ADELIC APPROXIMATION ON SPHERES

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Abstract. We establish an adelic version of Dirichlet's approximation theorem on spheres. Let K be a number field, E be a rigid adelic space over K and $q\colon E \to K$ be a quadratic form. Let v be a place of K and $\alpha \in E \otimes_K K_v$ such that $q(\alpha) = 1$. We produce an explicit constant c having the following property. If there exists $x \in E$ such that $q(x) = 1$ then, for any $T > c$, there exists $(\upsilon, \phi) \in E \times K$, with $\max(\|\upsilon\|_{E,v}, |\phi|_v) \leq T$ and $\max(\|\upsilon\|_{E,w}, |\phi|_w)$ controlled for any place w, satisfying $q(\upsilon) = \phi^2 \neq 0$ and $|q(\alpha\phi - \upsilon)|_v \leq c|\phi|_v/T$. This remains true for certain infinite algebraic extensions as well as for a compact set of places of K. Our statements generalize and improve on earlier results by Kleinbock & Merrill (2015) and Moshchevitin (2017). The proofs rely on the quadratic Siegel's lemma in a rigid adelic space obtained by the author and Rémond (2017).

1. Introduction

Let $n \geq 1$ be an integer and $q\colon \mathbb{R}^n \to \mathbb{R}$ be a positive-definite quadratic form. According to the Euclidean variant of Dirichlet's approximation theorem, for any $\alpha \in \mathbb{R}^n$ and any real number $T > 0$, there exists $(\upsilon, \phi) \in (\mathbb{Z}^n \times \mathbb{Z}) \setminus \{0\}$ such that

$$0 \leq \phi \leq T \quad \text{and} \quad q(\phi\alpha - \upsilon) \leq \frac{n(\det q)^{1/n}}{T^{2/n}},$$

where $\det q$ is the determinant of the symmetric matrix $A(q)$ associated to the quadratic form q (in the canonical basis of $\mathbb{R}^n$). Its proof consists of applying Minkowski's theorem to the lattice $\mathbb{Z}^n \times \mathbb{Z}$ endowed with the Euclidean structure $q(\phi\alpha - \upsilon) + a\phi^2$ for a well-chosen positive real number a (see Appendix a). In 2015, Kleinbock and Merrill published a similar statement in the particular case $q(x) = x_1^2 + \cdots + x_n^2$ but with the additional property $q(\upsilon) = \phi^2$ satisfied by the solution (υ, ϕ) [KM]. In their result $q(\phi\alpha - \upsilon)$ is bounded by $c(n)\phi/T$ for some positive constant $c(n)$. A generalization to any positive-definite quadratic form such that $A(q) \in \mathrm{M}_n(\mathbb{Z})$ was achieved in 2014 by Fishman et al. (see [FKMS, Theorem 5.1]). These results obtained by means of methods from dynamic systems were made explicit by Moshchevitin [Mo], with different arguments from the geometry of numbers.

The aim of this article is to improve these constants while simplifying the proofs and providing an adelic generalization. Our first result makes use of the Hermite constant γ_n in dimension n which is the greatest first minimum of unimodular lattices in the Euclidean space $\mathbb{R}^n$.

Theorem 1.1. *Let $q\colon \mathbb{R}^n \to \mathbb{R}$ be a positive-definite quadratic form such that $A(q)$ has integral coefficients. Assume that there exists $x \in \mathbb{Q}^n$ such that $q(x) = 1$. Then, for all $\alpha \in \mathbb{R}^n$ such that $q(\alpha) = 1$, for all real numbers $T \geq (2\gamma_n)^{n/2}\sqrt{(\det q)/2}$, there exist $\upsilon \in \mathbb{Z}^n$ and $\phi \in \mathbb{Z}$ with $1 \leq \phi \leq T$ satisfying*

$$q\left(\frac{\upsilon}{\phi}\right) = 1 \quad \textit{and} \quad q\left(\alpha - \frac{\upsilon}{\phi}\right) \leq \frac{2(2\gamma_n)^n \det q}{\phi T}.$$

Applying this statement to

$$T' = \max\left(T, \frac{(2\gamma_n)^n \det q}{2T}\right)$$

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for $T > 0$, which always satisfies the condition $T' \geq (2\gamma_n)^{n/2}\sqrt{(\det q)/2}$, leads to a variant where T is only assumed to be positive. Besides the real number $2(2\gamma_n)^n \det q$ is smaller than the $0.8n$-th root of the constant κ_q which is in [Mo, Theorem 1] (see Appendix b) but we were unable to determine whether the dependence in $\det q$ is optimal.

The proof consists of finding a small isotropic vector (υ, ϕ) of the quadratic form $Q(x, y) = q(x) - y^2$ using the quadratic Siegel's lemma obtained by the author and Rémond [GR2, Theorem 1.2]. In order to minimize the size of $q(\phi\alpha - \upsilon)$ we twist the product norm on $\mathbb{R}^n \times \mathbb{R}$ with a well-chosen isometry of Q, which is the argument at the heart of Kleinbock and Merrill's proof (written differently). Our proof is also inspired by Moshchevitin's proof, but we avoid any choice of basis. A generalization involving an algebraic extension K of $\mathbb{Q}$ and several (archimedean or ultrametric) places of K will be given in § 4. The main argument of the proof is the same as the one of Theorem 1.1, but some new difficulties appear since the condition $\phi \neq 0$ is not automatic when q is isotropic. That is why we prefer to start by proving this particular case.

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2. Proof of Theorem 1.1

Let $\alpha \in \mathbb{R}^n$ such that $q(\alpha) = 1$. Denote by $b\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the symmetric bilinear form associated to q.

2.1. **The Euclidean lattice.** Let $t \geq 1$ be a real number. For any $(x, y) \in \mathbb{R}^n \times \mathbb{R}$, let us consider

$$\mathcal{X} = \frac{1}{2}\left(\frac{1}{t} + t\right) b(x, \alpha) + \frac{1}{2}\left(\frac{1}{t} - t\right) y \quad \text{and} \quad \mathcal{Y} = \frac{1}{2}\left(\frac{1}{t} - t\right) b(x, \alpha) + \frac{1}{2}\left(\frac{1}{t} + t\right) y$$

as well as the linear map ξ defined by $\xi(x, y) = (x - b(x, \alpha)\alpha + \mathcal{X}\alpha, \mathcal{Y})$. It is an automorphism of $\mathbb{R}^n \times \mathbb{R}$ of determinant 1. Indeed, since $q(\alpha) \neq 0$, the q-orthogonal subspace $\{x \in \mathbb{R}^n \mid b(x, \alpha) = 0\}$ is a complement of $\mathbb{R}.\alpha$ in $\mathbb{R}^n$. The choice of a basis $e_1, \ldots, e_{n-1}$ of this hyperplane provides a basis $(e_1, 0), \ldots, (e_{n-1}, 0), (\alpha, 0), (0, 1)$ of $\mathbb{R}^n \times \mathbb{R}$ in which the matrix of ξ is written

$$\begin{pmatrix} \mathrm{I}_{n-1} & 0 \\ 0 & A \end{pmatrix} \quad \text{where} \quad A = \frac{1}{2}\begin{pmatrix} 1/t + t & 1/t - t \\ 1/t - t & 1/t + t \end{pmatrix} \quad \text{has determinant 1.}$$

Thus, the Euclidean norm $\|(x, y)\| = (q(x) + y^2)^{1/2}$ on $\mathbb{R}^n \times \mathbb{R}$ induces another norm $\|(x, y)\|_t = \|\xi(x, y)\|$. In that way, we get a Euclidean lattice $E_t = (\mathbb{Z}^n \times \mathbb{Z}, \|\cdot\|_t)$ whose covolume does not depend on t:

Lemma 2.1. *The covolume of E_t is equal to $\sqrt{\det q}$.*

Proof. The covolume of E_t is also that of $\xi(\mathbb{Z}^n \times \mathbb{Z})$ with respect to the norm $\|\cdot\|$, that is, $|\det \xi| \times \mathrm{vol}(\mathbb{Z}^n \times \mathbb{Z}, \|\cdot\|) = \sqrt{\det q}$. □

2.2. **The quadratic form.** Consider the regular quadratic form $Q(x, y) = q(x) - y^2$ on $\mathbb{Q}^n \times \mathbb{Q}$ which is isotropic by hypothesis. Using that $x - b(x, \alpha)\alpha$ is q-orthogonal to α, the equality $Q(\xi(x, y)) = Q(x, y)$ can be checked for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ with a direct calculation. In other words:

Lemma 2.2. *The map ξ is an isometry with respect to Q.*

At last, at every place p of $\mathbb{Q}$, we can consider the norm $\|B\|_p$ of the bilinear form B associated to Q defined by

$$\|B\|_\infty = \sup\left\{\frac{|B((x, y), (x', y'))|}{\|(x, y)\|_t \|(x', y')\|_t} \mid (x, y),\ (x', y') \in (\mathbb{R}^n \times \mathbb{R}) \setminus \{0\}\right\}$$

in the archimedean case and by $\|B\|_p = \max_{0 \leq i,j \leq n} |B(e_i, e_j)|_p$ where $\{e_0, \ldots, e_n\}$ is the canonical basis of $\mathbb{Z}^{n+1}$ and the absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ normalised with $|p|_p = p^{-1}$. By definition, the height $H(Q)$ of Q is the product of all the norms $\|B\|_p$ over the places p of $\mathbb{Q}$. Here we have the formula $B((x, y), (x', y')) = b(x, x') - yy'$ which immediately implies $\|B\|_p = 1$ for all prime numbers p since b has integral coefficients. Moreover, as ξ is a global isometry with respect to Q, we also have $\|B\|_\infty = \sup\{|B((x, y), (x', y'))| \mid \|(x, y)\| = \|(x', y')\| = 1\}$. Then, the Cauchy-Schwarz inequality applied to the positive-definite quadratic form $q(x) + y^2 = \|(x, y)\|^2$ gives $\|B\|_\infty = 1$. Hence, the height $H(Q)$ of Q equals 1.

2.3. **The quadratic Siegel's lemma.** Applying [GR2, Theorem 1.2] to the quadratic space (E_t, Q) over $\mathbb{Q}$, of dimension $n+1$ with $e = d = 1$, we obtain $(\upsilon, \phi) \in \mathbb{Z}^n \times \mathbb{N}_{\geq 1}$ such that $Q(\upsilon, \phi) = 0$ and

$$\|(\upsilon, \phi)\|_t \leq (2\gamma_n H(Q))^{n/2} H(E_t)$$

(the height of a vector of E_t equals the norm of a multiple of this element, see § 4.2). The height $H(E_t)$ of E_t is nothing but the covolume of $(\mathbb{Z}^n \times \mathbb{Z}, \|\cdot\|_t)$, that is, $\sqrt{\det q}$ by Lemma 2.1. Also note that the constant $c_{\mathbb{Q}}^{\mathrm{BV}}(n)$ in the original statement is simply $\gamma_n^{n/2}$ (see § 4.1). As $H(Q) = 1$ we get $\|(\upsilon, \phi)\|_t \leq (2\gamma_n)^{n/2} \sqrt{\det q}$ with $q(\upsilon/\phi) = 1$.

2.4. **Conclusion.** We observe that

$$q\left(\alpha - \frac{\upsilon}{\phi}\right) = \frac{2}{\phi}\left(\phi - b(\upsilon, \alpha)\right) = \frac{2}{\phi t}\left(\mathcal{Y} - \mathcal{X}\right)$$

where $\mathcal{X}, \mathcal{Y}$ are relative to (υ, ϕ). Hence, since $\mathcal{X}^2 + \mathcal{Y}^2 \leq q(\upsilon - b(\upsilon, \alpha)\alpha) + \mathcal{X}^2 + \mathcal{Y}^2 = \|(\upsilon, \phi)\|_t^2$, the Cauchy-Schwarz inequality provides the bound $q\left(\alpha - \upsilon/\phi\right) \leq 2\sqrt{2}\|(\upsilon, \phi)\|_t/\phi t$. Actually, since ξ is an isometry with respect to Q (Lemma 2.2), we have $Q(\xi(\upsilon, \phi)) = Q(\upsilon, \phi) = 0$ so $q(\upsilon - b(\upsilon, \alpha)\alpha + \mathcal{X}\alpha) = \mathcal{Y}^2$ and then $2\mathcal{Y}^2 = \|(\upsilon, \phi)\|_t^2$. We deduce $|\mathcal{X}| \leq |\mathcal{Y}|$ and, since $t \geq 1$,

$$\phi = \frac{1}{2}\left(t - \frac{1}{t}\right)\mathcal{X} + \frac{1}{2}\left(t + \frac{1}{t}\right)\mathcal{Y} \leq t \max\left(|\mathcal{X}|, |\mathcal{Y}|\right) \leq t\|(\upsilon, \phi)\|_t/\sqrt{2}.$$

We replace $\|(\upsilon, \phi)\|_t$ by its bound $(2\gamma_n)^{n/2} \sqrt{\det q}$ obtained in § 2.3 and we set $T = (t/\sqrt{2})\,(2\gamma_n)^{n/2} \sqrt{\det q}$ to end the proof of Theorem 1.1.

3. Extended statement

In Theorem 1.1 the form q plays two distinct roles since it is used both to define the set (ellipsoid) where the approximation takes place and to measure the quality of approximation. We can give a more general statement in which two quadratic forms appear: we will approximate points on $q = 1$ using another quadratic form q_0 to measure the size of the approximation. Here it is natural to retain the hypothesis that q_0 be positive-definite but we can relax the condition on q, allowing some indefinite forms.

Let E be a vector space over a field K and $q\colon E \to K$ a quadratic form. The isotropy index $i(q)$ of q is the maximal dimension of totally isotropic subspaces of q. The induced quadratic form $Q(x, y) = q(x) - y^2$ on $E \times K$ satisfies $i(Q) - i(q) \in \{0, 1\}$. In fact $i(Q) = i(q) + 1$ when the anisotropic part of q in the Witt decomposition takes the value 1. Given a positive-definite quadratic form $q_0\colon \mathbb{R}^n \to \mathbb{R}$, we denote by $\|\cdot\| = \sqrt{q_0}$ the associated Euclidean norm on $\mathbb{R}^n$ and by $\lambda_1 = \min\left\{\|\lambda\| \mid \lambda \in \mathbb{Z}^n \setminus \{0\}\right\}$ the first minimum of the Euclidean lattice $(\mathbb{Z}^n, \|\cdot\|)$. A quadratic form $q(x) = {}^{\mathrm{t}}xA(q)x$ associated with a symmetric matrix $A(q) \in \mathrm{M}_n(\mathbb{R})$ (not necessarily positive-definite) also inherits a norm by the formula

$$\|q\|_\infty = \max\left\{{}^{\mathrm{t}}xA(q)y \mid x, y \in \mathbb{R}^n,\ \|x\| = \|y\| = 1\right\}.$$

In this context we have the following statement.

Theorem 3.1. *Let $\alpha \in \mathbb{R}^n$ such that $q(\alpha) = 1$. Define*

$$\mathcal{T}_0 = n^{n/2}\left(2\max\left(1, \|q\|_\infty\right)\right)^{(n - i(q))/2} \|\alpha\| \sqrt{\det q_0}$$

and $\mathcal{T} = \max\left(\mathcal{T}_0^{1/(i(q)+1)}, (\sqrt{2}/\lambda_1)^{i(q)}\mathcal{T}_0\right)$. Assume that $A(q) \in \mathrm{M}_n(\mathbb{Z})$ and that $i(Q) > i(q)$ where $Q(x, y) = q(x) - y^2$. Then, for all real numbers $T \geq \mathcal{T}$, there exists $(\upsilon, \phi) \in \mathbb{Z}^n \times \mathbb{Z}$ satisfying $q(\upsilon) = \phi^2 \neq 0$,

$$\|\upsilon\|^2 + (\phi\|\alpha\|)^2 \leq \left(\|\alpha\|\|b(\cdot, \alpha)\|_{\mathrm{op},\infty} T\right)^2 \quad \text{and} \quad |q(\phi\alpha - \upsilon)| \leq \frac{2\sqrt{2}\mathcal{T}^2\phi}{T} \times \|b(\cdot, \alpha)\|_{\mathrm{op},\infty}.$$

Here $\|b(\cdot, \alpha)\|_{\mathrm{op},\infty}$ denotes the operator norm of the linear form $x \mapsto b(x, \alpha) = {}^{\mathrm{t}}xA(q)\alpha$ on $(\mathbb{R}^n, \|\cdot\|)$. It can be bounded by $\|q\|_\infty\|\alpha\|$. If the proof of Theorem 3.1 follows the same lines as those of Theorem 1.1, the new difficulty is to ensure $\phi \neq 0$ even though q is not assumed to be definite. To solve this problem, we introduce a maximal totally Q-isotropic sublattice Ω of $\mathbb{Z}^n \times \mathbb{Z}$ of small covolume and we distinguish two cases according to the value of the first minimum of Ω.

The proof is a special case of that of Theorem 4.1 (see § 4.5). Let us just say that the quantity n^n in $\mathfrak{T}_0$ is a bound for $\gamma_{i(q)+1}^{i(q)+1}\gamma_{n-i(q)}^{n-i(q)}$ which appears in $\mathfrak{T}_0$ during the proof (see Theorem 4.1 and the discussion in §4.7).

4. Adelic generalization

4.1. Let $K/\mathbb{Q}$ be an algebraic extension and $V(K)$ be its set of places. For $v \in V(K)$ we denote by K_v the topological completion of K at v and $|\cdot|_v$ is the unique absolute value on K_v such that $|p|_v \in \{1, p, p^{-1}\}$ for every prime number p. As explained in [GR1, § 2] (see also [Ga2, § 2.1]), $V(K)$ has a topology generated by the open and compact subset $V_v(K) = \{w \in V(K) \mid w_{|L} = v\}$ where L goes through all the number fields contained in K and $v \in V(L)$. It also has a Borel measure σ characterized by

$$\sigma(V_v(K)) = \frac{[L_v : \mathbb{Q}_v]}{[L : \mathbb{Q}]}.$$

The module $|f|$ of an integrable bounded function $f\colon V(K) \to (0, +\infty)$ such that $\{v \in V(K) \mid f(v) \neq 1\}$ is contained in a compact subset is the positive real number

$$|f| = \exp\left(\int_{V(K)} \log\left(f(v)\right) \,\mathrm{d}\sigma(v)\right).$$

When K is a number field, the hypothesis on f means that the set $\{v \in V(K) \mid f(v) \neq 1\}$ is finite and, in this case, the module of f is

$$|f| = \prod_{v \in V(K)} f(v)^{[K_v:\mathbb{Q}_v]/[K:\mathbb{Q}]}.$$

Given an integer $n \geq 1$, a place $v \in V(K)$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, write

$$m_v(x) = \begin{cases} \left(x_1^2 + \cdots + x_n^2\right)^{1/2} & \text{if } v \mid \infty \\ \max\left(x_1, \ldots, x_n\right) & \text{if } v \nmid \infty. \end{cases}$$

Then $|x|_v = m_v(|x_1|_v, \ldots, |x_n|_v)$ defines a norm on K_v^n for all $v \in V(K)$. Let $\mathbb{A}_K = K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}$ be the adèles of K. A rigid adelic space (of dimension n) is a n-dimensional K-vector space E endowed with norms $\|\cdot\|_{E,v}$ on $E_v = E \otimes_K K_v$ for all $v \in V(K)$, satisfying the following property: there exist an isomorphism $\varphi\colon E \to K^n$ and an adelic matrix $(A_v)_{v \in V(K)} \in \mathrm{GL}_n(\mathbb{A}_K)$ such that

$$\forall\, x \in E_v, \quad \|x\|_{E,v} = |A_v \varphi_v(x)|_v$$

where $\varphi_v = \varphi \otimes \mathrm{id}_{K_v}\colon E_v \to K_v^n$ is the natural extension of φ to E_v. The height $H(E)$ of E is the module of $v \mapsto |\det A_v|_v$ and the height $H_E(x)$ of $x \in E \setminus \{0\}$ is the module of $v \mapsto \|x\|_{E,v}$. The dual space $E^\vee$ has a rigid adelic structure given by the transpose map ${}^{\mathrm{t}}\varphi^{-1}\colon E^\vee \to K^n$ and $({}^{\mathrm{t}}A_v^{-1})_{v \in V(K)}$. Additionally, the product $E \times K$ has a natural rigid adelic structure given by the norms $\|(x, y)\|_{E \times K, v} = m_v(\|x\|_{E,v}, |y|_v)$ for all $(x, y) \in E_v \times K_v$.

To a rigid adelic space E over K can be attached several types of successive minima. Here we only use two of them: the minima of Roy-Thunder ($1 \leq i \leq n$)

$$\Lambda_i(E) = \lambda_i^\Lambda(E) = \inf\left\{\max\left(H_E(x_1), \ldots, H_E(x_i)\right) \mid \dim \mathrm{Vect}_K(x_1, \ldots, x_i) = i\right\}$$

and the minima of Bombieri-Vaaler $\lambda_i^{\mathrm{BV}}(E) = \inf\{r > 0 \mid \dim \mathrm{Vect}_K(E_r) \geq i\}$ where

$$E_r = \left\{ x \in E \mid \sup_{v \mid \infty} \|x\|_{E,v} \leq r \text{ and } \sup_{v \nmid \infty} \|x\|_{E,v} \leq 1 \right\}.$$

We can note that $\lambda_1^*(E) \leq \cdots \leq \lambda_n^*(E)$ for $* \in \{\Lambda, \mathrm{BV}\}$. They give rise to the following constants (possibly infinite), depending on a positive integer n:

$$c_K^\Lambda(n) = \sup_E \frac{\Lambda_1(E)^n}{H(E)} \quad \text{and} \quad c_K^{\mathrm{BV}}(n) = \sup_E \frac{\lambda_1^{\mathrm{BV}}(E)^n}{H(E)}$$

where E varies among the rigid adelic spaces over K of dimension n. It turns out that these constants provide variants of Minkowski's Second Theorem: for all $* \in \{\Lambda, \mathrm{BV}\}$, we have $\lambda_1^*(E) \cdots \lambda_n^*(E) \leq c_K^*(n) H(E)$ (see [GR1, Theorem 4.12]), which implies $(\lambda_1^*(E))^{n-1} \lambda_n^*(E) \leq c_K^*(n) H(E)$. Besides, if we set $c_1(K) = c_K^{\mathrm{BV}}(1)$, then $\Lambda_i(E) \leq \lambda_i^{\mathrm{BV}}(E) \leq c_1(K)\Lambda_i(E)$ for all

$1 \leq i \leq n$ [GR1, Proposition 4.8] and so $c_K^{\mathrm{BV}}(n) \leq c_1(K)^n c_K^{\Lambda}(n)$. For all $*$, we have $c_{\mathbb{Q}}^*(n) = \gamma_n^{n/2}$ and $c_K^*(n) \leq (n\delta_{K/\mathbb{Q}})^{n/2}$ when K is a number field of root discriminant $\delta_{K/\mathbb{Q}} = |\Delta_{K/\mathbb{Q}}|^{1/[K:\mathbb{Q}]}$ [GR1, Proposition 5.1]. It is also known [GR1, § 5.2] that $c_{\mathbb{Q}}^*(1) = 1$ and, for $n \geq 2$,

$$c_{\mathbb{Q}}^*(n) = \exp\left(\frac{n}{2}\left(\frac{1}{2} + \cdots + \frac{1}{n}\right)\right).$$

However, the constant $c_K^*(n)$ may be infinite when $n \geq 2$. For instance, this is the case when K is a Northcott field of infinite degree (see Corollary 1.2 and Proposition 4.10 of [GR1]). We say that K is a Siegel field if $c_K^{\Lambda}(n)$ is finite for all $n \geq 1$. At last, a quadratic space (E, q) is a rigid adelic space E endowed with a quadratic form $q\colon E \to K$. For $v \in V(K)$ and $b\colon E \times E \to K$ the symmetric bilinear form associated to q, the norm $\|q\|_v$ is the supremum of $|b(x,y)|_v / \|x\|_{E,v}\|y\|_{E,v}$ for nonzero $x, y \in E \otimes_K K_v$. The height $H(q)$ of q is the module of $v \mapsto \|q\|_v$ if $q \neq 0$ and 0 otherwise. We also write $H(1,q)$ for the module of $v \mapsto \max(1, \|q\|_v)$.

4.2. Example. Let us detail the case $K = \mathbb{Q}$. Let E be a rigid adelic space over $\mathbb{Q}$. The set

$$\Omega = \{x \in E \mid \forall p \in V(\mathbb{Q}) \setminus \{\infty\},\ \|x\|_{E,p} \leq 1\}$$

is a (full rank) lattice of the Euclidean space $(E \otimes_{\mathbb{Q}} \mathbb{R}, \|\cdot\|_{E,\infty})$. Conversely, any Euclidean lattice $(\Omega, \|\cdot\|)$ composed of a discrete subgroup Ω of a finite dimensional vector space over $\mathbb{R}$ and a Euclidean norm $\|\cdot\|$ on $\Omega \otimes_{\mathbb{Z}} \mathbb{R}$ gives rise to a rigid adelic space E over $\mathbb{Q}$ in the following way: let us consider $E = \Omega \otimes_{\mathbb{Z}} \mathbb{Q}$ and the isomorphism $\varphi\colon E \to \mathbb{Q}^n$ given by $\varphi(\sum_{i=1}^n x_i\omega_i) = (x_1, \ldots, x_n)$ where $(\omega_1, \ldots, \omega_n)$ is a $\mathbb{Z}$-basis of Ω. The norms $\|\cdot\|_{E,\infty} = \|\cdot\|$ on $\Omega \otimes_{\mathbb{Z}} \mathbb{R}$ and, for p prime, $\|\cdot\|_{E,p} = |\varphi(\cdot)|_p$ on $\Omega \otimes_{\mathbb{Z}} \mathbb{Q}_p$ provide a rigid adelic space structure to E where the underlying adelic matrix has coordinates: A_∞ is the transition matrix from $(\omega_1, \ldots, \omega_n)$ to an orthonormal basis of $(\Omega \otimes_{\mathbb{Z}} \mathbb{R}, \|\cdot\|)$ and A_p is the $n \times n$ identity matrix for p prime. Thus a rigid adelic space over $\mathbb{Q}$ is nothing other than a Euclidean lattice $(\Omega, \|\cdot\|)$. In this correspondence, the height $H(E)$ of E is $|\det A_\infty| = \sqrt{\mathrm{vol}(\Omega, \|\cdot\|)}$. Moreover the Roy-Thunder and Bombieri-Vaaler successive minima of E are the same ($\forall\, i,\ \Lambda_i(E) = \lambda_i^{\mathrm{BV}}(E)$) and also equal to the usual successive minima of the Euclidean lattice $(\Omega, \|\cdot\|)$. That explains why $c_{\mathbb{Q}}^*(n) = \gamma_n^{n/2}$ since, by definition, $\sqrt{\gamma_n}$ is the supremum of $\lambda_1(\Omega, \|\cdot\|)/\mathrm{vol}(\Omega, \|\cdot\|)^{1/n}$ over all Euclidean lattices $(\Omega, \|\cdot\|)$ of rank n. Lastly, it can be noted that for every $x \in E$ there exists $d_x \in \mathbb{Q} \setminus \{0\}$ such that $H_E(x) = \|d_x x\|$.

4.3. Main statements. Let $K/\mathbb{Q}$ be a Siegel field and let (E, q) be an adelic quadratic space over K of dimension $n \geq 1$. We present two statements according to the isotropy index of the quadratic form $Q(x,y) = q(x) - y^2$ on $E \times K$, which, as we have seen, is equal to $i(q)$ or $i(q) + 1$. Let us begin with the case $i(Q) = i(q) + 1$. For $v \in V(K)$, write $\epsilon_v = 1$ if $v \mid \infty$ and 0 otherwise.

Theorem 4.1. *Let $V \subset V(K)$ be a compact subset. Let $(\alpha_v, t_v)_{v \in V(K)} \in (E \times K) \otimes_K \mathbb{A}_K$ be such that $q(\alpha_v) = 1$ and $|t_v|_v > 1$ for all $v \in V$. Let $\alpha\colon V(K) \to \mathbb{R}$ the function defined by $\alpha(v) = \|\alpha_v\|_{E,v}$ if $v \in V$ and $\alpha(v) = 1$ if $v \notin V$. Let us assume that the quadratic form $Q(x,y) = q(x) - y^2$ on $E \times K$ has its isotropy index $i(Q)$ equal to $i(q) + 1$. Define*

$$\mathfrak{T}_0 = c_K^{\mathrm{BV}}(i(Q)) c_K^{\Lambda}(n + 1 - i(Q))\, (2H(1,q))^{(n+1-i(Q))/2}\, |\alpha| H(E)$$

and

$$\mathfrak{T} = \max\left(\mathfrak{T}_0^{1/i(Q)}, \left(\sqrt{2}/\lambda_1^{\mathrm{BV}}(E)\right)^{i(Q)-1} \mathfrak{T}_0\right).$$

We assume that $c_1(K)$ is finite (in particular $\mathfrak{T}$ is finite). For $v \in V$, define

$$T_v = (2\mathfrak{T})^{\epsilon_v} \|\alpha_v\|_{E,v} \|b(\cdot, \alpha_v)\|_{E^\vee, v} |t_v/2|_v.$$

Then, for all $\varepsilon > 0$, there exists $(\upsilon, \phi) \in E \times K$ satisfying $q(\upsilon) = \phi^2 \neq 0$ and such that:

$$\forall\, v \notin V, \quad m_v\left(\|\upsilon\|_{E,v}, |\phi|_v\right) \leq ((1+\varepsilon)\mathfrak{T})^{\epsilon_v}, \tag{1}$$

$$\forall\, v \in V, \quad m_v\left(\|\upsilon\|_{E,v}, |\phi|_v \|\alpha_v\|_{E,v}\right) \leq (1+\varepsilon)^{\epsilon_v}\, T_v \tag{2}$$

and

$$\forall\, v \in V, \quad |q(\phi\alpha_v - \upsilon)|_v \leq \left((1+\varepsilon) 2\sqrt{2}\mathfrak{T}^2\right)^{\epsilon_v} \left(\frac{|\phi|_v}{T_v}\right) \|\alpha_v\|_{E,v} \|b(\cdot, \alpha_v)\|_{E^\vee, v}^2. \tag{3}$$

The number $\|b(\cdot,\alpha_v)\|_{E^\vee,v}$ is the operator norm of the linear form $x\mapsto b(x,\alpha_v)$ on E_v. Additionally, the number $|\alpha|$ is the module of the map α (see the beginning of § 4.1). When K is a number field we can take $\varepsilon=0$. A discussion about the constant of $\mathcal{T}_0$ is given in §4.7.

Our second statement concerns the other case $i(Q)=i(q)$.

Theorem 4.2. *Consider*

$$\mathcal{T}_1=4\min\left(c_1(K),\frac{c_K^{\mathrm{BV}}(n+1-i(Q))}{c_K^{\Lambda}(n+1-i(Q))}\right)^2\left(\frac{\sqrt{2}}{\lambda_1^{\mathrm{BV}}(E)}\right)^{i(Q)}\mathcal{T}_0^2$$

(where $\mathcal{T}_0$ has been defined in the previous theorem). If $i(Q)=i(q)\geq 1$ then Theorem 4.1 remains true provided $\mathcal{T}_0$ in the definition of $\mathcal{T}$ is replaced by $\max\{\mathcal{T}_0,\mathcal{T}_1\}$.

These statements are part of what is commonly known as intrinsic approximation (on quadrics), with the particularity here of being simultaneous in several places of K at once.

4.4. Preparatory statements. In this part we prove three auxiliary results useful for the proofs of Theorems 4.1 and 4.2. The notation is that of these statements.

Since only the absolute value of t_v occurs, we can assume $t_v=|t_v|_v$ if $v\in V$ is archimedean. For $v\in V$, define $\mathcal{X}_v,\mathcal{Y}_v\in K_v$ with the formulas of § 2.1, where (α,t) is replaced by (α_v,t_v), and $b\colon E\times E\to K$ is still the symmetric bilinear form associated to q. Let E_t be the rigid adelic space $E\times K$ where each norm at $v\in V$ has been twisted in the following way:

$$\forall (x,y)\in E_v\times K_v,\quad \|(x,y)\|_{E_t,v}=m_v\left(\|x-b(x,\alpha_v)\alpha_v+\mathcal{X}_v\alpha_v\|_{E,v},\|\mathcal{Y}_v\alpha_v\|_{E,v}\right)$$

(when $v\notin V$, we have $\|(x,y)\|_{E_t,v}=m_v\left(\|x\|_{E,v},|y|_v\right)=\|(x,y)\|_{E\times K,v}$). So, to build this norm, we first modify the norm on $E\times K$ at v by multiplying the second component by $\alpha(v)$ and then we compose with the automorphism $\xi_v(x,y)=(x-b(x,\alpha_v)\alpha_v+\mathcal{X}_v\alpha_v,\mathcal{Y}_v)$ of $E_v\times K_v$, which has determinant 1. In particular we have $H(E_t)=|\alpha|H(E\times K)=|\alpha|H(E)$. Here are two properties of the norm $\|\cdot\|_{E_t,v}$.

Lemma 4.3. *For all $v\in V$, $x\in E\otimes_K K_v$ and $y\in K_v$, we have*

$$m_v(\|x\|_{E,v},\|y\alpha_v\|_{E,v})\leq 2^{\epsilon_v}|t_v/2|_v\left(\|b(\cdot,\alpha_v)\|_{E^\vee,v}\|\alpha_v\|_{E,v}\right)\|(x,y)\|_{E_t,v}.$$

Proof. The question is to bound the operator norm of ξ_v^{-1} when $E_v\times K_v$ is endowed with the norm $m_v(\|x\|_{E,v},\|y\alpha_v\|_{E,v})$. For any $(x,y)\in E_v\times K_v$, we have the formula

$$\xi_v^{-1}(x,y)=(x-b(x,\alpha_v)\alpha_v+\mathcal{X}'_v\alpha_v,\mathcal{Y}'_v)$$

where

$$\mathcal{X}'_v=\frac{1}{2}\left(t_v+\frac{1}{t_v}\right)b(x,\alpha_v)+\frac{1}{2}\left(t_v-\frac{1}{t_v}\right)y\quad\text{and}\quad \mathcal{Y}'_v=\frac{1}{2}\left(t_v-\frac{1}{t_v}\right)b(x,\alpha_v)+\frac{1}{2}\left(t_v+\frac{1}{t_v}\right)y.$$

When $v\in V$ is ultrametric, we have

$$\max\left(\|\mathcal{X}'_v\alpha_v\|_v,\|\mathcal{Y}'_v\alpha_v\|_{E,v}\right)\leq\max\left(\|x\|_{E,v},\|y\alpha_v\|_{E,v}\right)\times|t_v/2|_v\|b(\cdot,\alpha_v)\|_{E^\vee,v}\|\alpha_v\|_{E,v}$$

since $|t_v|_v\geq 1$ and

$$|b(x,\alpha_v)|_v\leq\|b(\cdot,\alpha_v)\|_{E^\vee,v}\|x\|_{E,v}\quad\text{and}\quad 1=|b(\alpha_v,\alpha_v)|_v\leq\|b(\cdot,\alpha_v)\|_{E^\vee,v}\|\alpha_v\|_{E,v}. \tag{4}$$

We easily deduce that the same bound holds for $\max\left(\|x-b(x,\alpha_v)\alpha_v+\mathcal{X}'_v\alpha_v\|_{E,v},\|\mathcal{Y}'_v\alpha_v\|_{E,v}\right)$, which gives the desired result. When v is archimedean, observe that

$$m_v\left(\|x-b(x,\alpha_v)\alpha_v+\mathcal{X}'_v\alpha_v\|_{E,v},\|\mathcal{Y}'_v\alpha_v\|_{E,v}\right)^2\leq\left(\|x\|_{E,v}+\|\left(\mathcal{X}'_v-b(x,\alpha_v)\right)\alpha_v\|_{E,v}\right)^2+\|\mathcal{Y}'_v\alpha_v\|_{E,v}^2.$$

We note

$$\|\left(\mathcal{X}'_v-b(x,\alpha_v)\right)\alpha_v\|_{E,v}\leq\frac{1}{2}\left(t_v+\frac{1}{t_v}-2\right)\|b(x,\alpha_v)\alpha_v\|_{E,v}+\frac{1}{2}\left(t_v-\frac{1}{t_v}\right)\|y\alpha_v\|_{E,v}$$

and

$$\|\mathcal{Y}'_v\alpha_v\|_{E,v}\leq\frac{1}{2}\left(t_v-\frac{1}{t_v}\right)\|b(x,\alpha_v)\alpha_v\|_{E,v}+\frac{1}{2}\left(t_v+\frac{1}{t_v}\right)\|y\alpha_v\|_{E,v}.$$

Then, using (4), we can factorize by the product $\|b(\cdot,\alpha_v)\|_{E^\vee,v}\|\alpha_v\|_{E,v}$ and we see that

$$\left(2m_v\left(\|x-b(x,\alpha_v)\alpha_v+\mathcal{X}'_v\alpha_v\|_{E,v},\|\mathcal{Y}'_v\alpha_v\|_{E,v}\right)/\|b(\cdot,\alpha_v)\|_{E^\vee,v}\|\alpha_v\|_{E,v}\right)^2$$

is bounded by

$$\left(\left(t_v+\frac{1}{t_v}\right)\|x\|_{E,v}+\left(t_v-\frac{1}{t_v}\right)\|y\alpha_v\|_{E,v}\right)^2+\left(\left(t_v-\frac{1}{t_v}\right)\|x\|_{E,v}+\left(t_v+\frac{1}{t_v}\right)\|y\alpha_v\|_{E,v}\right)^2.$$

We develop this expression and substitute the product $2\|x\|_{E,v}\|y\alpha_v\|_{E,v}$ by $\|x\|_{E,v}^2+\|y\alpha_v\|_{E,v}^2$ to finally obtain the desired bound $4t_v^2 m_v\left(\|x\|_{E,v},\|y\alpha_v\|_{E,v}\right)^2$. $\square$

When $y=0$ we can prove a better estimate, which does not depend on t_v.

Lemma 4.4. *For all $v\in V(K)$ and $x\in E\otimes_K K_v$, we have $\|x\|_{E,v}\leq 2^{\epsilon_v/2}\|(x,0)\|_{E_t,v}$.*

Proof. If $v\in V(K)\setminus V$ then $\|(x,0)\|_{E_t,v}=\|(x,0)\|_{E\times K,v}=\|x\|_{E,v}$ and the result is clear. Let $v\in V$ be an archimedean place and $x\in E\otimes_K K_v$. From the definition of the v-norm of E_t, the number $\|(x,0)\|_{E_t,v}^2$ equals

$$\left\|x-b(x,\alpha_v)\alpha_v+\frac{1}{2}\left(t_v+\frac{1}{t_v}\right)b(x,\alpha_v)\alpha_v\right\|_{E,v}^2+\left|\frac{1}{2}\Big(t_v-\frac{1}{t_v}\Big)b(x,\alpha_v)\right|_v^2\|\alpha_v\|_{E,v}^2.$$

Put $\theta=\frac{1}{2}\left(t_v+\frac{1}{t_v}-2\right)b(x,\alpha_v)\alpha_v$ and bound from below the first norm by $|\|x\|_{E,v}-\|\theta\|_{E,v}|$ (reverse triangle inequality). Also note that $t_v-1/t_v\geq t_v+1/t_v-2\geq 0$ since t_v is a real number greater than 1. In particular the norm of $(t_v-1/t_v)b(x,\alpha_v)\alpha_v/2$ is greater than or equal to $\|\theta\|_{E,v}$. We conclude with

$$\|(x,0)\|_{E_t,v}^2\geq\left(\|x\|_{E,v}-\|\theta\|_{E,v}\right)^2+\|\theta\|_{E,v}^2\geq\frac{\|x\|_{E,v}^2}{2}.$$

When $v\in V$ is ultrametric, the norm $\|(x,0)\|_{E_t,v}$ is

$$\max\left(\left\|x-b(x,\alpha_v)\alpha_v+\frac{1}{2}\Big(t_v+\frac{1}{t_v}\Big)b(x,\alpha_v)\alpha_v\right\|_{E,v},\left\|\frac{1}{2}\Big(t_v-\frac{1}{t_v}\Big)b(x,\alpha_v)\alpha_v\right\|_{E,v}\right).$$

Since $|t_v|_v>1$ we have $|t_v-1/t_v|_v=|t_v|_v=|t_v+1/t_v-2|_v$ so that

$$\|(x,0)\|_{E_t,v}=\max\left(\|x+\theta\|_{E,v},\|\theta\|_{E,v}\right)\geq\|x\|_{E,v}.$$

$\square$

At last, we also need the following statement.

Lemma 4.5. *The height of Q satisfies $H(Q)\leq H(1,q)$.*

Proof. Let B be the bilinear form associated to Q and $v\in V(K)$. Let $x,x'\in E_v$ and $y,y'\in K_v$. From the expression $B((x,y),(x',y'))=b(x,x')-yy'$, we get

$$|B((x,y),(x',y'))|_v\leq|b(x,x')|_v+|y|_v|y'|_v\leq\|q\|_v\|x\|_{E,v}\|x'\|_{E,v}+|y|_v|y'|_v$$

(the sum can be replaced by a maximum when v is ultrametric). We factorize by $\max(1,\|q\|_v)$ and we use the Cauchy inequality to obtain

$$|B((x,y),(x',y'))|_v\leq\max(1,\|q\|_v)\|(x,y)\|_{E\times K,v}\|(x',y')\|_{E\times K,v},$$

which implies $\|Q\|_v\leq\max(1,\|q\|_v)$ when $v\notin V$ since, in this case, $\|\cdot\|_{E_t,v}=\|\cdot\|_{E\times K,v}$. When $v\in V$, we observe that $B((x,y),(x',y'))=b(x,x')-b(y\alpha_v,y'\alpha_v)$ and so

$$|B((x,y),(x',y'))|_v\leq\|q\|_v m_v(\|x\|_{E,v},\|y\alpha_v\|_{E,v})m_v(\|x'\|_{E,v},\|y'\alpha_v\|_{E,v}).$$

Now, since B is invariant by ξ_v (Lemma 2.2), if we replace (x,y) and (x',y') by their images by ξ_v, we deduce $|B((x,y),(x',y'))|_v\leq\|q\|_v\|(x,y)\|_{E_t,v}\|(x',y')\|_{E_t,v}$ then $\|Q\|_v\leq\|q\|_v$. Thus, in all cases we have $\|Q\|_v\leq\max(1,\|q\|_v)$ which leads to $H(Q)\leq H(1,q)$. $\square$

4.5. Proof of Theorem 4.1. According to [GR2, Corollary 3.2], there exists a maximal Q-isotropic subspace $\{0\} \neq F \subset E_t$ (of dimension $i(Q)$) such that $(1+\varepsilon)^{-1/2n} \leq 2H(Q)\Lambda_1(E_t/F)^2$. Bounding from above $\Lambda_1(E_t/F)$ by $\left(c_K^{\Lambda}(n+1-i(Q))H(E_t/F)\right)^{1/(n+1-i(Q))}$ and using $H(E_t/F) = H(E_t)/H(F) = |\alpha|H(E)/H(F)$, we deduce the upper bound

$$H(F) \leq (1+\varepsilon)^{1/2} c_K^{\Lambda}(n+1-i(Q))\,(2H(Q))^{(n+1-i(Q))/2}\,|\alpha|H(E)$$

which leads to $H(F) \leq (1+\varepsilon)^{1/2}\mathfrak{T}_0/c_K^{\mathrm{BV}}(i(Q))$ with Lemma 4.5. Since $i(Q) > i(q)$, we have $F \not\subset E\times\{0\}$. To build the vector (υ,ϕ) of Theorem 4.1, we distinguish two cases.

(i) If $\lambda_1^{\mathrm{BV}}(F) < \lambda_1^{\mathrm{BV}}(E)/\sqrt{2}$, then we consider $0 < \varepsilon' \leq \varepsilon$ such that $(1+\varepsilon')^{1/2}\lambda_1^{\mathrm{BV}}(F) < \lambda_1^{\mathrm{BV}}(E)/\sqrt{2}$ and $(\upsilon,\phi) \in F\setminus\{0\}$ such that $\|(\upsilon,\phi)\|_{E_t,v} \leq \left((1+\varepsilon')^{1/2}\lambda_1^{\mathrm{BV}}(F)\right)^{\epsilon_v}$ for all $v \in V(K)$. By Lemma 4.4 and the choice of ε', we have $\phi \neq 0$. Moreover $\lambda_1^{\mathrm{BV}}(F) \leq \left(c_K^{\mathrm{BV}}(i(Q))H(F)\right)^{1/i(Q)}$, so that $\|(\upsilon,\phi)\|_{E_t,v} \leq ((1+\varepsilon)\mathfrak{T})^{\epsilon_v}$ for all $v \in V(K)$.

(ii) If $\lambda_1^{\mathrm{BV}}(F) \geq \lambda_1^{\mathrm{BV}}(E)/\sqrt{2}$, then we consider $(\upsilon,\phi) \in F$ such that $\phi \neq 0$ and $\|(\upsilon,\phi)\|_{E_t,v} \leq \left((1+\varepsilon)^{1/2}\lambda_{i(Q)}^{\mathrm{BV}}(F)\right)^{\epsilon_v}$ for all $v \in V(K)$. We can do that since $F \not\subset E\times\{0\}$ and every basis of F contains a vector whose last coordinate is nonzero. We bound

$$\lambda_{i(Q)}^{\mathrm{BV}}(F) \leq \frac{c_K^{\mathrm{BV}}(i(Q))H(F)}{\lambda_1^{\mathrm{BV}}(F)^{i(Q)-1}} \leq (1+\varepsilon)^{1/2}\mathfrak{T}.$$

Thus, in both cases, there exists $(\upsilon,\phi) \in E\times K$ such that $\phi \neq 0$, $q(\upsilon) = \phi^2$ (because $(\upsilon,\phi) \in F$ is Q-isotropic) and $\|(\upsilon,\phi)\|_{E_t,v} \leq ((1+\varepsilon)\mathfrak{T})^{\epsilon_v}$ for all $v \in V(K)$. These inequalities yield the first assertion of Theorem 4.1, since when $v \notin V$, we have $\|(\upsilon,\phi)\|_{E_t,v} = \|(\upsilon,\phi)\|_{E\times K,v} = m_v\left(\|\upsilon\|_{E,v},|\phi|_v\right)$. Now, let us consider $v \in V$. The second assertion of Theorem 4.1 is a direct consequence of Lemma 4.3 and the definition of T_v. At last, for (3), note that $q(\phi\alpha_v - \upsilon) = 2(\mathcal{Y}_v - \mathcal{X}_v)\phi/t_v$. From

$$|\mathcal{X}_v|_v = |b\left(\upsilon - b(\upsilon,\alpha_v)\alpha_v + \mathcal{X}_v\alpha_v,\alpha_v\right)|_v \leq \|b(\cdot,\alpha_v)\|_{E^\vee,v}\|\upsilon - b(\upsilon,\alpha_v)\alpha_v + \mathcal{X}_v\alpha_v\|_{E,v}$$

and $|\mathcal{Y}_v|_v = |b(\mathcal{Y}_v\alpha_v,\alpha_v)|_v \leq \|b(\cdot,\alpha_v)\|_{E^\vee,v}\|\mathcal{Y}_v\alpha_v\|_{E,v}$ we deduce

$$|\mathcal{Y}_v - \mathcal{X}_v|_v \leq (\sqrt{2})^{\epsilon_v}\|b(\cdot,\alpha_v)\|_{E^\vee,v}\|(\upsilon,\phi)\|_{E_t,v}$$

and so

$$|q\left(\phi\alpha_v - \upsilon\right)|_v \leq \left|\frac{2\phi}{t_v}\right|_v \|b(\cdot,\alpha_v)\|_{E^\vee,v}\left((1+\varepsilon)\sqrt{2}\mathfrak{T}\right)^{\epsilon_v}.$$

We conclude with the formula linking t_v and T_v. □

Theorem 3.1 can be deduced from Theorem 4.1: Choose $K = \mathbb{Q}$ and the singleton $V = \{\infty\}$ (archimedean place of $\mathbb{Q}$). Take $E = \mathbb{Q}^n$ with the norms $|\cdot|_p$ at $p \in V(\mathbb{Q})\setminus\{\infty\}$ and $\|x\|_{E,\infty} = \sqrt{q_0(x)}$. In other words E corresponds to the Euclidean lattice $(\mathbb{Z}^n,\sqrt{q_0})$ and, in particular, we have $H(E) = \sqrt{\det q_0}$ and $\lambda_1^{\mathrm{BV}}(E) = \lambda_1$. The integrality hypothesis on the coefficients of $A(q)$ gives $\|q\|_p \leq 1$ for all $p \neq \infty$ so that $H(1,q) = \max(1,\|q\|_\infty)$. Also choose $t_\infty = T/\mathfrak{T}$. The equality $i(Q) = i(q)+1$, as well as the bound $c_{\mathbb{Q}}^{\mathrm{BV}}(i(q)+1)c_{\mathbb{Q}}^{\Lambda}(n-i(q)) \leq n^n$ (see § 4.7), allows us to conclude. Theorem 1.1 also follows from Theorem 4.1 in the same way by further choosing $q_0 = q$. This implies $|\alpha| = \|\alpha\|_{E,\infty} = \|b(\cdot,\alpha)\|_{E^\vee,\infty} = \|q\|_\infty = H(1,q) = 1$. Additionally, since $i(Q) = 1$, we have both $\mathfrak{T} = \mathfrak{T}_0$ and $c_{\mathbb{Q}}^{\mathrm{BV}}(i(Q))c_{\mathbb{Q}}^{\Lambda}(n+1-i(Q)) = c_{\mathbb{Q}}^{\Lambda}(n) = \gamma_n^{n/2}$.

4.6. Proof of Theorem 4.2. The proof of Theorem 4.1 still works the same way when the maximal Q-isotropic subspace F introduced at the beginning of the proof satisfies $F \not\subset E\times\{0\}$. Thus, in this case, Theorem 4.2 is proved. So now we can assume that $F \subset E\times\{0\}$. We shall consider the image of F by a certain Q-isometry, where the image is not contained in $E\times\{0\}$, with which we shall apply the same method as Theorem 4.1. More precisely, we claim that there exist $a \in E_t/F$ and a maximal Q-isotropic subspace $F_a \subset E_t$ satisfying the following three conditions: (i) $F_a \not\subset E\times\{0\}$, ($ii$) $H_{E_t/F}(a) \leq 2(1+\varepsilon)^{1/4n}\lambda_{n+1-i(Q)}^{\mathrm{BV}}(E_t/F)$ and (iii) $H(F_a) \leq 2H(Q)H_{E_t/F}(a)^2H(F)$. Indeed, the space F_a comes from the key lemma of [GR2, § 3], whereas the element $a \in E_t/F$ is chosen in the same way as the beginning of the proof of [GR2, Theorem 7.1] (page 234 with, here, $Z(I) = E\times\{0\}$). Additionally, the minimum $\lambda_{n+1-i(Q)}^{\mathrm{BV}}(E_t/F)$ can be bounded in two different ways: either by $c_1(K)\Lambda_{n+1-i(Q)}(E_t/F)$ [GR1, Proposition 4.8] and then by $c_1(K)c_K^{\Lambda}(n+$

$1 - i(Q))H(E_t/F)/\Lambda_1(E_t/F)^{n-i(Q)}$ or, directly, by $c_K^{\mathrm{BV}}(n+1-i(Q))H(E_t/F)/\lambda_1^{\mathrm{BV}}(E_t/F)^{n-i(Q)}$ and then by $c_K^{\mathrm{BV}}(n+1-i(Q))H(E_t/F)/\Lambda_1(E_t/F)^{n-i(Q)}$ since $\lambda_1^{\mathrm{BV}}(E_t/F) \geq \Lambda_1(E_t/F)$. In both cases, we bound $\Lambda_1(E_t/F)$ from below by $(1+\varepsilon)^{-1/4n}\,(2H(Q))^{-1/2}$ (definition of F). Thus (with $i(Q) \geq 1$) $\lambda_{n+1-i(Q)}^{\mathrm{BV}}(E_t/F)$ is smaller than

$$(1+\varepsilon)^{(n-1)/4n} \min\left(c_1(K)c_K^{\Lambda}(n+1-i(Q)), c_K^{\mathrm{BV}}(n+1-i(Q))\right)(2H(Q))^{(n-i(Q))/2}\,\frac{H(E_t)}{H(F)}.$$

This information put in the previous estimate of $H(F_a)$ implies

$$\begin{aligned} H(F_a) &\leq (1+\varepsilon)^{1/2} \times 4\min\left(c_1(K)c_K^{\Lambda}(n+1-i(Q)), c_K^{\mathrm{BV}}(n+1-i(Q))\right)^2 \\ &\quad \times (2H(Q))^{n+1-i(Q)}\left(|\alpha|H(E)\right)^2/H(F). \end{aligned}$$

We have $H(F) \geq \lambda_1^{\mathrm{BV}}(F)^{i(Q)}/c_K^{\mathrm{BV}}(i(Q))$ and, since $F \subset E \times \{0\}$, we also have $\lambda_1^{\mathrm{BV}}(F) \geq \lambda_1^{\mathrm{BV}}(E \times \{0\})$ so $\lambda_1^{\mathrm{BV}}(F) \geq \lambda_1^{\mathrm{BV}}(E)/\sqrt{2}$ by Lemma 4.4. Reporting these estimates in the previous bound for $H(F_a)$, we obtain $H(F_a) \leq (1+\varepsilon)^{1/2}\mathcal{T}_1/c_K^{\mathrm{BV}}(i(Q))$ with Lemma 4.5. It is then enough to resume the demonstration of Theorem 4.1 by replacing F by F_a (and so $\mathcal{T}_0$ by $\mathcal{T}_1$) to conclude. □

4.7. The constant $c_K^{\mathrm{BV}}(i(Q))c_K^{\Lambda}(n+1-i(Q))$ in $\mathcal{T}_0$ is finite (only) when K is Siegel field with $c_1(K) < +\infty$. This happens e.g., when K is number field or $[\overline{K} : K] \leq 2$ or, also, when $K = \cup_n K_n$ is the union of a tower of number fields $(K_n)_{n\in\mathbb{N}}$ of bounded root discriminants [GR1, Lemma 5.8]. Additionally, the following estimates, valid for all $n \geq 1$ and $i \in \{0, \dots, n-1\}$, may be of interest:

(1) when K is a number field of root discriminant $\delta_{K/\mathbb{Q}}$, we have

$$c_K^{\mathrm{BV}}(i+1)c_K^{\Lambda}(n-i) \leq n^{n/2}\delta_{K/\mathbb{Q}}^{(n+1)/2},$$

(2) when $K = \overline{\mathbb{Q}}$, we have

$$c_{\overline{\mathbb{Q}}}^{\mathrm{BV}}(i+1)c_{\overline{\mathbb{Q}}}^{\Lambda}(n-i) \leq c_{\overline{\mathbb{Q}}}^{\Lambda}(n) \leq n^{n/2}.$$

The first one derives from $c_K^{\mathrm{BV}}(n) \leq (n\delta_{K/\mathbb{Q}})^{n/2}$ and the bound $(i+1)^{i+1}(n-i)^{n-i} \leq n^n$ (the function $a \mapsto (a+i)\log(a+i) - a\log a$ is increasing). The second bound is a consequence of the formula given for $c_{\overline{\mathbb{Q}}}^{*}(n)$ coupled with the estimate $aH_a + bH_b \leq (a+b-1)H_{a+b-1} + 1$ satisfied by the harmonic number $H_a = 1 + 1/2 + \cdots + 1/a$ for all positive integers a, b and proven by induction on b.

Appendix

a. Here we provide a proof of Dirichlet's approximation theorem given at the beginning of the introduction. First, observe that if $n = 1$ the statement is equivalent to the existence of a nonzero vector $(\upsilon, \phi) \in \mathbb{Z}^n \times \mathbb{Z}$ such that

$$0 \leq \phi \leq T \quad \text{and} \quad |\phi\alpha - \upsilon| \leq \frac{1}{T}$$

since the quadratic form q is $q(x) = (\det q)x^2$ ($x \in \mathbb{R}$), which is the classical Dirichlet's approximation theorem in $\mathbb{R}$. So, now we can assume $n \geq 2$. Let a be a positive real number and define the Euclidean norm

$$\|(x, y)\| = \left(q(y\alpha - x) + ay^2\right)^{1/2}$$

on $\mathbb{R}^n \times \mathbb{R}$. The pair $\Lambda = (\mathbb{Z}^n \times \mathbb{Z}, \|\cdot\|)$ is a Euclidean lattice of rank $n+1$ and covolume $(a\det q)^{1/2}$. Minkowski's First Theorem gives the existence of $(\upsilon, \phi) \in \mathbb{Z}^n \times \mathbb{Z}$, nonzero, such that $\|(\upsilon, \phi)\| \leq \sqrt{\gamma_{n+1}} \times \mathrm{covol}(\Lambda)^{1/(n+1)}$ where γ_{n+1} is the Hermite constant. Moreover, we can choose $\phi \geq 0$. Squaring we get

$$q(\phi\alpha - \upsilon) + a\phi^2 \leq \gamma_{n+1}(a\det q)^{1/(n+1)}.$$

Now we choose a such that

$$T^2 = \frac{\gamma_{n+1}(a\det q)^{1/(n+1)}}{a}, \quad \text{that is,} \quad a = (\gamma_{n+1})^{1+1/n}\frac{(\det q)^{1/n}}{T^{2+2/n}}.$$

Thus we obtain $0 \leq \phi \leq T$ and

$$q(\phi\alpha - \upsilon) \leq aT^2 = (\gamma_{n+1})^{1+1/n}\frac{(\det q)^{1/n}}{T^{2/n}}.$$

It remains to observe that $(\gamma_{n+1})^{1+1/n} \le n$: it is true for $n=2$ since $\gamma_3 = 2^{1/3}$. It is also true for $n \ge 3$ since $\gamma_{n+1} \le 1 + (n+1)/4$ (see [MH, p. 17]) and it is easy to check

$$\left(1 + \frac{n+1}{4}\right)^{1+1/n} \le n, \qquad \text{when } n \ge 3.$$

b. Here we compare the constant κ_q of [Mo, Theorem 1] with $\mathfrak{c}_q = 2(2\gamma_n)^n \det q$ which is in Theorem 1.1. We first recall that

$$\kappa_q = 6 \max\left(1, (n+1)^{n+1} 6^n 2^{n^2} \left(\frac{(n/2)!}{\pi^{n/2}}\right)^{n+1} (\det q)^{(n+1)/2}\right)^2$$

(taking into account $\mathfrak{o}_q = \mathrm{vol}(\{x \in \mathbb{R}^n \mid q(x) \le 1\}) = (\det q)^{-1/2} \pi^{n/2}/(n/2)!$). Since $\det q$ is a positive integer, the second part in the maximum is always greater than 1 so we can write $\kappa_q = \kappa(n)(\det q)^{n+1}$ with

$$\kappa(n) = 6^{2n+1} 4^{n^2} (n+1)^{2(n+1)} \left(\frac{(n/2)!}{\pi^{n/2}}\right)^{2(n+1)} \ge 9^n (n+1)^{2(n+1)} \left(\frac{2n}{e\pi}\right)^{n(n+1)}$$

(using $x! \ge (x/e)^x$ for $x > 0$). On the other hand we have $\mathfrak{c}_q \le 2(2+n/2)^n \det q$ using the bound $\gamma_n \le 1 + n/4$ recalled above. Then we deduce $\big(2(2\gamma_n)^n\big)^{0.8n} \le \kappa(n)$ (and so $\mathfrak{c}_q^{0.8n} \le \kappa_q$) since

$$\left(2\left(2 + \frac{n}{2}\right)^n\right)^{0.8} \le 9(n+1)^{2(1+1/n)} \left(\frac{2n}{e\pi}\right)^{n+1}.$$

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