almost surely convergent (Carleson Theorem) but there are possible divergence points.
- For a given β, what is the size of the set of points x for which $|S_n f(x)| \gg n^\beta$ i.o. ? (Aubry 2006)
- What is the behaviour of $S_n f$ for a generic function $f \in L^p$?
- Let $\beta(x)$ be the supremum of the *beta* such that $|S_n f(x)| \gg n^\beta$ i.o. and $E(\beta, f) = \{x \in \mathbb{T} \; ; \; \beta(x) = \beta\}$.
  If f is a generic function in $L^p(\mathbb{T})$,

  $$\text{for any } \beta \in [0, 1/p], \quad \dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta p.$$

  (Bayart, H., 2011)
- Always true when $p = 1$ (Bayart, H., 2012)
- What about $P_r * f(x) = \sum_{k=-\infty}^{+\infty} r^{|k|} \hat{f}(k) e^{ikx}$ when $r \to 1$?
- $h(re^{ix}) = P_r * f(x)$ is harmonic in the unit disk.
- $r \to 1$ corresponds to the radial convergence in the disk.

Harmonic functions in the ball B_{d+1}

The Poisson kernel:

$$P(x,\xi) = \frac{1-\|x\|^2}{\|x-\xi\|^{d+1}}.$$

- Bounded harmonic functions

$$h(x) = P[f](x) = \int_{\mathcal{S}_d} P(x,\xi)\, f(\xi) d\sigma(\xi) \quad \text{with} \quad f \in L^\infty(\mathcal{S}_d)$$

- Nonnegative harmonic functions

$$h(x) = P[\mu](x) = \int_{\mathcal{S}_d} P(x,\xi)\, d\mu(\xi) \quad \text{with} \quad \mu \in \mathcal{M}^+(\mathcal{S}_d)$$

- Harmonic functions with L^1 data

$$h(x) = P[f](x) = \int_{\mathcal{S}_d} P(x,\xi)\, f(\xi) d\sigma(\xi) \quad \text{with} \quad f \in L^1(\mathcal{S}_d)$$

Fatou's Theorem

- Fatou (1906) : if $f \in L^\infty(\mathbb{T})$, then

$$P_r * f(x) \to f(x) \quad \text{almost surely.}$$

- Generalizations (Hardy-Littlewood, Wiener, Bochner...)

$$P[\mu](ry) \to \frac{d\mu}{d\sigma}(y) \quad d\sigma\text{-almost surely} \quad \text{when } r \to 1.$$

- Hunt and Wheeden (1970) : If h is a nonnegative harmonic function in a Lipschitz domain $U \subset \mathbb{R}^n$, then h has a non tangential limit at almost every point of the boundary ∂U.

What about divergence points ?

An elementary upper bound for $|P[f](ry)|$ when $r \to 1$:

$$P(x,\xi) = \frac{1-\|x\|^2}{\|x-\xi\|^{d+1}} \le \frac{1-\|x\|^2}{(1-\|x\|)^{d+1}} \le \frac{2}{(1-\|x\|)^d}$$

$$|P[f](ry)| = \left|\int_{\mathcal{S}_d} \frac{1-\|ry\|^2}{\|ry-\xi\|^{d+1}} f(\xi)d\sigma(\xi)\right| \le \frac{2\|f\|_1}{(1-r)^d}$$

Question

Let $0 < \beta \le d$. What can we say about the size of the set of points y such that $|P[f](ry)| \approx (1-r)^{-\beta}$ when $r \to 1$?

Hausdorff dimension of exceptional sets

$0 < \beta < d$

$$\mathcal{E}(\beta, f) = \left\{ y \in \mathcal{S}_d;\ \limsup_{r \to 1} \frac{|P[f](ry)|}{(1-r)^{-\beta}} = +\infty \right\}$$

Theorem (Bayart, H.)

- *For any* $f \in L^1(\mathcal{S}_d)$, $\dim_{\mathcal{H}} \left(\mathcal{E}(\beta, f) \right) \leq d - \beta$.
- *If* $E \subset \mathcal{S}_d$ *is such that* $\dim_{\mathcal{H}}(E) < d - \beta$, *there exists* $f \in L^1(\mathcal{S}_d)$ *such that* $E \subset \mathcal{E}(\beta, f)$.

The first part was already obtained by Armitage (1981) in the context of the half upper space.

A more precise result

Let τ be a nonnegative nonincreasing function such that

$$\lim_{s\to 0^+} \tau(s) = +\infty, \quad \tau(s) \ll s^{-d} \quad \text{and} \quad \tau(s) \approx \tau(2s).$$

Define

$$\mathcal{E}(\tau, f) = \left\{ y \in \mathcal{S}_d;\ \limsup_{r\to 1} \frac{|P[f](ry)|}{\tau(1-r)} = +\infty \right\}.$$

Let ϕ be the gauge function defined by $\phi(s) = \tau(s)s^d$.

Theorem (Bayart, H.)

- *For any $f \in L^1(\mathcal{S}_d)$, $\mathcal{H}^\phi\big(\mathcal{E}(\tau, f)\big) = 0$.*
- *If $E \subset \mathcal{S}_d$ is such that $\mathcal{H}^\phi\big(E\big) = 0$, there exists $f \in L^1(\mathcal{S}_d)$ such that $E \subset \mathcal{E}(\tau, f)$.*

The Hardy-Littlewwod maximal inequality

$$P[\mu](x) = \int_{\mathcal{S}_d} P(x,\xi)\, d\mu(\xi)$$

$$\boxed{\sup_{r\in(0,1)} |P[\mu](ry)| \leq \sup_{\delta>0} \frac{|\mu|(\kappa(y,\delta))}{\sigma(\kappa(y,\delta))}}$$

where $\kappa(y,\delta) = \big\{\xi \in \mathcal{S}_d;\ \|\xi - y\| < \delta\big\}$.
$\kappa(y,\delta)$ is called a cap.

Lemma (a quantitative improvement - Bayart, H.)

Let $0 < r < 1$. There exists $\delta \geq 1 - r$ such that

$$\big|P[\mu](ry)\big| \leq C\frac{|\mu|(\kappa(y,\delta))}{\sigma(\kappa(y,\delta))},$$

where C is a constant independent of μ, r and y.

Dimension of $\mathcal{E}(\beta,\mu)$: the upper bound

$\tau(s) = s^{-\beta}$.

$$\mathcal{E}(\beta,\mu) = \left\{ y \in \mathcal{S}_d;\ \limsup_{r\to 1} \frac{|P[\mu](ry)|}{(1-r)^{-\beta}} = +\infty \right\}$$

$$\mathcal{E}_M = \left\{ y \in \mathcal{S}_d;\ \limsup_{r\to 1} \frac{|P[\mu](ry)|}{(1-r)^{-\beta}} > M \right\}.$$

Let $y \in \mathcal{E}_M$.
Using the previous lemma, we can find r_y as close to 1 as we want and a cap $\kappa_y = \kappa(y,\delta_y)$ with $\delta_y \geq 1 - r_y$

$$M(1-r_y)^{-\beta} < |P[\mu](r_y y)| \leq C \frac{|\mu|(\kappa_y)}{\sigma(\kappa_y)}.$$

δ_y goes to 0 when r_y goes to 1.

Dimension of $\mathcal{E}(\beta,\mu)$: the upper bound

$$(1-r_y)^{-\beta}\sigma(\kappa_y) < \frac{C}{M}|\mu|(\kappa_y).$$

By the Vitali covering lemma, we can find a family of disjoint caps $(\kappa_{y_j})_{j\in\mathbb{N}}$ such that $\mathcal{E}_M \subset \bigcup_i 5\kappa_{y_i}$.

$$\sum_i (1-r_{y_i})^{-\beta}\sigma(\kappa_{y_i}) \leq \frac{C}{M}\|\mu\|$$

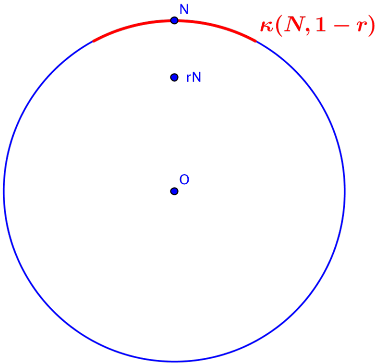
$$\sum_i \delta_{y_i}^{d-\beta} \leq \frac{C}{M}\|\mu\|$$

$$\mathcal{H}^{d-\beta}(\mathcal{E}_M) \leq \frac{C}{M}\|\mu\|$$

$$\boxed{\mathcal{H}^{d-\beta}(\mathcal{E}(\beta,\mu)) = 0}$$

Lower bound for the dimension : an elementary lemma

$$\text{If } r > 1/2, \qquad \int_{\kappa(N,1-r)} P(rN,\xi)\,d\sigma(\xi) \geq C$$



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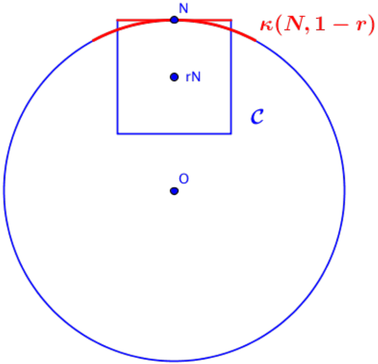

Lower bound for the dimension : the construction

Let E be such that $\mathcal{H}^{d-\beta}(E)=0$. Let $\mathcal{R}_j$ be a 2^{-j}-covering of E by caps such that

$$\sum_{\kappa\in\mathcal{R}_j}|\kappa|^{d-\beta}\leq 2^{-j}.$$

Define

$$\mathcal{C}_n=\left\{\kappa\in\bigcup_j\mathcal{R}_j;\ 2^{-(n+1)}<|\kappa|\leq 2^{-n}\right\}.$$

$$E\subset\limsup_n E_n\quad\text{where}\quad E_n=\bigcup_{\kappa\in\mathcal{C}_n}\kappa.$$

$$\sum_{n\geq 1}\sum_{\kappa\in\mathcal{C}_n}|\kappa|^{d-\beta}\leq\sum_{j\geq 1}\sum_{\kappa\in\mathcal{R}_j}|\kappa|^{d-\beta}\leq 1.$$

Choose $(\omega_n)_{n\geq 1}$ tending to infinity such that

$$\sum_{n\geq 1}\omega_n\sum_{\kappa\in\mathcal{C}_n}|\kappa|^{d-\beta}<+\infty.$$

Lower bound for the dimension : the function f

$$f = \sum_{n\geq 1} \omega_n 2^{-n\beta} \sum_{\kappa\in\mathcal{C}_n} \mathbb{1}_{4\kappa}$$

Let $y \in E_n = \bigcup_{\kappa\in\mathcal{C}_n} \kappa$.

Let $\kappa_0 \in \mathcal{C}_n$ such that $y \in \kappa_0$ and $r = 1 - 2^{-n}$.

$$\begin{aligned} P[f](ry) &\geq \omega_n 2^{-n\beta} \int_{4\kappa_0} P(ry,\xi)\, d\sigma(\xi) \\ &\geq \omega_n 2^{-n\beta} \int_{\kappa(y,2^{-n})} P(ry,\xi)\, d\sigma(\xi) \\ &\geq C\omega_n (1-r)^{-\beta}. \end{aligned}$$

$$E \subset \limsup_n E_n \subset \mathcal{E}(\beta, f)$$

The divergence index

Let $f \in L^1(\mathcal{S}_d)$ and $y_0 \in \mathcal{S}_d$.

$$\begin{aligned}\beta(y_0) &= \sup\left(\beta\ ;\ y_0 \in \mathcal{E}(\beta, f)\right)\\ &= \inf\left(\beta\ ;\ |P[f](ry_0)| = O((1-r)^{-\beta})\right)\\ &= \limsup_{r\to 1} \frac{\log|P[f](ry_0)|}{-\log(1-r)}\ .\end{aligned}$$

Level sets :

$$E(\beta, f) = \{y \in \mathcal{S}_d;\ \beta(y) = \beta\}\ .$$

The family $(\mathcal{E}(\beta, f))_\beta$ is a nonincreasing family of sets and the sets $(E(\beta, f))_\beta$ are disjoints.

Spectrum of singularities :

$$\beta \mapsto \dim_{\mathcal{H}}\left(E(\beta, f)\right)\ .$$

Multifractal behavior of $P[f]$

Of course,

$$\dim_{\mathcal{H}}\left(E(\beta,f)\right)\le d-\beta\ .$$

Theorem (Bayart, H.)

For quasi-all functions $f\in L^1(\mathcal{S}_d)$,

$$\forall\beta\in[0,d],\quad \dim_{\mathcal{H}}\left(E(\beta,f)\right)=d-\beta\ .$$

- Roughly speaking, for any β, $|P[f](ry)|\approx(1-r)^{-\beta}$ in a set with dimension $d-\beta$.
- "quasi-all" is related to the Baire category theorem.
- For such f we also have $\dim_{\mathcal{H}}\left(\mathcal{E}(\beta,f)\right)=d-\beta$.

The analogue of dyadic numbers in the sphere $\mathcal{S}_d$

There exists a sequence $(\mathcal{R}_n)_{n\geq 1}$ of finite subsets of $\mathcal{S}^d$ satisfying

- $\mathcal{R}_n \subset \mathcal{R}_{n+1}$;
- $\bigcup_{x\in\mathcal{R}_n} \kappa(x, 2^{-n}) = \mathcal{S}_d$;
- $\mathrm{card}\,(\mathcal{R}_n) \leq C2^{nd}$;
- For any x, y in $\mathcal{R}_n$, $x \neq y$, then $|x - y| \geq 2^{-n}$.

If $\alpha > 1$, let $N_{n,\alpha} = [n/\alpha] + 1$ and

$$D_{n,\alpha} = \bigcup_{x\in\mathcal{R}_{N_{n,\alpha}}} \kappa\left(x, 2^{-n}\right).$$

Proposition

$$\mathcal{H}^{d/\alpha}\left(\limsup_{n\to+\infty} D_{n,\alpha}\right) = +\infty.$$

Proof : mass transference principle.

Remark : we can replace n by a subsequence n_k.

In the way of saturating functions

$$f_n := \frac{1}{n+1} \sum_{N=1}^{n+1} \sum_{x \in \mathcal{R}_N} 2^{(n-N)d} \mathbb{1}_{\kappa(x, 2\cdot 2^{-n})}.$$

Proposition

$f_n \in L^1(\mathcal{S}_d)$ and $\|f_n\|_1 \leq C$.
Moreover, for any $\alpha > 1$, for any $y \in D_{n,\alpha}$,

$$P[f_n](r_n y) \geq \frac{C}{n} 2^{(n-N_{n,\alpha})d},$$

where $1 - r_n = 2^{-n}$, $N_{n,\alpha} = [n/\alpha] + 1$ and C is independent of n and α.

Remark : $2^{(n-N_{n,\alpha})d} \approx (1 - r_n)^{-\beta}$ if $\frac{d}{\alpha} = d - \beta$.

Construction of a dense sequence

Proposition

There exists a dense sequence $(h_n)_{n\geq 1}$ in $L^1(\mathcal{S}_d)$ such that for any $n \geq 1$, for any $\alpha > 1$ and any $y \in D_{n,\alpha}$,

$$P[h_n](r_n y) \geq \frac{C}{n^2} 2^{(n-N_{n,\alpha})d},$$

where $r_n = 1 - 2^{-n}$.

Let $(g_n)_{n\geq 1}$ be a sequence of continuous functions which is dense in $L^1(\mathcal{S}_d)$ ans such that $\|g_n\|_\infty \leq n$.

$$\boxed{h_n = \frac{1}{n} f_n + g_n}$$

The dense $\mathcal{G}_\delta$ set

The residual set we will consider is the dense G_δ-set

$$A = \bigcap_{k\geq 1} \bigcup_{n\geq k} B_{L^1}(h_n, \delta_n).$$

where δ_n is such that $\quad \|f\|_1 \leq \delta_n \Rightarrow \|P[f](r_n\cdot)\|_\infty \leq 1.$

If $\|f - h_n\|_1 < \delta_n$ and $y \in D_{n,\alpha}$,

$$\frac{\log|P[f](r_n y)|}{-\log(1-r_n)} \geq \left(d - \frac{N_{n,\alpha} d}{n}\right) + o(1).$$

$d - \frac{N_{n,\alpha} d}{n} \approx d - \frac{d}{\alpha} := \beta \quad$ if $\quad \frac{d}{\alpha} = d - \beta.$

The case of nonnegative harmonic functions

The set $\mathcal{H}^+(B_{d+1})$ of nonnegative harmonic functions in the ball B_{d+1} endowed with the topology of the locally uniform convergence is a closed cone in the space of all continuous functions in the ball : it satisfies Baire's property.

Theorem

For quasi-all nonnegative harmonic functions h in the unit ball B_{d+1}, for any $\beta \in [0, d]$,

$$\dim_{\mathcal{H}} \big(E(\beta, h) \big) = d - \beta$$

where

$$E(\beta, h) = \left\{ y \in \mathcal{S}_d \ ; \ \limsup_{r \to 1} \frac{\log h(ry)}{-\log(1-r)} = \beta \right\}.$$

Thank you for your attention