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Some explicit computations in Arakelov geometry of abelian varieties

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Abstract. Given a polarized complex abelian variety (A, L), a Gromov lemma makes a comparison between the sup and L^2 norms of a global section of L. We give here an explicit bound which depends on the dimension, degree and injectivity diameter of (A, L). It rests on a more general estimate for the jet of a global section of L. As an application we deduce some estimates of the maximal slope of the tangent and cotangent spaces of a polarized abelian variety defined over a number field. These results are effective versions of previous works by Masser and Wüstholz on one hand and Bost on the other. They also improve some similar statements established by Graftieaux in 2000.

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1. Introduction

Let (A, L) be a polarized abelian variety over a number field K. Its tangent space at the origin t_A is a finite dimensional K-vector space which can be endowed with a structure of Hermitian vector bundle over the ring of integers $\mathcal{O}_K$: the integral structure is given by the tangent space at the origin of the Néron model of A over $\mathcal{O}_K$ and the metrics at infinity are provided by the Riemann forms of the complex line bundles L_σ induced by L on each $A_\sigma = A \times_\sigma \operatorname{Spec} \mathbb{C}$ for $\sigma : K \hookrightarrow \mathbb{C}$. This feature allows to consider the successive Arakelov slopes of t_A in the sense of the 1995 Bourbaki seminar of Bost [Bo2].

The aim of our article is to provide some sharp and explicit bounds for the slopes of t_A, in terms of the dimension, Faltings height and degree of A relative to L. The estimates obtained in Sections 4.2 (upper bound for the maximal slope) and 4.3 (lower bound for the minimal slope) improve on previous results by Graftieaux [Gr] and some of them are even best possible with respect to L. For example, such results enable to have a basis of t_A made up of vectors whose height is well controlled (using connections between minima and slopes, see [Ga2, § 4]).

The main motivation for this text lies in the progress made during the last twenty years about the effective aspects of some problems of Diophantine geometry for abelian varieties. Indeed it can be observed that the emergence of the Faltings height of A in the proofs of the major theorems of [BG,Ga1,GR1] only rests on three basic tools:

- A matrix lemma which bounds from below the norms of the non-zero periods of A_σ for every $\sigma : K \hookrightarrow \mathbb{C}$,
- a formula for the Arakelov degree of the space of global sections of L equipped with a suitable Hermitian vector bundle structure over a finite extension of K,
- an explicit lower bound for the minimal slope of t_A.

If the origin of the first problem is located in an article of Masser [Mas], it has since been studied by several authors and an almost optimal solution was found by Autissier in 2013 [Au1]. Regarding the formula for the Arakelov degree of $\mathrm{H}^0(A, L)$, it has been established by Bost in 1996 [Bo1] (see the end of § 2.3.2). These first two tools allow to build the third one via evaluation at order 1 of global sections of L as Bost explained in [Bo2, § 5.3.4]. Following this, Graftieaux gave an explicit estimate for the minimal slope of t_A, still used until now. Furthermore, the latest accurate period theorems obtained by the author and Rémond [GR1] on one hand and the author and Bosser [BG] on the other

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hand brought to light the need for sharper statements, in particular from a numerical point of view. That is why it seemed useful to us to examine again the work of Graftieaux, taking into account the technological advances yielded by Autissier's matrix lemma. Here the use of the first jet evaluation map is done in a more direct way, without any reference to a so-called Shimura map, by applying a basic slope inequality. To do this, we have been led to consider the square of the polarization on A and to evaluate at a well-chosen torsion point of A so that the jet be onto. On the way, we revisit the so-called Gromov lemma (according to the terminology of Gillet and Soulé [GS, § 5.2.3]) which makes a comparison between the supremum norm of a section of L_σ (for any $\sigma : K \hookrightarrow \mathbb{C}$) and its Hermitian norm. This type of results turns out to be of great interest in other settings, such as the theory of linear forms in abelian logarithms [Ga1].

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2. Preliminary results

2.1

Let A be a complex abelian variety of dimension g and L be an ample and symmetric line bundle over A. Let t_{A} be the tangent space at the origin of A and $\exp_{\mathsf{A}} : t_{\mathsf{A}} \to \mathsf{A}$ be the exponential map of A. The period lattice $\Omega_{\mathsf{A}} = \ker \exp_{\mathsf{A}}$ of A is a free $\mathbb{Z}$-module with rank $2g$. An Appell-Humbert data for L is composed of a complex Hermitian form $H : t_{\mathsf{A}} \times t_{\mathsf{A}} \to \mathbb{C}$ (linear in the second variable), which is positive definite since L is ample and such that $\operatorname{Im} H(\Omega_{\mathsf{A}}, \Omega_{\mathsf{A}}) \subset \mathbb{Z}$, and a semicharacter $\chi : \Omega_{\mathsf{A}} \to (\{z \in \mathbb{C}\,;\ |z| = 1\}, \times)$ that is, for all $\omega, \xi \in \Omega_{\mathsf{A}}$, we have

$$\chi(\omega + \xi) = \chi(\omega)\chi(\xi) \exp(\pi i \operatorname{Im} H(\omega, \xi)).$$

The Hermitian form H, called the *Riemann form* of L, induces a Hermitian norm on the tangent space t_{A}: for all $z \in t_{\mathsf{A}}$, $\|z\|_{\mathsf{L}} = H(z, z)^{1/2}$. Define the factor of automorphy $a_{\mathsf{L}} : \Omega_{\mathsf{A}} \times t_{\mathsf{A}} \to \mathbb{C}$ of L by

$$\forall \omega \in \Omega_{\mathsf{A}}, \quad \forall z \in t_{\mathsf{A}}, \quad a_{\mathsf{L}}(\omega, z) = \chi(\omega) \exp\left(\pi H(\omega, z) + \frac{\pi}{2} H(\omega, \omega)\right)$$

and consider a trivialization $\phi\colon \exp_{\mathsf{A}}^* \mathsf{L} \to \mathcal{O}_{t_{\mathsf{A}}}$ compatible with this factor. Thus we get an isomorphism $s \mapsto \vartheta = \phi(\exp_{\mathsf{A}}^* s)$ which associates to a global section $s \in \mathrm{H}^0(\mathsf{A}, \mathsf{L})$ a holomorphic map $\vartheta : t_{\mathsf{A}} \to \mathbb{C}$ satisfying $\vartheta(\omega + z) = a_{\mathsf{L}}(\omega, z)\vartheta(z)$ for all $z \in t_{\mathsf{A}}$ and $\omega \in \Omega_{\mathsf{A}}$. The map

$$z \in t_{\mathsf{A}} \longmapsto |\vartheta(z)| \exp\left(-\frac{\pi}{2}\|z\|_{\mathsf{L}}^2\right)$$

is invariant by translation by elements of Ω_{A}. Hence, given $z \in t_{\mathsf{A}}$ and $x = \exp_{\mathsf{A}}(z) \in \mathsf{A}$, we may endow the fiber $\mathsf{L}_x = x^*\mathsf{L}$ with the (cubist) metric:

$$\forall s \in \mathrm{H}^0(\mathsf{A}, \mathsf{L}), \quad \|s(x)\|_{x^*\mathsf{L}} = |\vartheta(z)| \exp\left(-\frac{\pi}{2}\|z\|_{\mathsf{L}}^2\right).$$

That induces (at least) two norms on $\mathrm{H}^0(\mathsf{A}, \mathsf{L})$: the supremum norm $\|\cdot\|_\infty$ defined by

$$\forall s \in \mathrm{H}^0(\mathsf{A}, \mathsf{L}), \quad \|s\|_\infty = \sup\left(\|s(x)\|_{x^*\mathsf{L}}\,;\ x \in \mathsf{A}\right),$$

and the Hermitian norm $\|\cdot\|_2$ defined by

$$\forall s \in \mathrm{H}^0(\mathsf{A}, \mathsf{L}), \quad \|s\|_2 = \left(\int_{\mathsf{A}} \|s(x)\|_{x^*\mathsf{L}}^2 \,\mathrm{d}\mu(x)\right)^{1/2}$$

where μ is the normalized Haar measure on the compact group of complex points of A.

2.2 Jets of sections

2.2.1 Algebraic definition

Let $\mathcal{S}$ be a scheme and $\mathcal{A} \to \mathcal{S}$ be a scheme with base $\mathcal{S}$. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{A}$. We assume that there exists a closed immersion $\varepsilon \colon \mathcal{S} \to \mathcal{A}$. Let us denote by $\mathcal{I}$ the sheaf of ideals on $\mathcal{A}$ defined by ε and by $\Omega_{\mathcal{A}/\mathcal{S}} = \mathcal{I}/\mathcal{I}^2$ the $\mathcal{O}_{\mathcal{A}}$-module of relative differentials. Let ℓ be a nonnegative integer. When $\mathcal{A} \to \mathcal{S}$ is smooth along ε ($\mathcal{I}$ is regular), the inverse image $\varepsilon^*(\mathcal{I}^\ell/\mathcal{I}^{\ell+1})$ is isomorphic to the symmetric power $S^\ell(\varepsilon^*\Omega_{\mathcal{A}/\mathcal{S}})$. A section $s \in \mathrm{H}^0(\mathcal{A}, \mathcal{L})$ vanishes along ε up to order ℓ when $s \in \mathrm{H}^0(\mathcal{A}, \mathcal{I}^\ell \otimes_{\mathcal{O}_{\mathcal{A}}} \mathcal{L})$. In that case, the jet of s of order ℓ along ε, denoted by $\mathrm{jet}^\ell\, s(\varepsilon)$, is the image of s by the composed map

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{A}, \mathcal{I}^\ell \otimes \mathcal{L}) & \longrightarrow & \mathrm{H}^0(\mathcal{A}, \mathcal{I}^\ell/\mathcal{I}^{\ell+1} \otimes \mathcal{L}) \quad \text{Projection} \\ & & \downarrow \qquad\qquad \text{formula} \\ \mathrm{H}^0\big(\mathcal{S}, S^\ell(\varepsilon^*\Omega_{\mathcal{A}/\mathcal{S}}) \otimes \varepsilon^*\mathcal{L}\big) & \longleftarrow & \mathrm{H}^0\big(\mathcal{S}, \varepsilon^*(\mathcal{I}^\ell/\mathcal{I}^{\ell+1}) \otimes \varepsilon^*\mathcal{L}\big) \end{array}$$

2.2.2 Analytic description of the jets

Let us come back to the case of a complex abelian variety: $\mathcal{S} = \operatorname{Spec}\mathbb{C}$, $\mathcal{A} = \mathsf{A}$ and $\mathcal{L} = \mathsf{L}$. Let $x \in \mathsf{A}(\mathbb{C})$ and $z \in t_{\mathsf{A}}$ be a logarithm of x. This point can be viewed as a closed immersion $x \colon \operatorname{Spec}\mathbb{C} \to \mathsf{A}$ defined by the sheaf of ideals I. For $\ell \in \mathbb{N}$ and $s \in \mathrm{H}^0(\mathsf{A}, \mathsf{I}^\ell \otimes_{\mathcal{O}_{\mathsf{A}}} \mathsf{L})$, the element $\mathrm{jet}^\ell\, s(x)$ is the image of s in the quotient

$$\mathrm{H}^0\big(\operatorname{Spec}\mathbb{C}, x^*\big(\mathsf{I}^\ell/\mathsf{I}^{\ell+1} \otimes_{\mathcal{O}_{\mathsf{A}}} \mathsf{L}\big)\big) \simeq S^\ell(t_{\mathsf{A}}^\vee) \otimes_{\mathbb{C}} x^*\mathsf{L}.$$

This construction is compatible with the map $s \mapsto \vartheta$ of § 2.1. If we consider a basis $e = (e_1, \dots, e_g)$ of t_{A}, the jet of ϑ of order ℓ at z takes the form

$$\mathrm{jet}^\ell\, \vartheta(z) = \sum_i \frac{1}{i_1! \cdots i_g!} \left(\frac{\partial}{\partial z_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial z_g}\right)^{i_g} \vartheta(z_1 e_1 + \cdots + z_g e_g) \cdot (e_1^\vee)^{i_1} \cdots (e_g^\vee)^{i_g}$$

where $i = (i_1, \dots, i_g) \in \mathbb{N}^g$ is of length $|i| = i_1 + \cdots + i_g$ equal to ℓ. Let $\|\cdot\|$ be a norm on $S^\ell(t_{\mathsf{A}}^\vee)$. The trivialization of $\exp_{\mathsf{A}}^* \mathsf{L}$ which induces the isomorphism $s \mapsto \vartheta$ allows to connect the norm of $\mathrm{jet}^\ell\, s(x)$ (with respect to $S^\ell(t_{\mathsf{A}}^\vee) \otimes x^*\mathsf{L}$) to that of $\mathrm{jet}^\ell\, \vartheta(z)$ by the formula:

$$\|\mathrm{jet}^\ell\, s(x)\|_{S^\ell(t_{\mathsf{A}}^\vee)\otimes x^*\mathsf{L}} = \|\mathrm{jet}^\ell\, \vartheta(z)\| \cdot \|1\|_z = \|\mathrm{jet}^\ell\, \vartheta(z)\| \exp\Big(-\frac{\pi}{2}\|z\|_{\mathsf{L}}^2\Big). \tag{1}$$

Afterward the tangent space t_{A} will be endowed with a Hermitian inner product. If e is an orthonormal basis of t_{A}, the chosen norm on $S^\ell(t_{\mathsf{A}}^\vee)$ will be defined by:

$$\forall\, (x_i)_{|i|=\ell} \in \mathbb{C}^{\binom{\ell+g-1}{g-1}}, \qquad \Bigg\| \sum_{|i|=\ell} x_i (e_1^\vee)^{i_1} \cdots (e_g^\vee)^{i_g} \Bigg\|^2_{S^\ell(t_{\mathsf{A}}^\vee)} = \sum_{|i|=\ell} |x_i|^2 \cdot \frac{i_1! \cdots i_g!}{\ell!}.$$

2.3 Elements of slope theory

We make some very brief reminders about Hermitian vector bundles over the spectrum of the ring of integers $\mathcal{O}_K$ of a number field K and their slopes. In a second time we shall mention some classical statements about several Hermitian vector bundles attached to a polarized abelian variety. We refer to [Bo2, Appendix A] or to [Ga2] for more details.*

*Following the latter reference, we omit the traditional bar over Hermitian vector bundles and the hat over their degree and slopes.

2.3.1

A *Hermitian vector bundle* E over $\mathcal{O}_K$ is the data of a locally free $\mathcal{O}_K$-module $\mathcal{E}$ of finite type with Hermitian norms $\|\cdot\|_{E,v}\colon \mathcal{E}\otimes_v \mathbb{C} \to \mathbb{R}^+$ at archimedean places v of K. The $\mathcal{O}_K$-module $\mathcal{E}$ brings about norms $\|\cdot\|_{E,v}\colon \mathcal{E}\otimes_{\mathcal{O}_K} K_v \to \mathbb{R}^+$ at ultrametric places v of K, defined by:

$$\forall\, \xi \in \mathcal{E}\otimes_{\mathcal{O}_K} K_v, \quad \|\xi\|_{E,v} = \inf\,(|x|_v\ ;\ x\in K_v,\ \xi\in x.\mathcal{E}).$$

Here K_v is the completion of K at v equipped with the absolute value $|\cdot|_v$ normalized in such a way that $|p|_v \in \{1, p, p^{-1}\}$ for every prime p. The standard example is $\mathcal{E} = \mathcal{O}_K^n$ with $|(x_1,\dots,x_n)|_{2,v} = \big(\sum_i |x_i|_v^2\big)^{1/2}$ $\big(x_i\in K_v\big)$ at the archimedean places. In that case one has $|(x_1,\dots,x_n)|_{2,v} = \max_i |x_i|_v$ when v is ultrametric. Every Hermitian vector bundle E can be obtained from this canonical one via an adelic matrix $a=(a_v)_v \in \mathrm{GL}_n(\mathbb{A}_K)$ and after the choice of a K-basis $e=(e_1,\dots,e_n)$ of $\mathcal{E}\otimes_{\mathcal{O}_K} K$:

$$\forall\, x=(x_1,\dots,x_n)\in K_v^n \quad \|x_1e_1+\cdots+x_ne_n\|_{E,v} = |a_vx|_{2,v}.$$

The datum of E amounts to the data of a K-vector space (still denoted by) E of finite dimension (namely $\mathcal{E}\otimes_{\mathcal{O}_K} K$) with norms on the topological space $E\otimes_K K_v$ satisfying the above equality. A Hermitian subbundle of E is a subspace of E endowed with the restricted norms of E. The dual $E^\vee$ of a Hermitian vector bundle E is comprised of the dual space $\mathrm{Hom}_K(E,K)$ with the dual norms of $\|\cdot\|_{E,v}$'s. If $F\subset E$ is a subbundle, the quotient space E/F inherits of (quotient) norms from E, giving it a Hermitian vector bundle structure over $\mathcal{O}_K$. The *(normalized Arakelov) degree* of a Hermitian vector bundle E is

$$\deg E = -\frac{1}{[K:\mathbb{Q}]}\sum_{v \text{ place of } K} [K_v:\mathbb{Q}_v]\log|\det a_v|_v$$

and its *slope* is $\mu(E) = (\deg E)/\dim E$. It does not depend on the choice of (a,e) and the degree is invariant by finite extension of K. When $F\subset E$ is a subbundle, we have $\deg E/F = \deg E - \deg F$. The *canonical polygon* P_E of E is the piecewise linear function delimiting from above the convex hull of the set $(\dim E', \deg E')$ where E' runs over the non-zero subbundles of E. It satisfies the noteworthy duality relation: for all $0\le x\le n=\dim E$, we have $P_{E^\vee}(x) = P_E(n-x) - \deg E$. We also have $P_{E\otimes L}(x) = P_E(x) + x\deg L$ for any Hermitian line bundle L. The *maximal slope* of E is the real number $\mu_{\max}(E) = P_E(1)$, also equal to $\max\,(\mu(E')\ ;\ E'\subset E)$. These notions are invariant by scalar extension. A Hermitian vector bundle E such that $\mu_{\max}(E)=\mu(E)$ is called *semistable*. A K-linear map $\varphi\colon E\to F$ between two Hermitian vector bundles E and F over $\mathcal{O}_K$ induces local maps $\varphi_v\colon E\otimes_K K_v \to F\otimes_K K_v$ at the places v of K, having operator norms

$$\|\varphi\|_v = \sup\left(\frac{\|\varphi_v(x)\|_{F,v}}{\|x\|_{E,v}}\ ;\ x\in (E\otimes_K K_v)\setminus\{0\}\right).$$

By bringing together these norms, we form the *height* of φ:

$$h(\varphi) = \frac{1}{[K:\mathbb{Q}]}\sum_v [K_v:\mathbb{Q}_v]\log\|\varphi\|_v.$$

This number allows to compare the canonical polygons of E and F.

Proposition 2.1. *Let E and F be some Hermitian vector bundles over $\mathcal{O}_K$ and $\varphi\colon E\to F$ be a K-linear map. If φ is injective then, for all $x\in[0,\dim E]$, we have $P_E(x)\le P_F(x)+xh(\varphi)$. If φ is surjective then, for all $y\in[0,\dim F]$, we have $P_F(y)\le P_E(y+\dim\ker\varphi) + (\dim F - y)h(\varphi) - \deg E + \deg F$.*

Proof. Since the canonical polygon is linear on $[i-1,i]$, it is enough to prove these properties for x or y equal to an integer i. According to [Ga2, Theorem 43], the j-th slope $\mu_j(E) = P_E(j) - P_E(j-1)$ of E satisfies the inequality $\mu_j(E)\le\mu_j(F)+h(\varphi)$ when φ is injective and $j\in\{1,\dots,\dim E\}$. We add them for j ranging from 1 to i to get the first statement. The second one is a consequence: we apply the first result to the one-to-one dual map $\varphi^\vee\colon F^\vee\to E^\vee$, using the duality formula for the canonical polygon and noting that $\varphi^\vee$ has the same height as φ. □

2.3.2

Let A be an abelian variety over a number field K of dimension g. For every finite extension K'/K and every embedding $\sigma\colon K' \hookrightarrow \mathbb{C}$, we denote by A_σ the complex abelian variety deduced from $A_{K'} = A \times_K K'$ by scalar extension through σ. Over a certain finite extension K'/K, $A_{K'}$ is semistable. The sheaf of relative differentials $\omega_{\mathcal{A}'/\mathcal{O}_{K'}}$ of the Néron model $\mathcal{A}' \to \mathcal{O}_{K'}$ of $A_{K'}$ over K', endowed with the norms $\|s\|_\sigma^2 = 2^{-g} i^{g^2} \int_{A_\sigma} s \wedge \overline{s}$ is a Hermitian vector bundle over $\mathcal{O}_{K'}$ whose degree is the *(stable) Faltings height* $h_F(A)$ of A. Corollary 8.4 of [GR1] states that $h_F(A) \geq -g \log(\pi\sqrt{2})$, named *Bost's inequality* afterwards. Furthermore, given an ample and symmetric line bundle L on A (the pair (A, L) is said to be a polarized abelian variety), we can consider the Hermitian tangent bundle $(t_A, \|\cdot\|_L)$ over $\mathcal{O}_K$, also denoted t_A when there is no ambiguity on L, composed of the tangent space at the origin of the Néron model of A over $\operatorname{Spec}\mathcal{O}_K$ and with the norms $\|\cdot\|_{L_\sigma}$ induced by the Riemann forms of the complex line bundles $L_\sigma \to A_\sigma$ for all $\sigma\colon K \hookrightarrow \mathbb{C}$. Its degree is linked to the Faltings height of A by the formula

$$\deg(t_A, \|\cdot\|_L) = -\left(h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A, L)\right)$$

where $\mathrm{h}^0(A, L)$ denotes the dimension of $\mathrm{H}^0(A, L)$. In the same vein, the space of global sections $\mathrm{H}^0(A, L)$ of L can be equipped with a Hermitian vector bundle structure over a suitable finite extension K'/K. The integral model is composed of $\mathrm{H}^0(\mathcal{A}', \mathcal{L}')$ where $\mathcal{L}'$ is a cubist Hermitian line bundle over the Néron model $\mathcal{A}'$ of $A_{K'}$ over K'. The metrics at the complex embeddings σ of K' are the norms $\|\cdot\|_2$ on $\mathrm{H}^0(\mathcal{A}', \mathcal{L}') \otimes_\sigma \mathbb{C}$ defined at the end of § 2.1 (see [Ga1, § 4.2.2]). Building on deep results by Moret-Bailly, Bost proved that the Hermitian vector bundle $\mathrm{H}^0(\mathcal{A}', \mathcal{L}')$ (over $\mathcal{O}_{K'}$) is semistable with slope equal to $-(1/2)h_F(A) + (1/4)\log(\mathrm{h}^0(A, L)/(2\pi^2)^g)$ (see [Bo1, § 4.3] and [Bo2, p. 129]).

3. Gromov lemma for the jets

Let us start with a Gromov lemma in its initial acceptance (according to [GS, § 5.2.3]) before generalizing it to higher order jets.

3.1

Let (A, L) be a polarized complex abelian variety of dimension g. For $s \in \mathrm{H}^0(\mathsf{A}, \mathsf{L})$, we always have $\|s\|_2 \leq \|s\|_\infty$. A Gromov lemma makes a comparison in the other direction. The following statement is an unpublished result of Autissier. It involves the *injectivity diameter*

$$\rho(\mathsf{A}, \mathsf{L}) = \min\left(\|\omega\|_{\mathsf{L}}\,;\ \omega \in \Omega_{\mathsf{A}} \setminus \{0\}\right)$$

of (A, L), where $\|\cdot\|_{\mathsf{L}}$ is the Hermitian norm associated to the Riemann form of L.

Proposition 3.1. *Assume* (A, L) *is principally polarized. Then, for all* $s \in \mathrm{H}^0(\mathsf{A}, \mathsf{L})$*, we have*

$$\|s\|_\infty \leq \|s\|_2 \times \max\left(1, \frac{1}{\rho(\mathsf{A}, \mathsf{L})}\right)^{g/2} (1.7g)^{g/2}.$$

The proof is based on some estimates for explicit theta functions.

Proof. Let $\tau \in \mathrm{M}_g(\mathbb{C})$ be a Siegel-reduced element of the Siegel upper half plane such that Ω_{A} identifies with $\mathbb{Z}^g + \tau\mathbb{Z}^g$. Put $Y = \operatorname{Im}\tau$. Then the norm on t_{A} induced by L can be written as $\|z\|_{\mathsf{L}} = ({}^{\mathrm{t}}\overline{z}Y^{-1}z)^{1/2}$ and the one dimensional space $\mathrm{H}^0(\mathsf{A}, \mathsf{L})$ is generated by the theta function

$$\forall z \in \mathbb{C}^g, \quad \vartheta_\tau(z) = \exp\left(\frac{\pi}{2}{}^{\mathrm{t}}zY^{-1}z\right) \sum_{n \in \mathbb{Z}^g} \exp\left(i\pi\,{}^{\mathrm{t}}n\tau n + 2i\pi\,{}^{\mathrm{t}}nz\right).$$

It is known that $\|\vartheta_\tau\|_2 = \det(2Y)^{-1/4}$ (see, for instance, [GR1, Lemma 8.2 (1)]). Moreover [Au2, Lemma 8.2] yields $\|\vartheta_\tau\|_\infty \le \vartheta_{iY}(0) = \vartheta_{iY^{-1}}(0)(\det Y)^{-1/2}$, from which we deduce

$$\frac{\|\vartheta_\tau\|_\infty}{\|\vartheta_\tau\|_2} \le (2^g \det Y)^{1/4}\vartheta_{iY}(0) = 2^{g/4}(\vartheta_{iY}(0)\vartheta_{iY^{-1}}(0))^{1/2}.$$

Besides [Au2, Proposition 8.4] applied to Y and Y^{-1}, with $\lambda(Y) = \min_{n\in\mathbb{Z}^g\setminus\{0\}} {}^{\mathrm{t}}nYn \ge \sqrt{3}/2$ and $\lambda(Y^{-1}) \ge \rho(\mathsf{A},\mathsf{L})^2$, gives

$$\vartheta_{iY}(0)\vartheta_{iY^{-1}}(0) \le \left(\frac{g+2}{2}\right)^2 \left(\max\left(\frac{g+2}{\pi\sqrt{3}},1\right)\max\left(\frac{g+2}{2\pi\rho(\mathsf{A},\mathsf{L})^2},1\right)\right)^{g/2}.$$

We combine with the previous estimate to get, for all $s \in \mathrm{H}^0(\mathsf{A},\mathsf{L}) = \mathbb{C}.\vartheta_\tau$,

$$\|s\|_\infty \le \|s\|_2 \times \max\left(1, \frac{1}{\rho(\mathsf{A},\mathsf{L})}\right)^{g/2} \times \left(\frac{g+2}{2}\right)\left(2\max\left(\frac{g+2}{\pi\sqrt{3}},1\right)\max\left(\frac{g+2}{2\pi},1\right)\right)^{g/4}.$$

The latter constant is smaller than $(1.5g)^{g/2}$ when $g \ge 2$. Otherwise, if $g = 1$, we proceed as above except that we keep the exact value of $\vartheta_{iY}(0)$:

$$\frac{\|\vartheta_\tau\|_\infty}{\|\vartheta_\tau\|_2} \le (2Y)^{1/4}\sum_{n\in\mathbb{Z}} \exp\left(-\pi n^2 Y\right) =: \theta(Y).$$

Since $Y = \rho(\mathsf{A},\mathsf{L})^{-2}$ (see [GR1, Remark 3.3]), it is enough to prove that

$$\forall\, Y \ge \frac{\sqrt{3}}{2}, \quad (2\min(1,Y))^{1/4}\sum_{n\in\mathbb{Z}} \exp(-\pi n^2 Y) \le (1.7)^{1/2}.$$

If $Y \ge 1$ the left hand side is bounded by $\theta(1) \le \sqrt{1.7}$. Otherwise, we observe that θ is decreasing on the interval $[\sqrt{3}/2, 1]$ and so bounded by its value at $\sqrt{3}/2$, which satisfies $\theta(\sqrt{3}/2) \le \sqrt{1.7}$. Indeed the map $Y \mapsto Y^{3/4}\theta'(Y)$, which has a positive derivative on $[\sqrt{3}/2, 1]$, is increasing on this interval. The functional equation $\theta(Y) = \theta(1/Y)$ implies $\theta'(1) = 0$ and so $\theta'(Y) < 0$ for all $Y \in (\sqrt{3}/2, 1)$, yielding the decreasing property of θ. □

Proposition 3.1 is best possible with respect to $\rho(\mathsf{A},\mathsf{L})$. Indeed [Bo2, Inequality (C.9)] claims that $\|\vartheta_\tau\|_\infty \ge 1$, that is, $\|s\|_\infty \ge \|s\|_2 \det(2\,\mathrm{Im}\,\tau)^{1/4}$ for all $s \in \mathrm{H}^0(\mathsf{A},\mathsf{L})$. When $\dim \mathsf{A} = 1$, the imaginary part of τ is precisely $\rho(\mathsf{A},\mathsf{L})^{-2}$ as stated above. In this case, a global section s of the g-th external tensor power (B,M) of (A,L) satisfies $\|s\|_\infty \ge \|s\|_2(\sqrt{2}/\rho(\mathsf{A},\mathsf{L}))^{g/2}$. The relation $\rho(\mathsf{B},\mathsf{M}) = \rho(\mathsf{A},\mathsf{L})$ then shows that the exponent $g/2$ of the injectivity diameter in Proposition 3.1 cannot be improved in general. The same example also indicates that the numerical constant 1.7 is close to the best one, necessarily bounded from below by $\sqrt{2} > 1.4$.

3.2 Generalization to jets

The keystone of this article is the following statement.

Theorem 3.2. *Let (A,L) be a polarized complex abelian variety of dimension g. Let $x \in \mathsf{A}$, $\ell \in \mathbb{N}$ and $s \in \mathrm{H}^0(\mathsf{A},\mathsf{L})$ which vanishes at x up to order ℓ. Then we have*

$$\|\,\mathrm{jet}^\ell s(x)\|_{S^\ell(t_{\mathsf{A}}^\vee)\otimes x^*\mathsf{L}} \le \|s\|_2 \times \mathrm{h}^0(\mathsf{A},\mathsf{L})^{1/2}\max\left(1, \frac{1}{\rho(\mathsf{A},\mathsf{L})}\right)^{g/2}(5(g+\ell))^{g/2}\left(\frac{\pi^\ell}{\ell!}\right)^{1/2}$$

(the norm on $S^\ell(t_{\mathsf{A}}^\vee)$ is the one induced by $\|\cdot\|_{\mathsf{L}}$ as explained at the end of § 2.2.2).

Note that the dependence in $\mathrm{h}^0(\mathsf{A},\mathsf{L})$ is accurate (at least for $\ell = 0$) since we always have

$$\mathrm{h}^0(\mathsf{A},\mathsf{L})^{1/2} \le \sup\left(\frac{\|s\|_\infty}{\|s\|_2}\,;\ s \in \mathrm{H}^0(\mathsf{A},\mathsf{L})\backslash\{0\}\right).$$

Indeed, for any orthonormal basis $s_1,\dots,s_\nu$ of $\mathrm{H}^0(\mathsf{A},\mathsf{L})$, consider $b(x) = \sum_{i=1}^{\nu} \|s_i(x)\|^2_{x^*\mathsf{L}}$ for $x \in \mathsf{A}$, which does not depend on the choice of the basis. Let $x \in \mathsf{A}$ such that $b(x) = \sup_{y\in\mathsf{A}} b(y)$ and choose $s_1,\dots,s_\nu$ such that $s_i(x) = 0$ for $2 \le i \le \nu$. Then we have

$$\mathrm{h}^0(\mathsf{A},\mathsf{L}) = \nu = \int_{\mathsf{A}} b(y)\,\mathrm{d}\mu(y) \le b(x) \le \|s_1\|^2_\infty = \left(\frac{\|s_1\|_\infty}{\|s_1\|_2}\right)^2.$$

The proof of Theorem 3.2 is different from that of Proposition 3.1. It is inspired by the analytic estimates of [Via, § 3.1], partly derived from some lecture notes by Bost and it uses some ideas of Autissier and Rémond.

3.2.1

Let us begin with some auxiliary results. Given $\ell \in \mathbb{N}$, an element φ of the symmetric product $S^\ell(t_{\mathsf{A}}^\vee)$ is a sum of products of ℓ linear forms on t_{A}. It defines a polynomial map $z \in t_{\mathsf{A}} \mapsto \varphi(z)$ by evaluating at z the linear forms which compose φ. Given a real number $r > 0$, let us define the sesquilinear form on $S^\ell(t_{\mathsf{A}}^\vee)$ (linear on the right):

$$\forall\,\varphi,\psi \in S^\ell(t_{\mathsf{A}}^\vee) \quad \langle\varphi,\psi\rangle = \int_{B(0,r)} \overline{\varphi(t)}\,\psi(t)\exp\left(-\pi\|t\|^2_{\mathsf{L}}\right)\mathrm{d}t$$

where $B(0,r) = \{u \in t_{\mathsf{A}}\,;\ \|u\|_{\mathsf{L}} < r\}$ and $\mathrm{d}t$ is the normalized Haar measure of $(t_{\mathsf{A}}, \|\cdot\|_{\mathsf{L}})$.

Lemma 3.3. *The product $\langle\cdot,\cdot\rangle$ is a Hermitian inner product on $S^\ell(t_{\mathsf{A}}^\vee)$.*

The proof is straightforward. The group G of isometries of the Hermitian space $(t_{\mathsf{A}}, \|\cdot\|_{\mathsf{L}})$ acts on $S^\ell(t_{\mathsf{A}}^\vee)$ by composition on the right: if $g \in G$ and $\varphi \in t_{\mathsf{A}}^\vee$ then $\varphi \circ g \in t_{\mathsf{A}}^\vee$. By [Ig, § I.4], this action is irreducible. Moreover the norms $\|\cdot\|$ (associated to $\langle\cdot,\cdot\rangle$) and $\|\cdot\|_{S^\ell(t_{\mathsf{A}}^\vee)}$ are both invariant under the action of G. From these facts comes the following result.

Lemma 3.4. *There exists a constant $C > 0$, which depends on g, ℓ, r, such that $\|\cdot\| = C\|\cdot\|_{S^\ell(t_{\mathsf{A}}^\vee)}$.*

Proof. Let $a = \inf\left(\|\varphi\|\,;\ \varphi \in S^\ell(t_{\mathsf{A}}^\vee) \text{ and } \|\varphi\|_{S^\ell(t_{\mathsf{A}}^\vee)} = 1\right)$. Since the infimum is a minimum by continuity on a compact space, the set $S = \{\varphi \in S^\ell(t_{\mathsf{A}}^\vee)\,;\ \|\varphi\| = a\}$ is not empty. This set and, a fortiori, the subspace $\operatorname{Vect} S$ of $S^\ell(t_{\mathsf{A}}^\vee)$ generated by S are G-invariant. The irreducibility property gives $\operatorname{Vect} S = S^\ell(t_{\mathsf{A}}^\vee)$. Moreover if φ and ψ belong to S then $\langle\varphi,\psi\rangle = a\langle\varphi,\psi\rangle_\ell$ where $\langle\cdot,\cdot\rangle_\ell$ is the Hermitian inner product associated to the norm $\|\cdot\|_{S^\ell(t_{\mathsf{A}}^\vee)}$. Indeed, for all $\xi \in \mathbb{C}$, we have $\|\varphi + \xi\psi\| \ge a\|\varphi + \xi\psi\|_{S^\ell(t_{\mathsf{A}}^\vee)}$. Squaring and expanding, we get $\operatorname{Re}\xi\langle\varphi,\psi\rangle \ge a\operatorname{Re}\xi\langle\varphi,\psi\rangle_\ell$ and we conclude taking $\xi \in \{\pm 1, \pm i\}$. This property entails that the set

$$V = \left\{u \in S^\ell(t_{\mathsf{A}}^\vee)\,;\quad \forall\varphi \in S,\quad \langle\varphi,u\rangle = a\langle\varphi,u\rangle_\ell\right\}$$

is a linear subspace of $S^\ell(t_{\mathsf{A}}^\vee)$, containing S and so $\operatorname{Vect} S$. We deduce $V = S^\ell(t_{\mathsf{A}}^\vee)$ which leads to $\langle\varphi,u\rangle = a\langle\varphi,u\rangle_\ell$ for all $\varphi, u \in \operatorname{Vect} S = S^\ell(t_{\mathsf{A}}^\vee)$. We put $C = \sqrt{a}$ to conclude. □

Lemma 3.5. *The constant C in the previous lemma is characterized by*

$$C^2 = \frac{\ell!}{\pi^\ell(g-1+\ell)!}\int_0^{\pi r^2} x^{\ell+g-1}\exp(-x)\,\mathrm{d}x$$

(here $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}$).

Proof. Let $(e_1, \dots, e_g)$ be an orthonormal basis of $(t_{\mathsf{A}}, \|\cdot\|_{\mathsf{L}})$. The dual basis $(e_1^{\vee}, \dots, e_g^{\vee})$ yields an orthogonal basis $(e_1^{\vee})^{i_1} \cdots (e_g^{\vee})^{i_g}$, $i = (i_1, \dots, i_g) \in \mathbb{N}^g$ with $|i| = \ell$, of $S^{\ell}(t_{\mathsf{A}}^{\vee})$. We have

$$\|(e_1^{\vee})^{i_1} \cdots (e_g^{\vee})^{i_g}\|_{S^{\ell}(t_{\mathsf{A}}^{\vee})} = \left(\frac{i_1! \cdots i_g!}{\ell!}\right)^{1/2}$$

and, writing $t = t_1 e_1 + \cdots + t_g e_g$ with $(t_1, \dots, t_g) \in \mathbb{C}^g$, we also have

$$\|(e_1^{\vee})^{i_1} \cdots (e_g^{\vee})^{i_g}\|^2 = \int_{|t_1|^2 + \cdots + |t_g|^2 \le r^2} |t_1|^{2i_1} \cdots |t_g|^{2i_g} \exp\big(-\pi \|t\|_{\mathsf{L}}^2\big)\, \mathrm{d}t.$$

Thus we get the formula

$$C^2 = \frac{\ell!}{i_1! \cdots i_g!} \int_{\|t\|_{\mathsf{L}} \le r} |t_1|^{2i_1} \cdots |t_g|^{2i_g} \exp\big(-\pi \|t\|_{\mathsf{L}}^2\big)\, \mathrm{d}t$$

for all $(i_1, \dots, i_g) \in \mathbb{N}^g$ of length ℓ. Summing over all such i's and using the multinomial theorem, we obtain

$$\binom{g-1+\ell}{\ell} C^2 = \int_{B(0,r)} \|t\|_{\mathsf{L}}^{2\ell} \exp\big(-\pi \|t\|_{\mathsf{L}}^2\big)\, \mathrm{d}t.$$

To get the desired formula, we compute the latter integral with Fubini's theorem $\big(\int_{B(0,r)} = \int_0^r \big(\int_{\|t\|_{\mathsf{L}}=x}\big)\, \mathrm{d}x\big)$ using that the volume of the sphere of radius x:

$$\lim_{\varepsilon \to 0^+} \frac{\mathrm{vol}(B(0, x+\varepsilon)) - \mathrm{vol}(B(0,x))}{\varepsilon} = \mathrm{vol}(B(0,1)) \times \lim_{\varepsilon \to 0^+} \frac{(x+\varepsilon)^{2g} - x^{2g}}{\varepsilon}$$

equals $\frac{\pi^g}{g!} \times 2g x^{2g-1}$ (vol stands for the Lebesgue measure on $(t_{\mathsf{A}}, \|\cdot\|_{\mathsf{L}})$). We get

$$C^2 = \frac{2\pi^g \ell!}{(g-1+\ell)!} \int_0^r x^{2\ell+2g-1} \exp(-\pi x^2)\, \mathrm{d}x$$

and we conclude with the change of variables $x \mapsto y = \pi x^2$. □

3.2.2

Let $\mathcal{F}$ be a fundamental domain of t_{A} with respect to the period lattice Ω_{A}. Given a positive real number r and $z \in t_{\mathsf{A}}$, let us define $N(u) = \mathrm{Card}\, \Omega_{\mathsf{A}} \cap B(z-u, r)$ for any $u \in t_{\mathsf{A}}$ (where $B(z-u, r) = z - u + B(0, r)$ is the ball of radius r centered at $z - u$). This integer only takes a finite number of values.

Lemma 3.6. *For every integrable function $f : t_{\mathsf{A}} \to \mathbb{R}$, periodic with respect to Ω_{A}, we have*

$$\int_{B(z,r)} f(u)\, \mathrm{d}u = \int_{\mathcal{F}} f(u) N(u)\, \mathrm{d}u.$$

Proof. Let $\mathbf{1}$ denote the characteristic function of $B(z, r)$. By definition we have

$$N(u) = \sum_{\omega \in \Omega_{\mathsf{A}}} \mathbf{1}(u + \omega).$$

Since $\mathcal{F}$ is a fundamental domain, the translated $\mathcal{F} + \omega$ for $\omega \in \Omega_{\mathsf{A}}$ cover t_{A} and their interiors are disjoint sets. We deduce

$$\int_{B(z,r)} f(u)\, \mathrm{d}u = \sum_{\omega \in \Omega_{\mathsf{A}}} \int_{\mathcal{F}+\omega} f(u)\mathbf{1}(u)\, \mathrm{d}u = \int_{\mathcal{F}} \sum_{\omega \in \Omega_{\mathsf{A}}} f(u+\omega)\mathbf{1}(u+\omega)\, \mathrm{d}u.$$

It remains to observe that the latter sum can be written as $f(u)N(u)$ since $f(u+\omega) = f(u)$ for all $\omega \in \Omega_{\mathsf{A}}$. □

Proposition 3.7. *For any $z \in t_{\mathsf{A}}$, $\ell \in \mathbb{N}$ and $s \in \mathrm{H}^0(\mathsf{A}, \mathsf{L})$ which vanishes at $x = \exp_{\mathsf{A}}(z)$ up to order ℓ, we have*

$$\| \operatorname{jet}^{\ell} s(x) \|_{S^{\ell}(t_{\mathsf{A}}^{\vee}) \otimes x^* \mathsf{L}} \leq \|s\|_2 \times \left(\frac{\mathrm{h}^0(\mathsf{A}, \mathsf{L}) \max_{u \in \mathcal{F}} N(u)}{C^2} \right)^{1/2}$$

where C is the constant of Lemma 3.4.

Proof. Let μ denote the normalized Haar measure on A. The inverse measure $\exp_{\mathsf{A}}^* \mu$ is a Haar measure on t_{A}, which, by uniqueness up to a positive scalar, is proportional to the Lebesgue measure $\mathrm{d}z$ associated to $\| \cdot \|_{\mathsf{L}}$. Thus we have $\mathrm{d}z = \mathrm{h}^0(\mathsf{A}, \mathsf{L}) \exp_{\mathsf{A}}^* \mu$ since

$$\int_{\mathcal{F}} \mathrm{d}z = \mathrm{vol}(\mathcal{F}) = \mathrm{covol}(\Omega_{\mathsf{A}}, \| \cdot \|_{\mathsf{L}}) = \mathrm{h}^0(\mathsf{A}, \mathsf{L}).$$

Let $\vartheta : t_{\mathsf{A}} \to \mathbb{C}$ be the theta function associated to s (see § 2.1) and define $f(u) = |\vartheta(u)|^2 e^{-\pi \|u\|_{\mathsf{L}}^2}$ for all $u \in t_{\mathsf{A}}$. The definition of the norm $\| \cdot \|_2$ on $\mathrm{H}^0(\mathsf{A}, \mathsf{L})$ gives

$$\|s\|_2^2 = \int_{\mathsf{A}} \|s(x)\|_{x^*\mathsf{L}}^2 \,\mathrm{d}\mu(x) = \int_{\mathcal{F}} f(u) \big(\exp_{\mathsf{A}}^* \mu \big)(u) = \frac{1}{\mathrm{h}^0(\mathsf{A}, \mathsf{L})} \int_{\mathcal{F}} f(u) \,\mathrm{d}u.$$

With $\mathfrak{n} = \max_{u \in \mathcal{F}} N(u)$, we have $\mathfrak{n} \int_{\mathcal{F}} f(u) \,\mathrm{d}u \geq \int_{\mathcal{F}} f(u) N(u) \,\mathrm{d}u$ and Lemma 3.6 yields

$$\|s\|_2^2 \geq \frac{1}{\mathfrak{n} \mathrm{h}^0(\mathsf{A}, \mathsf{L})} \int_{B(z,r)} f(u) \,\mathrm{d}u = \frac{1}{\mathfrak{n} \mathrm{h}^0(\mathsf{A}, \mathsf{L})} \int_{B(0,r)} |\vartheta(z+u)|^2 e^{-\pi \|z+u\|_{\mathsf{L}}^2} \,\mathrm{d}u.$$

In the latter integral, we write $\|z+u\|_{\mathsf{L}}^2 = \|z\|_{\mathsf{L}}^2 + 2 \operatorname{Re} H(z,u) + \|u\|_{\mathsf{L}}^2$ (H is the Riemann form of L) and we expand the holomorphic function $\Theta(u) = \vartheta(z+u) e^{-\pi H(z,u)} = \sum_{i=(i_1, \ldots, i_g) \in \mathbb{N}^g} a_i u_1^{i_1} \cdots u_g^{i_g}$ in terms of the coordinates $u_1, \ldots, u_g$ of u in an orthonormal basis $(e_1, \ldots, e_g)$ of t_{A}. We get

$$\int_{B(0,r)} |\vartheta(z+u)|^2 e^{-\pi \|z+u\|_{\mathsf{L}}^2} \,\mathrm{d}u = e^{-\pi \|z\|_{\mathsf{L}}^2} \int_{B(0,r)} |\Theta(u)|^2 e^{-\pi \|u\|_{\mathsf{L}}^2} \,\mathrm{d}u$$

and, since ϑ vanishes at z up to order ℓ, we have $\operatorname{jet}^{\ell} \vartheta(z) = \operatorname{jet}^{\ell} \Theta(0)$. Using Bessel's inequality, we deduce that $\|s\|_2^2 \times \mathfrak{n} \mathrm{h}^0(\mathsf{A}, \mathsf{L}) \exp \big(\pi \|z\|_{\mathsf{L}}^2 \big)$ is bounded from below by

$$\sum_{i \in \mathbb{N}^g} |a_i|^2 \int_{B(0,r)} |u_1|^{2i_1} \cdots |u_g|^{2i_g} \exp \big(-\pi \|u\|_{\mathsf{L}}^2 \big) \,\mathrm{d}u$$

$$\geq \sum_{|i|=\ell} |a_i|^2 \|(e_1^{\vee})^{i_1} \cdots (e_g^{\vee})^{i_g}\|^2 = C^2 \sum_{|i|=\ell} |a_i|^2 \frac{i_1! \cdots i_g!}{\ell!} = C^2 \| \operatorname{jet}^{\ell} \Theta(0) \|_{S^{\ell}(t_{\mathsf{A}}^{\vee})}^2 = C^2 \| \operatorname{jet}^{\ell} \vartheta(z) \|_{S^{\ell}(t_{\mathsf{A}}^{\vee})}^2.$$

We conclude with formula (1) that yields the desired upper bound for the norm of $\operatorname{jet}^{\ell} s(x)$. □

3.2.3

Proof of Theorem 3.2. Let $\mathcal{F}$ be a fundamental domain of t_{A} and $z \in t_{\mathsf{A}}$ be a logarithm of the point x considered in Theorem 3.2. Let us choose $r > 0$ such that πr^2 is the median of the $\Gamma(\ell + g, 1)$ distribution, that is, such that

$$\int_0^{\pi r^2} x^{\ell+g-1} \exp(-x) \,\mathrm{d}x = \frac{(\ell+g-1)!}{2}.$$

In particular the constant C^2 of Lemma 3.5 equals $\ell!/(2\pi^{\ell})$. Besides [Ch, Theorem 1] provides the bound $\pi r^2 \leq \ell + g - 1 + \log 2$. We apply Proposition 3.7 to these data $\mathcal{F}, z, r$. To get Theorem 3.2, we have to prove that

$2\max_{u\in\mathcal{F}} N(u) \le \big(5(g+\ell)\max(1, 1/\rho(\mathsf{A},\mathsf{L}))\big)^g$. This follows from a result by Malikiosis [Mal, Theorem 1.2], which gives an upper bound of $N(u)$ in terms of the minima $\rho(\mathsf{A},\mathsf{L}) = \lambda_1(\Omega_{\mathsf{A}}) \le \cdots \le \lambda_{2g}(\Omega_{\mathsf{A}})$ of the Euclidean lattice $(\Omega_{\mathsf{A}}, \|\cdot\|_{\mathsf{L}})$:

$$\forall\, u \in \mathcal{F}, \qquad N(u) = \operatorname{Card}\Omega_{\mathsf{A}} \cap B(z-u, r) \le \prod_{i=1}^{2g} \left\lfloor \frac{2r}{\lambda_i(\Omega_{\mathsf{A}})} + 1 \right\rfloor$$

($\lfloor\cdot\rfloor$ refers to the integer part). We bound from below $\lambda_i(\Omega_{\mathsf{A}})$ by $\rho(\mathsf{A},\mathsf{L})$ for $1 \le i \le g$ and by 1 for $g+1 \le i \le 2g$ (thanks to the property $1 \le \lambda_i(\Omega_{\mathsf{A}})\lambda_{2g+1-i}(\Omega_{\mathsf{A}})$ [GR2, p. 2072]). We get

$$\max_{u\in\mathcal{F}} N(u) \le \left(\frac{2r}{\rho(\mathsf{A},\mathsf{L})} + 1\right)^g \lfloor 2r+1\rfloor^g \le \max\left(1, \frac{1}{\rho(\mathsf{A},\mathsf{L})}\right)^g ((2r+1)\lfloor 2r+1\rfloor)^g .$$

With the previous upper bound for r, the conclusion comes from

$$\max_{x\in\mathbb{N}_{\ge 1}} \frac{2}{x}\left(\frac{2}{\sqrt{\pi}}\sqrt{x-1+\log 2}+1\right)\left\lfloor \frac{2}{\sqrt{\pi}}\sqrt{x-1+\log 2}+1\right\rfloor \le 5$$

(the worst case for the size of the constant is $g = \ell = 1$ corresponding to $x = 2$). □.

When $\ell = 0$, we could reduce the constant 5 to 3.9 (leaving $2^{1/x}$ instead of the first 2 in the above maximum) where Proposition 3.1 yields 1.7 (but only when $\mathrm{h}^0(\mathsf{A},\mathsf{L}) = 1$) .

3.2.4 Remark

We could choose a (Hermitian) norm $\|\cdot\|_{t_{\mathsf{A}}}$ on t_{A} different from $\|\cdot\|_{\mathsf{L}}$ provided there exists a positive real number ξ such that $\|\cdot\|_{\mathsf{L}} = \xi\|\cdot\|_{t_{\mathsf{A}}}$. In that case, the norm on $S^\ell(t_{\mathsf{A}}^\vee)$ is that of induced by $\|\cdot\|_{t_{\mathsf{A}}}$ and the norm of the jet $\mathrm{jet}^\ell s(x)$ is simply multiplied by ξ^ℓ. This often happens in applications where $\|\cdot\|_{t_{\mathsf{A}}}$ is the norm associated with a polarization L_0 on A while $\mathsf{L} = \mathsf{L}_0^{\otimes n}$ (for some positive integer n). Then we have $\xi = \sqrt{n}$ and the minima of Ω_{A} satisfy $\lambda_i(\Omega_{\mathsf{A}}, \|\cdot\|_{\mathsf{L}}) = \sqrt{n}\lambda_i(\Omega_{\mathsf{A}}, \|\cdot\|_{\mathsf{L}_0})$. We keep this $\sqrt{n}$ in the Malikiosis bound of $N(u)$ which is used in the proof of Theorem 3.2. Let us introduce further an additional parameter $\alpha > 0$ to bound

$$\frac{2r}{\rho(\mathsf{A},\mathsf{L})} + 1 \le \left(\frac{2r}{\sqrt{n}} + \alpha\right)\max\left(\frac{1}{\alpha}, \frac{1}{\rho(\mathsf{A},\mathsf{L}_0)}\right).$$

Thus, for every $u \in \mathcal{F}$, we get

$$N(u) \le \max\left(\frac{1}{\alpha}, \frac{1}{\rho(\mathsf{A},\mathsf{L}_0)}\right)^g \left(\left(\frac{2r}{\sqrt{n}} + \alpha\right)\left\lfloor \frac{2r}{\sqrt{n}} + 1\right\rfloor\right)^g .$$

In this way, a numerically sharper version of Theorem 3.2 is: *For all* $\alpha > 0$, $x \in \mathsf{A}$, $\ell \in \mathbb{N}$, $n \in \mathbb{N}_{\ge 1}$ *and* $s \in \mathrm{H}^0(\mathsf{A},\mathsf{L})$ *which vanishes at* x *up to order* ℓ*, the norm of* $\mathrm{jet}^\ell s(x)$ *relative to* $S^\ell\big((t_{\mathsf{A}}, \|\cdot\|_{\mathsf{L}_0})^\vee\big) \otimes x^*\mathsf{L}$ *is smaller than*

$$\|s\|_2 \times \mathrm{h}^0(\mathsf{A},\mathsf{L})^{1/2} \max\left(\frac{1}{\alpha}, \frac{1}{\rho(\mathsf{A},\mathsf{L}_0)}\right)^{g/2} \times \left(\frac{2(\pi n)^\ell}{\ell!}\right)^{1/2} \times \left(\left(\frac{2r}{\sqrt{n}} + \alpha\right)\left\lfloor \frac{2r}{\sqrt{n}} + 1\right\rfloor\right)^{g/2}$$

with $r^2 = (\ell + g - 1 + \log 2)/\pi$.

4. Application to Arakelov geometry of abelian varieties

Henceforth, $\overline{K}$ is an algebraic closure of a number field K and (A, L) is a polarized abelian variety over K, with dimension $g \ge 1$.

4.1

Let $\mathsf{H} = \mathrm{H}^0(A, L^{\otimes 2}) \otimes_K \overline{K}$. For every $a \in A(\overline{K})$, the evaluation map $\delta_a^{(0)} \colon \mathsf{H} \to a^* L_{\overline{K}}^{\otimes 2}$, $s \mapsto s(a)$, is a non-zero $\overline{K}$-linear form since $L^{\otimes 2}$ is generated by global sections [Mu, p. 60]. Let $\delta_a^{(1)} \colon \ker \delta_a^{(0)} \to t_{A_{\overline{K}}}^{\vee} \otimes_{\overline{K}} a^* L_{\overline{K}}^{\otimes 2}$, $s \mapsto \mathrm{jet}^1 s(a)$, be the jet of order 1 at a. By duality, $\delta_a^{(1)}$ is surjective if and only if the complete linear system H separates tangent vectors at a: $\forall\, t \in t_{A_{\overline{K}}} \backslash \{0\}, \exists\, s \in \ker \delta_a^{(0)}$ such that $\mathrm{jet}^1 s(a)(t) \neq 0$. In other words, $\delta_a^{(1)}$ is surjective when the differential at a of the finite morphism $\phi \colon A_{\overline{K}} \to \mathbb{P}(\mathsf{H})$ built from H is injective. Such a a exists by generic smoothness of ϕ: there exists a non-empty open subset $U \subset A$ such that $\phi \colon U \to \phi(U)$ is étale. By density of torsion points, we can even assume that a is a torsion point of A. Then let us consider a Moret-Bailly model $(\mathcal{A}, \mathcal{L}_2, \varepsilon_a \colon \operatorname{Spec} \mathcal{O}_{K'} \to \mathcal{A})$ of $\big(A_{\overline{K}}, L_{\overline{K}}^{\otimes 2}, \{a\}\big)$ over a finite extension K'/K, in the sense of [Bo1, § 4.3.1]. The maps $\delta_a^{(0)}$ and $\delta_a^{(1)}$ associated to the polarized abelian variety $(A_{K'}, L_{K'}^{\otimes 2})$ and $a \in A(K')$ extend to homomorphisms of $\mathcal{O}_{K'}$-modules $\delta_0 \colon \mathrm{H}^0(\mathcal{A}, \mathcal{L}_2) \to \varepsilon_a^* \mathcal{L}_2$ and $\delta_1 \colon \ker \delta_0 \to t_{\mathcal{A}}^{\vee} \otimes \varepsilon_a^* \mathcal{L}_2$. The norms on $\mathcal{L}_2$ equip the modules $\ker \delta_0 \subset \mathrm{H}^0(\mathcal{A}, \mathcal{L}_2)$ and $\varepsilon_a^* \mathcal{L}_2$ of a structure of Hermitian vector bundle over $\mathcal{O}_{K'}$ (see § 2.3.2). Although they could endow $t_{\mathcal{A}}$ with a Hermitian vector bundle structure over $\mathcal{O}_{K'}$, using the Riemann forms of $(\mathcal{L}_2)_\sigma$'s ($\sigma : K' \hookrightarrow \mathbb{C}$), it seems more convenient to choose the structure given by $(t_A, \|\cdot\|_L)$ extended to K'. The important point is that the norms given by $(\mathcal{L}_2)_\sigma$ and L_σ satisfy the proportionality condition of Remark 3.2.4 (with $n = 2$). In this context the maps δ_0 and δ_1 have some heights which can be evaluated in a simple way thanks to Theorem 3.2.

Proposition 4.1. *For $g \geq 2$, we have*

$$h(\delta_0) \leq \frac{g}{4} \log \max \left(1, h_F(A) + \frac{1}{2} \log \mathrm{h}^0(A, L)\right) + \frac{1}{2} \log \mathrm{h}^0(A, L) + \frac{3g}{4} \log 5g$$

and

$$h(\delta_1) \leq \frac{g}{4} \log \max \left(1, h_F(A) + \frac{1}{2} \log \mathrm{h}^0(A, L)\right) + \frac{1}{2} \log \mathrm{h}^0(A, L) + \frac{3g}{4} \log 10.2g.$$

Moreover we can replace $\max\big(1, h_F(A) + (1/2) \log \mathrm{h}^0(A, L)\big)$ *by* $\max\big(1, h_F(A)\big)$ *in these bounds provided we change the couple* $(5, 10.2)$ *of numerical constants by* $(8.5, 18)$. *Besides, when $g \geq 16$, the couple* $(5, 10.2)$ *can also be substituted by* $(2.58, 2.84)$.

The proof of this statement relies on a so-called matrix lemma, involving the injectivity diameters $\rho(A_\sigma, L_\sigma)$ of the complex abelian varieties (A_σ, L_σ) (where σ is an embedding of K into $\mathbb{C}$), deduced from the polarized abelian variety (A, L) defined over K by scalar extension via σ.

Matrix Lemma. *We have*

$$\frac{1}{[K : \mathbb{Q}]} \sum_{\sigma \colon K \hookrightarrow \mathbb{C}} \frac{1}{\rho(A_\sigma, L_\sigma)^2} \leq \big(2.3 + 5.5g\big) \max \left(1, h_F(A) + \frac{1}{2} \log \mathrm{h}^0(A, L)\right).$$

In addition, the latter maximum can be replaced by $8 \max(1, h_F(A))$.

In the main, this result is due to Autissier [Au1]. More precisely this is a consequence of Corollary 1.4, ibid., where $\varepsilon = 1 - 6/(2.3\pi)$ is chosen. Then the general case is deduced from [GR1, Lemmas 3.4 and 3.5], in the same way as Proposition 3.6, ibid. The variant with $8 \max(1, h_F(A))$ arises from the first four lines of [GR1, p. 358] (Zarhin's trick).

Proof of Proposition 4.1. Given $\sigma : K' \hookrightarrow \mathbb{C}$, we apply Theorem 3.2 in the form described in Remark 3.2.4 (to get sharper numerical constants) to the polarized abelian variety $(\mathcal{A}_\sigma, (\mathcal{L}_2)_\sigma)$ at 0 for $\ell \in \{0, 1\}$. We choose $\mathsf{L}_0 = L_\sigma$, $n = 2$ and $\alpha = \sqrt{e/(2.3 + 5.5g)}$. The definition of $\mathcal{L}_2$ yields $\mathrm{h}^0(\mathcal{A}_\sigma, (\mathcal{L}_2)_\sigma) = 2^g \mathrm{h}^0(A, L)$ and we get

$$\|\delta_\ell\|_\sigma \leq \left(2^g \mathrm{h}^0(A, L)\right)^{1/2} \max\left(\frac{1}{\alpha}, \frac{1}{\rho(A_\sigma, L_\sigma)}\right)^{g/2} \times \left(2^{\ell+1} \pi^\ell\right)^{1/2} \left(\left(\sqrt{2}r + \alpha\right) \left\lfloor \sqrt{2}r + 1 \right\rfloor\right)^{g/2}$$

with $r^2 = (\ell + g - 1 + \log 2)/\pi$. As for the ultrametric norms of δ_ℓ, they are smaller than 1 since δ_ℓ is a homomorphism of $\mathcal{O}_{K'}$-modules, modules from which the norms have been defined. Now we make use of the elementary [BG, Lemma 3.19] to bound from above

$$\frac{1}{[K':\mathbb{Q}]}\sum_{\sigma\colon K'\hookrightarrow\mathbb{C}}\log\max\left(\frac{1}{\alpha},\frac{1}{\rho(A_\sigma,L_\sigma)}\right)\le\frac{1}{2}\log\max\left(\frac{e}{\alpha^2},\frac{1}{[K':\mathbb{Q}]}\sum_{\sigma\colon K'\hookrightarrow\mathbb{C}}\frac{1}{\rho(A_\sigma,L_\sigma)^2}\right).$$

The matrix lemma allows to replace the latter quantity by

$$\frac{1}{2}\log(2.3+5.5g)+\frac{1}{2}\log\max\left(1,h_F(A)+\frac{1}{2}\log \mathrm{h}^0(A,L)\right).$$

Then we obtain the desired estimate for the height of δ_ℓ with the constant

$$c_\ell(g)=\frac{g+\ell+1}{2}\log 2+\frac{g}{4}\log(2.3+5.5g)+\frac{\ell}{2}\log\pi+\frac{g}{2}\log\left\lfloor\left(\frac{2(\ell+g-1+\log 2)}{\pi}\right)^{1/2}+1\right\rfloor$$
$$+\frac{g}{2}\log\left(\left(\frac{2(\ell+g-1+\log 2)}{\pi}\right)^{1/2}+\left(\frac{e}{2.3+5.5g}\right)^{1/2}\right).$$

For the approximated form given in the proposition, we set apart the case $g=2$ for which we numerically compute the values of $c_0(2)$ and $c_1(2)$ (they determine the constants 5 and 10.2). Then, for $g\ge 3$, we use $\lfloor\xi\rfloor\le\xi$ and $\ell-1\le 0$ in the last two logarithms of $c_\ell(g)$. We obtain a new simplified constant $c'_\ell(g)$ for which the function $g\mapsto(c'_\ell(g)-(3g/4)\log g)/g$ is clearly decreasing and we estimate $c'_0(3)$ and $c'_1(3)$ to conclude. The same method applies when $g\ge 16$, distinguishing the cases $g=16$ and $g\ge 17$. As for the variant with $\max(1,h_F(A))$, it is the one we have in the matrix lemma. We multiply $2.3+5.5g$ by 8 (twice) in the constant $c_\ell(g)$ and we proceed as above. □

4.2 Maximal slope of the tangent space

Proposition 4.2. *The maximal slope of* $(t_A,\|\cdot\|_L)$ *satisfies*

$$\mu_{\max}(t_A)\le h_F(A)+\frac{1}{2}\log \mathrm{h}^0(A,L)+\frac{g}{2}\log\max\left(1,h_F(A)+\frac{1}{2}\log \mathrm{h}^0(A,L)\right)+\frac{3g}{2}\log 15.4g.$$

Moreover the same bound holds with $\max(1,h_F(A))$ *instead of* $\max\big(1,h_F(A)+(1/2)\log \mathrm{h}^0(A,L)\big)$ *if we change* 15.4 *by* 26.6. *When* $g\ge 16$ *we can substitute the numerical constant* 15.4 *by* 5.9.

Proof. When $g=1$ we have $\mu_{\max}(t_A)=\deg t_A=-h_F(A)-\frac{1}{2}\log \mathrm{h}^0(A,L)$ and the statement occurs from Bost's inequality $h_F(A)\ge-\log(\pi\sqrt{2})$. Thus, from now on, we may assume $g\ge 2$. Because the map δ_1 introduced in the previous paragraph is onto by construction, we can apply Proposition 2.1 with the dual map $\delta_1^\vee$, which is injective, and $x=1$:

$$\mu_{\max}\big(t_{\mathcal{A}}\otimes(\varepsilon_a^*\mathcal{L}_2)^\vee\big)\le\mu_{\max}\big((\ker\delta_0)^\vee\big)+h(\delta_1).$$

The maximal slope involved here on the left decomposes into a sum of the maximal slope of $t_{\mathcal{A}}$ (with the metrics described in the previous paragraph) and $\deg\varepsilon_a^*\mathcal{L}_2$. This last degree equals 0 since it is the Néron-Tate height of the torsion point a relative to $L^{\otimes 2}$ [Bo1, Theorem 4.10]. Hence we have $\mu_{\max}\big(t_{\mathcal{A}}\otimes(\varepsilon_a^*\mathcal{L}_2)^\vee\big)=\mu_{\max}(t_A)$. To control the other maximal slope, let us now apply the second part of Proposition 2.1 with the surjective restriction map $\mathrm{H}^0(\mathcal{A},\mathcal{L}_2)^\vee\to(\ker\delta_0)^\vee$, whose height is nonpositive, and $y=1$. This gives

$$\mu_{\max}\big((\ker\delta_0)^\vee\big)\le 2\mu_{\max}\big(\mathrm{H}^0(\mathcal{A},\mathcal{L}_2)^\vee\big)-\deg \mathrm{H}^0(\mathcal{A},\mathcal{L}_2)^\vee+\deg(\ker\delta_0)^\vee.$$

Since the duality reverses the sign of the degree, the sum of the two last quantities is the degree of $\mathrm{H}^0(\mathcal{A}, \mathcal{L}_2)/\ker\delta_0$. This Hermitian line bundle is isomorphic to $\varepsilon_a^*\mathcal{L}_2$ via the map induced by δ_0, hence its degree equals $\deg\varepsilon_a^*\mathcal{L}_2 + h(\delta_0) = h(\delta_0)$ [Ga2, Proposition 42]. The semistability of $\mathrm{H}^0(\mathcal{A}, \mathcal{L}_2)$ and the formula for its slope given at the end of § 2.3.2 lead to

$$\mu_{\max}\big(t_A, \|\cdot\|_L\big) \leq h_F(A) - \frac{1}{2}\log \mathrm{h}^0(A, L) + g\log\pi + h(\delta_0) + h(\delta_1).$$

We conclude with Proposition 4.1 and $g\log\pi + (3g/4)\log(5\times 10.2) \leq (3g/2)\log 15.4$. The other constants stem from the same inequality where (5, 10.2, 15.4) is replaced by (8.5, 18, 26.6) and, when $g \geq 16$, by (2.58, 2.84, 5.9). □

Taking the g-th external tensor power of an elliptic curve endowed with its principal polarization, we can prove that $(3g/2)\log 15.4g$ which is in the upper bound of $\mu_{\max}(t_A)$ cannot be replaced by a function smaller than g (using Bost's inequality). Furthermore, we can deduce from Proposition 4.2 some simpler but weaker estimates, getting rid of the logarithm with the following elementary result.

Lemma 4.3. *Let $x_0 \in \mathbb{R}$ and $c > 0$. Then, for every $x \in \mathbb{R}$, we have*

$$\log\max(1, x) - cx \leq \begin{cases} -cx_0 & \text{if } x_0 \leq x \leq 1,\\ \log\max(1, 1/c) - \max(1, c) & \text{if } x \geq 1.\end{cases}$$

Proof. When $x_0 \leq x \leq 1$, we have $\log\max(1, x) - cx = -cx \leq -cx_0$. When $x \geq 1$, we observe that the function $x \mapsto \log x - cx$ has a global maximum at $x = \max(1, 1/c)$. □

As a straightforward consequence of this lemma and Proposition 4.2, we deduce:

Corollary 4.4. *For all $\varepsilon > 0$, the maximal slope $\mu_{\max}(t_A)$ of $(t_A, \|\cdot\|_L)$ is bounded by*

$$(1+\varepsilon)\left(h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A, L)\right) + \frac{3g}{2}\log 15.4g + \frac{g}{2}\max\left(\varepsilon\log(2\pi^2), \log\frac{g}{2\varepsilon} - 1\right).$$

For instance, we have

$$\mu_{\max}(t_A) \leq 1.2h_F(A) + \log \mathrm{h}^0(A, L) + 2g\log 7.7g$$

and, when $g \geq 16$,

$$\mu_{\max}(t_A) \leq 1.5h_F(A) + \log \mathrm{h}^0(A, L) + 2g\log 3g.$$

Proof. Apply Lemma 4.3 with $x = h_F(A) + 0.5\log \mathrm{h}^0(A, L)$, $x_0 = -g\log(\pi\sqrt{2})$ and $c = 2\varepsilon/g$. That gives an upper bound for $\log\max(1, h_F(A) + (1/2)\log \mathrm{h}^0(A, L))$ and we conclude with Proposition 4.2. The next upper bound derives from the choice $\varepsilon = 0.2$ (setting apart the trivial case $g = 1$) whereas we choose $\varepsilon = 1/2$ in the last (with 5.9 instead of 15.4). □

Using the second assertion of Proposition 4.2, we can also prove

$$\mu_{\max}(t_A) \leq 1.2h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A, L) + 2g\log 11.5g.$$

With Zarhin's trick, the degree of A can even be removed from these estimates.

Corollary 4.5. *We have $\mu_{\max}\big(t_A, \|\cdot\|_L\big) \leq 12h_F(A) + 16g\log 24g$.*

Proof. The case $g = 1$ is treated with Bost's inequality, as at the beginning of the proof of Proposition 4.2. For $g \geq 2$, there exists a principal polarization L' on $Z(A) = A^4 \times \widehat{A}^4$ ($\widehat{A}$ is the dual abelian variety of A) such that, if $\imath$ denotes the injective map $A \hookrightarrow Z(A)$ on the first component, $L = \imath^* L'$ (see [Ré, Lemma 4.6]). In particular we get an isometric injection from $(t_A, \|\cdot\|_L)$ to $(t_{Z(A)}, \|\cdot\|_{L'})$ and $\mu_{\max}(t_A) \leq \mu_{\max}(t_{Z(A)})$. Since $\dim Z(A) \geq 16$ it remains now to apply the last estimate in the previous corollary to $(Z(A), L')$ whose Faltings height and dimension are eight times those of A in order to conclude. □

This estimate is optimal with respect to $\mathrm{h}^0(A, L)$ in the sense that there exists an abelian variety A but no unbounded from above function $f_A \colon \mathbb{N} \to \mathbb{R}$ such that $(\star)$ $\mu_{\max}(t_A) + f_A(\mathrm{h}^0(A, L)) \leq 0$ for any polarization L on A. Indeed, let us fix an elliptic curve E over K with its canonical polarization L_0. Define the abelian variety $A = E \times E$ endowed with $L_n = L_0^{\otimes n} \boxtimes L_0$ where n is any positive integer. Then $\mathrm{h}^0(A, L_n) = n$ and

$$\mu_{\max}(t_A) = \max\left(\mu_{\max}\big(t_E, \|\cdot\|_{L_0^{\otimes n}}\big),\ \mu_{\max}\big(t_E, \|\cdot\|_{L_0}\big)\right) = \mu_{\max}(t_E)$$

since $\mu_{\max}\big(t_E, \|\cdot\|_{L_0^{\otimes n}}\big) = \deg t_E - (\log n)/2$. Hence, if $(\star)$ were true for (A, L_n), the integer n should be bounded.

4.3 Maximal slope of the cotangent space

In this paragraph we bound from above the maximal slope of the dual of the Hermitian vector bundle $(t_A, \|\cdot\|_L)$. First, let us observe that this maximal slope equals to $P_{t_A}(g-1) - \deg t_A$. The choice of the metrics on $t_{\mathcal{A}}$ made in § 4.1 ensures that the canonical polygon of t_A is $P_{t_{\mathcal{A}} \otimes (\varepsilon_a^* \mathcal{L}_2)^\vee}$. Let us apply Proposition 2.1 twice: once with the injective map $\delta_1^\vee$ and again with the restriction map $\mathrm{H}^0(\mathcal{A}, \mathcal{L}_2)^\vee \to (\ker \delta_0)^\vee$ (like in the proof of Proposition 4.2). This gives

$$P_{t_{\mathcal{A}} \otimes (\varepsilon_a^* \mathcal{L}_2)^\vee}(g-1) \leq P_{(\ker \delta_0)^\vee}(g-1) + (g-1)h(\delta_1) \leq P_{\mathrm{H}^0(\mathcal{A}, \mathcal{L}_2)^\vee}(g) + h(\delta_0) + (g-1)h(\delta_1).$$

The semistability of $\mathrm{H}^0(\mathcal{A}, \mathcal{L}_2)$ allows to compute its canonical polygon and we get

$$\mu_{\max}(t_A^\vee) \leq -g\mu\left(\mathrm{H}^0(\mathcal{A}, \mathcal{L}_2)\right) - \deg t_A + h(\delta_0) + (g-1)h(\delta_1).$$

The formulas given in § 2.3.2 and Proposition 4.1 yield

$$\mu_{\max}(t_A^\vee) \leq \left(\frac{g}{2} + 1\right)\left(h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A, L)\right) + \frac{g^2}{4}\log\max\left(1, h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A, L)\right) + c(g)$$

where

$$c(g) = \frac{g^2}{2}\log\pi + \frac{3g}{4}(\log 5 + (g-1)\log 10.2 + g\log g) \leq \frac{3g^2}{4}\log 21.9g$$

(21.9 is an upper bound for $10.2 \times \pi^{2/3}$). As in Proposition 4.1, we can improve this estimate with respect to $\mathrm{h}^0(A, L)$ at the cost of a small loss on the constant:

$$\mu_{\max}(t_A^\vee) \leq \left(\frac{g}{2} + 1\right)\left(h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A, L)\right) + \frac{g^2}{4}\log\max(1, h_F(A)) + \frac{3g^2}{4}\log 38.7g.$$

Let us also state the more manageable following result.

Proposition 4.6. *We have*

$$\mu_{\max}(t_A^\vee) \leq (0.6g + 1)h_F(A) + \frac{1}{2}\left(\frac{g}{2} + 1\right)\log \mathrm{h}^0(A, L) + g^2\log 15.2g.$$

Proof. When $g = 1$ we have $\mu_{\max}(t_A^\vee) = h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A, L)$ and the statement comes from Bost's inequality. When $g \geq 2$, we apply Lemma 4.3 with $x = h_F(A)$, $x_0 = -g\log(\pi\sqrt{2})$ and $\mathrm{c} = 2/(5g)$. We substitute the upper bound of $\log\max\big(1, h_F(A)\big)$ obtained this way in the previous estimate for $\mu_{\max}(t_A^\vee)$ and we conclude noting that $g^2\log 15.2g$ is an upper bound for $(g^2/4)\max\big(0.2\log 2\pi^2, \log(5g/2e)\big) + (3g^2/4)\log 38.7g$ because the maximum is always attained for the second term (since $g \geq 2$) and $(5/2e)^{1/4} \times (38.7)^{3/4} \leq 15.2$. □

If we proceed in the same way using the bound with the numerical constant 21.9 instead of 38.7, we get

$$\mu_{\max}(t_A^\vee) \leq (0.6g+1)\left(h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A,L)\right) + g^2 \log 10g.$$

In terms of Deligne's normalization $h(A) = h_F(A) + \frac{g}{2}\log\pi$ of the Faltings height, we may replace the constant 10 by $10/\pi^{0.3} \leq 7.1$ and then compare with [Gr, Proposition 2.14] giving the estimate $\mu_{\max}(t_A^\vee) \leq (g+1)\big(h(A) + (1/2)\log \mathrm{h}^0(A,L)\big) + 2g^5 \log 2$, less accurate when $g \geq 2$. Contrary to the maximal slope of t_A, the one of $t_A^\vee$ cannot be bounded independently from $\mathrm{h}^0(A,L)$ since

$$h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A,L) = g\mu(t_A^\vee) \leq g\mu_{\max}(t_A^\vee).$$

However the same technics lead to the following statement.

Proposition 4.7. *We have*

$$\mu_{\max}(t_A^\vee) \leq (4g+1)h_F(A) + \frac{1}{2}\log \mathrm{h}^0(A,L) + 2g^2 \log\max(1, h_F(A)) + 6g^2 \log 98g.$$

Again this estimate is best possible with respect to $\mathrm{h}^0(A,L)$ by the same argument given for the optimality of Corollary 4.5.

Proof. As in the proof of Corollary 4.5, let us assume $g \geq 2$ and consider a principal polarization L' on $Z(A) = A^4 \times \widehat{A}^4$ such that $(t_A, \|\cdot\|_L)$ is seen as a subbundle of $(t_{Z(A)}, \|\cdot\|_{L'})$. In particular we have $P_{t_A}(x) \leq P_{t_{Z(A)}}(x)$ for all $0 \leq x \leq g$. Then $\mu_{\max}(t_A^\vee) + \deg t_A = P_{t_A}(g-1) \leq P_{t_{Z(A)}}(g-1)$. We estimate this last number following the same method as above, paying attention to the fact that $(\mathcal{A}, \mathcal{L}_2, \delta_0, \delta_1)$ are now built with $(Z(A), L')$ and therefore the dimension and Faltings height of the abelian variety are multiplied by 8. Then Proposition 4.7 follows with the constant

$$4g^2 \log\pi + 2g^2 \log 8 + 6g\log(2.58 \times 8g) + 6g(g-1)\log(2.84 \times 8g).$$

We conclude with $\pi^{2/3} \times 2 \times 2.84 \times 8 < 98$. □

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