Chapter II: Minima and Slopes of Rigid Adelic Spaces

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1 Introduction

We propose here a lecture on the geometry of numbers for normed (adelic) vector spaces over an algebraic extension of $\mathbb{Q}$. We shall define slopes and several type of minima for these objects and we shall compare them.

First, let us recall some basic notions of the classical geometry of numbers. Let Ω be a free $\mathbb{Z}$-module of rank $n \geq 1$ and let $\|\cdot\|$ be an Euclidean norm on $\Omega \otimes_{\mathbb{Z}} \mathbb{R}$. We shall say that the couple $(\Omega, \|\cdot\|)$ is an Euclidean lattice of rank n. To such a lattice are associated n positive real numbers, called the *successive minima* of $(\Omega, \|\cdot\|)$: for all $i \in \{1, \ldots, n\}$,

$$\lambda_i(\Omega, \|\cdot\|) = \min\{r > 0\,;\ \dim \mathrm{Vect}_{\mathbb{R}}(x \in \Omega\,;\ \|x\| \leq r) \geq i\}.$$

It is also the minimum of the set of $\max\{\|x_1\|, \ldots, \|x_i\|\}$ formed with linearly independent vectors $x_1, \ldots, x_i \in \Omega$. We have $0 < \lambda_1(\Omega, \|\cdot\|) \leq \cdots \leq \lambda_n(\Omega, \|\cdot\|)$. Given a $\mathbb{Z}$-basis $e_1, \ldots, e_n$ of Ω, the (co-)volume of Ω is the positive real number

$$\mathrm{vol}(\Omega) = \det\left(\langle e_i, e_j\rangle\right)^{1/2}_{1 \leq i, j \leq n}$$

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where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $\Omega\otimes_{\mathbb{Z}}\mathbb{R}$ associated to $\|\cdot\|$. Let us define

$$c_{\mathrm{I}}(n,\mathbb{Q}) = \sup \frac{\lambda_1(\Omega,\|\cdot\|)}{\mathrm{vol}(\Omega)^{1/n}}$$

and

$$c_{\mathrm{II}}(n,\mathbb{Q}) = \sup \left(\frac{\lambda_1(\Omega,\|\cdot\|)\cdots\lambda_n(\Omega,\|\cdot\|)}{\mathrm{vol}(\Omega)}\right)^{1/n}$$

where the suprema are taken over Euclidean lattices $(\Omega,\|\cdot\|)$ of rank n. The square $\gamma_n = c_{\mathrm{I}}(n,\mathbb{Q})^2$ is nothing but the famous Hermite constant. Its exact value is only known for $n \le 8$ and $n = 24$. It can also be characterized as the smallest positive real number c such that, for all $(a_0,\ldots,a_n)\in\mathbb{Z}^{n+1}\backslash\{0\}$, there exists $(x_0,\ldots,x_n)\in\mathbb{Z}^{n+1}\backslash\{0\}$ satisfying $a_0x_0+\cdots+a_nx_n=0$ and

$$\sum_{i=0}^{n} x_i^2 \le c\left(\sum_{i=0}^{n} a_i^2\right)^{1/n}.$$

Minkowski proved the following statement (see [19, § 51]):

Theorem (Minkowski) *For every positive integer n, we have $c_{\mathrm{I}}(n,\mathbb{Q}) = c_{\mathrm{II}}(n,\mathbb{Q}) \le \sqrt{n}$.*

We shall generalize this framework in the following manner:

$$\begin{aligned}
\mathbb{Q} &\longrightarrow \text{Algebraic extension } K/\mathbb{Q}\\
\text{Euclidean lattice } (\Omega,\|\cdot\|) &\longrightarrow \text{Rigid adelic space } E \text{ over } K\\
\text{Minimum } \lambda_i(\Omega,\|\cdot\|) &\longrightarrow \text{Minimum } \Lambda_i(E)\\
\text{Volume vol}(\Omega) &\longrightarrow \text{Height } H(E)\\
-\log \mathrm{vol}(\Omega)^{1/n} &\longrightarrow \text{Slope } \mu(E).
\end{aligned}$$

Actually, in the highly flexible world of rigid adelic spaces, there exist numerous types of possible successive minima, having an interest according to the problems addressed. To be over an algebraic extension of $\mathbb{Q}$ which is not necessarily finite brings some new perspectives, issues and results. In particular we shall explain how to compute the Hermite constants of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$.

2 Rigid Adelic Spaces

Let us begin with a Reader's Digest of [14, § 2].

2.1 Algebraic Extensions of $\mathbb{Q}$

Let $K/\mathbb{Q}$ be an algebraic extension. Let $V(K)$ be the set of places of K (equivalence classes of non trivial absolute values over K). We can write this set as the projective limit $\varprojlim_L V(L)$ over finite subextensions $\mathbb{Q} \subset L \subset K$ of K. The discrete topology on $V(L)$ induces a topology on $V(K)$ by projective limit. It coincides with the topology generated by the compact open subsets $V_v(K) = \{w \in V(K)\,;\ w_{|L} = v\}$ for $v \in V(L)$ and L varies among number fields contained in K. On $V(K)$ can be defined a Borel measure σ characterized by

$$\sigma(V_v(K)) = \frac{[L_v : \mathbb{Q}_v]}{[L : \mathbb{Q}]} \quad \text{for } v \in V(L)$$

($\mathbb{Q}_v = \mathbb{Q}_p$ or $\mathbb{R}$ depending on v, p-adic or archimedean). We have $\sigma(V_p(K)) = 1$ for all $p \in V(\mathbb{Q})$. For $v \in V(K)$ we denote by K_v the topological completion of K at v and $|\cdot|_v$ is the unique absolute value on K_v such that $|p|_v \in \{1, p, p^{-1}\}$ for every prime number p. Then the product formula is written

$$\forall x \in K\backslash\{0\}, \qquad \int_{V(K)} \log|x|_v \,\mathrm{d}\sigma(v) = 0.$$

Furthermore, the *adèles* of K is the tensor product $\mathbb{A}_K = K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}$ of K with the adèles of $\mathbb{Q}$:

$$\mathbb{A}_{\mathbb{Q}} = \left\{ (x_p)_p \in \prod_{p \in V(\mathbb{Q})} \mathbb{Q}_p\,;\, \{p \text{ prime};\ |x_p|_p > 1\} \text{ is finite} \right\}.$$

If K is a number field, $\mathbb{A}_K$ is the usual adèle ring and, for an arbitrary algebraic extension $K/\mathbb{Q}$, one has $\mathbb{A}_K = \bigcup_{L \subset K, [L:\mathbb{Q}]<\infty} \mathbb{A}_L$.

2.2 Rigid Adelic Spaces

From now on, the letter K always denotes an algebraic extension of $\mathbb{Q}$.

Definition 1 *An adelic space E is a K-vector space of finite dimension endowed with norms $\|\cdot\|_{E,v}$ on $E \otimes_K K_v$ for every $v \in V(K)$.*

The (adelic) *standard space* of dimension $n \geq 1$ is the vector space K^n endowed with the following norms:

$$\forall x = (x_1, \ldots, x_n) \in K_v^n, \quad |x|_v = \begin{cases} \left(|x_1|_v^2 + \cdots + |x_n|_v^2\right)^{1/2} & \text{if } v \mid \infty \\ \max\{|x_1|_v, \ldots, |x_n|_v\} & \text{if } v \nmid \infty. \end{cases}$$

Given an adelic space E over K and $v \in V(K)$, a basis $(e_1, \dots, e_n)$ of $E \otimes_K K_v$ is said to be *orthonormal* if, for all $(x_1, \dots, x_n) \in K_v^n$, we have $\|\sum_{i=1}^n x_i e_i\|_{E,v} = |(x_1, \dots, x_n)|_v$.

Definition 2 *A* rigid adelic space *is an adelic space E for which there exist an isomorphism $\varphi \colon E \to K^n$ and an adelic matrix $A = (A_v)_{v \in V(K)} \in \mathrm{GL}_n(\mathbb{A}_K)$ such that*

$$\forall x \in E \otimes_K K_v, \quad \|x\|_{E,v} = |A_v \varphi_v(x)|_v$$

where $\varphi_v = \varphi \otimes \mathrm{id}_{K_v} \colon E \otimes_K K_v \to K_v^n$ is the natural extension of φ to $E \otimes_K K_v$.

In looser terms, a rigid adelic space is a compact deformation of a standard space.

Remarks

(i) Actually, if E is a rigid adelic space over K of dimension n, for every isomorphism $\varphi \colon E \to K^n$, there exists $A \in \mathrm{GL}_n(\mathbb{A}_K)$, upper triangular, such that (φ, A) defines the adelic structure on E.
(ii) If $x \in E \backslash \{0\}$ there exists a number field $K_0 \subset K$ such that $A \in \mathrm{GL}_n(\mathbb{A}_{K_0})$ and $\varphi(x) \in K_0^n$. Thus, outside a compact subset of $V(K)$ (finite union of some $V_v(K)$ with $v \in V(K_0)$), we have $\|x\|_{E,v} = 1$ and A_v is an isometry.
(iii) A rigid adelic space is an adelic space with an orthonormal basis at each $v \in V(K)$ but the converse is not true.

Examples of Rigid Adelic Spaces

(i) K^n (standard space).
(ii) Let $(\Omega, \|\cdot\|)$ be an Euclidean lattice and $(e_1, \dots, e_n)$ a $\mathbb{Z}$-basis of Ω. We can consider $E_\Omega = \Omega \otimes_{\mathbb{Z}} \mathbb{Q}$ over $K = \mathbb{Q}$, endowed with the norm $\|\cdot\|$ at the archimedean place of $\mathbb{Q}$ and $\|\sum_{i=1}^n x_i e_i\|_{E_\Omega, p} = \max_{1 \le i \le n} \{|x_i|_p\}$ at every prime p ($x_i \in \mathbb{Q}_p$). This definition does not depend on the choice of the $\mathbb{Z}$-basis.
(iii) When K is a number field with ring of integers $\mathcal{O}_K$, we have a one-to-one correspondence between rigid adelic spaces over K and Hermitian vector bundles over $\mathrm{Spec}\,\mathcal{O}_K$. Indeed, let E be a rigid adelic space over K. The projective $\mathcal{O}_K$-module of finite type $\mathcal{E} = \left\{x \in E\,;\ \forall v \in V(K) \backslash V_\infty(K),\ \|x\|_{E,v} \le 1\right\}$ endowed with the Hermitian norms (invariant by complex conjugation) $\|\cdot\|_\sigma = \|\cdot\|_{E,v}$ at embeddings $\sigma \colon K \hookrightarrow \mathbb{C}$ with associated place $v = \{\sigma, \overline{\sigma}\}$ form a Hermitian vector bundle over $\mathrm{Spec}\,\mathcal{O}_K$. An example is given at the beginning of Sect. 2.2 in Chapter I.

Definition 3 *Given two adelic spaces E, F over K, a linear map $f \colon E \to F$ is an* isometry *if for all $v \in V(K)$ and $x \in E \otimes_K K_v$, we have $\|f_v(x)\|_{F,v} = \|x\|_{E,v}$, where $f_v = f \otimes \mathrm{id}_{K_v}$.*

The adelic spaces E and F will be called *isometric* if there exists an isomorphism $E \to F$ which is an isometry.

Operations on Adelic Spaces Let E, E' be adelic spaces over K and $F \subset E$ a vector subspace. One can consider the following adelic spaces:

Induced Structure F with norms $\|\cdot\|_{E,v}$ restricted to $F \otimes_K K_v$.

Quotient E/F with quotient norms

$$\|\overline{x}\|_{E/F,v} = \inf\left\{\|z\|_{E,v}\,;\ z \in E \otimes_K K_v,\ z = x \mod F \otimes_K K_v\right\}.$$

Dual $E^\vee = \mathrm{Hom}_K(E, K)$ (linear forms) with operator norms

$$\forall\, \ell \in E^\vee \otimes_K K_v, \quad \|\ell\|_{E^\vee,v} = \sup\left\{\frac{|\ell(z)|_v}{\|z\|_{E,v}}\,;\ z \in E \otimes_K K_v \backslash \{0\}\right\}.$$

Given an Euclidean lattice $(\Omega, \|\cdot\|)$, the dual of E_Ω corresponds to the dual lattice $\Omega^* = \{\varphi \in (\Omega \otimes_{\mathbb{Z}} \mathbb{R})^\vee\,;\ \varphi(\Omega) \subset \mathbb{Z}\}$ with the gauge[1] of the polar body $C^\circ = \{\varphi \in (\Omega \otimes_{\mathbb{Z}} \mathbb{R})^\vee\,;\ \varphi(C) \subset [-1, 1]\}$ of the unit ball $C = \{x \in \Omega \otimes_{\mathbb{Z}} \mathbb{R}\,;\ \|x\| \leq 1\}$.

(Hermitian) Direct Sum $E \oplus E'$ with norm at $v \in V(K)$ given by

$$\|(x, x')\|_{E\oplus E',v} = \begin{cases} \left(\|x\|^2_{E,v} + \|x'\|^2_{E',v}\right)^{1/2} & \text{if } v \mid \infty, \\ \max\left\{\|x\|_{E,v}, \|x'\|_{E',v}\right\} & \text{if } v \nmid \infty, \end{cases}$$

for all $x \in E \otimes_K K_v$ and $x' \in E' \otimes_K K_v$.

Operator Norm $\mathrm{Hom}_K(E, E')$ (linear maps) with

$$\|f\|_v = \sup\left\{\frac{\|f(x)\|_{E',v}}{\|x\|_{E,v}}\,;\ x \in E \otimes_K K_v \backslash \{0\}\right\}$$

for all $f \in \mathrm{Hom}_K(E, E') \otimes_K K_v$. Using the natural isomorphism $E \otimes_K E' \simeq \mathrm{Hom}_K(E^\vee, E')$, we get an adelic structure on $E \otimes E'$, denoted $E \otimes_\varepsilon E'$ in the sequel (the ε refers to the injective norm for tensor product of Banach spaces).

Tensor Product Assume $E = (\varphi, A)$ and $E' = (\varphi', A')$ are *rigid* adelic spaces. The tensor product $E \otimes_K E'$ is endowed with the (rigid) structure given by $(\varphi \otimes \varphi', A \otimes A')$. It is the same as saying that local orthonormal bases of $E \otimes_K K_v$ and $E' \otimes_K K_v$ give an orthonormal basis by tensor product.

Symmetric Power When $E = (\varphi, A)$ is a rigid adelic space and $i \in \mathbb{N}\backslash\{0\}$, the symmetric power $S^i E$ is endowed with $(S^i(\varphi), S^i(A))$. It corresponds to the quotient structure of the tensor norm by the natural surjection $E^{\otimes i} \to S^i E$. We

[1]Recall that the gauge of a set C is the function $j(x) = \inf\{\lambda > 0\,;\ x/\lambda \in \mathsf{C}\}$. When C is a symmetric compact convex set with non-empty interior in a vector space U, then the gauge defines a norm on U.

have $\|x^i\|_{S^i E,v} = \|x\|_{E,v}^i$ for all $x \in E \otimes_K K_v$. If $e_1, \dots, e_n$ is an orthonormal basis of $E \otimes_K K_v$, then the vectors $e_1^{i_1} \cdots e_n^{i_n}$ with $i_j \in \mathbb{N}$ and $i_1 + \cdots + i_n = i$ form an orthogonal basis of $S^i E$ and

$$\|e_1^{i_1} \cdots e_n^{i_n}\|_{S^i E,v} = \left(\frac{i_1! \cdots i_n!}{i!}\right)^{1/2} \quad \text{if } v \mid \infty \text{ and 1 otherwise.}$$

Exterior Power When $E = (\varphi, A)$ is a rigid adelic space with dimension n and $i \in \{1, \dots, n\}$, the exterior power $\bigwedge^i E$ is endowed with the rigid structure $(\bigwedge^i \varphi, \bigwedge^i A)$. Given $v \in V(K)$, an orthonormal basis $(e_1, \dots, e_n)$ of $E \otimes_K K_v$ induces an orthonormal basis $(e_{j_1} \wedge \cdots \wedge e_{j_i})_{1 \le j_1 < \cdots < j_i \le n}$ of $\bigwedge^i E \otimes_K K_v$. Note that it differs by a coefficient $\sqrt{i!}$ from the quotient norm $E^{\otimes i} \to \bigwedge^i E$. When $i = \dim E$, the exterior power $\bigwedge^i E$ is called the determinant of E and denoted by $\det E$.

Scalar Extension Let K'/K be an algebraic extension and $E = (\varphi, A)$ be a rigid adelic space. We endow $E \otimes_K K'$ with the rigid adelic structure given by $(\varphi \otimes \mathrm{id}_{K'}, A)$ where $\varphi \otimes \mathrm{id}_{K'} \colon E \otimes_K K' \to \left(K'\right)^n$ is induced by φ and A is viewed in $\mathrm{GL}_n(\mathbb{A}_{K'})$ by means of the diagonal embedding $\mathbb{A}_K \hookrightarrow \mathbb{A}_{K'}$. We denote by $E_{K'}$ the adelic space obtained in this way.

These definitions do not depend on the chosen couple (φ, A). Let us mention that every rigid adelic space E over K can be written as the scalar extension $E_0 \otimes_{K_0} K$ of a rigid adelic space E_0 over a number field K_0: choose K_0 such that $A \in \mathrm{GL}_n(\mathbb{A}_{K_0})$ and define $E_0 = \varphi^{-1}(K_0^n)$ with the structure given by $(\varphi_{|E_0}, A)$.

Theorem 4 *If E and E' are rigid adelic spaces, all these adelic structures are rigid except (in general) the one on $\mathrm{Hom}_K(E, E')$ and $E \otimes_\varepsilon E'$ (operator norms). Moreover the canonical isomorphisms $E \simeq \left(E^\vee\right)^\vee$ and, for $F \subset E$ a linear subspace, $E/F \simeq \left(F^\perp\right)^\vee$ (where $F^\perp$ denotes the annihilator $\left\{\ell \in E^\vee ;\ \ell(F) = \{0\}\right\}$) are isometries.*

Proof See [14, Proposition 3.6]. □

We can also prove that, given a rigid adelic space E over K with dimension n and $r \in \{0, \dots, n\}$, the pairing $\bigwedge^{n-r} E \otimes \bigwedge^r E \to \det E$, $x \otimes y \mapsto x \wedge y$, induces an isometric isomorphism

$$\bigwedge^{n-r} E \simeq (\det E) \otimes \left(\bigwedge^{r} E\right)^\vee.$$

Moreover the natural map $\bigwedge^r (E^\vee) \to \left(\bigwedge^r E\right)^\vee$, $\varphi_1 \wedge \cdots \wedge \varphi_r \mapsto (x_1 \wedge \cdots \wedge x_r \mapsto \varphi_1(x_1) \cdots \varphi_r(x_r))$ is an isomorphism of rigid adelic spaces (whereas it is false if we replace the exterior power by the symmetric power).

Height, Degree and Slope of Rigid Adelic Spaces Let E be a rigid adelic space over K defined by (φ, A).

- The *height of* E is the positive real number

$$H(E) = \exp \int_{V(K)} \log |\det A_v|_v \, \mathrm{d}\sigma(v).$$

  If $E = \{0\}$ one has $H(E) = 1$. This definition does not depend on the choice of (φ, A) and the integral converges since $|\det A_v|_v = 1$ for v outside a compact subset of $V(K)$.
- The *(Arakelov) degree of* E is

$$\deg E = -\log H(E) = -\int_{V(K)} \log |\det A_v|_v \, \mathrm{d}\sigma(v).$$

- The *slope of* E is $\mu(E) = \dfrac{\deg E}{\dim E}$ (only for $E \neq \{0\}$).

In the literature, a rigid adelic space is often denoted with a bar ($\overline{E}$ instead of E) and its degree and slope are accompanied by a hat ($\widehat{\deg}\,\overline{E}$ instead of $\deg E$). Also note that from the definitions, the height and degree of a rigid adelic space are those of its determinant.

Examples

(1) $H(K^n) = 1$, $\deg K^n = \mu(K^n) = 0$ for all $n \in \mathbb{N}\backslash\{0\}$.
(2) If $(\Omega, \|\cdot\|)$ is an Euclidean lattice, then $H(E_\Omega) = \mathrm{vol}(\Omega)$. Indeed, if $(e_1, \ldots, e_n)$ is a $\mathbb{Z}$-basis of Ω, we have $H(E_\Omega) = |\det A|$ where the matrix A characterizes the norm: for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $\|x_1 e_1 + \cdots + x_n e_n\| = |(x_1, \ldots, x_n)A|$ that is, $A^{\mathrm{t}}A = (\langle e_i, e_j \rangle)_{1 \leq i,j \leq n}$.
(3) If K is a number field we have $H(E) = \prod_{v \in V(K)} |\det A_v|_v^{\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}}$.

Proposition 5 *Let E and E' be rigid adelic spaces over K and $F \subset E$ a linear subspace endowed with its induced adelic structure. Then*

$$H(E/F) = \frac{H(E)}{H(F)} \qquad (\deg E = \deg F + \deg E/F)$$

$$H(E^\vee) = H(E)^{-1} \qquad (\deg E^\vee = -\deg E)$$

$$H(E \oplus E') = H(E)H(E') \qquad (\deg E \oplus E' = \deg E + \deg E')$$

$$H(E \otimes E') = H(E)^{\dim E'} H(E')^{\dim E} \qquad (\mu(E \otimes E') = \mu(E) + \mu(E'))$$

$$H(F^\perp) = \frac{H(F)}{H(E)} \qquad (\deg F^\perp = \deg F - \deg E).$$

If $n = \dim E$ *and* $i \in \{1, \ldots, n\}$*, we also have* $H\left(\bigwedge^i E\right) = H(E)^{\binom{n-1}{i-1}}$*, that is,* $\mu\left(\bigwedge^i E\right) = i\mu(E)$*. Moreover, for all* $i \in \mathbb{N}$*, we have*

$$\mu\left(S^i E\right) = i\mu(E) + \left(2\binom{i+n-1}{n-1}\right)^{-1} \sum_{\substack{(j_1,\ldots,j_n)\in\mathbb{N}^n \\ j_1+\cdots+j_n=i}} \log \frac{i!}{j_1!\cdots j_n!}.$$

Proof See [12, Lemma 7.3], [13, § 2.7], [14, Proposition 3.6]. □

Furthermore, height, degree and slope are invariant by scalar extension: if K'/K is algebraic, then $H(E_{K'}) = H(E)$, $\deg E_{K'} = \deg E$ and $\mu(E_{K'}) = \mu(E)$. Also note that we have an asymptotic estimate

$$\mu\left(S^i E\right) = i\mu(E) + \frac{i}{2}(H_n - 1)(1 + \mathrm{o}(1)) \quad \text{when } i \to +\infty$$

in terms of the harmonic number $H_n = \sum_{h=1}^{n} 1/h$ (see [12, Annex]). It may be viewed as a particular case of the arithmetic Hilbert-Samuel theorem (see Sect. 5 in Chapter III).

The following statement is the key result for the existence of the Harder-Narasimhan filtration of a rigid adelic space which shall be established later (see page 52).

Proposition 6 *Let* F *and* G *be linear subspaces of a rigid adelic space over* K*. Then*

$$H(F+G)H(F\cap G) \le H(F)H(G)$$

that is, $\deg F + \deg G \le \deg(F+G) + \deg F \cap G$.

Proof Let $\iota\colon F/F\cap G \to (F+G)/G$ be the natural isomorphism. For all $v \in V(K)$ and $x \in (F/F\cap G)\otimes_K K_v$, we have $\|\iota_v(x)\|_{(F+G)/G,v} \le \|x\|_{F/F\cap G,v}$ (here $\iota_v = \iota \otimes \mathrm{id}_{K_v}$). In particular, if $e_1, \ldots, e_m$ is an orthonormal basis of $(F/F\cap G)\otimes_K K_v$, then

$$\begin{aligned}
&\|(\det \iota_v)(e_1 \wedge \cdots \wedge e_m)\|_{\det(F+G)/G,v} \\
&= \|\iota_v(e_1) \wedge \cdots \wedge \iota_v(e_m)\|_{\det(F+G)/G,v} \\
&\underset{\substack{\uparrow \\ \text{Hadamard inequality}}}{\le} \prod_{i=1}^{m} \|\iota_v(e_i)\|_{(F+G)/G,v} \le \prod_{i=1}^{m} \|e_i\|_{F/F\cap G,v} = 1.
\end{aligned}$$

In other words, the operator norm $\| \det \iota \|_v$ of $\det \iota$ at v is smaller than 1. Thus, using Proposition 5, we get

$$\begin{aligned}\frac{H(F+G)H(F\cap G)}{H(F)H(G)} &= \frac{H((F+G)/G)}{H(F/F\cap G)} \\ &= H\left((\det F/F\cap G)^{\vee}\otimes \det((F+G)/G)\right) \\ &= \exp\int_{V(K)} \log\|\det\iota\|_v \,\mathrm{d}\sigma(v) \le \exp 0 = 1. \qquad \square\end{aligned}$$

A slightly more natural proof can be obtained from Proposition 42.

Heights of Points Let E be an adelic space over K.

Definition 7 *We shall say that the adelic space E is* integrable *if, for all $x \in E\backslash\{0\}$, the function $V(K) \to \mathbb{R}$, $v \mapsto \log\|x\|_{E,v}$ is σ-integrable.*

A rigid adelic space is integrable as well as ε-tensor products of finitely many rigid adelic spaces. Indeed we have:

Lemma 8 *Let E be rigid adelic space and F be an integrable adelic space over K. Then $E \otimes_\varepsilon F$ is integrable.*

Proof Using the isometric isomorphism $E \otimes_\varepsilon F \simeq \mathrm{Hom}(E^\vee, F)$, it amounts to proving that $\widetilde{f}\colon v \mapsto \log\|f\|_v$ is σ-integrable for every $f \in \mathrm{Hom}(E^\vee, F)\backslash\{0\}$, that is, this function is Borel and its absolute value has finite integral. For the measurability, choose a number field $K_0 \subset K$ such that E, F, f are defined over K_0. Then $\widetilde{f}$ is the composite of $v \mapsto v_{|K_0}$ and $v_0 \in V(K_0) \mapsto \log\|f\|_{v_0}$. This latter function is measurable since every subset of $V(K_0)$ (endowed with its discrete topology) is measurable. As for the restriction map $v \mapsto v_{|K_0}$, it is continuous by definition of the topology put on $V(K)$. Thus $\widetilde{f}$ is Borel and we shall now prove that $\int_{V(K)} |\widetilde{f}| < +\infty$. Let $(e_1, \dots, e_n)$ be a K-basis of E. Since E is rigid, there exists $a = (a_p)_{p\in V(\mathbb{Q})} \in \mathbb{A}_{\mathbb{Q}}^{\times}$ such that, for all $v \in V(K)$ above $p \in V(\mathbb{Q})$ and all $x = \sum_{i=1}^n x_i e_i \in E \otimes_K K_v$, we have

$$|a_p|_v^{-1} \max_{1\le i\le n}\{|x_i|_v\} \le \|x\|_{E,v} \le |a_p|_v \max_{1\le i\le n}\{|x_i|_v\}.$$

Since $f(x) = x_1 f(e_1) + \cdots + x_n f(e_n)$, the triangle inequality yields $\|f\|_v \le |b_p|_v \max_{1\le i\le n}\{\|f(e_i)\|_{F,v}\}$ where $b_\infty = na_\infty$ and, if p is prime number, $b_p = a_p$. Moreover, since $f \ne 0$, one can choose $m \in \{1, \dots, n\}$ such that $f(e_m) \ne 0$ and we bound $\|f\|_v^{-1} \le \|e_m\|_{E,v}/\|f(e_m)\|_{F,v} \le |b_p|_v/\|f(e_m)\|_{F,v}$. Thus $|\widetilde{f}(v)| = |\log\|f\|_v| = \log\max\left\{\|f\|_v, \|f\|_v^{-1}\right\}$ is bounded above by

$$\log|b_p|_v + \max_{1\le i\le n}\left\{\log\|f(e_i)\|_{F,v}, -\log\|f(e_m)\|_{F,v}\right\}.$$

In this bound, we can restrict to indices i such that $f(e_i) \neq 0$. Since F is integrable, then each function appearing in the maximum is σ-integrable. We conclude with the fact that the maximum of a finite number of σ-integrable functions is still σ-integrable (since $|\max\{a,b\}| \leq |a| + |b|$). □

The integrability condition is the minimal condition which allows to define the height of a vector of an adelic space.

Definition 9 *Let E be an integrable adelic space over K and $x \in E$.* The height *$H_E(x)$ is the nonnegative real number:*

$$H_E(0) = 0 \quad \text{and if } x \neq 0, \quad H_E(x) = \exp \int_{V(K)} \log \|x\|_{E,v} \, \mathrm{d}\sigma(v).$$

The product formula entails that H_E is a projective height, that is, $H_E(\lambda x) = H_E(x)$ for all $\lambda \in K \backslash \{0\}$.

Examples

1. For all $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n \backslash \{0\}$, one has

$$H_{\mathbb{Q}^n}(x) = \left(x_1^2 + \cdots + x_n^2\right)^{1/2} \gcd(x_1, \ldots, x_n)^{-1}.$$

2. Let $(\Omega, \|\cdot\|)$ be an Euclidean lattice and $x \in E_\Omega$. Then there exists $d_x \in \mathbb{Q} \backslash \{0\}$ such that $H_{E_\Omega}(x) = \|d_x x\|$.
3. When E is a rigid adelic space of dimension 1, one has $H_E(x) = H(E)$ for all $x \in E \backslash \{0\}$.
4. When K is a number field, one has

$$\forall x \in E, \quad H_E(x) = \prod_{v \in V(K)} \|x\|_{E,v}^{[K_v:\mathbb{Q}_v]/[K:\mathbb{Q}]}.$$

5. Let F be the hyperplane $a_1 x_1 + \cdots + a_n x_n = 0$ of K^n (given by $(a_1, \ldots, a_n) \in K^n \backslash \{0\}$). Then $H(F) = H_{K^n}(a_1, \ldots, a_n)$ (since $H(F) = H(F^\perp)$).

Note that when E is a rigid adelic space, the height H_E is invariant by scalar extension: for all $x \in E$, for every algebraic extension K'/K, one has $H_{E \otimes_K K'}(x) = H_E(x)$.

Proposition 10 (Convexity Inequality for Heights) *Let N be a positive integer and $E_1, \ldots, E_N$ be integrable adelic spaces over K. Then the direct sum $E_1 \oplus \cdots \oplus E_N$ is integrable. Moreover, for all $(x_1, \ldots, x_N) \in E_1 \oplus \cdots \oplus E_N$, we have*

$$\left(\sum_{i=1}^N H_{E_i}(x_i)^2\right)^{1/2} \leq H_{E_1 \oplus \cdots \oplus E_N}(x_1, \ldots, x_N).$$

Proof For the integrability, we can restrict to $N = 2$. Observe that for positive real numbers a, b, we have

$$|\log(a+b)| \leq \log 2 + \log\max\left\{a, \frac{1}{a}\right\} + \log\max\left\{b, \frac{1}{b}\right\}$$
$$= \log 2 + |\log a| + |\log b|$$

and $|\log\max\{a, b\}| \leq |\log a| + |\log b|$. Applying this to $a = \|x_1\|^2_{E_1,v}$ and $b = \|x_2\|^2_{E_2,v}$ the result comes from the definition of $E_1 \oplus E_2$. As for the height inequality, we proceed as in [13, Lemma 2.2]. Applying Jensen inequality on the probability space $(V_\infty(K), \sigma)$ to the convex function $u\colon \mathbb{R} \to \mathbb{R}$, $u(x) = \log(1 + e^x)$, we get

$$1 + \exp\int_{V_\infty(K)} \log f \leq \exp\int_{V_\infty(K)} \log(1 + f)$$

for every nonnegative function f. By direct induction, we have

$$\sum_{i=1}^{N} \exp\int_{V_\infty(K)} \log f_i \leq \exp\int_{V_\infty(K)} \log(f_1 + \cdots + f_N)$$

for all nonnegative functions $f_1, \ldots, f_N$. Choosing $f_i(v) = \|x_i\|^2_{E_i,v}$ we get the convexity inequality but with only the archimedean part of the heights. To complete with the ultrametric part, we multiply both sides by $\exp\int_{V(K)\setminus V_\infty(K)} \log\max\{f_1, \ldots, f_N\}$ and we bound from below this number by $\exp\int_{V(K)\setminus V_\infty(K)} \log f_i$ for each i. □

3 Minima and Slopes

3.1 Successive Minima

Let E be a rigid adelic space with dimension $n \geq 1$. We denote $\Lambda_1(E) = \inf\{H_E(x)\,;\ x \in E\setminus\{0\}\}$. We define three types of successive minima associated to E (still others exist in the literature, see [14]) which have been respectively inspired by the articles [8, 23] and [25]. Let $i \in \{1, \ldots, n\}$.

Bost-Chen minima:

$$\Lambda^{(i)}(E) = \sup\{\Lambda_1(E/F)\,;\ F \subset E \text{ linear subspace, } \dim F \leq i-1\}$$

Roy-Thunder minima:

$$\Lambda_i(E) = \inf\{\max\{H_E(x_1), \ldots, H_E(x_i)\}\,;\ \dim \mathrm{Vect}_K(x_1, \ldots, x_i) = i\}$$

Zhang minima: $Z_i(E) = \inf\left\{\sup_{x\in S} H_E(x)\,;\ S \subset E,\ \dim \mathrm{Zar}(S) \geq i\right\}$
Here $\mathrm{Zar}(S)$ means the Zariski closure of $K.S = \{ax\,;\ a \in K,\ x \in S\}$ and its dimension is the one of the scheme over $\mathrm{Spec}\, K$ defined by the algebraic set $\mathrm{Zar}(S)$. We have

$$\begin{array}{ccccccccc} 0 < \Lambda^{(1)}(E) & \leq & \Lambda^{(2)}(E) & \leq \cdots \leq & \Lambda^{(n)}(E) & < \infty \\ \| & & \wedge\!\mid & & \wedge\!\mid & \\ \Lambda_1(E) & \leq & \Lambda_2(E) & \leq \cdots \leq & \Lambda_n(E) & < \infty \\ \| & & \wedge\!\mid & & \wedge\!\mid & \\ Z_1(E) & \leq & Z_2(E) & \leq \cdots \leq & Z_n(E) & \leq \infty \end{array}$$

A field K is a *Northcott field* if, for all positive real number B, the set $\{x \in K\,;\ H_{K^2}(1, x) \leq B\}$ is finite (for instance, any number field or, according to [5], $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \ldots)$ are Northcott fields). It can be proved that, for every integer $n \geq 2$, for every rigid adelic space E with $\dim E = n$, we have $Z_n(E) < \infty$ if and only if K is *not* a Northcott field (see [14, Proposition 4.4]).

Examples

Let n be a positive integer and $i, j \in \{1, \ldots, n\}$.

– We have $\Lambda^{(i)}(K^n) = \Lambda_i(K^n) = 1$.
– If K contains infinitely many roots of unity (e.g., $K = \overline{\mathbb{Q}}$), then $Z_i(K^n) = \sqrt{i}$ (consequence of the convexity inequality for heights, see Proposition 10).
– Let $A_n = \{(x_0, \ldots, x_n) \in K^{n+1}\,;\ \sum_{\ell=0}^n x_\ell = 0\} \subset K^{n+1}$. Then $\Lambda_i\left(\bigwedge^j A_n\right) = \sqrt{j+1}$.

In the following, in order to unify notation, we shall sometimes use $\lambda_i^*(E)$ with $* \in \{\mathrm{BC}, \Lambda, Z\}$ to indicate $\lambda_i^{\mathrm{BC}}(E) = \Lambda^{(i)}(E)$, $\lambda_i^{\Lambda}(E) = \Lambda_i(E)$ or $\lambda_i^{Z}(E) = Z_i(E)$.

Basic Properties Let E be a rigid adelic space with dimension n and let $i \in \{1, \ldots, n\}$.

1. For any non-zero linear subspace $F \subset E$, we have $\lambda_i^*(E) \leq \lambda_i^*(F)$ for all $* \in \{\mathrm{BC}, \Lambda, Z\}$ and $i \leq \dim F$.
2. For every algebraic extension K'/K and every $* \in \{\mathrm{BC}, \Lambda, Z\}$, we have $\lambda_i^*(E_{K'}) \leq \lambda_i^*(E)$.

The latter property is quite easy to prove (see [14, Lemma 4.22]) except, maybe, for $* = \mathrm{BC}$, for which we provide a proof (suggested by G. Rémond): Let us assume that there exists i such that $\lambda_i^{\rm BC}(E_{K'}) > \lambda_i^{\rm BC}(E)$. We choose it as small as possible and we consider a subspace $F \subset E_{K'}$, (necessarily) with dimension $i-1$, such that $\Lambda_1(E_{K'}/F) > \lambda_i^{\rm BC}(E)$. Let $G \subset E$ be a subspace with maximal dimension satisfying $G \otimes_K K' \subset F$. We have $\dim G \leq i-1$ and so $\lambda_i^{\rm BC}(E) \geq \Lambda_1(E/G)$. Then let us consider $x \in E \backslash G$ such that $\Lambda_1(E_{K'}/F) > H_{E/G}(x \bmod G)$. We have $x \notin F$ otherwise $G \oplus K.x$ has dimension greater than $\dim G$ with $(G \oplus K.x) \otimes_K K' \subset F$. Therefore, we have $H_{E_{K'}/F}(x \bmod F) \geq \Lambda_1(E_{K'}/F) > H_{E/G}(x \bmod G)$, contradicting the fact that the w-norm of $x \bmod F$ is smaller than the w-norm of $x \bmod G$, for all $w \in V(K')$, since $E_{K'}/F$ is a quotient of $(E/G) \otimes_K K'$.

In the following result it is convenient to put $\Lambda_0(E) = 0$ when E is an (integrable) adelic space.

Proposition 11 *Let N be a positive integer and let $E_1, \ldots, E_N$ be integrable adelic spaces over K. Then, for all $i \in \{1, \ldots, \sum_{h=1}^N \dim E_h\}$, we have*

$$\Lambda_i(E_1 \oplus \cdots \oplus E_N) = \min \max \left\{ \Lambda_{a_1}(E_1), \ldots, \Lambda_{a_N}(E_N) \right\}$$

where the minimum is taken over all integers $a_h \in [0, \dim E_h]$, $1 \leq h \leq N$, such that $\sum_{h=1}^N a_h = i$.

In particular, $\Lambda_1(E_1 \oplus \cdots \oplus E_N) = \min\{\Lambda_1(E_1), \ldots, \Lambda_1(E_N)\}$.

Proof Fix $(a_1, \ldots, a_N)$ as above. For each $j \in \{1, \ldots, N\}$ such that $a_j \neq 0$, let $x_1^{(j)}, \ldots, x_{a_j}^{(j)}$ be linearly independent vectors of E_j. Then $\{x_h^{(j)}\,;\ 1 \leq h \leq a_j,\ 1 \leq j \leq N\}$ forms a free family of i vectors of $E := E_1 \oplus \cdots \oplus E_N$. Thus, by definition of Λ_i, we get

$$\Lambda_i(E) \leq \max \left\{ H_E\left(x_h^{(j)}\right)\,;\ 1 \leq h \leq a_j,\ 1 \leq j \leq N \right\}.$$

The infimum of the right hand side when all $x_h^{(j)}$ vary is precisely $\max\left\{\Lambda_{a_1}(E_1), \ldots, \Lambda_{a_N}(E_N)\right\}$ and, then, we can take the infimum over $(a_1, \ldots, a_N)$ to obtain $\Lambda_i(\bigoplus_{j=1}^N E_j) \leq \min \max_j \left\{\Lambda_{a_j}(E_j)\right\}$. For the reverse inequality, consider $x_h^{(j)} \in E_h$ for all $j \in \{1, \ldots, i\}$ and $h \in \{1, \ldots, N\}$ such that the vectors $X_j = (x_1^{(j)}, \ldots, x_N^{(j)})$'s are linearly independent. In particular $X_1 \wedge \cdots \wedge X_i \neq 0$ and, writing this vector as a sum of $x_{\tau(1)}^{(1)} \wedge \cdots \wedge x_{\tau(i)}^{(i)}$ over functions $\tau\colon \{1, \ldots, i\} \to \{1, \ldots, N\}$, we deduce the existence of τ such that $\{x_{\tau(1)}^{(1)}, \ldots, x_{\tau(i)}^{(i)}\}$ is a free family. For each $h \in \{1, \ldots, N\}$, let n_h be the number of $u \in \{1, \ldots, i\}$ such that $\tau(u) = h$ (the integer n_h may be zero). We have $\sum_h n_h = i$ and the vector space generated by $\{x_h^{(j)}\,;\ 1 \leq j \leq i\}$ has dimension at least n_h. From Proposition 10, we get $H_E(X_j) \geq \max_{1 \leq h \leq N} H_{E_h}(x_h^{(j)})$ for all $j \in \{1, \ldots, i\}$, so

$$\max_{1 \leq j \leq i} H_E(X_j) \geq \max_{1 \leq j \leq i} \max_{1 \leq h \leq N} H_{E_h}(x_h^{(j)}) \geq \max_{1 \leq h \leq N} \Lambda_{n_h}(E_h).$$

We then conclude by bounding from below the latter maximum by $\min_{\sum_h a_h = i} \max_h \Lambda_{a_h}(E_h)$. □

3.2 Slopes

In this paragraph, we define the canonical polygon of a rigid adelic space, which gives birth to its successive slopes. These notions have their origin in the works by Stuhler [24] and Grayson [15] (inspired by the article [16] of Harder-Narasimhan). Later on, they have been developed by Bost in two lectures given at the Institut Henri Poincaré in 1997 and 1999 and in his articles [6, 7], then extended in different ways in [12, 1, 10, 8]. Let E be a rigid adelic space over K and $n = \dim E$.

Lemma 12 *There exists a positive constant $c(E)$ such that $H(F) \geq c(E)$ for every linear subspace $F \subset E$.*

Proof Let (φ, A) be a couple defining the adelic structure of E. There exists $a = (a_p)_{p \in V(\mathbb{Q})} \in \mathbb{A}_{\mathbb{Q}}^{\times}$ such that, for all $v \in V(K)$ above $p \in V(\mathbb{Q})$ and for all $x \in E \otimes_K K_v$,

$$|a_p|_v^{-1} |\varphi_v(x)|_v \leq \|x\|_{E,v} \leq |a_p|_v |\varphi_v(x)|_v.$$

Define $|a| = \exp \int_{V(K)} \log |a_p|_v \, d\sigma(v) \geq 1$. For every subspace $F \subset E$ with dimension ℓ, we have $H(F) \geq |a|^{-\ell} H(\varphi(F)) = |a|^{-\ell} H(\det \varphi(F))$. Since $\det \varphi(F)$ is a non-zero vector of $\bigwedge^{\ell} K^n$, which is isometric to $K^{\binom{n}{\ell}}$, we have $H(\det \varphi(F)) \geq 1$ and the conclusion follows with $c(E) = |a|^{-n}$. □

In other words, the set $\{\deg F\,;\ F \subset E\}$ is bounded from above. This result allows to define some positive real numbers associated to E: for all $i \in \{0, \dots, n\}$,

$$\sigma_i(E) = \inf\{H(F)\,;\ F \text{ linear subspace of } E \text{ and } \dim F = i\}.$$

For instance, $\sigma_0(E) = 1$, $\sigma_1(E) = \Lambda_1(E)$ and $\sigma_n(E) = H(E)$. Note that we have $\sigma_{n-1}(E) = \Lambda_1(E^{\vee}) H(E)$ and, more generally, $\sigma_{n-i}(E) = \sigma_i(E^{\vee}) H(E)$ which comes from the isometry $E/F \simeq (F^{\perp})^{\vee}$ (Theorem 4 and Proposition 5). We also have $\sigma_i(E) \geq \Lambda_1(\bigwedge^i E)$. Lemma 12 justifies the following

Definition 13 *Let $P_E : [0, n] \to \mathbb{R}$ denote the piecewise linear function delimiting from above the convex hull of the set*

$$\left\{ (\dim F, \deg F) \in \mathbb{R}^2\,;\ F \text{ linear subspace of } E \right\}.$$

We shall call P_E the canonical polygon *of E.*

Naturally, the latter convex hull can be replaced by the one of the (finite) set $\{(i, -\log \sigma_i(E))\,;\ i \in \{0, \dots, n\}\}$. By definition, the function P_E is a concave function which satisfies $P_E(0) = 0$ and its slopes

$$\mu_i(E) = P_E(i) - P_E(i-1) \qquad (i \in \{1, \dots, n\})$$

form a nonincreasing sequence $\mu_1(E) \geq \mu_2(E) \geq \cdots \geq \mu_n(E)$, called the *successive slopes of* E. The greatest slope $\mu_1(E)$ is also denoted $\mu_{\max}(E)$ and the smallest slope $\mu_n(E)$ is $\mu_{\min}(E)$. This terminology is also justified by the following key result.

Lemma 14 *For every rigid adelic space E over K, we have*

$$\mu_{\max}(E) = \max\{\mu(F)\,;\ F \neq \{0\} \text{ linear subspace of } E\}.$$

More precisely, there exists a (single) subspace of E, denoted E_{des}, such that $\mu(E_{\text{des}}) = \mu_{\max}(E)$ and E_{des} contains every linear subspace $F \subset E$ satisfying $\mu(F) = \mu_{\max}(E)$.

The subscript “des” refers to the word *destabilizing*. The proof follows the one of [8, Proposition 2.2].

Proof Let us temporarily denote by c the supremum of slopes $\mu(F)$ when F runs over non-zero linear subspaces of E. This is a real number by Lemma 12. Actually, if $m = \dim F$, we have

$$\mu(F) = \frac{\deg F}{m} \leq \frac{P_E(m)}{m} = \frac{\mu_1(E) + \cdots + \mu_m(E)}{m} \leq \mu_1(E)$$

and so $c \leq \mu_1(E)$. On the other hand, for every linear subspace $F \subset E$, we have $\deg F \leq (\dim F)c$. Since $m \mapsto mc$ is a concave (linear) function we deduce $P_E(m) \leq mc$ for all $m \in [0, n]$ ($n = \dim E$). Thus $\mu_1(E) = P_E(1) \leq c$ and we get $\mu_1(E) = c = \sup\{\mu(F)\,;\ \{0\} \neq F \subset E\}$. Let us now prove the existence of E_{des}. We proceed by induction on n. The statement is clear for $n = 1$ since $\mu_1(E) = \mu(E)$ in this case. Assume the existence of the destabilizing rigid adelic space when the dimension of the ambient space is at most $n - 1$. Let E be of dimension n. If $\mu_1(E) = \mu(E)$, then $E_{\text{des}} := E$ is the winner. Otherwise the set $\{F \subset E\,;\ F \neq \{0\} \text{ and } \mu(F) > \mu(E)\}$ is non-empty and we can choose F in it with maximal dimension. By induction hypothesis (and since $\dim F \leq n-1$), there exists F_{des} such that $\mu(F_{\text{des}}) = \mu_{\max}(F)$ and such that, for every linear subspace $G \subset F$ with $\mu(G) = \mu_{\max}(F)$, we have $G \subset F_{\text{des}}$. Let G be a non-zero linear subspace of E. If $G \not\subset F$, then $\dim(F + G) > \dim F$ and, by maximality property of $\dim F$, we have $\mu(F + G) \leq \mu(E)$. Replacing this information in the inequality $\deg F + \deg G \leq \deg(F + G) + \deg F \cap G$ given by Proposition 6, we get

$$(\dim F)\mu(F) + (\dim G)\mu(G) \leq (\dim(F + G))\mu(E) + (\dim F \cap G)\mu_{\max}(F)$$

and so $(\dim G)\mu(G)$ is bounded above by

$$\dim(F+G)\underbrace{(\mu(E)-\mu(F))}_{<0}+(\dim(F+G)-\dim F)\underbrace{\mu(F)}_{\leq\mu_{\max}(F)},$$
$$+(\dim F\cap G)\,\mu_{\max}(F)$$

which implies $\mu(G)<\mu_{\max}(F)$. If $G\subset F$ we have $\mu(G)\leq\mu_{\max}(F)$. Thus, every non-zero linear subspace of E has its slope at most $\mu_{\max}(F)$ and so $\mu_{\max}(E)=\mu_{\max}(F)$. Then the space $E_{\text{des}}:=F_{\text{des}}$ has the required properties. □

Definition 15 *A rigid adelic space E is* semistable *if $\mu(E)=\mu_{\max}(E)$ (that is, $E_{\text{des}}=E$).*

In this case, the canonical polygon is a straight line. For instance, K^n and A_n (defined on page 48) are semistable (see [13, p. 580] for A_n). Lemma 14 allows to define a unique filtration of E composed of linear subspaces $\{0\}=E_0\subsetneq E_1\subsetneq\cdots\subsetneq E_N=E$ such that E_{i+1}/E_i is semistable for every $i\in\{0,1,\ldots,N-1\}$: the first spaces $E_0,\ldots,E_i$ being chosen, take E_{i+1} satisfying $E_{i+1}/E_i=(E/E_i)_{\text{des}}$. This filtration is called the *Harder-Narasimhan filtration* of E (shortened in *HN-filtration* thereafter). By definition we have $\mu(E_{i+1}/E_{i-1})<\mu(E_i/E_{i-1})$ and, using $\deg E_{i+1}/E_i=\deg E_{i+1}/E_{i-1}-\deg E_i/E_{i-1}$, we deduce that

$$\mu\left(E_N/E_{N-1}\right)<\mu\left(E_{N-1}/E_{N-2}\right)<\cdots<\mu(E_1).$$

Theorem 16 *Let $E_0=\{0\}\subsetneq E_1\subsetneq\cdots\subsetneq E_N=E$ be the HN-filtration of E. Let $m_i=\dim E_i$. Then $m_1,\ldots,m_{N-1}$ are (exactly) the points at which P_E is not differentiable and $P_E(m_i)=\deg E_i$ for all $i\in\{0,\ldots,N\}$. Moreover, for all $i\in\{1,\ldots,N\}$ and $j\in\{1,\ldots,m_i-m_{i-1}\}$, we have $\mu_{m_{i-1}+j}(E)=\mu(E_i/E_{i-1})$.*

The proof will use the following result (here $n=\dim E$).

Lemma 17 *Let $x\in[0,n]$ such that P_E is not differentiable at x. Then x is an integer and there exists a unique linear subspace $F_x\subset E$ with dimension x such that $P_E(x)=\deg F_x$. Moreover, if P_E is not differentiable at $y\leq x$, then $F_y\subset F_x$.*

Proof By definition of P_E, which is a linear function on each interval $(h,h+1)$ for $h\in\{0,\ldots,n-1\}$, the real number x is necessarily an integer. Since $F_0=\{0\}$ and $F_n=E$ we may assume $x\in\{1,\ldots,n-1\}$. The construction of P_E and its non differentiability at x entail

$$P_E(x)=\sup\{\deg F\ ;\ F \text{ linear subspace of } E \text{ with dimension } x\}.$$

Then, let us choose some linear subspaces A and B of E, with dimension x, such that $P_E(x)\leq\deg A+\varepsilon$ and $P_E(x)\leq\deg B+\varepsilon$ where

$$\varepsilon=\frac{1}{4}\min\left\{\frac{P_E(\delta)-P_E(i)}{\delta-i}-\frac{P_E(j)-P_E(h)}{j-h}\right\},$$

the minimum being taken over integers i, δ, h, j satisfying $0 \leq i < \delta \leq h < j \leq n$ and $(P_E(\delta) - P_E(i))/(\delta - i) - (P_E(j) - P_E(h))/(j - h) \neq 0$ (in particular $\varepsilon > 0$ by concavity of P_E). Defining $j = \dim(A + B)$ and $i = \dim A \cap B$ and using $\deg A + \deg B \leq \deg(A + B) + \deg A \cap B$ (Proposition 6), we get $2P_E(x) - 2\varepsilon \leq P_E(j) + P_E(i)$ which, if $x \neq i$, implies

$$\frac{P_E(x) - P_E(i)}{x - i} - \frac{P_E(j) - P_E(x)}{j - x} \leq 2\varepsilon \quad \text{since } x - i = j - x.$$

Since P_E is not differentiable at x, the left hand side is positive, contradicting the definition of ε. Thus we have $x - i = j - x = 0$, that is, $A = B$. We proved that there exists $\varepsilon > 0$ such that the set $\{A \subset E\,;\ \dim A = x \text{ and } P_E(x) \leq \deg A + \varepsilon\}$ is a singleton $\{F_x\}$. The same approach with $A = F_x$ and $B = F_y$ demonstrates $\dim(F_x + F_y) = \dim F_x$ and so $F_y \subset F_x$ when $y \leq x$. □

Proof of Theorem 16 Let $f_0 = 0 < f_1 < \cdots < f_M = n$ be the abscissae for which P_E is not differentiable and $F_0 = \{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_M$ the corresponding subspaces given by Lemma 17. For every linear subspace $F \subset E$ and every $i \in \{1, \ldots, M\}$ such that $F \not\subset F_{i-1}$, the concavity of P_E yields

$$\frac{P_E(\dim(F + F_{i-1})) - P_E(f_{i-1})}{\dim(F + F_{i-1}) - f_{i-1}} \leq \frac{P_E(f_i) - P_E(f_{i-1})}{f_i - f_{i-1}}$$

and this inequality is strict if $\dim(F + F_{i-1}) > f_i$. Bounding from below $P_E(\dim(F + F_{i-1}))$ by $\deg(F + F_{i-1})$ we get $\mu\left((F + F_{i-1})/F_{i-1}\right) \leq \mu\left(F_i/F_{i-1}\right)$ which proves that $\mu\left(F_i/F_{i-1}\right) = \mu_{\max}(E/F_{i-1})$. The equality can occur only if $\dim(F + F_{i-1}) \leq f_i$ and so $F_i/F_{i-1} = (E/F_{i-1})_{\mathrm{des}}$. Thus the sequence $(F_i)_i$ satisfies the same definition as the HN-filtration of E and, by unicity, it is the same: $N = M$ and $F_i = E_i$ and $f_i = m_i$ for all i. As for the equality $\mu_{m_{i-1}+j}(E) = \mu(E_i/E_{i-1})$, it comes from the fact that $\mu_{m_{i-1}+1}(E) = \cdots = \mu_{m_i}(E)$ (since P_E is a line on $[m_{i-1}, m_i]$) and

$$\begin{aligned}\sum_{j=1}^{m_i - m_{i-1}} \mu_{m_{i-1}+j}(E) &= \sum_{j=1}^{m_i - m_{i-1}} P_E(m_{i-1} + j) - P_E(m_{i-1} + j - 1) \\ &= P_E(m_i) - P_E(m_{i-1}) \\ &= \deg\left(E_i/E_{i-1}\right) = (m_i - m_{i-1})\mu\left(E_i/E_{i-1}\right). \end{aligned}$$ □

From Theorem 16 can be deduced a minimax formula for $\mu_i(E)$.

Proposition 18 *Let E be a rigid adelic space over K and $i \in \{1, \ldots, \dim E\}$. Then*

$$\mu_i(E) = \sup_A \inf_B \mu\left(A/B\right) = \inf_B \sup_A \mu\left(A/B\right)$$

where $B \subset A$ run over linear subspaces of E with $\dim B \leq i - 1 < \dim A$.

Proof Let $E_0 = \{0\} \subsetneq E_1 \subsetneq \cdots \subsetneq E_N = E$ be the HN-filtration of E. Let $h \in \{0, \ldots, N-1\}$ such that $\dim E_h \leq i - 1 < \dim E_{h+1}$. Let A be a linear subspace of E with dimension $\geq i$ (in particular $A \not\subset E_h$). Using Theorem 16, Lemma 14 and Proposition 6, we get

$$\mu_i(E) = \mu_{\max}(E/E_h) \geq \mu((A + E_h)/E_h) \geq \mu(A/(A \cap E_h))$$

which is greater than $\inf\{\mu(A/B)\,;\ B \subset A \text{ and } \dim B \leq i-1\}$. Taking the supremum over A, we obtain $\mu_i(E) \geq \alpha := \sup_A \inf_B \mu(A/B)$. On the other hand, the concavity of P_E implies

$$\mu\left(E_{h+1}/B\right) \geq \frac{P_E(\dim E_{h+1}) - P_E(\dim B)}{\dim E_{h+1} - \dim B} \geq \mu\left(E_{h+1}/E_h\right) = \mu_i(E)$$

for any linear subspace $B \subset E_{h+1}$ with dimension $< i$. We conclude using $\alpha \geq \inf_B \mu\left(E_{h+1}/B\right)$. The same method works with $\inf_B \sup_A \mu\left(A/B\right)$. □

In particular we have $\mu_n(E) = \mu_{\min}(E) = \inf\{\mu\left(E/F\right)\,;\ F \subsetneq E\}$ (where $n = \dim E$). Actually the infimum is a minimum as the next proposition and Lemma 14 prove it.

The following statement summarizes several properties of the canonical polygon of a rigid adelic space over an algebraic extension K.

Proposition 19 *Let E be a rigid adelic space over K with dimension n.*

(1) *If L is a rigid adelic space over K with dimension 1, then, for all $x \in [0, n]$, we have $P_{E \otimes L}(x) = P_E(x) + x \deg L$. In particular, for all $i \in \{1, \ldots, n\}$, we have $\mu_i(E \otimes L) = \mu_i(E) + \deg L$.*
(2) *For all $x \in [0, n]$, we have $P_{E^\vee}(x) = P_E(n - x) - \deg E$. In particular, for all $i \in \{1, \ldots, n\}$, we have $\mu_i(E^\vee) = -\mu_{n+1-i}(E)$.*
(3) *Let K'/K be an algebraic extension. Then we have $P_{E_{K'}} = P_E$. In particular, for all $i \in \{1, \ldots, n\}$, we have $\mu_i(E_{K'}) = \mu_i(E)$.*

The last property means that the μ_i's are *absolute minima* (that is, over an algebraic closure of K). The similar feature is not true in general for λ_i^*'s. Moreover this proposition can be restated in terms of the HN-filtration $E_0 = \{0\} \subset E_1 \subset \cdots \subset E_N = E$ of E: The HN-filtrations of $E \otimes_K L$, $E^\vee$, $E_{K'}$ are (respectively) $(E_i \otimes_K L)_i$, $(E_{N-i}^\perp)_i$ and $((E_i)_{K'})_i$.

Proof

(1) Since P_E is a linear function on each interval $[i, i+1]$, $i \in \{0, \ldots, n-1\}$, it is enough to prove the equality for $x = m \in \{0, \ldots, n\}$. For every subspace $F \subset E$ with dimension m, we have $\dim F \otimes_K L = m$ and $\deg F + m \deg L = \deg F \otimes_K L \leq P_{E \otimes_K L}(m)$. So $\deg F \leq P_{E \otimes_K L}(m) - m \deg L$ and since the function $m \mapsto P_{E \otimes_K L}(m) - m \deg L$ is concave we deduce $P_E(m) \leq P_{E \otimes_K L}(m) - m \deg L$. The reverse inequality is obtained replacing E by $E \otimes_K L$ and L by $L^\vee$ (using the fact that $L \otimes_K L^\vee$ is isometric to K). The equality for the

i-th slopes arises from this equality relating the canonical polygons and from the definition of μ_i.

(2) As previously, it is enough to prove the equality for $x = m \in \{0, \ldots, n\}$. For a subspace $F \subset E$ with dimension m, the isometric isomorphism $E/F \simeq \left(F^{\perp}\right)^{\vee}$ yields $\deg F - \deg E = \deg F^{\perp}$ and $\deg F \leq \deg E + P_{E^{\vee}}(n-m)$. Then we deduce $P_E(m) \leq \deg E + P_{E^{\vee}}(n-m)$ since the right hand side is a concave function of m. For the reverse inequality, replace E by $E^{\vee}$, m by $n-m$ and use $\left(E^{\vee}\right)^{\vee} \simeq E$ (Theorem 4).

(3) For every subspace $F \subset E$ with dimension m, we have $\deg F = \deg F \otimes_K K'$ and so $\deg F \leq P_{E_{K'}}(m)$ and then $P_E(m) \leq P_{E_{K'}}(m)$. It implies that in order to prove the reverse inequality, we may assume that K'/K is Galois. Let $\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_N = E \otimes_K K'$ be the HN-filtration of $E_{K'}$ and $d_i = \dim F_i$. Let $e_1, \ldots, e_n$ be a K-basis of E. For every $\tau \in \mathrm{Gal}(K'/K)$, the correspondence $\iota_\tau \colon E \otimes_K K' \to E \otimes_K K'$ which sends $\sum_{i=1}^n x_i e_i$ ($x_i \in K'$) to $\sum_{i=1}^n \tau(x_i) e_i$ is a bijection that preserves dimension and degree of subspaces of $E_{K'}$. Using Theorem 16 and Lemma 17, we deduce $\iota_\tau(F_i) = F_i$ for all i. Now, let us fix $i \in \{1, \ldots, N\}$. Even if it means permuting the vectors $e_1, \ldots, e_n$, we can find a K'-basis $f_1, \ldots, f_{d_i}$ of F_i and scalars $\alpha_{j,h} \in K'$ for all $1 \leq j \leq d_i$ and $d_i + 1 \leq h \leq n$ such that

$$(\star) \qquad \forall\, j \in \{1, \ldots, d_i\}, \qquad f_j = e_j + \sum_{h=d_i+1}^{n} \alpha_{j,h} e_h$$

(Gaussian elimination). The Galois closure K'_0 of the field generated by K and all algebraic numbers $\alpha_{j,h}$'s is both a subfield of K' and a finite extension of K. So we can consider its normalized trace function $\mathrm{Tr}\colon K'_0 \to K$ ($\mathrm{Tr}(1) = 1$). Since $\iota_\tau(F_i) = F_i$ for all $\tau \in \mathrm{Gal}(K'_0/K)$, the vector

$$\mathrm{Tr}\, f_j := \frac{1}{[K'_0 : K]} \sum_{\tau \in \mathrm{Gal}(K'_0/K)} \iota_\tau(f_j)$$

belongs to F_i for all $j \in \{1, \ldots, d_i\}$. With $(\star)$, this element can also be written $e_j + \sum_{h=d_i+1}^n \mathrm{Tr}(\alpha_{j,h}) e_h$ which implies, in particular, $\mathrm{Tr}\, f_j \in E$ and the family $\{\mathrm{Tr}\, f_1, \ldots, \mathrm{Tr}\, f_{d_i}\}$ is free. Thus the linear subspace $G_i := \mathrm{Vect}_K\left(\mathrm{Tr}\, f_1, \ldots, \mathrm{Tr}\, f_{d_i}\right)$ of F_i has the same dimension d_i as F_i and so $F_i = G_i \otimes_K K'$. We deduce $P_{E_{K'}}(d_i) = \deg F_i = \deg G_i \leq P_E(d_i)$, hence $P_{E_{K'}}(d_i) = P_E(d_i)$. Since $P_{E_{K'}}$ is linear on $[d_i, d_{i+1}]$ and $P_E \leq P_{E_{K'}}$ are both concave functions on this interval, we get the equality $P_E = P_{E_{K'}}$ on $[d_i, d_{i+1}]$ and then on $[0, n]$ by varying i. □

In general P_E is difficult to compute, starting with its first value $P_E(1) = \mu_{\max}(E)$. To conclude this paragraph, let us outline an application of the above results to direct sums of rigid adelic spaces, obtaining a counterpart to the formula for $\Lambda_i(E_1 \oplus \cdots \oplus$

E_N) given by Proposition 11. In the following statement, the term $\mu_a(E)$ is $+\infty$ if $a < 1$ and $-\infty$ if $a > \dim E$.

Theorem 20 *Let E, F be some rigid adelic spaces over K and $(E_\ell)_{\ell\in\{0,\dots,N\}}$ and $(F_h)_{h\in\{0,\dots,M\}}$ their respective HN-filtrations. Then the HN-filtration of $E \oplus F$ is formed with some subspaces of the shape $E_\ell \oplus F_h$, beginning with $\{0\}$. To go from one notch to the next, the rule is the following:*

$$E_\ell \oplus F_h \longrightarrow \begin{cases} E_{\ell+1} \oplus F_h & \text{if} \quad \mu\left(E_{\ell+1}/E_\ell\right) > \mu\left(F_{h+1}/F_h\right) \\ E_{\ell+1} \oplus F_{h+1} & \text{if} \quad \mu\left(E_{\ell+1}/E_\ell\right) = \mu\left(F_{h+1}/F_h\right) \\ E_\ell \oplus F_{h+1} & \text{if} \quad \mu\left(E_{\ell+1}/E_\ell\right) < \mu\left(F_{h+1}/F_h\right). \end{cases}$$

In particular we have $\mu_i(E \oplus F) = \max\limits_{\substack{a,b\in\mathbb{N}\\ a+b=i}} \min\{\mu_a(E), \mu_b(F)\}$ *for all* $i \in \mathbb{N}$.

A straightforward induction yields a formula for the i-th slope of a general direct sum: Let N be a positive integer and $E_1, \dots, E_N$ be some rigid adelic spaces over K. Then, for all i, we have

$$\mu_i(E_1 \oplus \cdots \oplus E_N) = \max_{\substack{a_1,\dots,a_N\in\mathbb{N}\\ a_1+\cdots+a_N=i}} \min\left\{\mu_{a_1}(E_1), \dots, \mu_{a_N}(E_N)\right\}.$$

The key statement for proving Theorem 20 is

Lemma 21 *Let A, B be rigid adelic spaces over K. Then we have*

$$(A \oplus B)_{\text{des}} = \begin{cases} A_{\text{des}} & \text{if} \ \ \mu_{\max}(A) > \mu_{\max}(B) \\ A_{\text{des}} \oplus B_{\text{des}} & \text{if} \ \ \mu_{\max}(A) = \mu_{\max}(B) \\ B_{\text{des}} & \text{if} \ \ \mu_{\max}(A) < \mu_{\max}(B). \end{cases}$$

From this lemma we deduce at once the well-known formula

$$\mu_{\max}(A \oplus B) = \max\{\mu_{\max}(A), \mu_{\max}(B)\}.$$

Moreover, it can be checked that the map $A^\vee \oplus B^\vee \to (A \oplus B)^\vee$, $(\varphi, \psi) \mapsto ((a, b) \mapsto \varphi(a) + \psi(b))$ is an (isometric) isomorphism of rigid adelic spaces. In particular their maximal slopes are equal. We then deduce the equality

$$\mu_{\min}(A \oplus B) = \min\{\mu_{\min}(A), \mu_{\min}(B)\}$$

with Proposition 19 and Lemma 21.

Proof of Lemma 21 First note that since A and B are linear subspaces of $A \oplus B$, their maximal slopes are at most the maximal slope of $A \oplus B$ (Lemma 14). Now let us consider $C = (A \oplus B)_{\text{des}}$, $C_A = C \cap A$ and $C^B = \text{Im}(C \to B)$ where $C \to B$ is the restriction to C of the second projection $A \oplus B \to B$. The linear space C^B is isometrically isomorphic to $(A + C)/A$. Then, if C_A and C^B are not reduced to $\{0\}$, Proposition 6 implies

$$\mu(C) \leq \frac{n_A \mu(C_A) + n^B \mu(C^B)}{n_A + n^B}$$

where $n_A = \dim C_A$ and $n^B = \dim C^B$. Since $C_A \subset A$ and $C^B \subset B$, we deduce $\mu_{\max}(A \oplus B) = \mu(C) \leq \max\{\mu_{\max}(A), \mu_{\max}(B)\}$. In view of the lower bound for $\mu(C)$ mentioned at the beginning, it is necessarily an equality and so $\mu(C) = \mu(C_A) = \mu(C^B) = \mu_{\max}(A) = \mu_{\max}(B)$. From Lemma 14 we deduce $C^B \subset B_{\text{des}}$ and the same reasoning with C^A (we still have C^A and C_B non-zero) proves that $C^A \subset A_{\text{des}}$. Finally we get $C \subset A_{\text{des}} \oplus B_{\text{des}}$ and, since this latter space has slope equal to $\mu_{\max}(A \oplus B)$, we have $C = A_{\text{des}} \oplus B_{\text{des}}$. If $C_A = \{0\}$, then $\mu(C) \leq \mu(C^B) \leq \mu_{\max}(B)$; so there is equality and $B_{\text{des}} \subset C \xrightarrow{\sim} C^B \subset B_{\text{des}}$. This proves that $C = B_{\text{des}}$. Besides we have $\mu_{\max}(B) > \mu_{\max}(A)$ since otherwise there would be equality and we should have $A_{\text{des}} \subset C$, then $A_{\text{des}} \subset C_A$, contradicting $C_A = \{0\}$. When $C_A \neq \{0\}$ but $C^B = \{0\}$ we have $C = C \cap A$ and $\mu(C) = \mu_{\max}(A)$, so $A_{\text{des}} = C$. Here again we have $\mu_{\max}(A) > \mu_{\max}(B)$ since otherwise we should have $B_{\text{des}} \subset C$, contradicting $C^B = \{0\}$. □

To prove Theorem 20 we shall use the fact that if E' is a non-zero linear subspace of a rigid adelic space E, then $\mu_i(E') \leq \mu_i(E)$ for all i (consequence of Proposition 18).

Proof of Theorem 20 We build the HN-filtration of $E \oplus F$ step by step. Let us suppose that we got the notch $E_\ell \oplus F_h$ where the integers ℓ, h may be zero. According to the construction of the HN-filtration (p. 52), the next step in the filtration of $E \oplus F$ is its subspace G, (strictly) containing $E_\ell \oplus F_h$ such that

$$G/E_\ell \oplus F_h = (E \oplus F/E_\ell \oplus F_h)_{\text{des}} = (E/E_\ell \oplus F/F_h)_{\text{des}}\,.$$

Then we apply the previous lemma to $A = E/E_\ell$ and $B = F/F_h$. It gives the first part of Theorem 20 by observing that $A_{\text{des}} = E_{\ell+1}/E_\ell$ and $B_{\text{des}} = F_{h+1}/F_h$. Let us now establish the formula for the i-th slope of $E \oplus F$. Define $\delta(i) = \max_{a+b=i} \min\{\mu_a(E), \mu_b(F)\}$ and fix $a \in \{1, \ldots, \dim E\}$ and $b \in \{1, \ldots, \dim F\}$ such that $a + b = i$ for some $i \in \{1, \ldots, \dim E + \dim F\}$. There exists $\ell \in \{0, \ldots, N-1\}$ (resp. $h \in \{0, \ldots, M-1\}$) such that $\mu_a(E) = \mu(E_{\ell+1}/E_\ell)$

(resp. $\mu_b(F) = \mu(F_{h+1}/F_h)$). Besides we have $a \in \{\dim E_\ell + 1, \dots, \dim E_{\ell+1}\}$ and $b \in \{\dim F_h + 1, \dots, \dim F_{h+1}\}$. Then we have

$$\begin{aligned}\mu_i(E \oplus F) &\geq \mu_{\dim E_{\ell+1} + \dim F_{h+1}}(E \oplus F) \\ &\geq \mu_{\dim E_{\ell+1} + \dim F_{h+1}}(E_{\ell+1} \oplus F_{h+1}) = \mu_{\min}(E_{\ell+1} \oplus F_{h+1}) \\ &= \min\{\mu_{\min}(E_{\ell+1}), \mu_{\min}(F_{h+1})\} = \min\{\mu_a(E), \mu_b(F)\}.\end{aligned}$$

If a or b equals i, this inequality remains true since $\mu_i(E \oplus F) \geq \max\{\mu_i(E), \mu_i(F)\}$. Thus, for all i, we have $\mu_i(E \oplus F) \geq \delta(i)$. Observe now that $i \mapsto \delta(i)$ is a nonincreasing function since, for $a + b = i + 1$, we have

$$\min\{\mu_a(E), \mu_b(F)\} \leq \begin{cases} \min\{\mu_{a-1}(E), \mu_b(F)\} & \text{if } a \geq 1, \\ \min\{\mu_a(E), \mu_{b-1}(F)\} & \text{if } a = 0. \end{cases}$$

Starting from $E_\ell \oplus F_h$, let us call G the next notch in the HN-filtration of $E \oplus F$. We now prove that $\mu_i(E \oplus F) = \delta(i)$ for $i \in \{\dim E_\ell + \dim F_h + 1, \dots, \dim G\}$. The crucial observation is that either $F_h = \{0\}$ or $F_h \neq \{0\}$ and the appearance of F_h at the notch $E_\ell \oplus F_h$ was caused by the fact that $\ell = 0$ and $\mu(F_h/F_{h-1}) > \mu_{\max}(E)$ or that $\ell \geq 1$ and the slope $\mu(F_h/F_{h-1})$ is greater than $\mu(E_m/E_{m-1}) = \mu_{\dim E_m}(E)$ for some integer $1 \leq m \leq \ell$. In every case, we have $\mu_{\dim F_h}(F) > \mu_{\dim_{E_{\ell+1}}}(E)$. The same reasoning with E_ℓ gives $\mu_{\dim_{E_\ell}}(E) > \mu_{\dim F_{h+1}}(F)$. Now, if $\mu(E_{\ell+1}/E_\ell) > \mu(F_{h+1}/F_h)$, then $G = E_{\ell+1} \oplus F_h$ and

$$\begin{aligned}\delta(i) \geq \delta(\dim G) \geq \min\left\{\mu_{\dim_{E_{\ell+1}}}(E), \mu_{\dim F_h}(F)\right\} &= \mu_{\dim_{E_{\ell+1}}}(E) \\ &= \mu_i(E \oplus F).\end{aligned}$$

The two other possibilities for G are treated in the same way, which allows to conclude. □

4 Comparisons Between Minima and Slopes

4.1 Lower Bounds

The following inequality is as simple as fundamental. It is an extension of the fact that $n \in \mathbb{Z}$ *and* $n \neq 0$ *implies* $1 \leq |n|$ and can be seen as a variant of the Liouville inequality in transcendence theory. Let E be a rigid adelic space over K with dimension n.

Proposition 22 *We have* $1 \leq \Lambda_1(E) \exp \mu_1(E)$.

Proof Observe that for every $x \in E\backslash\{0\}$, we have $-\log H_E(x) = \deg K.x \leq P_E(1) = \mu_1(E)$. We conclude using the definition of $\Lambda_1(E)$ as the infimum of $H_E(x)$ over $x \in E\backslash\{0\}$. □

Corollary 23 *For all $i \in \{1, \ldots, n\}$, we have $1 \leq \Lambda^{(i)}(E) \exp \mu_i(E)$.*

In particular $1 \leq \lambda_i^*(E) \exp \mu_i(E)$ for all $* \in \{\mathrm{BC}, \Lambda, Z\}$ (since $\Lambda^{(i)}(E) \leq \Lambda_i(E) \leq Z_i(E)$).

Proof Let $i \in \{1, \ldots, n\}$ and $E_0 = \{0\} \subsetneq E_1 \subsetneq \cdots \subsetneq E_N = E$ be the HN-filtration of E. Consider the index h such that $\dim E_h \leq i-1 < \dim E_{h+1}$, so that the maximal slope of E/E_h is equal to $\mu_i(E)$ (Theorem 16). We apply the previous proposition to E/E_h and we conclude bounding from above $\Lambda_1(E/E_h)$ by $\Lambda^{(i)}(E)$. □

Corollary 24 *We have $H(E) \leq \prod_{i=1}^n \Lambda^{(i)}(E)$.*

Proof We multiply the inequalities of the previous corollary and we use $\sum_{i=1}^n \mu_i(E) = \deg E = -\log H(E)$. □

Often, the weaker *Hadamard inequality* $H(E) \leq \Lambda_1(E) \cdots \Lambda_n(E)$ is used.

4.2 Upper Bounds

Let us recall that $\lambda_i^*(E) = \Lambda^{(i)}(E)$, $\Lambda_i(E)$ or $Z_i(E)$ according to $* = \mathrm{BC}$, Λ or Z. Given a positive integer n, let us define several constants:

- $c_{\mathrm{I}}(n, K) = \sup\limits_{\dim E = n} \Lambda_1(E) H(E)^{-1/n} = \sup\limits_{\dim E = n} \Lambda_1(E) \exp \mu(E)$
- $c_{\mathrm{II}}^*(n, K) = \sup\limits_{\dim E = n} \left(\dfrac{\lambda_1^*(E) \cdots \lambda_n^*(E)}{H(E)} \right)^{1/n}$
- $\forall\, i \in \{1, \ldots, n\}, \quad c_i^*(n, K) = \sup\limits_{\dim E = n} \lambda_i^*(E) \exp \mu_i(E)$.

Here the suprema are taken upon all the rigid adelic spaces over K with dimension n. As in [14, § 4.8], we can prove that it is enough to consider hyperplanes of the standard space K^{n+1} (instead of E) to obtain the same numbers. Note that these constants can be infinite (see below).

Some Simple Observations Here n is a positive integer.

1. The number $c_{\mathrm{I}}(n, \mathbb{Q}) = c_{\mathrm{II}}^{\Lambda}(n, \mathbb{Q})$ is the square root of the Hermite constant γ_n mentioned at the beginning of the text.
2. We have $1 \leq c_{\mathrm{I}}(n, K) \leq c_{\mathrm{II}}^{\mathrm{BC}}(n, K) \leq c_{\mathrm{II}}^{\Lambda}(n, K) \leq c_{\mathrm{II}}^{Z}(n, K)$.
3. We have $c_{\mathrm{II}}^*(n, K)^n \leq \prod_{i=1}^n c_i^*(n, K)$.
4. For all $i \in \{1, \ldots, n\}$, we have $1 \leq c_i^*(n, K) \leq c_{\mathrm{II}}^*(n, K)^n$.
5. The function $n \mapsto c_{\mathrm{I}}(n, K)^n$ is nondecreasing.

To prove this last property, we can take $E \oplus L$ with $\dim L = 1$ and $H(L) = \Lambda_1(E)$. To my knowledge it is not known whether $n \mapsto c_{\mathrm{I}}(n, K)$ is nondecreasing, even for $K = \mathbb{Q}$. Nevertheless we shall see it is true when $K = \overline{\mathbb{Q}}$. In the opposite direction, we have the

Theorem (Mordell Inequality) *For every integer $n \geq 2$, we have*

$$c_{\mathrm{I}}(n+1, K) \leq c_{\mathrm{I}}(n, K)^{n/(n-1)}.$$

Proof Let E be a rigid adelic space of dimension $n+1$. Let ε be a positive real number and $x \in E\backslash\{0\}$ such that $H_E(x) \leq \Lambda_1(E)+\varepsilon$. The hyperplane $F = \{x\}^{\perp} \subset E^{\vee}$ satisfies $\Lambda_1(E^{\vee}) \leq \Lambda_1(F) \leq c_{\mathrm{I}}(n, K) H(F)^{1/n}$. Since $F \simeq (E/K.x)^{\vee}$, we have $H(F) = H_E(x)/H(E) \leq \Lambda_1(E)/H(E) + \varepsilon/H(E)$. Replacing this bound in the previous inequality and letting $\varepsilon \to 0$ leads to

$$\Lambda_1\left(E^{\vee}\right) \leq c_{\mathrm{I}}(n, K) \left(\frac{\Lambda_1(E)}{H(E)}\right)^{1/n}.$$

Applying this estimate to $E^{\vee}$ instead of E and combining both inequalities we obtain $\Lambda_1(E) \leq c_{\mathrm{I}}(n, K)^{n/(n-1)} H(E)^{1/(n+1)}$. □

With a bit more pain, one can also prove that $c_{\mathrm{II}}^{Z}(n, K) \leq c_{\mathrm{II}}^{Z}(2, K)^{2^n}$ (see [14, Proposition 4.14]). Let us also mention the analogue of Minkowski's theorem: *For every positive integer n, we have $c_{\mathrm{I}}(n, K) = c_{\mathrm{II}}^{\Lambda}(n, K)$ (in particular $c_{\mathrm{I}}(n, K) = c_{\mathrm{II}}^{\mathrm{BC}}(n, K)$).* The proof is based on a deformation metric argument. To a rigid adelic space E over K, we associate another rigid adelic space E' such that $\Lambda_1(E') \geq 1$ and $H(E') = H(E)\,(\Lambda_1(E) \cdots \Lambda_n(E))^{-1}$ (see [14, Theorem 4.12]).

Definition 25 *An algebraic extension $K/\mathbb{Q}$ is called a* Siegel field *if $c_{\mathrm{II}}^{\Lambda}(n, K) < \infty$ for all $n \geq 1$.*

With the previous observations, K is a Siegel field if and only if $c_{\mathrm{I}}(2, K) < \infty$. In a more elementary approach, K is a Siegel field if and only if it has the following property: There exists a positive real number α such that, for every $(a, b, c) \in K^3\backslash\{0\}$ there exists $(x, y, z) \in K^3\backslash\{0\}$ such that $ax + by + cz = 0$ and $H_{K^3}(x, y, z) \leq \alpha\, H_{K^3}(a, b, c)^{1/2}$.

Examples of Siegel Fields

(i) $\mathbb{Q}$, number fields (Minkowski),
(ii) $\overline{\mathbb{Q}}$ (Zhang [25] and Roy & Thunder [23]),
(iii) Hilbert class field towers of number fields [14, § 5.5].

Note that a finite extension of a Siegel field is still a Siegel field.

The following result comes from [14, Theorem 1.1 and Corollary 1.2].

Theorem 26

(1) $\forall n \geq 2$, $c_{\mathrm{II}}^{Z}(n, K) < \infty$ *if and only if K is a Siegel field of infinite degree.*

(2) *A Northcott field is a Siegel field if and only if it is a number field.*

The second claim is a direct consequence of the first one: If K is both a Northcott and a Siegel field, then $Z_i(E) = \infty$ for all $i \in \{2, \ldots, \dim E\}$ and so $c_{\mathbb{II}}^{Z}(n, K) = \infty$ as soon as $n \geq 2$ and by (1), K is a number field. Besides the implication $\Rightarrow$ in (1) is easy enough: use $c_{\mathbb{II}}^{\Lambda}(n, K) \leq c_{\mathbb{II}}^{Z}(n, K)$ and $Z_i(E) = \infty$ for $2 \leq i \leq \dim E$ when K is a number field. So the striking part of Theorem 26 is that it suffices to be a Siegel field of infinite degree to have $c_{\mathbb{II}}^{Z}(2, K) < \infty$. The proof rests on a deformation metric argument at some ultrametric place, much more subtle than for Minkowski theorem (see [14, § 4.6]). To be a little more precise, let us define the impurity index $u(K)$ of an algebraic extension $K/\mathbb{Q}$. If $v \in V(K)\backslash V_\infty(K)$ we denote by $\sigma(v)$ the measure of the singleton $\{v\}$, by p_v the prime number associated to v, by e_v the ramification index at v and by f_v its residual degree. The *impurity index* of K is

$$u(K) := \sup_{N \geq 1} \inf \left\{ p_v^{\sigma(v)/e_v}\,;\ v \in V(K)\backslash V_\infty(K),\ p_v^{f_v} \geq N \right\}$$

with the conventions: $p_v^{\sigma(v)/e_v} = 1$ if $e_v = +\infty$ (or $\sigma(v) = 0$) and $p_v^{f_v} \geq N$ is true if $f_v = +\infty$. We can check that $u(K) < +\infty$ if and only if $[K : \mathbb{Q}] = \infty$. Note also that $u(\overline{\mathbb{Q}}) = 1$ and that, for every real number B, there exists an algebraic extension K such that $B < u(K) < +\infty$.

Proposition 27 *Let E be a rigid adelic space over K with dimension n. For each $i \in \{1, \ldots, n\}$, let α_i be a real number such that $0 < \alpha_i < Z_i(E)$. Then there exists a rigid adelic space E' over K with dimension n such that*

$$\frac{\alpha_1 \cdots \alpha_n}{H(E)} \leq \frac{\big(u(K)\Lambda_1(E')\big)^n}{H(E')}.$$

This proposition leads to the bound $c_{\mathbb{II}}^{Z}(n, K) \leq u(K) c_{\mathrm{I}}(n, K)$ for all $n \geq 1$, thus yielding the first equivalence in Theorem 26.

In short, every constant $c_{\mathrm{I}}(n, K)$, $c_{\mathbb{II}}^{*}(n, K)$, $c_i^{*}(n, K)$ is finite when K is a Siegel field of infinite degree and, if $* \neq Z$, this remains true when K is a number field. In everyday life it is useful to have some concrete bounds for $c_{\mathbb{II}}^{*}(n, K)$. In general it seems to be a difficult problem. Let us mention two cases (see [14, § 5.1 and § 5.2]):

(1) If K is number field with root discriminant $\delta_{K/\mathbb{Q}} = |\Delta_{K/\mathbb{Q}}|^{1/[K:\mathbb{Q}]}$, then

$$c_{\mathrm{I}}(n, K) = c_{\mathbb{II}}^{\Lambda}(n, K) \leq \big(n\delta_{K/\mathbb{Q}}\big)^{1/2}.$$

(2) If $K = \overline{\mathbb{Q}}$, then

$$c_{\mathrm{I}}(n, \overline{\mathbb{Q}}) = c_{\mathbb{II}}^{\mathrm{BC}}(n, \overline{\mathbb{Q}}) = c_{\mathbb{II}}^{\Lambda}(n, \overline{\mathbb{Q}}) = c_{\mathbb{II}}^{Z}(n, \overline{\mathbb{Q}}) = \exp\left(\frac{H_n - 1}{2}\right)$$

where $H_n = \sum_{i=1}^{n} 1/i$.

The numbers $c_{\mathrm{I}}(n, \overline{\mathbb{Q}})$ are the only Hermite constants computed for every positive integer n, a situation which contrasts with the classical case $K = \mathbb{Q}$.

Let us now discuss in more details the constants $c_i^*(n, K)$.

Proposition 28 *We have* $c_1^*(n, K) = \max_{1 \le i \le n} c_{\mathrm{I}}(i, K)$ *for every integer* $n \ge 1$.

Proof The constant $c_1^*(n, K)$ does not depend on $*$ since $\lambda_1^*(E) = \Lambda_1(E)$ and so $c_1^*(n, K) = \sup_{\dim E = n} \Lambda_1(E) \exp \mu_1(E)$. Let us consider a rigid adelic space E over K with dimension n and E_{des} its destabilizing space. If $d = \dim E_{\mathrm{des}}$ we have $\Lambda_1(E) \le \Lambda_1(E_{\mathrm{des}})$ and $\mu_1(E) = \mu(E_{\mathrm{des}})$. So we get

$$\Lambda_1(E) \exp \mu_1(E) \le \Lambda_1(E_{\mathrm{des}}) \exp \mu(E_{\mathrm{des}}) \le c_{\mathrm{I}}(d, K) \le \max_{1 \le i \le n} c_{\mathrm{I}}(i, K).$$

Conversely, let F be a rigid adelic space of dimension $i \in \{1, \ldots, n\}$ and $G = L^{\oplus(n-i)}$ where L is a rigid adelic line with $\Lambda_1(L) = \Lambda_1(F)$. Then we have $\dim F \oplus G = n$, $\Lambda_1(F \oplus G) = \min\{\Lambda_1(F), \Lambda_1(G)\} = \Lambda_1(F)$ (Proposition 11) and $\mu(F) \le \mu_1(F \oplus G)$ (since F is a subspace of $F \oplus G$). We get

$$\Lambda_1(F) \exp \mu(F) \le \Lambda_1(F \oplus G) \exp \mu_1(F \oplus G) \le c_1^*(n, K)$$

and so $c_{\mathrm{I}}(i, K) \le c_1^*(n, K)$. □

Proposition 29 *For every positive integer* n *and* $* \in \{\mathrm{BC}, \Lambda, Z\}$*, we have* $c_1^*(n, K) \le c_2^*(n, K) \le \cdots \le c_n^*(n, K)$.

The proof rests on two auxiliary results.

Lemma 30 *We have* $c_i^*(n, K) \le c_i^*(n+1, K)$ *for every integer* $i \in \{1, \ldots, n\}$.

Proof Let E be a rigid adelic space of dimension n. For any rigid adelic line L, we have $\min\{\lambda_i^*(E), \Lambda_1(L)\} \le \lambda_i^*(E \oplus L)$. Indeed, for $* = \mathrm{BC}$ and $F \subset E$ of dimension $\le i - 1$,

$$\min\{\Lambda_1(E/F), \Lambda_1(L)\} = \Lambda_1\left((E/F) \oplus L\right) = \Lambda_1\left((E \oplus L) / (F \oplus \{0\})\right)$$

is bounded above by $\lambda_i^{\mathrm{BC}}(E \oplus L)$ and we make $\Lambda_1(E/F) \to \lambda_i^{\mathrm{BC}}(E)$. For $* = \Lambda$ or Z we use Proposition 10: Consider a subset $S \subset E \oplus L$ such that $\mathrm{Vect}(S)$ or $\mathrm{Zar}(S)$ has dimension $\ge i$. Either $S \subset E$ and we have $\sup_{x \in S} H_{E \oplus L}(x) = \sup_{x \in S} H_E(x) \ge \lambda_i^*(E)$. Or there exists $a \in E$ and $\ell \in L \backslash \{0\}$ such that $(a, \ell) \in S$ and then $\sup_{x \in S} H_{E \oplus L}(x) \ge H_{E \oplus L}(a, \ell) \ge H_L(\ell) = \Lambda_1(L)$. In any case, we get the wanted inequality for $\lambda_i^*(E \oplus L)$. Next, choosing L such that $\Lambda_1(L) = \lambda_i^*(E)$, we get $\lambda_i^*(E) \le \lambda_i^*(E \oplus L)$. Moreover we have $\mu_i(E) \le \mu_i(E \oplus L)$ (see the comment before the proof of Theorem 20, on page 57). So

$$\lambda_i^*(E) \exp \mu_i(E) \le \lambda_i^*(E \oplus L) \exp \mu_i(E \oplus L) \le c_i^*(n+1, K)$$

and Lemma 30 follows. □

Lemma 31 *The function $n \mapsto c_n^*(n, K)$ is nondecreasing.*

Proof Let L be a rigid adelic line over K such that $\Lambda_1(L) = \exp(-\mu_{\min}(E))$. We have $\lambda_{n+1}^*(E \oplus L) \geq \lambda_n^*(E)$ and, by Theorem 20, we have $\mu_{\min}(E \oplus L) = \min\{\mu_{\min}(E), \mu_{\min}(L)\} = \mu_{\min}(E)$ since $\mu_{\min}(L) = \deg L = \mu_{\min}(E)$. We get

$$\lambda_n^*(E) \exp \mu_{\min}(E) \leq \lambda_{n+1}^*(E \oplus L) \exp \mu_{\min}(E \oplus L) \leq c_{n+1}^*(n+1, K).$$

□

Proof of Proposition 29 We proceed by induction on n assuming $c_i^*(j, K) \leq c_{i+1}^*(j, K)$ is true for all integers $j \leq n-1$ and $i \leq j-1$. Let E be a rigid adelic space of dimension n and an integer $0 \leq i \leq n-1$. If $\mu_i(E) = \mu_{i+1}(E)$, then $\lambda_i^*(E) \exp \mu_i(E) \leq \lambda_{i+1}^*(E) \exp \mu_{i+1}(E) \leq c_{i+1}^*(n, K)$. Otherwise the HN-filtration $\{0\} \subsetneq E_1 \subsetneq \cdots \subsetneq E_N = E$ of E is not trivial (that is, $N \geq 2$).

- If $i + 1 \leq \dim E_{N-1}$, then $\mu_i(E) = \mu_i(E_{N-1})$ (since the restriction of the canonical polygon of E to the interval $[0, \dim E_{N-1}]$ equals $P_{E_{N-1}}$). We deduce

$$\begin{aligned} \lambda_i^*(E) \exp \mu_i(E) &\leq \lambda_i^*(E_{N-1}) \exp \mu_i(E_{N-1}) \\ &\leq c_i^*(\dim E_{N-1}, K) \\ &\underset{\substack{\uparrow \\ \text{Induction hypothesis}}}{\leq} c_{i+1}^*(\dim E_{N-1}, K) \underset{\substack{\uparrow \\ \text{Lemma 30}}}{\leq} c_{i+1}^*(n, K). \end{aligned}$$

- If $i + 1 > \dim E_{N-1}$, then $\dim E_{N-1} = i$ since $\mu_j(E) = \mu_{\min}(E)$ for $j \geq \dim E_{N-1} + 1$ and $\mu_i(E) \neq \mu_{i+1}(E)$ (see Theorem 16). Thus we get $\mu_{i+1}(E) = \mu_{\min}(E)$ and $\mu_i(E_{N-1}) = \mu_{\min}(E_{N-1}) = \mu_i(E)$. We deduce

$$\begin{aligned} \lambda_i^*(E) \exp \mu_i(E) &\leq \lambda_i^*(E_{N-1}) \exp \mu_i(E_{N-1}) \\ &\leq c_i^*(i, K) \underset{\substack{\uparrow \\ \text{Lemma 31}}}{\leq} c_{i+1}^*(i+1, K) \underset{\substack{\uparrow \\ \text{Lemma 30}}}{\leq} c_{i+1}^*(n, K). \end{aligned}$$

In any case, we have $\lambda_i^*(E) \exp \mu_i(E) \leq c_{i+1}^*(n, K)$, which implies Proposition 29. □

Actually, for $* = \mathrm{BC}$, we have $c_1^{\mathrm{BC}}(n, K) = \cdots = c_n^{\mathrm{BC}}(n, K)$. Indeed, for every linear subspace $F \subsetneq E$, we have

$$\Lambda_1(E/F) \exp \mu_{\max}(E/F) \leq c_1(n, K) \ (= c_1^{\mathrm{BC}}(n, K)).$$

Using Proposition 18, we get $\Lambda_1(E/F) \exp \mu_{\min}(E) \leq c_1(n, K)$ and, taking the supremum over F allows to replace $\Lambda_1(E/F)$ by $\Lambda^{(n)}(E)$. In the end, we obtain $c_n^{\mathrm{BC}}(n, K) \leq c_1(n, K)$ and the equality follows from Proposition 29. In summary we have

$$
\begin{array}{rlllll}
c_{\mathrm{I}}(n,K) = & c_{\mathrm{II}}^{\mathrm{BC}}(n,K) & = c_{\mathrm{II}}^{\Lambda}(n,K) & & & \\
& \quad\rotatebox{-90}{$\leq$} & & & & \\
& c_1^{\mathrm{BC}}(n,K) & = c_2^{\mathrm{BC}}(n,K) & = \cdots = & c_n^{\mathrm{BC}}(n,K) & \\
& \quad\parallel & \quad\rotatebox{-90}{$\leq$} & & \quad\rotatebox{-90}{$\leq$} & \\
& c_1^{\Lambda}(n,K) & \leq c_2^{\Lambda}(n,K) & \leq \cdots \leq & c_n^{\Lambda}(n,K) & \\
& \quad\parallel & \quad\rotatebox{-90}{$\leq$} & & \quad\rotatebox{-90}{$\leq$} & \\
& c_1^{Z}(n,K) & \leq c_2^{Z}(n,K) & \leq \cdots \leq & c_n^{Z}(n,K) & \\
& \quad\parallel & & & & \\
& \max\limits_{1\leq i\leq n} c_{\mathrm{I}}(i,K) & & & &
\end{array}
$$

Other Relations Between These Constants

Proposition 32 *Let E be a rigid adelic space of dimension n over K. Then, for every $m \in \{1,\ldots,n\}$ and every $* \in \{\mathrm{BC}, \Lambda, Z\}$, we have*

(1) $\lambda_1^*(E)\cdots\lambda_m^*(E)\exp P_E(m) \leq c_{\mathrm{II}}^*(n,K)^n$,
(2) $\lambda_1^*(E)\cdots\lambda_m^*(E) \leq c_{\mathrm{II}}^*(n,K)^m H(E)^{m/n}$.

Question 33 Is the first bound true with $c_{\mathrm{II}}^*(n,K)^m$ on the right hand side?

Since $P_E(m) \geq m\mu(E)$, a positive answer to this question would improve both (1) and (2) of the proposition. A weak form of Question 33 might be:

Question 34 Is $c_i^*(n,K) \leq c_{\mathrm{II}}^*(n,K)^i$ for $1 \leq i \leq n$?

One can prove that these last two questions have affirmative answers if $n \mapsto c_{\mathrm{II}}^*(n,K)$ is a nondecreasing function.

Proof of Proposition 32 For the first inequality, we use the definition of $c_{\mathrm{II}}^*(n,K)$ and $\lambda_i^*(E)\exp\mu_i(E) \geq 1$ for every $i \in \{m+1,\ldots,n\}$ (Corollary 23). We get the result with $\deg E - \sum_{i=m+1}^{n}\mu_i(E) = P_E(m)$. As for the second inequality, we still use the definition of $c_{\mathrm{II}}^*(n,K)$ but, i being as above, we bound from below $\lambda_i^*(E)$ by $\big(\lambda_1^*(E)\cdots\lambda_m^*(E)\big)^{1/m}$. □

Regarding the second point of the proposition, one can prove that if, for $i \in \{1,\ldots,n\}$, we denote by

$$a_i^*(n,K) = \sup_{\dim E = n} \frac{\big(\lambda_1^*(E)\cdots\lambda_i^*(E)\big)^{1/i}}{H(E)^{1/n}}$$

(E varies among rigid adelic spaces over K of dimension n), then $c_{\mathrm{I}}(n,K) = a_1^*(n,K) \leq a_2^*(n,K) \leq \cdots \leq a_n^*(n,K) = c_{\mathrm{II}}^*(n,K)$. In particular, when $c_{\mathrm{I}}(n,K) = c_{\mathrm{II}}^*(n,K)$, all these constants are equal and $c_{\mathrm{II}}^*(n,K)$ is the best constant in Proposition 32-2. There exist other constants in the literature such as the *Rankin* *constant* associated to two integers $1 \leq m \leq n$ and to an algebraic extension K:

$$R(m,n,K) = \sup\left\{\frac{\sigma_m(E)}{H(E)^{m/n}}\,;\ \dim E = n\right\}$$

(recall $\sigma_m(E) = \inf\{H(F)\,;\ F \subset E,\ \dim F = m\}$, see page 50). We leave the following properties as an exercise, whose solution can be found in the book [17, § 2.8] by Martinet. Here $1 \leq i \leq m \leq n$ are integers and $R(m,n,K)$ is shortened in $R(m,n)$ since K is fixed.

(a) $R(1,n) = c_{\mathrm{I}}(n,K)$,
(b) $R(m,n) = R(n-m,n)$,
(c) $R(m,n) \leq c_{\mathrm{I}}(n,K)^m$,
(d) $R(i,n) \leq R(i,m)R(m,n)^{i/m}$ (Generalization of Mordell inequality),
(e) $a_i^*(n,K) \leq a_i^*(m,K)R(m,n)^{1/m}$.

4.3 Transference Theorems

Let E be a rigid adelic space of dimension n over K and let $i \in \{1,\dots,n\}$. A *transference theorem* gives an upper bound of $\lambda_i^*(E)\lambda_{n-i+1}^*(E^\vee)$ for $* \in \{\mathrm{BC}, \Lambda, Z\}$. To establish such a theorem, we are therefore naturally led to introduce the following quantity:

$$t_i^*(n,K) := \sup\left\{\lambda_i^*(E)\lambda_{n-i+1}^*(E^\vee)\,;\ \dim E = n\right\}$$

where E varies among rigid adelic spaces over K with dimension n. From the definition, we get $t_i^*(n,K) = t_{n-i+1}^*(n,K)$ and $t_i^{\mathrm{BC}}(n,K) \leq t_i^{\Lambda}(n,K) \leq t_i^{Z}(n,K)$. Also note that Proposition 19 implies that the product

$$\lambda_i^*(E)\lambda_{n-i+1}^*(E^\vee) = \left(\lambda_i^*(E)\exp\mu_i(E)\right)\left(\lambda_{n-i+1}^*(E^\vee)\exp\mu_{n-i+1}\left(E^\vee\right)\right)$$

is greater than 1 (Corollary 23) and at most $c_i^*(n,K)c_{n-i+1}^*(n,K)$. Moreover we have

$$\lambda_i^*(E)\exp\mu_i(E) \leq \lambda_i^*(E)\lambda_{n-i+1}^*(E^\vee) \quad \text{and so,} \quad c_i^*(n,K) \leq t_i^*(n,K).$$

That proves that $t_i^*(n,K) < \infty$ is a real number as soon as K is a Siegel field for $* \in \{\mathrm{BC}, \Lambda\}$ or a Siegel field of infinite degree for $* = Z$. We do not know if we can expect a polynomial bound in n for $t_i^*(n,K)$ when K is a Siegel field. In particular can we bound $t_i^Z(n,\overline{\mathbb{Q}})$ polynomially in n? In a very optimistic view, we would like to answer positively to the

Question 35 Is $t_i^*(n,K) \leq c_{\mathrm{II}}^*(n,K)^2$ true?

The square might be justified by several observations. Firstly, this inequality (and even the equality) is true for $n = 2$ and $* = \Lambda$ (Theorem 37 below). Then, for $K = \overline{\mathbb{Q}}$, we have $c_{\mathrm{II}}^{Z}(n, \overline{\mathbb{Q}})^2 = \exp(H_n - 1) \simeq n$ whereas $t_i^{Z}(n, \overline{\mathbb{Q}}) \geq Z_i(\overline{\mathbb{Q}}^n) Z_{n-i+1}(\overline{\mathbb{Q}}^n) = \sqrt{i(n-i+1)}$ is greater than $n/2$ for $i = \lfloor (n+1)/2 \rfloor$. That proves we cannot replace the square by a lower exponent in Question 35. Moreover, for $* = \mathrm{BC}$, since $c_i^{\mathrm{BC}}(n, K) = c_{n-i+1}^{\mathrm{BC}}(n, K) = c_1(n, K)$ (see page 63), we have $t_i^{\mathrm{BC}}(n, K) \leq c_1(n, K)^2$. However $c_{\mathrm{II}}^{\mathrm{BC}}(n, K) = c_{\mathrm{I}}(n, K) \leq c_1(n, K)$ and so we only got a weak version of Question 35. At last we have the following result valid for any number field and proved by Banaszczyk for $K = \mathbb{Q}$ in his article [2]. The upper bound for $t_i^{\Lambda}(n, K)$ given hereunder is not too far from $c_{\mathrm{II}}^{\Lambda}(n, K)^2 = c_{\mathrm{I}}(n, K)^2$ if we take into account the inequalities

$$\frac{n\delta_{K/\mathbb{Q}}^{(1-1/n)}}{25 \max\{1, \log \delta_{K/\mathbb{Q}}\}^{2/n}} \leq c_{\mathrm{II}}^{\Lambda}(n, K)^2 \leq n\delta_{K/\mathbb{Q}}$$

where the number $\delta_{K/\mathbb{Q}} = \left|\Delta_{K/\mathbb{Q}}\right|^{1/[K:\mathbb{Q}]}$ is the root discriminant of K (see [14, Proposition 5.2]).

Theorem 36 *When K is a number field, we have $t_i^{\Lambda}(n, K) \leq n\delta_{K/\mathbb{Q}}$.*

Let us outline the proof. The problem is to reduce to the case $K = \mathbb{Q}$ and to apply Banaszczyk's theorem. For this, we use the scalar restriction $\mathrm{Res}_{K/\mathbb{Q}} E$ of a rigid adelic space E over a number field K (for a more general finite extension L/K, see [14, Lemma 4.24]). It is the rigid adelic space over $\mathbb{Q}$ built from the space E viewed as a $\mathbb{Q}$-vector space (with dimension $[K : \mathbb{Q}] \dim E$) endowed with

$$\|x\|_{\mathrm{Res}_{K/\mathbb{Q}} E, \infty} = \left(\sum_{v \in V_\infty(K)} [K_v : \mathbb{Q}_v] \|x\|_{E,v}^2 \right)^{1/2}$$

at the archimedean place ∞ of $\mathbb{Q}$ and $\|x\|_{\mathrm{Res}_{K/\mathbb{Q}} E, p} = \max_{v|p} \|x\|_{E,v}^2$ at ultrametric places p of $\mathbb{Q}$. By [14, Lemma 4.29], the height of $\mathrm{Res}_{K/\mathbb{Q}} E$ is

$$H\left(\mathrm{Res}_{K/\mathbb{Q}} E\right) = H(E)^{[K:\mathbb{Q}]} \left|\Delta_{K/\mathbb{Q}}\right|^{(\dim E)/2}.$$

Associated to K we also have its differential (rigid) adelic space ω_K over K whose underlying space is $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{Q})$ viewed as a K-vector space with the scalar multiplication: $\lambda.\varphi(x) = \varphi(\lambda.x)$ for $\lambda, x \in K$ and $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{Q})$. The trace $\mathrm{Tr}_{K/\mathbb{Q}}$ is a basis of ω_K ($\dim \omega_K = 1$) and it allows to define rigid adelic metrics on ω_K by stating $\|\mathrm{Tr}_{K/\mathbb{Q}}\|_v = 1$ if $v \in V_\infty(K)$ and, otherwise, $\|\mathrm{Tr}_{K/\mathbb{Q}}\|_v$ is

$$\inf\left\{ |\lambda|_v \,;\ \lambda \in K_v \setminus \{0\} \text{ and } \lambda^{-1} \mathrm{Tr}_{K/\mathbb{Q}} \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{K_v}, \mathbb{Z}_p) \right\}.$$

It is known that $H(\omega_K) = \delta_{K/\mathbb{Q}}^{-1}$ (see [20, p. 219]). Besides, given a rigid adelic space E over K, the $\mathbb{Q}$-linear map $E^\vee \otimes_K \omega_K \to \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{Q})$, $\varphi \otimes \lambda \mapsto \lambda \circ \varphi$ induces an (isometric) isomorphism of rigid adelic spaces over $\mathbb{Q}$:

$$\mathrm{Res}_{K/\mathbb{Q}}\left(E^\vee \otimes_K \omega_K\right) \simeq \left(\mathrm{Res}_{K/\mathbb{Q}} E\right)^\vee$$

(see [9, Proposition 3.2.2]). With these reminders being done, we can now easily prove Theorem 36. Corollary 4.28 of [14] gives $\Lambda_i(E) \leq [K : \mathbb{Q}]^{-1/2}\Lambda_{(i-1)[K:\mathbb{Q}]+1}(\mathrm{Res}_{K/\mathbb{Q}} E)$. Applying this inequality with $E^\vee \otimes_K \omega_K$ and using $\Lambda_{n-i+1}(E^\vee \otimes_K \omega_K) = \Lambda_{n-i+1}(E^\vee)H(\omega_K)$ (since ω_K is a line), we deduce that $\Lambda_i(E)\Lambda_{n-i+1}(E^\vee)$ is bounded above by

$$\frac{\delta_{K/\mathbb{Q}}}{[K:\mathbb{Q}]} \times \Lambda_{i[K:\mathbb{Q}]}\left(\mathrm{Res}_{K/\mathbb{Q}} E\right) \Lambda_{(n-i)[K:\mathbb{Q}]+1}\left(\left(\mathrm{Res}_{K/\mathbb{Q}} E\right)^\vee\right)$$

and the last product of minima is at most $[K : \mathbb{Q}]n$ by [2, Theorem 2.1].

Pekker's Theorem In this paragraph we build on the work of Pekker [21] about $t_i^\Lambda(n, \overline{\mathbb{Q}})$ to generalize it to any Siegel field. It allows to give some general upper bounds for $t_i^\Lambda(n, K)$ exponential in n, as in the following result.

Theorem 37 *Let $1 \leq i \leq n$ be integers. Then, for any algebraic extension $K/\mathbb{Q}$, we have*

(1) $t_1^\Lambda(2, K) = t_2^\Lambda(2, K) = c_{\mathrm{I}}(2, K)^2$
(2) $t_i^\Lambda(n, K) \leq t_1^\Lambda(i, K)t_1^\Lambda(n-i+1, K)$
(3) *For $n \geq 2$, $t_1^\Lambda(n, K) \leq t_1^\Lambda(n-1, K)t_1^\Lambda(2, K)$*
(4) $t_i^\Lambda(n, K) \leq t_1^\Lambda(2, K)^{n-1}$

Question 38 Do we have similar results for $t_i^Z(n, K)$?

For the proof of Theorem 37 we need an auxiliary result.

Lemma 39 *Let E be a rigid adelic space of dimension n over K. Let ε be a positive real number. Then there exists a hyperplane $F \subset E$ such that*

$$H(F) \leq (1+\varepsilon)\sigma_{n-1}(E) \quad \text{and} \quad \Lambda_n(E)H(F) \leq (1+\varepsilon)H(E)t_1^\Lambda(n, K).$$

Proof Let $\varphi \in E^\vee \backslash \{0\}$ such that $H_{E^\vee}(\varphi) \leq (1+\varepsilon)\Lambda_1(E^\vee)$ and consider $F = \mathrm{Ker}\,\varphi$. The first bound comes from $H(F) = H_{E^\vee}(\varphi)H(E)$ (see Proposition 5) and $\sigma_{n-1}(E) = \Lambda_1(E^\vee)H(E)$ (see page 50). The other one uses in addition the definition of $t_1^\Lambda(n, K)$. □

Proof of Theorem 37

(1) When E is a rigid adelic space with dimension 2, we have $\Lambda_1(E) = \sigma_{2-1}(E) = \Lambda_1(E^\vee)H(E)$ so $\Lambda_1(E)\Lambda_2(E^\vee) = \Lambda_1(E^\vee)\Lambda_2(E^\vee)/H(E^\vee)$. We conclude with Minkowski theorem:

$$t_1^\Lambda(2, K) = \sup_{\dim E=2} \frac{\Lambda_1(E)\Lambda_2(E)}{H(E)} = c_{\mathrm{II}}^\Lambda(2, K)^2 = c_{\mathrm{I}}(2, K)^2.$$

(2) Let E be a rigid adelic space over K with dimension n and let G be a linear subspace of dimension $i - 1$ of $E^\vee$. We apply Lemma 39 to $G^\perp$ (viewed as a subspace of E): there exists a hyperplane $A \subset G^\perp$ such that $\Lambda_{n-i+1}(G^\perp)H(A) \le (1+\varepsilon)H(G^\perp)t_1^\Lambda(n-i+1, K)$. Let us apply again (in the same way) Lemma 39 to $A^\perp \subset E^\vee$: there exists a hyperplane $B \subset A^\perp$ such that $\Lambda_i(A^\perp)H(B) \le (1+\varepsilon)t_1^\Lambda(i, K)H(A^\perp)$. We have $\dim B = i - 1$ and $H(B) \ge \sigma_{i-1}(E^\vee)$. Moreover $\Lambda_{n-i+1}(G^\perp) \ge \Lambda_{n-i+1}(E)$ and $\Lambda_i(A^\perp) \ge \Lambda_i(E^\vee)$. Multiplying the above inequalities given by Lemma 39 we get

$$\Lambda_{n-i+1}(E)\Lambda_i(E^\vee) \le (1+\varepsilon)^2 t_1^\Lambda(n-i+1, K)t_1^\Lambda(i, K) \times \frac{H(A^\perp)H(G^\perp)}{H(B)H(A)}.$$

By Proposition 5 the latter quotient equals $H(G)/H(B)$ and so it is at most $H(G)/\sigma_{i-1}(E^\vee)$. We conclude by letting $H(G) \to \sigma_{i-1}(E^\vee)$ and $\varepsilon \to 0$.

(3) Let E be a rigid adelic space over K with dimension n. Let $\varphi, \psi \in E^\vee$ be linearly independent linear forms. Define $V = \operatorname{Ker}\varphi$ and $W = \operatorname{Ker}\psi$. By Lemma 39, there exist hyperplanes $V' \subset V$ and $W' \subset W$ such that $\Lambda_{n-1}(V)H(V') \le (1+\varepsilon)H(V)t_1^\Lambda(n-1, K)$ and $\Lambda_{n-1}(W)H(W') \le (1+\varepsilon)H(W)t_1^\Lambda(n-1, K)$. Moreover we have $H(V)/H_{E^\vee}(\varphi) = H(W)/H_{E^\vee}(\psi) = H(E)$. By hypothesis $V \neq W$ and so $V + W = E$ which gives $\Lambda_n(E) \le \max\{\Lambda_{n-1}(V), \Lambda_{n-1}(W)\}$. The latter maximum is attained for one of the two spaces V or W and we denote by G the corresponding hyperplane V' or W'. Hence we get

$$\Lambda_n(E)H(G) \le (1+\varepsilon)H(E)t_1^\Lambda(n-1, K)\max\{H_{E^\vee}(\varphi), H_{E^\vee}(\psi)\}.$$

Choosing φ and ψ such that their heights be at most $(1+\varepsilon)\Lambda_2(E^\vee)$ and using $H(G^\perp) = H(G)/H(E)$, we get $\Lambda_n(E)H(G^\perp) \le (1+\varepsilon)^2 t_1^\Lambda(n-1, K)\Lambda_2(E^\vee)$. Multiplying both sides by $\Lambda_1(E^\vee)$, we find that the quantity $\Lambda_n(E)\Lambda_1(E^\vee)$ is at most

$$(1+\varepsilon)^2 t_1^\Lambda(n-1, K) \times \frac{\Lambda_1(E^\vee)\Lambda_2(E^\vee)}{H(G^\perp)}$$
$$\le (1+\varepsilon)^2 t_1^\Lambda(n-1, K) \times \frac{\Lambda_1(G^\perp)\Lambda_2(G^\perp)}{H(G^\perp)}$$

and, lastly, smaller than $(1+\varepsilon)^2 t_1^{\Lambda}(n-1, K) c_{\mathrm{II}}^{\Lambda}(2, K)^2$. We conclude with the first statement of Theorem 37 and $\varepsilon \to 0$.

(4) As a direct consequence of (3), we get $t_1^{\Lambda}(n, K) \le t_1^{\Lambda}(2, K)^{n-1}$. Then the changeover to $t_i^{\Lambda}(n, K)$ arises from point (2) of Theorem 37. □

The equality $\Lambda_1(E) = \Lambda_1(E^{\vee}) H(E)$ when $\dim E = 2$ allows to prove

$$\sup\{\Lambda_1(E)\Lambda_1(E^{\vee})\,;\ \dim E = 2\} = c_{\mathrm{I}}(2, K)^2.$$

In general, the definition of the number $c_{\mathrm{I}}(n, K)$ provides the upper bound $\sup\{\Lambda_1(E)\Lambda_1(E^{\vee})\,;\ \dim E = n\} \le c_{\mathrm{I}}(n, K)^2$, but, when $K = \mathbb{Q}$ and $n = 3$, Bergé and Martinet proved that the equality is no longer true [4, Proposition 2.13 (iii)]. Notwithstanding this, when $K = \mathbb{Q}$ and for every $n \ge 1$, a result by Banaszczyk [3, Lemma 5], based on Siegel's mean value theorem, states that

$$\sup\{\Lambda_1(E)\Lambda_1(E^{\vee})\,;\ \dim E = n\} > \frac{n}{2\pi e}$$

(see also Sect. 4.2 in Chapter IV).

Question 40 Do we have a similar lower bound, true for any K and $n \ge 1$, with a function growing to infinity with n, or even linear in n?

5 Heights of Morphisms and Slope-Minima Inequalities

5.1. Until now we have considered only *rigid* adelic spaces. Nevertheless it may be useful (or even crucial) to work with $\mathrm{Hom}_K(E, F)$ endowed with the operator norms, which is not Hermitian in general.

Definition 41 *Let E and F be adelic spaces over K such that $E^{\vee} \otimes_{\varepsilon} F$ is integrable. The height of $\varphi \in \mathrm{Hom}_K(E, F)\setminus\{0\}$ is*

$$h(\varphi) = h(E, F; \varphi) = \int_{V(K)} \log \|\varphi\|_{E^{\vee}\otimes_{\varepsilon} F, v}\, \mathrm{d}\sigma(v).$$

We may also use $H(\varphi) = \exp h(\varphi)$. Here, as defined on page 41,

$$\|\varphi\|_{E^{\vee}\otimes_{\varepsilon} F, v} = \sup\left\{\frac{\|\varphi(x)\|_{F,v}}{\|x\|_{E,v}}\,;\ x \in (E \otimes_K K_v)\setminus\{0\}\right\}$$

is the operator norm. Note that if $E' \subset E$ is a linear subspace, then $h(E', F; \varphi_{|E'}) \le h(E, F; \varphi)$. When E and F are rigid adelic spaces over K, there is also the Hilbert-Schmidt height for φ built with $\|\varphi\|_{E^{\vee}\otimes F, v}$, which is greater than $h(\varphi)$. In this paragraph, our aim is to compare minima and slopes of two (rigid) adelic spaces connected by a linear map. In the following results, E and F are some rigid adelic spaces over K and $\varphi\colon E \to F$ a linear map.

Proposition 42 *If $\varphi\colon E \to F$ is an isomorphism, then*

(1) $\deg E = \deg F + h(\det E, \det F; \det\varphi)$,
(2) $\mu(E) \leq \mu(F) + h(\varphi)$.

Proof (1) By hypothesis $\det\varphi\colon \det E \to \det F$ is an isomorphism between rigid adelic lines and, for all $v \in V(K)$ and $x \in (\det E) \otimes_K K_v \backslash \{0\}$, we have

$$|\det\varphi|_v = \frac{\|\det\varphi(x)\|_{\det F, v}}{\|x\|_{\det E, v}}.$$

We take logarithms and we integrate over v to conclude. (2) The second statement is a direct consequence of the first one and of Hadamard's inequality $|\det\varphi|_v \leq \|\varphi\|^{\dim E}_{E^\vee \otimes_\varepsilon F, v}$. □

Theorem 43 *If $\varphi\colon E \to F$ is injective, then*

$$\mu_{\max}(E) \leq \mu_{\max}(F) + h(\varphi) \quad \text{and} \quad \Lambda_1(F) \leq \Lambda_1(E)H(\varphi).$$

More generally, if $\varphi \neq 0$, then for all $i \in \{1, \ldots, \operatorname{rk}\varphi\}$ and $* \in \{\mathrm{BC}, \Lambda, Z\}$ we have

$$\mu_{i+\dim \operatorname{Ker}\varphi}(E) \leq \mu_i(F) + h(\varphi) \quad \text{and} \quad \lambda_i^*(F) \leq \lambda^*_{i+\dim \operatorname{Ker}\varphi}(E)H(\varphi).$$

Proof First assume that φ is injective. Let $E_0 \subset E$ be a non-zero linear subspace and $F_0 = \varphi(E_0)$. Since φ is injective the induced map $\widetilde{\varphi}\colon E_0 \to F_0$ is an isomorphism and, by Proposition 42,

$$\mu(E_0) \leq \mu(F_0) + h(E_0, F_0; \widetilde{\varphi}) \leq \mu_{\max}(F) + h(\varphi).$$

Taking the supremum over E_0 on the left hand side leads to the first maximal slopes inequality. As for the inequality for the first minima, if $x \in E\backslash\{0\}$, then $\varphi(x) \in F\backslash\{0\}$ so $\Lambda_1(F) \leq H_F(\varphi(x)) \leq H_E(x)H(\varphi)$ and we take the infimum over x to replace $H_E(x)$ by $\Lambda_1(E)$. Now just assume $\varphi \neq 0$. Let $F_0 \subset F$ be a linear subspace with dimension $\leq i-1$ and $A \subset E$ a linear subspace such that $\dim A \geq i + \dim\ker\varphi$. We have $\dim\varphi^{-1}(F_0) \leq \dim \operatorname{Ker}\varphi + i - 1$ and $\dim\varphi(A) \geq i$. Moreover the induced map $\overline{\varphi}\colon A/A\cap\varphi^{-1}(F_0) \to (\varphi(A)+F_0)/F_0$ is an isomorphism. Using Proposition 42 and $h(\overline{\varphi}) \leq h(\varphi)$, we get $\mu(A/A\cap\varphi^{-1}(F_0)) \leq \mu((\varphi(A)+F_0)/F_0) + h(\varphi)$, from which we deduce

$$\begin{aligned}&\inf\{\mu(A/B)\,;\ B \subset A \text{ and } \dim B \leq i + \dim\ker\varphi - 1\}\\&\qquad \leq \sup\{\mu(F_1/F_0)\,;\ F_0 \subset F_1 \subset F \text{ and } \dim F_1 \geq i\} + h(\varphi).\end{aligned}$$

We conclude with Proposition 18, taking the supremum over A on the left hand side and the infimum over F_0 on the right. As for the analogous inequality for λ_i^*, we distinguish the three cases $* = \mathrm{BC}, \Lambda, Z$. For $* = \mathrm{BC}$, we proceed as above: $\Lambda_1(F/F_0) \leq \Lambda_1(E/E_0)H(\overline{\varphi})$ (where $E_0 = \varphi^{-1}(F_0)$ and $\overline{\varphi}\colon E/E_0 \to F/F_0$ is the map induced by φ). Since $\dim E_0 \leq \dim \operatorname{Ker}\varphi + i - 1$ we have $\Lambda_1(E/E_0) \leq \Lambda^{(i+\dim \operatorname{Ker}\varphi)}(E)$. So

$$\Lambda^{(i)}(F) = \sup_{\dim F_0 \leq i-1} \Lambda_1(F/F_0) \leq \Lambda^{(i+\dim \operatorname{Ker}\varphi)}(E)H(\varphi).$$

For $* = \Lambda$ or Z we get an injective map from φ making the quotient by $\operatorname{Ker}\varphi$, which yields $\lambda_i^*(F) \leq \lambda_i^*(E/\operatorname{Ker}\varphi)H(\varphi)$ and we conclude with $\lambda_i^*(E/\operatorname{Ker}\varphi) \leq \lambda^*_{i+\dim \operatorname{Ker}\varphi}(E)$ (for instance, for $* = \Lambda$ it means that if $\{e_1, \ldots, e_{i+\dim \operatorname{Ker}\varphi}\} \subset E$ is a free family, then at least i of the images of the vectors e_j in $E/\operatorname{Ker}\varphi$ are also linearly independent). □

Corollary 44 *Let $\varphi\colon E \to F$ be a linear map.*

(1) *If $\varphi \neq 0$, then $\mu_{\min}(E) \leq \mu_{\max}(F) + h(\varphi)$ and $\lambda_1^*(F) \leq \lambda^*_{\dim E}(E)H(\varphi)$.*
(2) *If φ is surjective, then $\mu_{\min}(E) \leq \mu_{\min}(F) + h(\varphi)$ and $\lambda^*_{\dim F}(F) \leq \lambda^*_{\dim E}(E)H(\varphi)$.*
(3) *If φ is surjective, then $\mu_{\max}(F) \leq \deg F - (\dim F - 1)\mu_{\min}(E) + (\dim F - 1)h(\varphi)$.*

Proof (1) Take $i = \operatorname{rk}\varphi$ in Theorem 43, bound from below i by 1 and use $\dim E = \dim \operatorname{Ker}\varphi + \operatorname{rk}\varphi$. (2) Same method but keep $i = \operatorname{rk}\varphi$ which is equal to $\dim F$ since φ is surjective. (3) Observe $\deg F = \mu_{\max}(F) + \sum_{i=2}^{\dim F} \mu_i(F) \geq \mu_{\max}(F) + (\dim F - 1)\mu_{\min}(F)$ and use (2). □

One can prove that if φ is injective, then for all $i \in \{1, \ldots, \dim E\}$, one has

$$(\star) \qquad P_E(i) \leq P_F(i) + h\Big(\bigwedge^i E, \bigwedge^i F; \bigwedge^i \varphi\Big).$$

For this, observe that, for all $v \in V(K)$, the function

$$i \mapsto \Big\|\bigwedge^i \varphi\Big\|_v \Big/ \Big\|\bigwedge^{i-1} \varphi\Big\|_v$$

is a nonincreasing function, so $\left(h(\bigwedge^i \varphi) - h(\bigwedge^{i-1} \varphi)\right)_{1 \leq i \leq \operatorname{rk}\varphi}$ is a decreasing sequence and $i \mapsto P_F(i) + h\left(\bigwedge^i E, \bigwedge^i F; \bigwedge^i \varphi\right)$ is a concave function. The case $i = 1$ in $(\star)$ corresponds to the first statement of Theorem 43.

To conclude this part, let us prove a variant of the first bound in Corollary 44 where we replace the height built with the operator norms by the Hilbert-Schmidt height $h_{\mathrm{HS}}(\varphi) := \log H_{E^\vee \otimes F}(\varphi)$ of φ.

Proposition 45 *Let* $\varphi\colon E \to F$ *be a linear map. Then*

$$\mu_{\min}(E) + \frac{1}{2}\log \operatorname{rk}\varphi \le \mu_{\max}(F) + h_{\mathrm{HS}}(\varphi).$$

This statement derives almost immediately from the following result.

Lemma 46 *If* $\varphi\colon E \to F$ *is an isomorphism, then*

$$\mu(E) + \frac{1}{2}\log \dim E \le \mu(F) + h_{\mathrm{HS}}(\varphi).$$

Proof With Proposition 42, it amounts to proving

$$h(\det E, \det F; \det\varphi) \le n\left(h_{\mathrm{HS}}(\varphi) - \frac{1}{2}\log n\right)$$

where n denotes the dimension of E. If $v \in V(K)$ is an ultrametric place, we simply bound $|\det\varphi|_v \le \|\varphi\|_v^n = \|\varphi\|_{E^\vee\otimes F,v}^n$. If v is archimedean, we use the Hermitian adjoint φ_v^* of $\varphi_v\colon E\otimes_K K_v \to F\otimes_K K_v$ to write $|\det\varphi|_v = \det\left(\varphi_v^*\varphi_v\right)^{1/2}$. We conclude with the inequality of arithmetic and geometric means applied to the eigenvalues of the positive operator $\varphi_v^*\varphi_v$:

$$n\left(\det\varphi_v^*\varphi_v\right)^{1/n} \le \operatorname{Tr}\left(\varphi_v^*\varphi_v\right) = \|\varphi\|_{E^\vee\otimes F,v}^2. \qquad \square$$

We use this lemma with the isomorphism $\overline{\varphi}\colon E/\operatorname{Ker}\varphi \to \operatorname{Im}\varphi$ and the bounds $\mu_{\min}(E) \le \mu(E/\operatorname{Ker}\varphi)$ and $\mu(\operatorname{Im}\varphi) \le \mu_{\max}(F)$ (Proposition 18) as well as $h_{\mathrm{HS}}(\overline{\varphi}) = h_{\mathrm{HS}}(\varphi)$ to get Proposition 45.

When $K = \mathbb{Q}$ or $K = \overline{\mathbb{Q}}$, we have $\log\Lambda_1(F) \le -\mu(F) + \frac{1}{2}\log\dim F$ since $c_{\mathrm{I}}(n, K) \le \sqrt{n}$ (see page 61). In particular, for every non-zero $\varphi\colon E \to F$, the same technique gives $\log\Lambda_1(\operatorname{Im}\varphi) + \mu_{\min}(E) \le h_{\mathrm{HS}}(\varphi)$ and so $\log\Lambda_1(F) + \mu_{\min}(E) \le \log\Lambda_1(E^\vee\otimes F)$ for all rigid adelic spaces E, F over $\overline{\mathbb{Q}}$ (see [13, Theorem 1.3] for $K = \overline{\mathbb{Q}}$). This leads us to the last part of our course.

5.2 *Tensor Product*

We will conclude this lecture by raising the problem of the behaviour of minima and slopes (only the first ones) with respect to tensor product. It has been seen that $\mu(E\otimes F) = \mu(E) + \mu(F)$ for every rigid adelic spaces E and F over K. What happens for $\Lambda_1(E\otimes F)$ and $\mu_1(E\otimes F)$? Let us start with two inequalities, always true:

$$\Lambda_1(E\otimes F) \le \Lambda_1(E)\Lambda_1(F) \quad \text{and} \quad \mu_{\max}(E) + \mu_{\max}(F) \le \mu_{\max}(E\otimes F).$$

To prove the first one, we can observe that, for all $v \in V(K)$, $x \in E \otimes_K K_v$, $y \in F \otimes_K K_v$, we have $\|x \otimes y\|_{E\otimes F,v} \leq \|x\|_{E,v}\|y\|_{F,v}$. That gives $H_{E\otimes F}(e \otimes f) \leq H_E(e)H_F(f)$ for $e \in E$ and $f \in F$. When e and f are not zero, $e \otimes f$ is not zero either and $H_{E\otimes F}(e \otimes f) \geq \Lambda_1(E \otimes F)$, leading to the first bound. As for the second one, we can note that

$$\mu_{\max}(E) + \mu_{\max}(F) = \mu(E_{\text{des}}) + \mu(F_{\text{des}}) = \mu(E_{\text{des}} \otimes F_{\text{des}}) \leq \mu_{\max}(E \otimes F).$$

So the problem is whether these inequalities are equalities. For Λ_1 the answer is *no, in general*. Actually it has been proved by Steinberg that, for any integer $n \geq 292$, there exists a rigid adelic space E over $\mathbb{Q}$ with dimension n such that $\Lambda_1(E \otimes E) \neq \Lambda_1(E)^2$ [18, p. 47]. Coulangeon obtained similar results for some imaginary quadratic fields K [11]. The author and Rémond proved that for every integers n, m both ≥ 2, there exist some rigid adelic spaces E and F over $\overline{\mathbb{Q}}$ with $\dim E = n$ and $\dim F = m$ such that $\Lambda_1(E \otimes F) \neq \Lambda_1(E)\Lambda_1(F)$ [13, Theorem 1.5]. Actually all the difficulty of the proof lies in the case $n = m = 2$. Here we shall give a different proof, due to Gaël Rémond, not yet published.

Proposition 47 *There exists a rigid adelic plane E over $\overline{\mathbb{Q}}$ such that $\Lambda_1(E \otimes E^\vee) \neq \Lambda_1(E)\Lambda_1(E^\vee)$.*

Proof Let us recall that $c_{\mathrm{I}}(2, \overline{\mathbb{Q}}) = \exp \frac{H_2 - 1}{2} = \exp \frac{1}{4}$ (see page 61). Let us choose $0 < \varepsilon < 1$ such that $e(1-\varepsilon)^4 > 2$. Let E be a rigid adelic plane over $\overline{\mathbb{Q}}$ such that $\Lambda_1(E) \geq (1-\varepsilon)H(E)^{1/2}c_{\mathrm{I}}(2, \overline{\mathbb{Q}})$ (definition of $c_{\mathrm{I}}(2, \overline{\mathbb{Q}})$). Since $\dim E = 2$ we have $\Lambda_1(E^\vee)/H(E^\vee)^{1/2} = \Lambda_1(E)/H(E)^{1/2}$. From this equality, we deduce

$$\Lambda_1(E)\Lambda_1(E^\vee) = \left(\frac{\Lambda_1(E)}{H(E)^{1/2}}\right)^2 \geq (1-\varepsilon)^2 \exp \frac{1}{2}.$$

Furthermore, considering a basis $\{e_1, e_2\}$ of E and the identity map $x = e_1 \otimes e_1^\vee + e_2 \otimes e_2^\vee$ we have $\Lambda_1(E \otimes E^\vee) \leq H_{E\otimes E^\vee}(x) = \sqrt{2}$. The choice of ε makes it possible to conclude $\Lambda_1(E \otimes E^\vee) < \Lambda_1(E)\Lambda_1(E^\vee)$. □

The heart of the proof is a lower bound for $\Lambda_1(E)\Lambda_1(E^\vee)$ which is compared to $\Lambda_1(E \otimes E^\vee) \leq \sqrt{n}$. Using Banaszczyk's lower bound given before Question 40, we can easily prove that $\Lambda_1(E)\Lambda_1(E^\vee) \neq \Lambda_1(E \otimes E^\vee)$ when $K = \mathbb{Q}$ and $\sqrt{n} < n/(2\pi e)$, that is for $n \geq 292$ (the integer 292 is the upper part of $(2\pi e)^2$). In this way we obtain a variant of Steinberg's result.

It may be the phenomenon enlighted by Proposition 47 does not occur for the maximal slope.

Bost's Conjecture *For all rigid adelic spaces E and F over K, we have $\mu_{\max}(E \otimes F) = \mu_{\max}(E) + \mu_{\max}(F)$.*

The field K is not important here since we can freely replace it by (one of) its algebraic closure, due to the invariance by scalar extension of the maximal slope (Proposition 19). This conjecture is known to be true when $\dim E \times \dim F \leq$

9 (see [8]) or, as recently proved by Rémond, when there is a group acting isometrically on $E \otimes_K \overline{K}$ such that this vector space is a direct sum of some non isomorphic irreducible vector subspaces [22]. Here we shall prove a weaker result, also due to Bost and Chen (ibid.). We recall that $H_n = 1 + 1/2 + \cdots + 1/n$ is the harmonic number.

Theorem 48 *Let E and F be some rigid adelic spaces over K and $n = \dim E$. Then we have*

$$\mu_{\max}(E \otimes F) \leq \mu_{\max}(E) + \mu_{\max}(F) + \frac{H_n - 1}{2}.$$

Lemma 49 *For any rigid adelic space E and integrable adelic space F over K, we have*

$$\Lambda_1(F) \leq \Lambda^{(\dim E)}(E)\Lambda_1(E^{\vee} \otimes_{\varepsilon} F).$$

Proof As for the first statement of Corollary 44 with $* =$ BC, extended to an integrable space (same proof), we have $\Lambda_1(F) \leq \Lambda^{(\dim E)}(E) H(\varphi)$ for every non-zero $\varphi \in E^{\vee} \otimes_{\varepsilon} F$. We conclude making $H(\varphi)$ tend to $\Lambda_1(E^{\vee} \otimes_{\varepsilon} F)$. □

Lemma 50 *For all rigid adelic spaces A and B, we have*

$$\exp\{-\mu_{\max}(A) - \mu_{\max}(B)\} \leq \Lambda_1(A \otimes_{\varepsilon} B).$$

Proof If $\varphi \in A \otimes_{\varepsilon} B = \mathrm{Hom}(A^{\vee}, B)$ and $\varphi \neq 0$, we saw $\mu_{\min}(A^{\vee}) \leq \mu_{\max}(B) + h(\varphi)$ (Corollary 44). The left hand side equals to $-\mu_{\max}(A)$ (Proposition 19). We conclude with making $h(\varphi)$ tend to $\log \Lambda_1(A \otimes_{\varepsilon} B)$. □

In particular, for every rigid adelic space E, we have $\Lambda_1(E \otimes_{\varepsilon} E_{\mathrm{des}}^{\vee}) = 1$. Indeed, by this lemma, the first minimum is greater than 1 but it is also at most 1 since the injection map $E_{\mathrm{des}} \hookrightarrow E$ has height at most 1.

Lemma 51 *For all rigid adelic spaces A, B, E over K, we have*

$$1 \leq \Lambda_1(E^{\vee} \otimes_{\varepsilon} A \otimes_{\varepsilon} B)\Lambda^{(\dim E)}(E) \exp\{\mu_{\max}(A) + \mu_{\max}(B)\}.$$

Proof Replace F by $A \otimes_{\varepsilon} B$ in Lemma 49 and apply Lemma 50. □

Lemma 52 *For all rigid adelic spaces E and F over K, we have*

$$\mu_{\max}(E \otimes F) \leq \mu_{\max}(F) + \log \Lambda^{(\dim E)}(E^{\vee}).$$

Proof Let us replace E by its dual $E^\vee$ and take $A = F$ and $B = (E \otimes F)^\vee_{\text{des}}$ in Lemma 51. We get

$$1 \leq \Lambda_1(E \otimes_\varepsilon F \otimes_\varepsilon (E \otimes F)^\vee_{\text{des}}) \Lambda^{(\dim E)}(E^\vee) \\ \times \exp\{\mu_{\max}(F) - \mu_{\max}(E \otimes F)\}.$$

Then, since the operator norm is smaller than the Hilbert-Schmidt norm, we can bound from above the first minimum on the right by $\Lambda_1(E \otimes F \otimes_\varepsilon (E \otimes F)^\vee_{\text{des}}) = 1$. □

Proof of Theorem 48 Let us apply Lemma 52 and use the inequality

$$\Lambda^{(\dim E)}(E^\vee) \exp \mu_{\min}(E^\vee) \leq c_n^{\mathrm{BC}}(n, \overline{\mathbb{Q}})$$

(definition of the constant on the right). By Proposition 19, we have $\mu_{\min}(E^\vee) = -\mu_{\max}(E)$ and we saw that $c_n^{\mathrm{BC}}(n, \overline{\mathbb{Q}})$ is equal to the number $\exp((H_n - 1)/2)$ (see pages 61 and 63). □

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