Rational equivalence for Poisson polynomial algebras [1]

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Abstract

The purpose of these lectures is to provide an overview of the rational equivalence for polynomial Poisson algebras arising from deformation theories. More precisely, we are interested in the algebraic study of some commutative polynomial algebras S appearing as semiclassical limits of noncommutative polynomial algebras U, with a Poisson structure on S derived from the commutation bracket in U. In the significant situations, U can be of "classical" type (differential operator algebras, enveloping algebras,...) or "quantum" type (algebraic quantum groups,...), giving rise to two main types of Poisson structures on S ("Poisson-Weyl" type or "Poisson-quantum" type). This Poisson structure can be canonically extended to the field of fractions Q of S and our general problem concerns the separation or the classification of the fields of rational functions Q up to Poisson isomorphism.

The first part is devoted to exposing the necessary notions on Poisson algebras and their fields of fractions, with examples, basic properties and first general results about Poisson-rational equivalence.

The second part deals with the situation where S is the symmetric algebra $S(\mathfrak{g})$ of a finite dimensional complex Lie algebra $\mathfrak{g}$, equipped with the Poisson bracket deduced from the Lie bracket on $\mathfrak{g}$. Then the question of the rational equivalence or not of $S(\mathfrak{g})$ with a Poisson-Weyl algebra appears as a Poisson-analogue of the classical problem of Gel'fand-Kirillov about the noncommutative rational equivalence for the enveloping algebra $U(\mathfrak{g})$. Positive answers are given for nilpotent $\mathfrak{g}$ (M. Vergne 1972) and more recently solvable $\mathfrak{g}$ (P. Tauvel and R. Yu, 2010).

A similar formulation makes sense for quantum algebras, with results (K.R. Goodearl and S. Launois, 2011) concerning wide classes of Poisson algebras defined as semiclassical limits of quantized coordinate rings. In this context, the main tools are a general algorithmical method to reduce by localization the algebraic rules in the definition of Poisson brackets, and the study of some suitable actions of algebraic tori by Poisson automorphisms on the algebras under consideration.

The third part concerns the fields of invariants Q^G under the action of a group G of Poisson automorphisms of S. In the philosophy of Noether's problem in classical invariant theory, an improvement (taking in consideration the Poisson structure) of a theorem of Miyata can be proved to obtain a Poisson-isomorphism of Q^G with a field of fractions of a Poisson-Weyl algebra, in the case of the Kleinian surfaces (J. Baudry 2009), and for diagonalizable actions of G. Exploratory results about the Poisson-quantum situation in dimension two lead to consider the action of Poisson-quantum automorphisms in the Cremona group.

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1 Poisson structures on rational functions fields

1.1 Poisson polynomial algebras

1.1.1 Basic notions on Poisson structures

Definition. A commutative $\Bbbk$-algebra A is a Poisson algebra when there exists a bilinear skew-symmetric map $\{\cdot,\cdot\} : A \times A \to A$ satisfying the two conditions:

- Leibniz rule: $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in A$;
- Jacobi identity : $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$ for all $a, b, c \in A$.

Then the Poisson bracket $\{\cdot,\cdot\}$ defines a structure of Lie algebra on A and acts as a biderivation. It's clear that a Poisson bracket on a finitely generated algebra A is entirely determined by the values of $\{x_i, x_j\}$ for $i < j$ where $x_1, \dots, x_N$ generate A.

Example 1. The commutative polynomial algebra in two variables $S = \Bbbk[x, y]$ is a Poisson algebra for the bracket defined on the generators by:

$$\{x, y\} = 1, \tag{1}$$

or equivalently for any $P, Q \in S$:

$$\{P, Q\} = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x}\frac{\partial P}{\partial y} = P'_1Q'_2 - Q'_1P'_2. \tag{2}$$

More generally, for any $n \geq 1$, $S = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_n]$ is a Poisson algebra for the "symplectic" bracket defined on the generators by:

$$\{x_i, y_j\} = \delta_{i,j} \text{ and } \{x_i, x_j\} = \{y_i, y_j\} = 0, \quad \text{for all } 1 \leq i, j \leq n, \tag{3}$$

or equivalently for any $P, Q \in S$:

$$\{P, Q\} = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}\frac{\partial Q}{\partial y_i} - \frac{\partial Q}{\partial x_i}\frac{\partial P}{\partial y_i}. \tag{4}$$

We refer to this Poisson algebra as the Poisson-Weyl algebra, denoted by $\mathbb{S}_n(\Bbbk)$.

Example 2. For any $\lambda \in \Bbbk$, the commutative polynomial algebra $S = \Bbbk[x, y]$ is a Poisson algebra for the "multiplicative" bracket defined on the generators by:

$$\{x, y\} = \lambda xy. \tag{5}$$

More generally, for any $n \geq 2$ and for any $n \times n$ antisymmetric matrix $\boldsymbol{\lambda} = (\lambda_{ij})$ with entries in $\Bbbk$, $S = \Bbbk[x_1, \dots, x_n]$ is a Poisson algebra for the bracket defined on the generators by:

$$\{x_i, x_j\} = \lambda_{i,j} x_i x_j \quad \text{for all } 1 \leq i, j \leq n. \tag{6}$$

We refer to this Poisson algebras as the Poisson-quantum plane and Poisson-quantum space respectively, denoted by $\mathbb{P}_2^{\lambda}(\Bbbk)$ and $\mathbb{P}_n^{\boldsymbol{\lambda}}(\Bbbk)$.

EXAMPLE 3. Let F be a fixed element of the polynomial algebra in three variables $S = \Bbbk[x, y, z]$; then there exists a Poisson bracket on S defined for any $P, Q \in S$ by :

$$\begin{aligned}\{P, Q\} &= \mathrm{Jac}(P, Q, F) \\ &= (P_2'Q_3' - Q_2'P_3')F_1' + (P_3'Q_1' - Q_3'P_1')F_2' + (P_1'Q_2' - Q_1'P_2')F_3'.\end{aligned}$$

The brackets on the generators are then $\{x, y\} = F_3'$, $\{y, z\} = F_1'$, $\{z, x\} = F_2'$.

More generally, one can prove (see [7] for complete detailed calculations) that $S = \Bbbk[x_1, \ldots, x_N]$ is a Poisson algebra for the Poisson bracket defined (when $N \geq 3$) for any $P, Q \in S$ by: $\{P, Q\} = \mathrm{Jac}(P, Q, F_1, \ldots, F_{N-2})$, where $F_1, \ldots, F_{N-2}$ are arbitrary chosen polynomials in S.

REMARK 1 : *Poisson structure on quotient algebras.* An ideal I of a Poisson algebra A is a Poisson ideal when $\{a, x\} \in I$ for any $a \in A, x \in I$; in this case, we also have $\{x, a\} \in I$ and the trivial observation $\{a, b\} - \{a', b'\} = \{a - a', b\} + \{a', b - b'\}$ for all $a, b \in A$ allows to define on the algebra A/I the induced bracket $\{\overline{a}, \overline{b}\} = \overline{\{a, b\}}$.

REMARK 2 : *Poisson structure on localized algebras.* Let X be a multiplicative set containing 1 in a Poisson algebra A. Then there exists exactly one Poisson bracket $\{\cdot, \cdot\}$ on the localization $X^{-1}A$ extending the bracket of A. It is given by:

$$\{as^{-1}, bt^{-1}\} = \{a, b\}s^{-1}t^{-1} - \{a, t\}bs^{-1}t^{-2} - \{s, b\}as^{-2}t^{-1} + \{s, t\}abs^{-2}t^{-2}$$

for any $a, b \in A, s, t \in X$. In particular, if A is a domain, the Poisson bracket on A extends canonically in a Poisson bracket on the field of fractions of A.

REMARK 3 : *Poisson structure on invariant algebras.* Let G be a group of algebra automorphisms of a Poisson algebra A. An element $g \in G$ is said to be a Poisson automorphism when $g\{a, b\} = \{g(a), g(b)\}$ for all $a, b \in A$. If any $g \in G$ is a Poisson automorphism, then the Poisson bracket of two elements of the invariant algebra A^G also lies in A^G. We say that A^G is a Poisson subalgebra of A.

REMARK 4 : *Poisson center.* The Poisson center of a Poisson algebra A is defined as the set $\mathrm{Z_P}(A) = \{c \in A\,; \{c, a\} = 0$ for any $a \in A\}$. It is a Poisson subalgebra of A satisfying $\{\mathrm{Z_P}(A), A\} \subseteq \mathrm{Z_P}(A)$.

EXERCISE 1. Prove that the unique Poisson bracket on $\Bbbk[x]$ is the trivial one.

Prove that, for any polynomial $P \in \Bbbk[x, y]$, there exists a Poisson bracket on $\Bbbk[x, y]$ defined by $\{x, y\} = P$.

Prove that, for any polynomials $P, Q, R \in \Bbbk[x, y, z]$, there exists a Poisson bracket on $\Bbbk[x, y, z]$ defined by $\{x, y\} = R$, $\{y, z\} = P$, $\{z, x\} = Q$ if and only if, denoting $\mathrm{curl}\,(P, Q, R) = (R_2' - Q_3', P_3' - R_1', Q_1' - P_2')$, we have $(P, Q, R) \cdot \mathrm{curl}\,(P, Q, R) = 0$.

EXERCISE 2. Prove that the group $\mathrm{SL}_2(\Bbbk)$ acts linearly by Poisson automorphisms on the Poisson-Weyl algebra $\mathbb{S}_1(\Bbbk)$, defined by:

$$g.x = \alpha x + \beta y \quad \text{and} \quad g.y = \gamma x + \delta y, \quad \text{for any } g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\Bbbk).$$

Prove more generally that the symplectic group $\mathrm{Sp}_n(\Bbbk)$ acts linearly by Poisson automorphisms on the Poisson-Weyl algebra $\mathbb{S}_n(\Bbbk)$.

EXERCISE 3. Prove that the Laurent polynomial algebra $\Bbbk[x^{\pm1}, y^{\pm1}]$ is a Poisson algebra for the Poisson bracket defined by (5), with λ fixed in $\Bbbk$. We name it the Poisson-quantum torus, denoted by $\mathbb{T}_2^\lambda(\Bbbk)$.

Prove that the group $\mathrm{SL}_2(\mathbb{Z})$ acts by Poisson automorphisms on $\mathbb{T}_2^\lambda(\Bbbk)$, defined by:

$$g.x = x^a y^b \quad \text{and} \quad g.y = x^c y^d, \quad \text{for any } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

1.1.2 Poisson-Ore polynomial algebras

We need firstly some precisions about the terminology.

TERMINOLOGY. Let A be a Poisson algebra. A derivation of A is a $\Bbbk$-linear map $\sigma : A \to A$ such as $\sigma(ab) = \sigma(a)b + a\sigma(b)$ for all $a, b \in A$. A Poisson derivation of A is a derivation σ of A satisfying moreover:

$$\sigma(\{a, b\}) = \{\sigma(a), b\} + \{a, \sigma(b)\} \quad \text{for all } a, b \in A. \tag{7}$$

If σ is a Poisson derivation of A, a Poisson σ-derivation of A is a derivation δ of A satisfying moreover:

$$\delta(\{a, b\}) = \{\delta(a), b\} + \{a, \delta(b)\} + \sigma(a)\delta(b) - \delta(a)\sigma(b) \quad \text{for all } a, b \in A. \tag{8}$$

EXAMPLES.

a. For any fixed element $u \in A$, the map $\sigma_u : a \mapsto \{u, a\}$ is a Poisson derivation; such σ_u are called Hamiltonian derivations.

b. If $\sigma = 0$, a Poisson σ-derivation is just a Poisson derivation.

c. It is an easy exercise to check that, for any Poisson derivation σ and fixed element $u \in A$, the map $\delta : A \to A$ defined by $\delta(a) = \sigma(a)\, u - \{u, a\}$ is a Poisson σ-derivation. Such Poisson σ-derivations will be called here Hamiltonian Poisson σ-derivations.

We expose now a canonical way to extend a Poisson structure from an algebra A to the polynomial algebra $A[x]$. This construction introduced by Oh in [27] is somewhat similar to the classical noncommutative notion of Ore polynomial algebra.

PROPOSITION AND DEFINITION (S.-Q. Oh). *Let A be a Poisson algebra. Let σ, δ be $\Bbbk$-linear maps on A. Then the polynomial ring $A[x]$ becomes a Poisson algebra with Poisson bracket defined by $\{a, b\} = \{a, b\}_A$ and*

$$\{x, a\} = \sigma(a)x + \delta(a) \quad \textit{for all } a \in A \tag{9}$$

if and only if σ is a Poisson derivation and δ is a Poisson σ-derivation. In this case, the Poisson algebra $A[x]$ is said to be a Poisson-Ore extension of A and is denoted by $A[x]_{\sigma,\delta}$.

PROOF. Suppose that $A[x]$ is a Poisson algebra satisfying (9). By identification in the Leibniz identity:

$$\sigma(ab)x + \delta(ab) = \{x, ab\} = a\{x, b\} + \{x, a\}b = (a\sigma(b) + \sigma(a)b)x + a\delta(b) + \delta(a)b,$$

for all $a, b \in A$, the maps σ, δ are derivations of A. On the same way, Jacobi identity gives:

$$\begin{aligned}0 &= \{x, \{a, b\}\} + \{\{x, b\}, a\} + \{b, \{x, a\}\} \\ &= \sigma(\{a, b\})x + \delta(\{a, b\}) + \{\sigma(b)x, a\} + \{\delta(b), a\} + \{b, \sigma(a)x\} + \{b, \delta(a)\} \\ &= \big[\sigma(\{a, b\}) - \{\sigma(a), b\} - \{a, \sigma(b)\}\big]\, x \\ &\qquad + \delta(\{a, b\}) - \{\delta(a), b\} - \{a, \delta(b)\} - \sigma(a)\delta(b) + \delta(a)\sigma(b).\end{aligned}$$

Hence σ and δ satisfy (7) and (8). Conversely, assume now that (7) and (8) are satisfied. We define a $\mathbb{k}$-bilinear map $\{.,.\} : A[x] \times A[x] \to A[x]$ by

$$\{ax^n, bx^m\} = (\{a, b\} + mb\sigma(a) - na\sigma(b))x^{m+n} + (mb\delta(a) - na\delta(b))x^{m+n-1} \tag{10}$$

for all monomials ax^n et bx^m in $A[x]$. Note that the case $m = 0, n = 1, b = 1$ is just (9). It is clear from (10) that $\{f, g\} = -\{g, f\}$ for all $f, g \in A[x]$. Since σ and δ are derivations of A, the $\mathbb{k}$-linear maps $g \mapsto \{f, g\}$ and $g \mapsto \{g, f\}$ are derivations of $A[x]$ for any fixed $f \in A[x]$. Finally, it follows by induction on ℓ, m, n (see [27] for more details) from (7) and (8) that

$$\{\{ax^\ell, bx^m\}, cx^n\} + \{\{bx^m, cx^n\}, ax^\ell\} + \{\{cx^n, ax^\ell\}, bx^m\} = 0$$

for all $a, b, c \in A$ and all nonnegative integers ℓ, m, n, which is enough to prove Jacobi identity. □

REMARK 1. This proposition shows in particular that, if $A[x]$ is a Poisson algebra such that A is a Poisson subalgebra and $\{A, x\} \subset Ax + A$, then $A[x]$ is a Poisson-Ore extension $A[x]_{\sigma,\delta}$ for some well chosen σ, δ.

REMARK 2. The construction of Poisson-Ore extensions can be easily iterated. We start with the commutative polynomial algebra $\mathbb{k}[x_1]$. This is a Poisson algebra for the trivial Poisson bracket. Hence conditions (7) and (8) are trivially satisfied for any derivations σ_2, δ_2 in $\mathbb{k}[x_1]$ and we can consider the two variables Poison-Ore polynomial algebra $\mathbb{k}[x_1][x_2]_{\sigma_2,\delta_2}$ with Poisson bracket defined by (9) from relation $\{x_1, x_2\} = \sigma(x_2)x_2 + \delta(x_2)$. By iteration, we can consider iterated Poisson-Ore extensions:

$$\mathbb{k}[x_1][x_2]_{\sigma_2,\delta_2}[x_3]_{\sigma_3,\delta_3} \dots [x_n]_{\sigma_n,\delta_n} \tag{11}$$

with σ_i a Poisson derivation of $A_i = \mathbb{k}[x_1][x_2]_{\sigma_2,\delta_2} \dots [x_{i-1}]_{\sigma_{i-1},\delta_{i-1}}$ and δ_i a Poisson σ_i derivation of A_i, for any $i = 2, \dots, n$.

EXAMPLE 1. The Poisson-Weyl algebra introduced in example 1 of 1.1.1 is an iterated Poisson-Ore extension:

$$\mathbb{S}_n(\mathbb{k}) = \mathbb{k}[y_1, \dots, y_n][x_1]_{0,\partial_{y_1}}[x_2]_{0,\partial_{y_2}} \dots [x_n]_{0,\partial_{y_n}}, \tag{12}$$

the bracket being trivial on $\Bbbk[y_1,\dots,y_n]$ and satisfying $\{x_i,y_j\}=0\times y_i+\partial_{y_i}(y_j)$ for all $1\le i,j\le n$.

Example 2. The Poisson-quantum space introduced in example 2 of 1.1.1 is an iterated Poisson-Ore extension:

$$\mathbb{P}_n^{\boldsymbol{\lambda}}(\Bbbk)=\Bbbk[x_1]_{\sigma_1,0}[x_2]_{\sigma_2,0}\dots[x_n]_{\sigma_n,0}, \tag{13}$$

with $\sigma_i=\sum_{1\le j<i}\lambda_{ij}x_j\partial_{x_j}$, the bracket being defined from $\{x_i,x_j\}=\lambda_{ij}x_jx_i+0$.

Remark 3. The localization of a Poisson-Ore extension $A[x]_{\sigma,\delta}$ with respect to the powers of x is by remark 2 of 1.1.1 a Poisson algebra for the bracket extended by relation

$$\{x^{-1},a\}=-\sigma(a)x^{-1}-\delta(a)x^{-2}\quad\text{for any } a\in A. \tag{14}$$

We denote it by $A[x^{\pm1}]_{\sigma,\delta}$.

Exercise 1. Let A be a Poisson algebra, σ a Poisson derivation of A and δ an Hamiltonian Poisson σ-derivation (in the sense of point c in the terminology above). Prove that, up to a change of variable, $A[x]_{\sigma,\delta}=A[x']_{\sigma,0}$.

Exercise 2. Let A be a Poisson algebra, σ a Poisson derivation of A and δ a Poisson σ-derivation of A. Check that $\sigma(\mathrm{Z_P}(A))\subseteq\mathrm{Z_P}(A)$. We suppose moreover that A is a field K. Prove that, if there exists some $c\in\mathrm{Z_P}(K)$ such that $\sigma(c)\neq0$, then δ is the Hamiltonian Poisson σ-derivation determined by $u=\delta(c)\sigma(c)^{-1}\in K$, and then, up to a change of variable, $K[x]_{\sigma,\delta}=K[x']_{\sigma,0}$.

Exercise 3 (see proposition 2.9 of [36]). Let K be a Poisson algebra which is a field and δ a non Hamiltonian Poisson derivation of K. Prove that the Poisson-Ore algebra $S=K[x]_{0,\delta}$ is Poisson simple (i.e. there is no nonzero proper Poisson ideal in S). [Hint: suppose that there exists a Poisson ideal I of S such that $I\neq(0)$ and $I\neq S$; define $n=\min(\deg p\,;\,p\in I)\ge1$ and take $p\in I$ of degree n with leading coefficient 1; for any $a\in K$, compute $\{p,a\}\in I$ and deduce that $\{p,a\}=0$; conclude that δ is Hamiltonian, so a contradiction].

Exercise 4. Consider the Poisson-Weyl algebra $\mathbb{S}_n(\Bbbk)=\Bbbk[x_1,\dots,x_n,y_1,\dots,y_n]$ with Poisson bracket (3). For any $1\le i\le n$, set $w_i=x_iy_i$. Prove that:

$$\{x_i,w_i\}=x_i,\quad\{x_i,w_j\}=\{w_i,w_j\}=\{x_i,x_j\}=0\text{ if } j\neq i,$$

Prove that the localization of $\mathbb{S}_n(\Bbbk)$ with respect of the powers of the x_i's is the localized Poisson-Ore iterated extension:

$$\mathbb{S}'_n(\Bbbk):=\Bbbk[w_1,w_2,\dots,w_n][x_1^{\pm1}]_{\partial_{w_1},0}[x_2^{\pm1}]_{\partial_{w_2},0}\cdots[x_n^{\pm1}]_{\partial_{w_n},0}.$$

Show that, setting $t_i=-x_i^{-1}w_i$ for any $1\le i\le n$, we also have:

$$\mathbb{S}'_n(\Bbbk):=\Bbbk[x_1^{\pm1},x_2^{\pm1}\dots x_n^{\pm1}][t_1]_{0,\partial_{x_1}}[t_2]_{0,\partial_{x_2}}\cdots[t_n]_{0,\partial_{x_n}}.$$

1.1.3 Deformations and semiclassical limits

We fix a commutative principal ideal domain R containing $\Bbbk$ and an element $h \in R$ with hR a maximal ideal of R. Suppose that B is a non necessarily commutative torsion free R-algebra for which the quotient $A := B/hB$ is commutative. Thus any $u, v \in B$ satisfy $(u + hB)(v + hB) = (v + hB)(u + hB)$ and then $[u, v] := uv - vu \in hB$. We denote by $\gamma(u, v)$ the element of B defined by $[u, v] = h\gamma(u, v)$. We set:

$$\{\overline{u}, \overline{v}\} = \overline{\gamma(u, v)} \quad \text{for any } \overline{u}, \overline{v} \in A.$$

This is independent of the choice of u, v.

> If $u' = u + hw$ with $w \in B$, we have $[u', v] = [u, v] + h[w, v]$ since h is central; thus $h\gamma(u', v) = h\gamma(u, v) + h^2\gamma(w, v)$, then $\gamma(u', v) = \gamma(u, v) + h\gamma(w, v)$ and so $\overline{\gamma(u', v)} = \overline{\gamma(u, v)}$. The result follows by antisymmetry.

This defines a Poisson bracket on A.

> Jacobi identity holds for $[\cdot, \cdot]$, thus for $\gamma(\cdot, \cdot)$ because h is central, and then for $\{\cdot, \cdot\}$. Using again the centrality of h, the Leibniz rule for $\{\cdot, \cdot\}$ follows from $[uv, w] = u[v, w] + [u, w]v$ for all $u, v, w \in B$.

Definitions. With the previous data and notation, we say that:

- the noncommutative algebra B is a *quantization* of the Poisson algebra A, and A is the *semiclassical limit* of B,
- for any $\lambda \in \Bbbk$ such that the central element $h - \lambda$ of B is non invertible in B, the algebra $A_\lambda := B/(h - \lambda)B$ is a *deformation* of the Poisson algebra A.

$$\begin{tikzcd}
A = B/hB \arrow[rr, dotted, bend left, "\text{quantization}"] \arrow[dr, squiggly, "\text{deformation}"'] & & B \arrow[ll, "\text{semiclassical limit}"] \arrow[dl] \\
& A_\lambda = B/(h-\lambda)B &
\end{tikzcd}$$

Example 1. Let $\mathfrak{g}$ be a complex finite dimensional Lie algebra. Let B be the homogenized enveloping algebra $U_h(\mathfrak{g})$ of $\mathfrak{g}$, that is B is the $\mathbb{C}[h]$-algebra with generators a basis $\{x_1, \dots, x_n\}$ of $\mathfrak{g}$ and relations: $x_i x_j - x_j x_i = h\,[x_i, x_j]_{\mathfrak{g}}$. It's clear that B is a quantization of the algebra $\mathscr{A} = \mathbb{C}[x_1, \dots, x_n] \simeq S(\mathfrak{g}) \simeq \mathscr{O}(\mathfrak{g}^*)$ with the so-called Kirillov-Kostant-Souriau Poisson bracket defined on the generators by $\{x_i, x_j\} = [x_i, x_j]_{\mathfrak{g}}$, and that the enveloping algebra $U(\mathfrak{g}) \simeq B/(h-1)B$ is a deformation of $\mathscr{A}$.

$$\begin{tikzcd}
S(\mathfrak{g}) \arrow[rr, dotted, bend left, "\text{quantization}"] \arrow[dr, rightsquigarrow, "\text{deformation}"'] & & U_h(\mathfrak{g}) \arrow[ll, "\text{s.c. limit}"] \arrow[dl] \\
& U(\mathfrak{g}) &
\end{tikzcd}$$

EXAMPLE 2. Let $A_n(\Bbbk)$ be the n-th Weyl algebra, i.e. the algebra generated over $\Bbbk$ by $2n$ generators $q_1, \dots, q_n$, $p_1, \dots, p_n$ with relations:

$$[p_i, q_i] = 1, \quad [p_i, q_j] = [p_i, p_j] = [q_i, q_j] = 0 \text{ for } i \neq j, \tag{15}$$

where $[\,.\,,\,.\,]$ is the commutation bracket (i.e. $[a, b] = ab - ba$ for all $a, b \in A_n(\Bbbk)$).
The monomials $(q_1^{i_1} \dots q_n^{i_n} p_1^{j_1} \dots p_n^{j_n})_{i_1,\dots,i_n,j_1,\dots,j_n \in \mathbb{N}}$ are a $\Bbbk$-left basis of the algebra $A_n(\Bbbk)$. For any nonnegative integer m, denote by $\mathscr{F}_m$ the $\Bbbk$-vector space generated in $A_n(\Bbbk)$ by monomials $q_1^{i_1} \dots q_n^{i_n} p_1^{j_1} \dots p_n^{j_n}$ such that $i_1 + \dots + i_n + j_1 + \dots + j_n \leq m$. We have:

$$\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \dots \subseteq A_n(\Bbbk), \qquad A_n(\Bbbk) = \textstyle\bigcup_{i \geq 0} \mathscr{F}_i, \qquad \mathscr{F}_i \mathscr{F}_j \subseteq \mathscr{F}_{i+j}.$$

In other words, $\mathscr{F} = (\mathscr{F}_i)_{i \geq 0}$ is a filtration of $A_n(\Bbbk)$, called the Bernstein filtration. We define $\mathscr{G}_i = \mathscr{F}_i / \mathscr{F}_{i-1}$ and the $\Bbbk$-vector space $\mathrm{gr}_{\mathscr{F}}(A_n(\Bbbk)) := \bigoplus_{i \geq 0} (\mathscr{F}_i / \mathscr{F}_{i-1})$, with convention $\mathscr{F}_{-1} = 0$, is an algebra for the product defined on each homogeneous component $\mathscr{G}_i := \mathscr{F}_i / \mathscr{F}_{i-1}$ (and then linearly extended) by:

$$(a_i + \mathscr{F}_{i-1})(a_j + \mathscr{F}_{j-1}) = a_i a_j + \mathscr{F}_{i+j-1}, \text{ for all } i, j \geq 0,\ a_i \in \mathscr{F}_i,\ a_j \in \mathscr{F}_j \qquad (\star).$$

We have $\mathscr{G}_i \mathscr{G}_j \subseteq \mathscr{G}_{i+j}$ for all nonnegative integers i, j, and an easy calculation from relations (15) shows that the graded algebra $S := \mathrm{gr}_{\mathscr{F}}(A_n(\Bbbk))$ is isomorphic to the commutative polynomial algebra in $2n$ variables $\Bbbk[x_1, \dots, x_n, y_1, \dots, y_n]$. Using the fact that $[\mathscr{F}_i, \mathscr{F}_j] \subseteq \mathscr{F}_{i+j-1}$ for all $i, j \geq 0$, we can apply the semiclassical limit process, with $R = \Bbbk[h]$ and B be the Rees algebra of $A_n(\Bbbk)$ related to the filtration $\mathscr{F}$, to build a Poisson bracket on S:

$$\{a_i + \mathscr{F}_{i-1}, a_j + \mathscr{F}_{j-1}\} = [a_i, a_j] + \mathscr{F}_{i+j-2} \text{ for all } i, j \geq 0,\ a_i \in \mathscr{F}_i,\ a_j \in \mathscr{F}_j. \qquad (\star\star)$$

PROOF. By definition, B is the subalgebra $\bigoplus_{i \geq 0} \mathscr{F}_i h^i$ in the algebra $A_n(\Bbbk)[h] = A_n(R)$ with $R = \Bbbk[h]$, with central indeterminate h.

• The linear map $\varphi : B \to S$ defined by $a_i h^i \mapsto a_i + \mathscr{F}_{i-1}$ is clearly surjective and a morphism of algebras for the product $(\star)$ in S. It is clear that $hB \subset \ker \varphi$ because $\varphi(h) = 1 + \mathscr{F}_0 = 0$ in S. Conversely, take $f = a_0 + a_1 h + \dots + a_i h^i$ in B with $a_0, a_1, \dots, a_i$ be elements of $\mathscr{F}_0, \mathscr{F}_1, \dots, \mathscr{F}_i$ respectively and suppose that $\varphi(f) = 0$. Then: $a_0 + \mathscr{F}_{-1} = 0$, $a_1 + \mathscr{F}_0 = 0, \dots, a_i + \mathscr{F}_{i-1} = 0$, that means $a_0 = 0, a_1 \in \mathscr{F}_0, \dots, a_i \in \mathscr{F}_{i-1}$. Thus $f = h(a_1 + a_2 h + \dots + a_i h^{i-1})$. We conclude that $\ker \varphi = hB$ and $\widetilde{\varphi} : B/hB \to S$ is an isomorphism of algebras.

• The linear map $\psi : B \to A_n(\Bbbk)$ defined by $a_i h^i \mapsto a_i$ is clearly surjective and a morphism of algebras. We have $\psi(h-1) = 1 - 1 = 0$ in $A_n(\Bbbk)$ thus $(h-1)B \subset \ker \psi$.

Now consider $f = a_0 + a_1 h + \cdots + a_i h^i \in B$ with $a_0, a_1, \ldots, a_i$ be elements of $\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_n$ respectively such that $\psi(t) = 0$. Then: $a_0 + a_1 + \cdots + a_i = 0$. Therefore $f = a_0(1-h^i) + a_1 h(1-h^{i-1}) + \cdots + a_{i-1}h^{i-1}(1-h) \in (h-1)B$. Hence $\ker \psi = (h-1)B$ and $\widetilde{\psi} : B/(h-1)B \to A_n(\Bbbk)$ is an isomorphism of algebras.

- For $a_i \in \mathscr{F}_i$, $a_j \in \mathscr{F}_j$ we have $[a_i h^i, a_j h^j] = [a_i, a_j]h^{i+j} = h\gamma(a_i h^i, a_j h^j)$ with notation $\gamma(a_i h^i, a_j h^j) = [a_i, a_j]h^{i+j-1} = a_{i+j-1}h^{i+j-1}$, where $a_{i+j-1} := [a_i, a_j]$ lies in $\mathscr{F}_{i+j-1}$ because of the property $[\mathscr{F}_i, \mathscr{F}_j] \subseteq \mathscr{F}_{i+j-1}$ of the Bernstein filtration. Thus the Poisson bracket defined on B/hB by deformation process is given by $\{a_i h^i + hB, a_j h^j + hB\} = [a_i, a_j]h^{i+j-1} + hB$ whose image by $\widetilde{\varphi}$ is $(\star\star)$. $\square$

The calculation of this bracket of S on elements of $\mathscr{F}_1$ gives $\{x_i, x_j\} = [p_i, p_j] = 0$, $\{y_i, y_j\} = [q_i, q_j] = 0$ and $\{x_i, y_j\} = [p_i, q_j] = \delta_{i,j}$. We conclude that S is Poisson isomorphic to the Poisson-Weyl algebra $\mathbb{S}_n(\Bbbk)$ defined in example 1 of 1.1.1.

$$\begin{tikzcd}
\mathbb{S}_n(\Bbbk) \simeq \mathrm{gr}_{\mathscr{F}} A_n(\Bbbk) \simeq B/hB \arrow[dr, squiggly, "\text{deformation}"'] \arrow[rr, dotted, shift left, "\text{quantization}"] & & \mathrm{Rees}_{\mathscr{F}}(A_n(\Bbbk)) = B \arrow[ll, "\text{semiclassical limit}"] \arrow[dl] \\
& A_n(\Bbbk) \simeq B/(h-1)B &
\end{tikzcd}$$

In the "classical" examples 1 and 2, the principal ideal domain R was just $\Bbbk[h]$. In quantum algebra construction, a standard choice is $R = \Bbbk[q, q^{-1}]$ and $h = q - 1$ when the relations laws depend on a single quantification parameter. In the multiparamater cases, it is more convenient to take $R = \Bbbk[[h]]$. We will use the notation $e(\alpha) = \exp(\alpha h) = \sum_{i \geq 0} \frac{1}{i!}\alpha^i h^i$ for any $\alpha \in \Bbbk$, observing that $e(\alpha + \beta) = e(\alpha)e(\beta)$.

Example 3. Let $\boldsymbol{\lambda} = (\lambda_{ij})$ a $n \times n$ antisymmetric matrix with entries in $\Bbbk$. Denoting $q_{ij} = e(\lambda_{ij}) \in R := \Bbbk[[h]]$, the matrix $\boldsymbol{q} = e(\boldsymbol{\lambda}) = (q_{ij})$ is multiplicatively antisymmetric with entries in R. We can consider the multiparameter quantized coordinate ring of affine n-space (or simply quantum n-space) $B := \mathscr{O}_{\boldsymbol{q}}(R^n) = R_{\boldsymbol{q}}[t_1, \ldots, t_n]$, which is by definition the algebra generated over R by n generators $t_1, \ldots, t_n$ with relations:

$$t_i t_j = q_{ij} t_j t_i \quad \text{for all } 1 \leq i, j \leq n. \tag{16}$$

Observe that hR is maximal in R and B is a free R-module. Denoting by $S = \Bbbk[x_1, \ldots, x_n]$ the commutative polynomial algebra in n indeterminates over $\Bbbk$, the additive map $\varphi : B \to S$ defined by $\varphi(r(h)t_1^{m_1} \ldots t_n^{m_n}) = r(0)x_1^{m_1} \ldots x_n^{m_n}$ is a ring homomorphism (because $e(\lambda_{ij})(0) = 1$), surjective, with $\ker \varphi = hB$. We compute:

$$[t_i, t_j] = (q_{ij} - 1)t_j t_i = \sum_{m \geq 1} \tfrac{\lambda_{ij}^m h^m}{m!} t_j t_i = h\big(\lambda_{ij} t_j t_i + \sum_{m \geq 1} \tfrac{\lambda_{ij}^{m+1} h^m}{(m+1)!} t_j t_i\big).$$

Hence, applying the semiclassical limit process, we introduce $\gamma(t_i, t_j) := h^{-1}[t_i, t_j] \in B$ and define in $A \simeq B/hB$ the Poisson bracket:

$$\{x_i, x_j\} = \{\overline{t_i}, \overline{t_j}\} = \overline{\gamma(t_i, t_j)} = \lambda_{ij} x_i x_j \quad \text{for any } 1 \leq i, j \leq n.$$

We recover relations (6) and conclude that S is Poisson isomorphic to the Poisson-quantum space $\mathbb{P}_n^{\boldsymbol{\lambda}}(\Bbbk)$ defined in example 2 of 1.1.1.

$$\mathbb{P}_n^{\boldsymbol{\lambda}}(\Bbbk) \simeq B/hB \underset{\text{semiclassical limit}}{\overset{\text{quantization}}{\rightleftarrows}} \mathscr{O}_{e(\boldsymbol{\lambda})}(\Bbbk[[h]]^n) = B$$

Example 4. Let $\boldsymbol{p} = (p_{ij})$ a $n \times n$ antisymmetric matrix with entries in $\Bbbk$, and $\lambda \in \Bbbk$. Denoting $q_{ij} = e(p_{ij}) \in R := \Bbbk[[h]]$, the matrix $\boldsymbol{q} = e(\boldsymbol{\lambda}) = (q_{ij})$ is multiplicatively antisymmetric with entries in R. We also introduce $\mu = e(\lambda) \in R$ and consider the multiparameter quantum $n \times n$ matrix algebra $B := \mathscr{O}_{\boldsymbol{q},\mu}(M_n(R))$, which is by definition the noncommutative algebra generated over R by n^2 generators t_{ij}, $1 \leq i, j \leq n$, with relations:

$$t_{lm}t_{ij} = \begin{cases} q_{li}q_{jm}\, t_{ij}t_{lm} + (\mu - 1)q_{li}\, t_{im}t_{lj} & \text{if } l > i \text{ and } m > j, \\ \mu\, q_{li}q_{jm}\, t_{ij}t_{lm} & \text{if } l > i \text{ and } m \leq j, \\ q_{jm}\, t_{ij}t_{lm} & \text{if } l = i \text{ and } m > j. \end{cases} \tag{17}$$

The standard single parameter relations are recovered when $\mu = q^{-2}$ and $q_{ij} = q$ for all $i > j$. When $\mu = 1$, we just have a multiparameter quantum affine n^2-space $\mathscr{O}_{\boldsymbol{q}}(R^{n^2})$ in the sense of previous example, for suitable $\boldsymbol{q}$. It is easy to observe that B is a free R-module (for instance viewing B as an noncommutative iterated Ore extension of R). By calculations similar to those in example 3, we check that $B/hB = \mathscr{O}(M_n(\Bbbk))$ is the commutative algebra $B/hB = \mathscr{O}(M_n(\Bbbk))$ in n^2 generators x_{ij} $(1 \leq i, j \leq n)$ over $\Bbbk$, and the Poisson structure deduced on it by the semiclassical limit process is defined by:

$$\{x_{lm}, x_{ij}\} = \begin{cases} (p_{li} + p_{jm})x_{ij}x_{lm} + \lambda x_{im}x_{lj} & \text{if } l > i \text{ and } m > j \\ (\lambda + p_{li} + p_{jm})x_{ij}x_{lm} & \text{if } l > i \text{ and } m \leq j \\ p_{jm}x_{ij}x_{lm} & \text{if } l = i \text{ and } m > j. \end{cases} \tag{18}$$

Using notation $\mathscr{O}(M_n(\Bbbk))_{\boldsymbol{p},\lambda}$ for this Poisson algebra, we have:

$$\mathscr{O}(M_n(\Bbbk))_{\boldsymbol{p},\lambda} \simeq B/hB \underset{\text{semiclassical limit}}{\overset{\text{quantization}}{\rightleftarrows}} \mathscr{O}_{e(\boldsymbol{p}),e(\lambda)}(M_n(\Bbbk[[h]]^n)) = B$$

Remark. An important observation on relations (18) is the fact that, if $(i,j) <_{\text{lex}} (l,m)$, then $x_{ij}x_{lm} = ax_{lm} + b$ where a, b are polynomials in the generators x_{kh} where $(k,h) <_{\text{lex}} (l,m)$. Hence, when the x_{ij}'s are adjoined in lexicographic order, the Poisson algebra $\mathscr{O}(M_n(\Bbbk))_{\boldsymbol{p},\lambda}$ can be described as an iterated Poisson-Ore extension:

$$\mathscr{O}(M_n(\Bbbk))_{\boldsymbol{p},\lambda} = \Bbbk[x_{11}][x_{12}]_{\sigma_{12},\delta_{12}}[x_{13}]_{\sigma_{13},\delta_{13}} \ldots [x_{nn}]_{\sigma_{nn},\delta_{nn}}, \tag{19}$$

where σ_{lm} is the Poisson derivation and δ_{lm} is the Poisson σ_{lm}-derivation of the subalgebra generated by the x_{ij}'s with $(i,j) <_{\text{lex}} (l,m)$ defined, for any $1 \leq l, m \leq n$, by:

$$\sigma_{lm}(x_{ij}) = \begin{cases} (p_{li}+p_{jm})x_{ij} & \text{if } l > i \text{ and } m > j, \\ (\lambda + p_{li}+p_{jm})x_{ij} & \text{if } l > i \text{ and } m \leq j \\ p_{jm}x_{ij} & \text{if } l = i \text{ and } m > j, \end{cases} \tag{20}$$

$$\delta_{lm}(x_{ij}) = \begin{cases} \lambda x_{im}x_{lj} & \text{if } l > i \text{ and } m > j, \\ 0 & \text{in other cases.} \end{cases} \tag{21}$$

Examples 5. In the continuation of examples 3 and 4, the article [20] studies Poisson analogues of some others significant families of multiparameter quantum algebras (quantum symplectic spaces, quantum even or odd-dimensional euclidian spaces, quantum symmetric and quantum antisymmetric matrices), which are described in terms of semiclassical limits and as iterated Poisson-Ore extensions.

1.2 Rational equivalence of polynomial Poisson algebras

Let A and A' be two Poisson algebras which are domains, with respective fields of fractions K and K' equipped with the canonical extensions of the Poisson brackets. We say that A and A' are rationally equivalent (as Poisson algebras) is there exists a (ring) isomorphism $\varphi : K \to K'$ which is a Poisson morphism, i.e. $\varphi(\{a,b\}_K) = \{\varphi(a), \varphi(b)\}_{K'}$ for all $a, b \in K$.

Remark. In most cases studied in the following, we will have $A = \Bbbk[x_1, \ldots, x_n]$ and $A' = \Bbbk[x'_1, \ldots, x'_n]$ with Poisson brackets defined on the generators:

$$\{x_i, x_j\}_A = f_{ij}(x_1, \ldots, x_n) \quad \text{and} \quad \{x'_i, x'_j\}_{A'} = g_{ij}(x'_1, \ldots, x'_n)$$

for some $f_{ij} \in A$, $g_{ij} \in A'$ for all $1 \leq i, j \leq n$. Then a necessary and sufficient condition for A and A' to be rationally equivalent is the existence of elements $y_1, \ldots, y_n$ in $K = \Bbbk(x_1, \ldots, x_n)$ such that:

$$K = \Bbbk(y_1, \ldots, y_n) \quad \text{and} \quad \{y_i, y_j\}_K = g_{ij}(y_1, \ldots, y_n) \quad \text{for all} \quad 1 \leq i, j \leq n$$

which allows to make explicit a Poisson isomorphism $K \to K'$ (defined by $y_i \mapsto x'_i$ for any $1 \leq i \leq n$).

1.2.1 Poisson-Ore rational functions fields

Notation. Suppose that A is a domain with field of fractions K. If σ is a Poisson derivation and δ is a σ-derivation of A, their canonical extensions into derivations of K also satisfy (7) and (8) for all $a, b \in K$ and for the Poisson bracket extended to K by remark 2 of 1.1.1. Hence we can consider the Poisson-Ore extension $B = K[x]_{\sigma,\delta}$. Applying again remark 2 of 1.1.1 to $\operatorname{Frac} B$, we obtain a Poisson-Ore structure on the field of rational functions $K(x)$ and we naturally denote it by:

$$\operatorname{Frac}(A[x]_{\sigma,\delta}) = \operatorname{Frac}(K[x]_{\sigma,\delta}) = K(x)_{\sigma,\delta}. \tag{22}$$

LEMMA. *We suppose here that the base field $\Bbbk$ is of characteristic zero. Let K be a commutative field with a structure of Poisson algebra over $\Bbbk$. Let σ be a Poisson derivation of K. If σ is not Hamiltonian, then the Poisson center of the field $K(x)_{\sigma,0}$ is:*

$$\mathrm{Z_P}(K(x)_{\sigma,0}) = \mathrm{Z_P}(K) \cap \ker \sigma.$$

PROOF. Let f be a nonzero element of $K(x)$. We can develop it canonically in $K(\!(x)\!)$ as a Laurent series $f = \sum_{n \geq m} a_n x^n$ where $m \in \mathbb{Z}$ is the valuation of f and $a_n \in K$ for any $n \geq m$ with $a_m \neq 0$. Since $\{x, a\} = \sigma(a)x$ for any $a \in K$, the Poisson bracket on $K(x)$ can be extended naturally to $K(\!(x)\!)$. In particular, for any $a \in K$, we have:

$$\{f, a\} = \sum_{n \geq m} \{a_n x^n, a\} = \sum_{n \geq m} (\{a_n, a\}x^n + n a_n \{x, a\} x^{n-1}) = \sum_{n \geq m} (\{a_n, a\} + n a_n \sigma(a)) x^n.$$

If $f \in \mathrm{Z_P}(K(x)_{\sigma,0})$, then $\{f, a\} = 0$ for any $a \in K$ and therefore $n a_n \sigma(a) = \{a, a_n\}$. Since σ is not Hamiltonian and $\operatorname{char} \Bbbk = 0$, it follows that $a_n = 0$ for any $n \neq 0$. In other words, $f = a_0 \in K$, and more precisely $f \in \mathrm{Z_P}(K)$. Now the condition $\{x, f\} = 0$ implies $\sigma(f) = 0$. $\square$

REMARK. It is proved similarly in proposition 2.9 of [36] that: *if δ is a non Hamiltonian Poisson derivation of K, then the Poisson center of $K(x)_{0,\delta}$ is $\mathrm{Z_P}(K(x)_{0,\delta}) = \mathrm{Z_P}(K) \cap \ker \delta$.*

EXAMPLE 1 (*first Poisson-Weyl field*). We consider, as in example 1 of 1.1.1 or (12), the first Poisson-Weyl algebra $\mathbb{S}_1(\Bbbk)$. Its field of fractions (with the extended Poisson structure) will be named here the first Poisson-Weyl field and denoted by $\mathbb{F}_1(\Bbbk)$:

$$\mathbb{F}_1(\Bbbk) = \operatorname{Frac} \mathbb{S}_1(\Bbbk) = \Bbbk(y)(x)_{0,\partial_y} = \Bbbk(x)(y)_{0,-\partial_x}. \tag{23}$$

It would be useful in many circumstances to give another presentation of $\mathbb{F}_1(\Bbbk)$. Set $w = xy$; of course $\Bbbk(x, y) = \Bbbk(x, w)$ and relation $\{x, y\} = 1$ becomes $\{x, w\} = x$. Hence:

$$\mathbb{F}_1(\Bbbk) = \operatorname{Frac} \mathbb{S}_1(\Bbbk) = \Bbbk(w)(x)_{\partial_w,0} = \Bbbk(x)(w)_{0,-x\partial x}. \tag{24}$$

Applying the lemma above, we obtain:

$$\text{if } \Bbbk \text{ is of characteristic zero, then } \mathrm{Z_P}(\mathbb{F}_1(\Bbbk)) = \Bbbk. \tag{25}$$

EXAMPLE 2 (*first Poisson-quantum field*). We consider, as in example 2 of 1.1.1 or (13), the Poisson-quantum plane $\mathbb{P}_2^\lambda(\Bbbk)$ for some $\lambda \in \Bbbk$. Its field of fractions (with the extended Poisson structure) will be named here the first Poisson-quantum field parametrized by λ and denoted by $\mathbb{Q}_2^\lambda(\Bbbk)$:

$$\mathbb{Q}_2^\lambda(\Bbbk) = \operatorname{Frac} \mathbb{P}_2^\lambda(\Bbbk) = \Bbbk(y)(x)_{\lambda y \partial_y, 0}. \tag{26}$$

Applying the lemma above, we obtain:

$$\text{if } \Bbbk \text{ is of characteristic zero, and } \lambda \neq 0, \text{ then } \mathrm{Z_P}(\mathbb{Q}_2^\lambda(\Bbbk)) = \Bbbk. \tag{27}$$

Then we can give the following classification up to rational equivalence of quadratic Poisson polynomial algebras in two variables (see also exercise 1 below).

Proposition. *We suppose here that* $\Bbbk$ *is algebraically closed. Let* $S = \Bbbk[x,y]$ *with a Poisson bracket defined from* $\{x,y\} = P(x,y)$ *for some polynomial* P *of total degree* ≤ 2. *Then,* $\operatorname{Frac} S = \Bbbk(x,y)$ *is Poisson isomorphic to* $\mathbb{F}_1(\Bbbk)$ *or to* $\mathbb{Q}_2^{\lambda}(\Bbbk)$ *for some* $\lambda \in \Bbbk$.

Proof. The proof is by direct calculation of adapted rational generators. We denote $P(x,y) = \alpha x^2 + \beta yx + \gamma y^2 + \lambda x + \mu y + \eta$ with $\alpha, \beta, \gamma, \lambda, \mu, \eta \in \Bbbk$.

• *Case 1.1:* $\alpha = 0$ and $\beta \neq 0$. The element $x' := x + (\beta y + \lambda)^{-1}(\gamma y^2 + \mu y + \eta)$ satisfies $\Bbbk(y)[x] = \Bbbk(y)[x']$ with $\{x', y\} = (\beta y + \lambda)x'$. Then $y' := \beta y + \lambda$ satisfies $\Bbbk(y)[x'] = \Bbbk(y')[x']$ and $\{x', y'\} = \beta x' y'$. We conclude that $\Bbbk(x,y) = \Bbbk(x', y')$ is Poisson isomorphic to $\mathbb{Q}_2^{\beta}(\Bbbk)$.

• *Case 1.2:* $\alpha = \beta = 0$ and $\lambda \neq 0$. The element $x' := x + \lambda^{-1}(\gamma y^2 + \mu y + \eta)$ satisfies $\Bbbk[y,x] = \Bbbk[y,x']$ with $\{x', y\} = \lambda x'$. Then $y' := \lambda^{-1}(x')^{-1} y$ satisfies $\Bbbk(x')[y] = \Bbbk(x')[y']$ and $\{x', y'\} = 1$. We conclude that $\Bbbk(x,y) = \Bbbk(x', y')$ is Poisson isomorphic to $\mathbb{F}_1(\Bbbk)$.

• *Case 1.3:* $\alpha = \beta = \lambda = 0$. The element $x' := (\gamma y^2 + \mu y + \eta)^{-1} x$ satisfies $\Bbbk(y)[x] = \Bbbk(y)[x']$ with $\{x', y\} = 1$. We conclude that $\Bbbk(x,y) = \Bbbk(x', y)$ is Poisson isomorphic to $\mathbb{F}_1(\Bbbk)$.

We suppose in the following that $\alpha \neq 0$. We have: $\{x,y\} = \alpha(x + \frac{\beta y + \lambda}{2\alpha})^2 - \frac{(\beta y+\lambda)^2}{4\alpha} + \gamma y^2 + \eta y + \mu$. The element $x' := x + \frac{\beta y + \lambda}{2\alpha}$ satisfies $\Bbbk[y,x] = \Bbbk[y,x']$ and we compute:

$$\{x', y\} = \{x, y\} = \alpha x'^2 - \tfrac{\beta^2 - 4\alpha\gamma}{4\alpha} y^2 - \tfrac{\beta\lambda - 2\alpha\eta}{2\alpha} y - \tfrac{\lambda^2 - 4\alpha\mu}{4\alpha}.$$

• *Case 2.1:* $\beta^2 \neq 4\alpha\gamma$. Then this bracket can be rewritten:

$$\{x', y\} = \alpha x'^2 - \tfrac{\beta^2 - 4\alpha\gamma}{4\alpha}\left[y^2 + 2\tfrac{\beta\lambda - 2\,\alpha\eta}{\beta^2 - 4\alpha\gamma}\, y\right] - \tfrac{\lambda^2 - 4\alpha\mu}{4\alpha}.$$

The element $y' := y + \frac{\beta\lambda - 2\alpha\eta}{\beta^2 - 4\alpha\gamma}$ satisfies $\Bbbk[y, x'] = \Bbbk[y', x']$ and $\{x', y'\} = \{x', y\} = \alpha x'^2 - \zeta y'^2 + \xi$, where $\zeta = \frac{\beta^2 - 4\alpha\gamma}{4\alpha} \in \mathbb{C}^{\times}$ and $\xi \in \mathbb{C}$ deduced from the resting terms. We choose $\alpha', \zeta' \in \mathbb{C}^{\times}$ square roots of α, ζ and we define $x'' := \alpha' x' - \zeta' y'$ and $y'' := \alpha' x' + \zeta' y'$, so that $\Bbbk[y', x'] = \Bbbk[y'', x'']$ with relation $\{x'', y''\} = 2\alpha'\zeta'\{x', y'\} = 2\alpha'\zeta' x'' y'' + 2\alpha'\zeta'\xi$. A last change of variable $x''' := x'' + \xi y''^{-1}$ leads to $\Bbbk(y'')(x'') = \Bbbk(y'')(x''')$ and $\{x''', y''\} = 2\alpha'\zeta' x''' y''$. We conclude that $\Bbbk(x,y) = \Bbbk(x''', y'')$ is Poisson isomorphic to $\mathbb{Q}_2^{2\alpha'\zeta'}(\Bbbk)$.

• *Case 2.2:* $\beta^2 = 4\alpha\gamma$ and $\beta\lambda \neq 2\alpha\eta$. We have $\{x', y\} = \frac{2\alpha\eta - \beta\lambda}{2\alpha} y + \alpha x'^2 - \frac{\lambda^2 - 4\alpha\mu}{4\alpha}$. By a first polynomial change of variable $y' := y + (\alpha x'^2 - \frac{\lambda^2 - 4\alpha\mu}{4\alpha})(\frac{2\alpha}{\beta\lambda - 2\alpha\eta})$, we have $\Bbbk[y, x'] = \Bbbk[y', x']$ and $\{x', y'\} = \frac{2\alpha\eta - \beta\lambda}{2\alpha} y'$. Setting $x'' := (\frac{2\alpha}{\beta\lambda - 2\alpha\eta}) y'^{-1} x'$, we obtain $\Bbbk(y')[x'] = \Bbbk(y')[x'']$ and $\{x'', y'\} = 1$. Hence $\Bbbk(x,y) = \Bbbk(x'', y')$ is Poisson isomorphic to $\mathbb{F}_1(\Bbbk)$.

• *Case 2.3:* $\beta^2 = 4\alpha\gamma$ and $\beta\lambda = 2\alpha\eta$. The bracket $\{x', y\} = \alpha x'^2 - \frac{\lambda^2 - 4\alpha\mu}{4\alpha}$ reduces to $\{x', y'\} = 1$ by the rational change of variable $y' = (\alpha x'^2 - \frac{\lambda^2 - 4\alpha\mu}{4\alpha})^{-1} y$, and we conclude that $\Bbbk(x,y) = \Bbbk(x', y')$ is Poisson isomorphic to $\mathbb{F}_1(\Bbbk)$. □

Example 3 (*Poisson-Weyl fields*). The n-th Poisson-Weyl field $\mathbb{F}_n(\Bbbk)$ is the field of fractions of the n-th Poisson-Weyl algebra defined in example 1 of 1.1.1 or by (12):

$$\mathbb{F}_n(\Bbbk) = \operatorname{Frac} \mathbb{S}_n(\Bbbk) = \Bbbk(y_1, \ldots, y_n)(x_1)_{0,\partial_{y_1}}(x_2)_{0,\partial_{y_2}} \ldots (x_n)_{0,\partial_{y_n}}, \tag{28}$$

The products $w_i = x_i y_i$ for all $1 \le i \le n$ satisfy the relations

$$\{x_i, w_i\} = x_i, \quad \{x_i, w_j\} = \{w_i, w_j\} = 0 \text{ if } j \ne i, \tag{29}$$

and provide an alternative presentation as a field of fractions of an iterated Poisson-Ore extension:

$$\mathbb{F}_n(\Bbbk) = \Bbbk(w_1, w_2, \dots, w_n)(x_1)_{\partial_{w_1},0}(x_2)_{\partial_{w_2},0} \cdots (x_n)_{\partial_{w_n},0}. \tag{30}$$

If we replace $\Bbbk$ by a purely transcendental extension $K = \Bbbk(z_1, z_2, \dots, z_t)$ of degree t of $\Bbbk$, the Poisson field $\mathbb{F}_n(K)$ is denoted by $\mathscr{F}_{n,t}(\Bbbk)$. By convention, we set $\mathscr{F}_{0,t}(\Bbbk) = K$. To sum up:

$$\mathscr{F}_{n,t}(\Bbbk) = \mathbb{F}_n(\Bbbk(z_1, \dots, z_t)) \quad \text{for all} \ \ t \ge 0, n \ge 0. \tag{31}$$

One can prove using inductively the lemma above that:

$$\text{if } \ \Bbbk \text{ is of characteristic zero, then } \ \mathrm{Z_P}(\mathscr{F}_{n,t}(\Bbbk)) = \Bbbk(z_1, \dots, z_t). \tag{32}$$

Example 4 (*Poisson-quantum fields*). For any $n \times n$ antisymmetric matrix $\boldsymbol{\lambda} = (\lambda_{ij})$ with entries in $\Bbbk$, the n-th Poisson-quantum field $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ is the field of fractions of the n-th Poisson-quantum space defined in example 2 of 1.1.1 or by (13):

$$\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk) = \Bbbk(x_1)_{\sigma_1,0}(x_2)_{\sigma_2,0} \dots (x_n)_{\sigma_n,0}, \tag{33}$$

with $\sigma_i = \sum_{1 \le j < i} \lambda_{ij} x_j \partial_{x_j}$, traducing the brackets $\{x_i, x_j\} = \lambda_{ij} x_j x_i + 0$.

It will be useful in the following to observe that the bracket of two monomials in $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ can be expressed by:

$$\{\boldsymbol{x}^a, \boldsymbol{x}^b\} = \sum_{1 \le l,m \le n} a_l b_m \lambda_{lm} \boldsymbol{x}^{a+b} = (a\, \boldsymbol{\lambda}\, b^{\mathrm{tr}})\, \boldsymbol{x}^{a+b} \quad \text{for all } a, b \in \mathbb{Z}^n, \tag{34}$$

with the global notation $\boldsymbol{x}^a = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ for $a = (a_1, a_2, \dots, a_n)$.

Exercise 1 (example in degree 3 on the plane). Let $S = \Bbbk[x, y]$ with bracket:

$$\{x, y\} = \lambda(y - \alpha)(y - \beta)x,$$

for some $\alpha, \beta, \lambda \in \Bbbk, \lambda \ne 0$. Prove that $\operatorname{Frac} S = \Bbbk(x, y)$ is Poisson isomorphic to $\mathbb{F}_1(\Bbbk)$ when $\alpha = \beta$, and to $\mathbb{Q}_2^{\lambda'}(\Bbbk)$ for $\lambda' = \lambda(\alpha - \beta)$ when $\alpha \ne \beta$.

Hint: in the first case, take $\Bbbk(x, y) = \Bbbk(x, y')$ for $y' = (y - \alpha)x$ and then $\Bbbk(x, y') = \Bbbk(x', y')$ for $x' = \lambda^{-1} x y'^{-2}$; in the second case, take $\Bbbk(x, y) = \Bbbk(x, y'')$ for $y'' = (y - \alpha)(y - \beta)^{-1} x$.

Exercise 2 (2×2 matrices). Let S be the Poisson algebra defined on the coordinate algebra $\Bbbk[x, y, z, t]$ of the 2×2 matrices with entries in $\Bbbk$ (see [27], or above example 4 of 1.1.3 with particular values $n = 2, p_{12} = -2 = -p_{21}, \lambda = -4$) by:

$$\begin{array}{lll} \{x, y\} = 2xy, & \{x, z\} = 2xz, & \{x, t\} = 4yz, \\ \{y, z\} = 0, & \{y, t\} = 2yt, & \{z, t\} = 2zt. \end{array}$$

Prove that $F = \operatorname{Frac} S$ is Poisson isomorphic to $\mathbb{Q}_2^\lambda(K)$ for $\lambda = 2$ and K a purely transcendental extension of degree 2 of $\Bbbk$.

Hint: check that $u := xt - yz \in \mathrm{Z_P}(S)$ and $v := y^{-1}z \in \mathrm{Z_P}(F)$; deduce that $F = \Bbbk(u,v)(y)(x)_{\sigma,0}$ for $\sigma = 2y\partial_y$.

Exercise 3 (semi-classical limits of 3-Calabi-Yau algebras). Two types of Poisson algebras arise from the classification of [8]:

the "jordanian" $A = \Bbbk[x,y,z]$ with $\{z,y\} = 2xz$, $\{z,x\} = 0$, $\{y,x\} = x^2$,

the "multiplicative" $S = \Bbbk[x,y,z]$ with $\{x,y\} = xy$, $\{y,z\} = yz$, $\{z,x\} = zx$.

Prove that $\operatorname{Frac} A$ is Poisson isomorphic to $\mathscr{F}_{1,1}(\Bbbk)$, and $\operatorname{Frac} S$ is Poisson isomorphic to $\mathbb{Q}_2^\lambda(K)$ for $\lambda = 1$ and K a purely transcendental extension of degree 1 of $\Bbbk$.

Hint: check that $u := x^2z$ lies in the Poisson center of A, and $v := xyz$ lies in the Poisson center of S.

1.2.2 Rational classification and rational separation results

In all this section, we suppose that the base field $\Bbbk$ is of characteristic zero.

We address here the question of when Poisson-Weyl fields or Poisson-quantum fields are Poisson isomorphic or not. Most results come from [20] and can be viewed as Poisson analogues of noncommutative similar problems studied in [1], [29] or [31]. We start with the following observation:

Observation. *Poisson-Weyl fields $\mathscr{F}_{n,t}(\Bbbk)$ and $\mathscr{F}_{m,s}(\Bbbk)$ are Poisson isomorphic if and only if $m = n$ and $s = t$.*

Proof. By (31) and (32), $2n+t$ is the transcendence degree of $\mathscr{F}_{n,t}(\Bbbk)$ and t is the transcendence degree of its Poisson center. □

The same question for Poisson-quantum fields is of course more difficult since dimensional invariants are not sufficient to take into consideration the complexity of the n^2 parameters of the matrix $\boldsymbol{\lambda}$ encoding the Poisson structure. We will see that it is necessary in this case to consider the set of all matrices $B\,\boldsymbol{\lambda}\,B^{\mathrm{tr}}$ for $B \in M_n(\mathbb{Z})$ as an invariant of $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$. The first step is the following key result:

Proposition 1. *Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ two $n\times n$ antisymmetric matrices with entries in $\Bbbk$. If there exists $A \in \mathrm{GL}_n(\mathbb{Z})$ such that $\boldsymbol{\mu} = A\boldsymbol{\lambda}A^{\mathrm{tr}}$, then $\mathbb{Q}_n^{\boldsymbol{\mu}}(\Bbbk)$ and $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ are Poisson isomorphic.*

Proof. Let $a_1, \dots, a_n$ be the rows of A, set $y_i = \boldsymbol{x}^{a_i}$ for $1 \le i \le n$ and observe using (34) that:

$$\{y_i, y_j\}_{\boldsymbol{\lambda}} = (a_i\boldsymbol{\lambda}a_j^{\mathrm{tr}})y_iy_j = \mu_{ij}y_iy_j \quad \text{for all } 1 \le i,j \le n. \tag{35}$$

A being invertible, the x_i's lie in $\Bbbk(y_1, \dots, y_n)$, so the y_i's are algebraically independent over $\Bbbk$ and $\Bbbk(y_1, \dots, y_n) = \Bbbk(x_1, \dots, x_n)$. The $\Bbbk$-algebra automorphism ϕ defined by $y_i \mapsto x_i$ for any $1 \le i \le n$ becomes a Poisson isomorphism $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk) \to \mathbb{Q}_n^{\boldsymbol{\mu}}(\Bbbk)$ since we have:

$$\phi(\{y_i, y_j\}_{\boldsymbol{\lambda}}) = \phi(\mu_{ij}y_iy_j) = \mu_{ij}\phi(y_i)\phi(y_j) = \mu_{ij}x_ix_j = \{x_i, x_j\}_{\boldsymbol{\mu}} = \{\phi(y_i), \phi(y_j)\}_{\boldsymbol{\mu}},$$

□

NOTATIONS. We introduce for a Poisson-quantum field $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ the two following adaptations of the noncommutative rational invariants of [1].

(i) $B_{\boldsymbol{\lambda}}$ is the $\Bbbk$-vector space $\mathscr{B}_{\boldsymbol{\lambda}} \cap \Bbbk$, where $\mathscr{B}_{\boldsymbol{\lambda}} = \{\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk), \mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)\}$ is the $\Bbbk$-subspace of $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ generated by all brackets $\{f, g\}$ for $f, g \in \mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$.

(ii) $C_{\boldsymbol{\lambda}}$ is the subset $\mathscr{C}_{\boldsymbol{\lambda}} \cap M_n(\Bbbk)$, where $\mathscr{C}_{\boldsymbol{\lambda}}$ is the subset of $M_n(\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk))$ whose elements are all matrices $(\{f_i, f_j\} f_i^{-1} f_j^{-1})_{1 \le i,j \le n}$ for all nonzero elements $f_1, \ldots, f_n$ in $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$.

PROPOSITION 2. *Let $\boldsymbol{\lambda}$ be a $n \times n$ antisymmetric matrix with entries in $\Bbbk$. Then:*

$$B_{\boldsymbol{\lambda}} = (0) \quad \textit{and} \quad C_{\boldsymbol{\lambda}} = \{A\, \boldsymbol{\lambda}\, A^{\mathrm{tr}}\,;\ A \in M_n(\mathbb{Z})\}.$$

PROOF. Here $K = \mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk) = \Bbbk(x_1, \ldots, x_n)$ with Poisson bracket (6), or more generally (34). Put the lexicographic order on $\mathbb{Z}^n$ and denote by L the corresponding Hahn-Laurent power series field in $x_1, \ldots, x_n$. The elements of L are formal series $\sum_{a \in I} \alpha_a \boldsymbol{x}^a$ where I is a well-ordered subset of $\mathbb{Z}^n$ and the coefficients α_a lie in $\Bbbk$. The field L contains the subring of Laurent polynomials $\Bbbk[x_1^{\pm}, \ldots, x_n^{\pm 1}]$ (corresponding to the series whose support is finite) and then its field of fractions K. The Poisson bracket (6) extends to L by setting:

$$\{f, g\} = \sum_{i,j=1}^{n} \lambda_{ij} x_i x_j \partial_{x_i}(f) \partial_{x_j}(g) \quad \text{for all} \quad f, g \in L,$$

which is well-defined in L because the supports of $x_i \partial_{x_i}(f)$ and $x_j \partial_{x_j}(g)$ are contained in those of f and g.

• Let $f = \sum_{a \in I} \alpha_a \boldsymbol{x}^a$ and $g = \sum_{b \in J} \beta_b \boldsymbol{x}^b$ be two elements in L, for I, J well-ordered subset of $\mathbb{Z}^n$ and $\alpha_a, \beta_b \in \Bbbk$. Since $x_i \partial_{x_i}(\alpha_a \boldsymbol{x}^a) = a_i \alpha_a \boldsymbol{x}^a$, we have:

$$\{f, g\} = \sum_{i,j=1}^{n} \lambda_{ij} \big(\sum_{a \in I} a_i \alpha_a \boldsymbol{x}^a\big)\big(\sum_{b \in J} b_j \beta_b \boldsymbol{x}^b\big) = \sum_{a \in I, b \in J} \big(\sum_{i,j=1}^{n} \lambda_{ij} a_i b_j\big) \alpha_a \beta_b \boldsymbol{x}^{a+b}. \tag{36}$$

Denoting by $\pi : L \to \Bbbk$ the $\Bbbk$-linear map that gives the constant term (i.e. the coefficient of $\boldsymbol{x}^{\mathbf{0}}$ in the development), we deduce that:

$$\pi(\{f, g\}) = \sum_{\substack{a \in I, j \in J \\ a+b=\mathbf{0}}} \big(\sum_{i,j=1}^{n} \lambda_{ij} a_i b_j\big) \alpha_a \beta_b = \sum_{a \in I \cap (-J)} \big(-\sum_{i,j=1}^{n} \lambda_{ij} a_i a_j\big) \alpha_a \beta_{-a}.$$

Since λ is antisymmetric, we have $\lambda_{ij} a_i a_j + \lambda_{ji} a_j a_i = 0$ for all $i, j = 1, \ldots, n$ and then each of the sums $\sum_{i,j=1}^{n} \lambda_{ij} a_i a_j$ is zero. Thus $\pi(\{f, g\}) = 0$. We conclude that $B_{\boldsymbol{\lambda}} = (0)$.

• The identity (35) doesn't depend on the the property for A to be invertible or not. Therefore we have $A \boldsymbol{\lambda} A^{\mathrm{tr}} \in M_n(\Bbbk)$ for any $A \in M_n(\mathbb{Z})$. Hence it suffices to show that for any $(y_1, \ldots, y_n) \in (L^\times)^n$, the matrix $\big(\pi(\{y_i, y_j\}(y_i y_j)^{-1})\big)$ has the form $A \boldsymbol{\lambda} A^{\mathrm{tr}}$ for some $A \in M_n(\mathbb{Z})$.

Write each $y_i = \sum_{a \in I(i)} \alpha_{i,a} \boldsymbol{x}^a$ where $I(i)$ is a well-ordered subset of $\mathbb{Z}^n$ with minimum element $m(i)$ and coefficients $\alpha_{i,a}$ in $\Bbbk$ such that $\alpha_{i,m(i)} \neq 0$. Then $y_i^{-1} = \sum_{b \in J(i)} \beta_{i,b} \boldsymbol{x}^b$ where $J(i)$ is a well-ordered subset of $\mathbb{Z}^n$ with minimum element $-m(i)$ and coefficients $\beta_{i,b}$ in $\Bbbk$ satisfying $\beta_{i,-m(i)} = \alpha_{i,m(i)}^{-1}$. For all $1 \le i, j \le n$, any element $c \in \mathbb{Z}^n$ in the support of the series $\{y_i, y_j\}$ satisfies $c \ge m(i) + m(j)$, see (36), and so applying (34), we have:

$$\pi(\{y_i, y_j\}(y_i y_j)^{-1}) = \{\boldsymbol{x}^{m(i)}, \boldsymbol{x}^{m(j)}\} \boldsymbol{x}^{-m(i)-m(j)} = m(i) \boldsymbol{\lambda} m(j)^{\mathrm{tr}}.$$

Finally $\left(\pi(\{y_i, y_j\}(y_iy_j)^{-1})\right)_{1\leq i,j\leq n} = A\boldsymbol{\lambda}A^{\mathrm{tr}}$ where A is the matrix with rows $m(1),\dots,m(n)$. $\square$

Corollary 1. *Any Poisson-quantum field $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ is not Poisson isomorphic to any Poisson-Weyl field $\mathscr{F}_{m,t}(\Bbbk)$ such that $m \geq 1$.*

Proof. It follows from equality $B_{\boldsymbol{\lambda}} = (0)$ that $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ doesn't contain elements a, b such that $\{a, b\} = 1$. $\square$

Observe that the proof of the corollary shows more generally that a Poisson algebra containing two elements a, b such that $\{a, b\} = 1$ cannot be embedded in a Poisson-quantum field.

Corollary 2. *Let λ, μ be two elements of $\Bbbk$. Then $\mathbb{Q}_2^{\lambda}(\Bbbk)$ and $\mathbb{Q}_2^{\mu}(\Bbbk)$ are Poisson isomorphic if and only if $\lambda = \pm\mu$.*

Proof. Assume $\lambda = -\mu$. The $\Bbbk$-algebra automorphism ϕ of $\Bbbk(x, y)$ defined by $x \mapsto x, y \mapsto y^{-1}$ becomes a Poisson isomorphism of $\mathbb{Q}_2^{\lambda}(\Bbbk)$ onto $\mathbb{Q}_2^{\mu}(\Bbbk)$ because

$$\{\phi(x), \phi(y)\}_{\boldsymbol{\mu}} = \{x, y^{-1}\}_{\boldsymbol{\mu}} = -\{x, y\}_{\boldsymbol{\mu}}y^{-2} = -\mu xy^{-1} = \lambda xy^{-1} = \lambda\phi(x)\phi(y) = \phi(\{x, y\}_{\boldsymbol{\lambda}}).$$

Conversely assume that $\mathbb{Q}_2^{\lambda}(\Bbbk)$ and $\mathbb{Q}_2^{\mu}(\Bbbk)$ are Poisson isomorphic. By second point of proposition 2, we have $\{A\boldsymbol{\lambda}A^{\mathrm{tr}}\,;\, A \in M_2(\mathbb{Z})\} = \{B\boldsymbol{\mu}B^{\mathrm{tr}}\,;\, B \in M_2(\mathbb{Z})\}$, with $\boldsymbol{\lambda} = \begin{pmatrix} 0, \lambda \\ -\lambda, 0 \end{pmatrix}$ and $\boldsymbol{\mu} = \begin{pmatrix} 0, \mu \\ -\mu, 0 \end{pmatrix}$. From which we see that $\mathbb{Z}\lambda = \mathbb{Z}\mu$, and finally $\lambda = -\mu$. $\square$

Thus, the answer to the question of the converse of proposition 1 above is complete for $n = 2$. For arbitrary n, three significant particular situations are addressed on the following theorem.

Theorem (K. Goodearl, S. Launois). *Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two antisymmetric $n \times n$ matrices with entries in $\Bbbk$. Assume that one of the following holds:*

(i) $\boldsymbol{\lambda} \in \mathrm{GL}_n(\Bbbk)$;

(ii) *the additive subgroup of $\Bbbk$ generated by the n^2 entries of $\boldsymbol{\lambda}$ is cyclic;*

(iii) *the additive subgroup of $\Bbbk$ generated by the n^2 entries of $\boldsymbol{\lambda}$ is free abelian of rank $\frac{n(n-1)}{2}$.*

Then, $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ and $\mathbb{Q}_n^{\boldsymbol{\mu}}(\Bbbk)$ are Poisson isomorphic if and only if there exists $A \in \mathrm{GL}_n(\mathbb{Z})$ such that $\boldsymbol{\mu} = A\boldsymbol{\lambda}A^{\mathrm{tr}}$

Proof. We follow [20]. Sufficiency is given by proposition 1 independently of assumptions (i), (ii) or (iii). So we suppose that $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ and $\mathbb{Q}_n^{\boldsymbol{\mu}}(\Bbbk)$ are Poisson isomorphic. Hence by second point of proposition 2, there exists $A, B \in M_n(\mathbb{Z})$ such that $\boldsymbol{\mu} = A\boldsymbol{\lambda}A^{\mathrm{tr}}$ and $\boldsymbol{\lambda} = B\boldsymbol{\mu}B^{\mathrm{tr}}$. Then $\boldsymbol{\lambda} = (BA)\boldsymbol{\lambda}(BA)^{\mathrm{tr}}$.

(i) Assume $\boldsymbol{\lambda} \in \mathrm{GL}_n(k)$. The identity $\boldsymbol{\lambda} = BA\boldsymbol{\lambda}(BA)^{\mathrm{tr}}$ implies $\det(BA)^2 = 1$, so $A, B \in \mathrm{GL}_n(\mathbb{Z})$.

(ii) Assume that there exists some $\lambda \in \Bbbk$ such that $\sum_{1\le i,j\le n} \mathbb{Z}\lambda_{ij} = \mathbb{Z}\lambda$. If $\lambda = 0$, then $\boldsymbol{\lambda} = \mathbf{0}$; the Poisson bracket on $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ is the trivial one, and similarly the Poisson bracket on $\mathbb{Q}_n^{\boldsymbol{\mu}}(\Bbbk)$ by isomorphism, i.e. $\boldsymbol{\mu} = \mathbf{0}$. Finally $\boldsymbol{\lambda} = I_n \boldsymbol{\mu} I_n^{\rm tr}$.
We suppose now that $\lambda \neq 0$. Since $\lambda_{ij} \in \lambda\mathbb{Z}$ for all $1 \le i, j \le n$, the matrix $\lambda^{-1}\boldsymbol{\lambda}$ is antisymmetric with integer entries. By a classical argument of linear algebra on integer matrices[2], its rank is even (denote it by $2r$) and there exists a matrix $C \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$C(\lambda^{-1}\boldsymbol{\lambda})C^{\rm tr} = \begin{pmatrix} 0 & d_1 & 0 & \cdots & & & \cdots & 0 \\ -d_1 & 0 & & & & & & \vdots \\ 0 & & \ddots & & & & & \\ \vdots & & & 0 & d_r & & & \\ & & & -d_r & 0 & & & \\ & & & & & 0 & & \\ \vdots & & & & & & \ddots & \vdots \\ 0 & \cdots & & & & & \cdots & 0 \end{pmatrix} := \begin{pmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \text{with } \Lambda \in \mathrm{GL}_{2r}(\mathbb{Z}),$$

for some nonzero integers $d_1, \ldots, d_r$ (satisfying moreover $d_k | d_{k+1}$ for $k = 1, 2, \ldots r-1$). Since $C \in \mathrm{GL}_n(\mathbb{Z})$ we may replace $\boldsymbol{\lambda}$ by $C(\lambda^{-1}\boldsymbol{\lambda})C^{\rm tr}$ and there is no loss of generality assuming that $\boldsymbol{\lambda} = \left(\begin{smallmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right)$. The equations $\boldsymbol{\mu} = A\boldsymbol{\lambda}A^{\rm tr}$ and $\boldsymbol{\lambda} = B\boldsymbol{\mu}B^{\rm tr}$ imply that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same rank $2r$, and that $\sum_{1\le i,j\le n} \mathbb{Z}\mu_{ij} = \sum_{1\le i,j\le n} \mathbb{Z}\lambda_{ij} = \mathbb{Z}\lambda$. On the same way, we assume without loss of generality that $\boldsymbol{\mu} = \left(\begin{smallmatrix} M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right)$ for a suitable matrix $M \in \mathrm{GL}_{2r}(k)$. Write A, B in block form as:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where $A_{11}, B_{11} \in M_{2r}(\mathbb{Z})$. The equalities $\boldsymbol{\mu} = A\boldsymbol{\lambda}A^{\rm tr}$ and $\boldsymbol{\lambda} = B\boldsymbol{\mu}B^{\rm tr}$ become :

$$\begin{pmatrix} M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} A_{11}\Lambda A_{11}^{\rm tr} & A_{11}\Lambda A_{21}^{\rm tr} \\ A_{21}\Lambda A_{11}^{\rm tr} & A_{21}\Lambda A_{21}^{\rm tr} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} B_{11}MB_{11}^{\rm tr} & B_{11}MB_{21}^{\rm tr} \\ B_{21}MB_{11}^{\rm tr} & B_{21}MB_{21}^{\rm tr} \end{pmatrix}.$$

A first consequence is that $M = A_{11}\Lambda A_{11}^{\rm tr}$, and then $\boldsymbol{\mu} = E\boldsymbol{\lambda}E^{\rm tr}$ with $E := \left(\begin{smallmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & I_{n-2r} \end{smallmatrix}\right) \in M_n(\mathbb{Z})$. A second consequence is that $\Lambda = (B_{11}A_{11})\Lambda(B_{11}A_{11})^{\rm tr}$ and then (using the determinant) $A_{11} \in \mathrm{GL}_{2r}(\mathbb{Z})$. We conclude that E lies in $\mathrm{GL}_n(\mathbb{Z})$ and the proof of (ii) is complete.

(iii) We suppose now that the abelian group $G := \sum_{1\le i,j\le n} \mathbb{Z}\lambda_{ij}$ is free of rank $n(n-1)/2$. Since $\boldsymbol{\lambda}$ is antisymmetric, G is generated by the λ_{ij}'s for $i < j$, the assumption of rank $n(n-1)/2$ implies that $B = \{\lambda_{ij}\,;\, 1 \le i < j \le n\}$ is a $\mathbb{Z}$-basis of G. As noted in the proof of part (ii), we have also $G = \sum_{1\le i,j\le n} \mathbb{Z}\mu_{ij}$ and then $C = \{\mu_{ij}\,;\, 1 \le i < j \le n\}$ is another $\mathbb{Z}$-basis of G.
Identify $\boldsymbol{\lambda}$ with the $\Bbbk$-linear map $\Bbbk^n \to \Bbbk^n$ given by left multiplication of $\boldsymbol{\lambda}$ on column vectors. Let $a = (a_1, \ldots, a_n)$ an element of $\mathbb{Z}^n \cap \ker \boldsymbol{\lambda}$. We have $\lambda_{12}a_2 + \cdots + \lambda_{1n}a_n = 0$. Since $\lambda_{12}, \ldots, \lambda_{1n}$ are $\mathbb{Z}$-linearly independent, it follows $a_2 = \cdots = a_n = 0$. Similarly, $\lambda_{21}a_1 + \lambda_{22}a_2 + \cdots + \lambda_{2n}a_n = \lambda_{21}a_1 = 0$ implies $a_1 = 0$ since $\lambda_{21} = -\lambda_{12} \neq 0$ (as an element of the basis B). We conclude that $\mathbb{Z}^n \cap \ker \boldsymbol{\lambda} = \{\mathbf{0}\}$. Since $\boldsymbol{\lambda} = BA\boldsymbol{\lambda}(BA)^{\rm tr}$, we deduce $\mathbb{Z}^n \cap \ker(AB)^{\rm tr} = \{\mathbf{0}\}$. But $(BA)^{\rm tr}$ is an integer matrix, so we obtain $\det(BA)^{\rm tr} \neq 0$ and thus $\det(BA) \neq 0$. Write $(BA) = (d_{ij})_{1\le i,j\le n}$ and compare entries in equation $\boldsymbol{\lambda} = BA\boldsymbol{\lambda}(BA)^{\rm tr}$:

$$\lambda_{ij} = \sum_{1\le l,m\le n} d_{il}\lambda_{lm}d_{jm} = \sum_{1\le l<m\le n} d_{il}\lambda_{lm}d_{jm} + \sum_{1\le m<l\le n} d_{il}\lambda_{lm}d_{jm} = \sum_{1\le l<m\le n} (d_{il}d_{jm} - d_{im}d_{jl})\lambda_{lm}$$

[2]see for instance theorem IV.1 in: M. Newman, *Integral Matrices*, Academic Press, New-York, 1972

for all $1 \leq i, j \leq n$. The λ_{lm}'s such that $l < m$ being $\mathbb{Z}$-linearly independent, we obtain:

$$d_{il}d_{jm} - d_{im}d_{jl} = \delta_{il}\delta_{jm} \qquad \text{for all } 1 \leq i < j \leq n \text{ and } 1 \leq l < m \leq n.$$

It follows from Laplace relations that all the 2×2 and larger minors of BA for which the row and column index sets differ must vanish. In particular, this implies that the adjoint matrix $D = \operatorname{adj}(BA)$ is diagonal. Since $BAD = \det(AB)I_n$ and $\det(AB) \neq 0$, we conclude that BA is diagonal. The equation $\boldsymbol{\lambda} = BA\boldsymbol{\lambda}(BA)^{\mathrm{tr}}$ reduces to $\lambda_{ij} = d_{ii}\lambda_{ij}d_{jj}$ for all i, j whence $d_{ii}d_{jj} = 1$ for all $i < j$. Since $n \geq 2$ and the d_{ii}'s are integers, $d_{ii} = \pm 1$ for all i, whence $BA \in \mathrm{GL}_n(\mathbb{Z})$. We conclude that $A \in \mathrm{GL}_n(\mathbb{Z})$, and the proof is complete □

To the best of our knowledge, the natural question of a similar result for any arbitrary antisymmetric matrices $\boldsymbol{\lambda}, \boldsymbol{\mu}$ remains still open.

2 Poisson analogues of Gel'fand-Kirillov problem

2.1 Rational equivalence for semiclassical limits of enveloping algebras

The framework of this section is the Kirillov-Kostant-Souriau Poisson structure defined in example 1 p. 6. We fix a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{k}$ (supposed of characteristic zero). The symmetric algebra $S(\mathfrak{g})$ is the quotient $S(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x)$ of the tensor algebra $T(\mathfrak{g})$. For any $\mathbb{k}$-basis $(x_1, \ldots, x_n)$ of $\mathfrak{g}$, the monomials $(x_1^{m_1} \ldots x_n^{m_n})_{m_1,\ldots,m_n \in \mathbb{N}}$ are a $\mathbb{k}$-basis of $S(\mathfrak{g})$. Hence $S(\mathfrak{g})$ is isomorphic to the commutative[3] polynomial algebra $\mathbb{k}[x_1, \ldots, x_n]$. The Poisson bracket on $S(\mathfrak{g})$ is defined from:

$$\{x, y\} = [x, y]_{\mathfrak{g}} \qquad \text{for all } x, y \in \mathfrak{g},$$

extended by Leibniz rule into:

$$\{x, x_1^{m_1} \ldots x_n^{m_n}\} = \sum_{i=1}^{n} m_i \big(\textstyle\prod_{j \neq i} x_j^{m_j}\big) x_i^{m_i - 1}\{x, x_i\} \quad \text{for all } x \in \mathfrak{g},\ m_1, \ldots, m_n \in \mathbb{N}, \tag{37}$$

and finally by linearity and antisymmetry to define $\{x, y\}$ for all $x, y \in S(\mathfrak{g})$. Following remark 2 p. 2, the Poisson structure also extends canonically to the field of fractions $\mathbb{k}(x_1, \ldots, x_n)$. We denote by $L(\mathfrak{g})$ the field $\operatorname{Frac} S(\mathfrak{g})$ with this extended Poisson structure.

2.1.1 Problem

We start with some toy examples.

Example 1 (Solvable case). We consider the nonabelian twodimensional Lie algebra $\mathfrak{g}_2 = \mathbb{k}x \oplus \mathbb{k}y$, the symmetric algebra $S = S(\mathfrak{g}_2) = \mathbb{k}[x, y]$ and its field of fractions

[3] Similarly, for the enveloping algebra $U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y]_{\mathfrak{g}})$, the Poincaré-Birkhoff-Witt theorem asserts that $U(\mathfrak{g})$ is the noncommutative polynomial algebra with $\mathbb{k}$-basis $(x_1^{m_1} \ldots x_n^{m_n})_{m_1,\ldots,m_n \in \mathbb{N}}$ and multiplication defined from the commutation law $xy - yx = [x, y]_{\mathfrak{g}}$ for $x, y \in \mathfrak{g}$.

$L(\mathfrak{g}_2) = \mathbb{k}(x, y)$ with the Poisson bracket defined from $\{x, y\} = x$. Introducing (see proposition 1.2.1) the rational variable $y' = yx^{-1}$ which satisfies $\mathbb{k}(x, y) = \mathbb{k}(x, y')$ and $\{x, y'\} = 1$, we conclude that $L(\mathfrak{g}_2)$ is Poisson isomorphic to $\mathbb{F}_1(\mathbb{k})$.

Example 2 (Nilpotent case). We consider the 3-dimensional Heisenberg Lie algebra $\mathfrak{sl}_3^+ = \mathbb{k}x \oplus \mathbb{k}y \oplus \mathbb{k}z$, the symmetric algebra $S = S(\mathfrak{sl}_3^+) = \mathbb{k}[z, y, x]$ and its field of fractions $L(\mathfrak{sl}_3^+) = \mathbb{k}(x, y, z)$ with the Poisson bracket defined from $\{x, z\} = \{y, z\} = 0,\ \{x, y\} = z$. $L(\mathfrak{sl}_3^+)$ is the Poisson-Ore rational functions field $\mathbb{k}(z, y)(x)_{0,\delta}$ where the Poisson bracket on $\mathbb{k}(z, y)$ is the trivial one and $\delta = z\partial_y$. Then $L(\mathfrak{sl}_3^+) = \mathbb{k}(z, y, x')$ for $x' = xz^{-1}$ with $\{x', z\} = \{y, z\} = 0,\ \{x', y\} = 1$. Hence $L(\mathfrak{sl}_3^+)$ is Poisson isomorphic to $\mathscr{F}_{1,1}(\mathbb{k})$.

> Exercise. Set $\mathfrak{h}_n$ the $(2n + 1)$-dimensional Heisenberg Lie algebra, the symmetric algebra $S(\mathfrak{h}_n) = \mathbb{k}[z, y_1 \ldots, y_n, , x_1 \ldots, x_n]$ and $L(\mathfrak{h}_n) = \mathbb{k}(z, y_1 \ldots, y_n, , x_1 \ldots, x_n)$ its field of fractions, with Poisson bracket defined from $\{x_i, z\} = \{y_i, z\} = 0$ and $\{x_i, y_j\} = \delta_{i,j}z$ for $1 \leq i, j \leq n$. Prove that $L(\mathfrak{h}_n)$ is Poisson isomorphic to $\mathscr{F}_{n,1}(\mathbb{k})$.

Example 3 (Semisimple case). We consider here the Lie algebra $\mathfrak{sl}_2 = \mathbb{k}e \oplus \mathbb{k}h \oplus \mathbb{k}f$, the symmetric algebra $S(\mathfrak{sl}_2) = \mathbb{k}[e, h, f]$ and its field of fractions $L(\mathfrak{sl}_2) = \mathbb{k}(e, h, f)$ with the Poisson bracket defined from $\{e, f\} = h,\ \{h, e\} = 2e,\ \{h, f\} = -2f$. The Casimir element $\omega = 4ef + h^2 + 1$ lies in the Poisson center of $S(\mathfrak{sl}_2)$. Observing that $e = \frac{1}{4}(\omega - h^2 - 1)f^{-1}$, we have $L(\mathfrak{sl}_2) = \mathbb{k}(\omega, h, f)$ with Poisson structure $L(\mathfrak{sl}_2) = \mathbb{k}(\omega, h)(f)_{\sigma,0}$ where the Poisson bracket on $\mathbb{k}(\omega, h)$ is the trivial one and $\sigma = 2\partial_h$ is a nonzero derivation (i.e. a non Hamiltonian Poisson derivation) of $\mathbb{k}(\omega, h)$. Hence $\mathrm{Z_P}(L(\mathfrak{sl}_2)) = \mathbb{k}(\omega)$ by lemma 1.2.1 and, up to the change of h into $h' = \frac{1}{2}h$, we conclude using (30) that $L(\mathfrak{sl}_2)$ is Poisson isomorphic to $\mathscr{F}_{1,1}(\mathbb{k})$.

Problem. Let $\mathfrak{g}$ be an algebraic Lie algebra over the base field $\mathbb{k}$ (supposed of characteristic zero). Let $S(\mathfrak{g})$ be its symmetric algebra with the Kirillov-Kostant-Souriau Poisson structure, and $L(\mathfrak{g})$ its field of fractions.

Do we have a Poisson isomorphism $L(\mathfrak{g}) \simeq \mathscr{F}_{m,t}(\mathbb{k})$ for some integers $m, t \geq 0$?

By (31) and (32), such an isomorphism implies that $L(\mathfrak{g})$ is purely transcendental of degree $2m + t$ over $\mathbb{k}$, and the Poisson center of $L(\mathfrak{g})$ is purely transcendental of degree t over $\mathbb{k}$.

This question naturally appears as a Poisson formulation of the well known Gel'fand-Kirillov problem.

> Comment. This is not about to give here a comprehensive presentation of this deep and difficult problem. We recall only that the original article [19] makes the conjecture that, for any finite dimensional algebraic Lie algebra $\mathfrak{g}$ over $\mathbb{k}$ of characteristic zero, the skewfield of fractions of the enveloping algebra $U(\mathfrak{g})$ is $\mathbb{k}$-isomorphic to the field of fractions $\mathscr{D}_{m,t}(\mathbb{k})$ of the Weyl algebra $A_m(K)$ over some purely transcendental extension K of $\mathbb{k}$ of transcendence degree t. The conjecture was proved by I. M. Gel'fand and A. A. Kirillov in [19] for nilpotent $\mathfrak{g}$, for $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$

(1966). It was confirmed to by true for solvable $\mathfrak{g}$ in three simultaneous independent papers (1973-1974) by A. Joseph, by J. C. McConnell, and by W. Borho, P. Gabriel and R. Rentschler. After several partial results, a step was taken with the first counterexamples produced by J. Alev, A. Ohms and M. Van den Bergh (1996). The situation where $\Bbbk$ is of characteristic p was studied by J.-M. Bois (2006). A major step forward is the recent article [30] by A. Premet which proves, using by reduction modulo p results for modular Lie algebras, that the original conjecture in characteristic zero fails for simple $\mathfrak{g}$ of types B_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8 or F_4. We refer for details to [30], [38], [36] or [13] I.2.11, and their bibliographies.

It is addressed by M. Vergne in [38], which gives a positive answer in the case where $\mathfrak{g}$ is nilpotent. Another reference on the topic is the article [23] by B. Kostant and N. Wallach. Our goal here is to outline the methods used by P. Tauvel and R. Yu in [36] for the case where $\mathfrak{g}$ is solvable and in a generalized form, i.e. not only for $S(\mathfrak{g})$ but also for all quotients $S(\mathfrak{g})/Q$ by a prime Poisson ideal Q.

Observation (about the condition on $\mathfrak{g}$ to be algebraic or not). We adapt to our Poisson context the last examples of [19].

(1) Let $\mathfrak{g}$ be the solvable Lie algebra $\mathfrak{g} = \Bbbk x \oplus \Bbbk y \oplus \Bbbk z$ with $[x,y] = y$, $[y,z] = 0$ and $[x,z] = \alpha z$ for some fixed nonzero scalar α. It is clear that $S(\mathfrak{g}) = \Bbbk[y,z][x]_{0,\delta}$ for $\delta = y\partial_y + \alpha z\partial_z$. Applying the remark in 1.2.1 p. 11, the Poisson center of $L(\mathfrak{g}) = \operatorname{Frac} S(\mathfrak{g})$ is $\mathrm{Z_P}(L(\mathfrak{g})) = \Bbbk(y,z) \cap \ker \delta$. The derivation δ being homogeneous with action on monomials defined by $\delta(y^n z^m) = (n + \alpha m) y^n z^m$, two cases may occur:

(i) $\alpha = \frac{p}{q}$ for relatively prime nonzero integers p, q. Then $\ker \delta = \Bbbk(t)$ for $t := y^{-p} z^q$. Denoting by u, v integers such that $pu + qv = 1$, we can define $s := y^v z^u$. Since $z = t^v s^p$ and $y = t^{-u} s^q$, we have $\Bbbk(y,z) = \Bbbk(s,t)$. Therefore $L(\mathfrak{g}) = \Bbbk(s,t)(x)_{0,\delta}$ with $\{t,x\} = \{t,s\} = 0$ and $\{x,s\} = \delta(s) = (v+\alpha u)s$, with $v+\alpha u \neq 0$. Replacing x by $x' = (v+\alpha u)^{-1} s^{-1} x$, we conclude that $L(\mathfrak{g}) = \Bbbk(t,s)(x')_{0,\partial_s}$ is Poisson isomorphic to $\mathscr{F}_{1,1}(\Bbbk)$.

(ii) $\alpha \notin \mathbb{Q}$. Then $\ker \delta = \Bbbk$. Hence $L(\mathfrak{g})$ is of transcendence degree 3 over its Poisson center. Consequently, $L(\mathfrak{g})$ cannot be Poisson isomorphic to a Poisson-Weyl field.

As proved in example 24.8.4 of [35], the Lie algebra $\mathfrak{g}$ is algebraic in the case (i) and non algebraic in the case (ii). So: *the answer to the problem p. 19 can be negative for non algebraic solvable Lie algebras.*

(2) Let $\mathfrak{h}$ be the solvable Lie algebra $\mathfrak{h} = \Bbbk x \oplus \Bbbk y \oplus \Bbbk z \oplus \Bbbk t$ with: $[x,t] = (1+\alpha)t$, $[y,t] = [z,t] = 0$, $[x,y] = y$, $[y,z] = t$ and $[x,z] = \alpha z$ for some fixed nonzero scalar α. For any α, a direct calculation proves that the elements: $x_1 := yt^{-1}$, $y_1 := z$, $x_2 := (1+\alpha)^{-1} t$ and $y_2 := \alpha yzt^{-2} - xt^{-1}$ satisfy $\Bbbk(x,y,z,t) = \Bbbk(x_1,y_1,x_2,y_2)$ and $\{x_1,y_1\} = \{x_2,y_2\} = 1$ and $\{x_1,x_2\} = \{y_1,y_2\} = \{x_1,y_2\} = \{x_2,y_1\} = 0$. So $L(\mathfrak{h})$ is Poisson isomorphic to $\mathscr{F}_{2,0}(\Bbbk)$ for any $\alpha \in \Bbbk^\times$. However, since $\mathfrak{h}/\Bbbk t \simeq \mathfrak{g}$, the Lie algebra $\mathfrak{h}$ is not algebraic for $\alpha \notin \mathbb{Q}$. So: *the answer to the problem p. 19 can be positive for non algebraic solvable Lie algebras.*

2.1.2 The case of solvable Lie algebras

We present only the broad outlines of the study [36] by P. Tauvel and R. Yu and we refer the interested reader to the original article and the papers cited in references. Many of them concern the study of the same question for enveloping algebras and it seemed difficult to give in these notes all details and complete proofs for many specialized Lie-theoretical results. Through the end of this section, $\Bbbk$ is algebraically closed of characteristic zero.

(i) The Poisson semi-center of the prime quotients of $S(\mathfrak{g})$***.*** Let $\mathfrak{g}$ a finite dimensional Lie algebra over $\Bbbk$. The adjoint action $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is defined by $x\cdot y=\mathrm{ad}_x(y)$ for all $x,y\in\mathfrak{g}$, with notation $\mathrm{ad}_x:\mathfrak{g}\to\mathfrak{g}, y\mapsto[x,y]_{\mathfrak{g}}$. For any $x\in\mathfrak{g}$, the endomorphism ad_x extends by Leibniz rule into the Hamiltonian Poisson derivation $\sigma_x:S(\mathfrak{g})\mapsto S(\mathfrak{g}), y\mapsto \{x,y\}$ of $S(\mathfrak{g})$, in the sense of the first example p. 3. Hence $\mathfrak{g}$ acts on $S(\mathfrak{g})$ by:

$$\mathfrak{g}\times S(\mathfrak{g})\to S(\mathfrak{g}),\ (x,y)\mapsto x\cdot y=\sigma_x(y). \tag{38}$$

The invariant algebra $S(\mathfrak{g})^{\mathfrak{g}}:=\{y\in S(\mathfrak{g})\,;\,\{x,y\}=0 \text{ for all } x\in\mathfrak{g}\}$ is no more (by Leibniz rule) that the Poisson center $\mathrm{Z_P}(S(\mathfrak{g}))=\{y\in S(\mathfrak{g})\,;\,\{x,y\}=0 \text{ for all } x\in S(\mathfrak{g})\}$.

• Let Q be an ideal of $S(\mathfrak{g})$. It is clear (as above) that Q is $\mathfrak{g}$-invariant [i.e. $\sigma_x(Q)\subseteq Q$ for all $x\in\mathfrak{g}$] if and only if Q is a Poisson ideal [i.e. $\sigma_x(Q)\subseteq Q$ for all $x\in S(\mathfrak{g})$].
In this case $S(\mathfrak{g})/Q$ is a Poisson algebra (see remark 4 in 1.1.1). For any $x\in\mathfrak{g}$, the map $\overline{\sigma}_x:S(\mathfrak{g})/Q\to S(\mathfrak{g})/Q,\ a\mapsto\{\overline{x},a\}$ is well defined (with $\overline{\sigma}_x(a)=\overline{\{x,y\}}$ for $a=\overline{y}$) and is an Hamiltonian Poisson derivation.
To simplify, we denote $B:=S(\mathfrak{g})/Q$. The $\mathfrak{g}$-action on B is defined by:

$$x\cdot a=\overline{\sigma}_x(a)=\{\overline{x},a\}\quad \text{for } x\in\mathfrak{g},\ a\in B. \tag{39}$$

Reasoning as above, the Poisson center of B is:

$$\mathrm{Z_P}(B)=B^{\mathfrak{g}}=\{a\in B\,;\,x\cdot a=0 \text{ for any } x\in\mathfrak{g}\}=\textstyle\bigcap_{x\in\mathfrak{g}}\ker\overline{\sigma}_x. \tag{40}$$

A semi-invariant for this action is an element $a\in B$ such that $\mathfrak{g}\cdot a\in\Bbbk a$. For any nonzero semi-invariant element a, there exists a linear form $\lambda\in\mathfrak{g}^*$ such that $x\cdot a=\lambda(x)a$ for any $x\in\mathfrak{g}$. We say that λ is the weight of a. We denote

$$E(B):=\text{ the set of nonzero semi-invariants of } B. \tag{41}$$

Conversely, for any $\lambda\in\mathfrak{g}^*$, we set:

$$B_\lambda:=\{a\in B\,;\,x\cdot a=\lambda(x)a \text{ for any } x\in\mathfrak{g}\}, \tag{42}$$

which is a subspace of B. An linear form λ such that $B_\lambda\neq(0)$ is a weight of the action. We denote the set of weights by

$$\Lambda:=\{\lambda\in\mathfrak{g}^*\,;\,B_\lambda\neq(0)\}, \tag{43}$$

which is stable by addition [since $a \in B_\lambda$ and $b \in B_\mu$ implies $ab \in B_{\lambda+\mu}$]. We define the Poisson semi-center of B by:

$$\mathrm{SZ_P}(B) := \sum_{\lambda\in\mathfrak{g}^*} B_\lambda = \text{the subspace of } B \text{ generated by } E(B). \tag{44}$$

Since $B_0 = \mathrm{Z_P}(B)$, we always have:

$$\mathrm{Z_P}(B) \subseteq \mathrm{SZ_P}(B) \tag{45}$$

and the equality holds when $\mathfrak{g}$ is nilpotent[4] (because in this case, the $\mathfrak{g}$-action on B is locally nilpotent; see also 4.3.2 in [17]).

• We suppose moreover that Q is a prime ideal. We denote $L := \mathrm{Frac}\, B = \mathrm{Frac}\,(S(\mathfrak{g})/Q)$. The $\mathfrak{g}$-action extends to L:

$$x \cdot ab^{-1} = \overline{\sigma}_x(ab^{-1}) = \big(\{\overline{x}, a\}b - a\{\overline{x}, b\}\big)b^{-2} \quad \text{for } x \in \mathfrak{g},\ a, b \in B,\ b \neq 0. \tag{46}$$

and we consider in the same way as above:

$L_\lambda := \{c \in L\,;\, x \cdot c = \lambda(x)c \text{ for any } x \in \mathfrak{g}\}$, for all $\lambda \in \mathfrak{g}^*$,
$\Lambda' := \{\lambda \in \mathfrak{g}^*\,;\, L_\lambda \neq (0)\}$, which is an additive subgroup of $\mathfrak{g}^*$,
$\mathrm{Z_P}(L) := L_0 = L^{\mathfrak{g}} = \{c \in L\,;\, x \cdot c = 0 \text{ for any } x \in \mathfrak{g}\}$ = the Poisson center of L,
$E(L) := (\bigcup_{\lambda\in\mathfrak{g}^*} L_\lambda) \setminus \{0\}$ = the set of nonzero semi-invariants of L,
$\mathrm{SZ_P}(L) := \sum_{\lambda\in\mathfrak{g}^*} L_\lambda$ = the subspace generated by $E(L)$ = the Poisson semi-center of L.

• We suppose now that $\mathfrak{g}$ is solvable[5]. This assumption occurs in the proof of the following lemma, which links together the weight in Λ and Λ'.

Lemma. *Any $c \in E(L)$ can be written $c = ab^{-1}$ with $a, b \in E(B)$. More precisely, for $\lambda \in g^*$ and $c \in L_\lambda$, there exist $\mu \in \mathfrak{g}^*$, $a \in B_{\lambda+\mu}$ and $b \in B_\mu$ such that $c = ab^{-1}$.*

Proof. Let λ be an element of $\mathfrak{g}^*$ and $c \in L_\lambda$. Let I be the set of $u \in B$ such that $uc \in B$. It is clear that I is a nonzero ideal of B. We claim that I is $\mathfrak{g}$-stable: for any $x \in \mathfrak{g}$, we have $(x \cdot u)c = x \cdot (uc) - u(x \cdot c) = x \cdot (uc) - \lambda(x)uc$, with $uc \in B$ and then $x \cdot (uc) \in B$, so $(x \cdot u)c \in B$, and consequently $x \cdot u \in I$. So I is a $\mathfrak{g}$-submodule of B.
Denoting by V_n the image by the canonical map $S(\mathfrak{g}) \to S(\mathfrak{g})/Q$ of the subspace of polynomials of total degree $\leq n$, it follows from (37) that each V_n is stable under the $\mathfrak{g}$-action. Hence denoting $W_n = I \cap V_n$ for any integer $n \geq 0$, the $\mathfrak{g}$-module I appears as the increasing union of the finite dimensional $\mathfrak{g}$-submodules W_n. By Lie's theorem[6], each of them admits a flag of $\mathfrak{g}$-submodules.

[4]The central descending series of a Lie algebra $\mathfrak{g}$ is the decreasing chain of ideals: $C^1(\mathfrak{g}) = \mathfrak{g}$, $C^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$, …, $C^{i+1}(\mathfrak{g}) = [\mathfrak{g}, C^i(\mathfrak{g})]$, … By definition, $\mathfrak{g}$ is nilpotent if $C^k(\mathfrak{g}) = (0)$ for some positive integer k.

[5]The derived series of a Lie algebra $\mathfrak{g}$ is the decreasing chain of ideals: $D^0(\mathfrak{g}) = \mathfrak{g}$, $D^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$, …, $D^{i+1}(\mathfrak{g}) = [D^i(\mathfrak{g}), D^i(\mathfrak{g})]$, … By definition, $\mathfrak{g}$ is solvable if $D^k(\mathfrak{g}) = (0)$ for some nonnegative integer k.

[6]Let (V, σ) a finite-dimensional representation of a solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic 0. Then there exists a flag $(0) = V_0 \subset V_1 \subset \cdots \subset V_n = V$, with $\dim V_i = i$, such that $\sigma(\mathfrak{g})(V_i) \subseteq V_i$ for any $0 \leq i \leq n$; (see for instance [36] or [16]).

In particular, there exists some nonzero element b in I such that $\mathfrak{g}\cdot b = \Bbbk b$. In other words there exist $b \in I, b \neq 0$ and $\mu \in \mathfrak{g}^*$ such that $b \in B_\mu$. The element $a := bc$ satisfies $b \in B$ because $b \in I$ and $x \cdot a = (x \cdot b)c + b(x \cdot c) = \mu(x)bc + \lambda(x)bc$. Hence $a \in B_{\lambda+\mu}$. $\square$

• This lemma is a crucial ingredient for a sequence of technical and non obvious lemmas (5.6 to 5.11 in [36]) which leads to the following two propositions (see also 4.3.5 in [17]).

Proposition 1. *The Poisson semi-center* $\mathrm{SZ_P}(B)$ *is a Poisson subalgebra of* B*, whose Poisson structure is the trivial one.*

For the second proposition, we first observe that the lemma implies that Λ' is the additive subgroup of $\mathfrak{g}^*$ generated by Λ. So we can consider the subspace of elements of $\mathfrak{g}$ which are annihilated by all weights of the $\mathfrak{g}$-action on B, or equivalently on L:

$$\widehat{\mathfrak{g}} := \bigcap_{\lambda\in\Lambda} \ker\lambda = \bigcap_{\lambda\in\Lambda'} \ker\lambda. \tag{47}$$

Let $x, y \in \mathfrak{g}$ and $\lambda \in \Lambda$. There exists $a \in B_\lambda$, $a \neq 0$. In particular, we have $[x, a] = \lambda(x)a$, $[y, a] = \lambda(y)a$ and, using Jacobi identity: $\lambda([x, y])a = [[x, y], a] = -[[y, a], x] - [[a, x], y] = -[\lambda(y)a, x] + [\lambda(x)a, y] = 0$. Therefore $[x, y] \in \ker\lambda$. Hence we have proved:

$$[\mathfrak{g}, \mathfrak{g}] \in \widehat{\mathfrak{g}}.$$

In particular, $\widehat{\mathfrak{g}}$ is an ideal of $\mathfrak{g}$. We consider $\widehat{Q} := Q \cap S(\widehat{\mathfrak{g}})$ and $\widehat{B} := S(\widehat{\mathfrak{g}})/\widehat{Q}$ identified with a subalgebra of B. More generally, for $(z_1, \ldots, z_r)$ a $\Bbbk$-basis of a complementary subspace of $\widehat{\mathfrak{g}}$ in $\mathfrak{g}$, the subspace $\widehat{\mathfrak{g}} + \Bbbk z_1 + \cdots + \Bbbk z_i$ is an ideal of $\mathfrak{g}$ for any $1 \leq i \leq r$ and it can be proved (see 5.13 in [36]; see also 14.4.5 in [25] in the context of enveloping algebras) that, with the notations of 1.1.2 for Poisson-Ore extensions:

Proposition 2. *We have the Poisson-algebra isomorphism:*

$$B \simeq \widehat{B}[z_1]_{0,\delta_1}[z_2]_{0,\delta_2} \ldots [z_r]_{0,\delta_r},$$

where δ_i *is the Hamiltonian derivation determined by* z_i*, with:*

$$\mathrm{SZ_P}(B) \subset \mathrm{SZ_P}(\widehat{B}) = \mathrm{Z_P}(\widehat{B}).$$

• Another application of this lemma is the next proposition. Denoting $m = \dim\mathfrak{g}$, we can consider (see 19.4.5 in [35]) a Jordan-Hölder chain $(0) = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_m = \mathfrak{g}$ of ideals of $\mathfrak{g}$ with $\dim\mathfrak{g}_i = i$ for any $0 \leq i \leq m$. Each $\mathfrak{g}_i/\mathfrak{g}_{i-1}$ being a one-dimensional $\mathfrak{g}$-module, it defines a linear form $\nu_i \in \mathfrak{g}^*$, for any $1 \leq i \leq m$ (if $(y_1, \ldots, y_m)$ is a basis of $\mathfrak{g}$ such that $(y_1, \ldots, y_i)$ is a basis of $\mathfrak{g}_i$ for any $1 \leq i \leq m$, we have $[x, y_i] = \nu_i(x)y_i + \mathfrak{g}_{i-1}$ for all $x \in \mathfrak{g}$). By Jordan-Hölder theorem, the composition factors $\mathfrak{g}_i/\mathfrak{g}_{i-1}$ are uniquely determined up to order and isomorphism (see [25], 0.1.3 and 14.4.10, or [36], 6.1, or [17], 1.2.6); hence the set $\{\nu_1, \ldots, \nu_m\}$ is independent of the choice of chains of ideals. These eigenvalues are called the Jordan-Hölder weights of $\mathfrak{g}$ on $\mathfrak{g}$. Then it can be proved ([36] lemme 6.2, see also [25] lemma 14.4.10) that:

$$\Lambda \subset \nu_1\mathbb{N} + \cdots \nu_m\mathbb{N}. \tag{48}$$

It follows by the above lemma that $\Lambda' \subset \nu_1\mathbb{Z}+\cdots\nu_m\mathbb{Z}$. We have seen that Λ' is a subgroup of $\mathfrak{g}^*$, and $\operatorname{char}\Bbbk = 0$ implies that $\mathfrak{g}^*$ is $\mathbb{Z}$-torsion free. Hence Λ' is free of finite rank, say with a basis $(\mu_1,\dots,\mu_\ell)$. Applying again the lemma, there exist $\lambda_1,\dots,\lambda_s\in\Lambda$ such that $\mu_i=\sum_{j=1}^s n_{ij}\lambda_j$ for some $n_{ij}\in\mathbb{Z}$. Let us choose an nonzero element $e_i\in B_{\lambda_i}$ for each $1\le i\le s$. The product $e:=e_1\dots e_s$ generates a multiplicative subset in B and we consider the Poisson algebra B_e. In the group of units of B_e, generated by $e_1,\dots,e_s$, there are monomials $w_1,\dots,w_\ell$ which are semi-invariants of weight $\mu_1,\dots,\mu_\ell$ respectively. Denoting by W the multiplicative subgroup of B_e generated by $w_1,\dots,w_\ell$, we can see that, for any $w\in W$, the set of all nonzero semi-invariants in B_e having the same weight as w is $w\mathrm{Z_P}(B_e)$. It is now possible to describe the semi-center of B_e:

Proposition 3. *There exist a nonzero semi-invariant element e in B, and nonzero semi-invariants elements $w_1,\dots,w_\ell$ in B_e algebraically independent over $\mathrm{Z_P}(B_e)$, such that:*
$$\mathrm{SZ_P}(B_e)=\mathrm{Z_P}(B_e)[w_1^{\pm1},\dots,w_\ell^{\pm1}].$$

We refer for more details to 6.2 and 11.1 in [36] (see also 14.4.12 in [25] for a noncommutative analogue).

Exercise 1. Prove that, with notation $L=\operatorname{Frac}B$ and $\widehat{L}=\operatorname{Frac}\widehat{B}$, we have:
$$\mathrm{Z_P}(L)\subseteq\operatorname{Frac}(\mathrm{SZ_P}(B))\subseteq\operatorname{Frac}(\mathrm{Z_P}(\widehat{B}))=\mathrm{Z_P}(\widehat{L}).$$
[Hint: use the lemma p. 22 for L_0 to check that $\mathrm{Z_P}(L)\subseteq\operatorname{Frac}(\mathrm{SZ_P}(B))$, and similarly $\mathrm{Z_P}(\widehat{L})\subseteq\operatorname{Frac}(\mathrm{Z_P}(\widehat{B}))$. Observing that the inclusion $\operatorname{Frac}(\mathrm{Z_P}(\widehat{B}))\subseteq\mathrm{Z_P}(\widehat{L})$ is trivial, use proposition 2 to achieve the proof].

Exercise 2. We consider the example (1) p. 20, i.e. $\mathfrak{g}=\Bbbk x\oplus\Bbbk y\oplus\Bbbk z$ with $[x,y]=y$, $[y,z]=0$ and $[x,z]=\alpha z$ for some fixed nonzero scalar α. We take $Q=0$ so that $B=S(\mathfrak{g})=\Bbbk[y,z][x]_{0,\delta}$ with $\delta=y\partial_y+\alpha z\partial_z$, and $L=\operatorname{Frac}B=L_{\mathfrak{g}}=\operatorname{Frac}S(\mathfrak{g})$.

a) Prove that, if $\alpha\notin\mathbb{Q}$, each distinct monomial in y,z is an eigenvector for a different eigenvalue, and $E(B)=\{\lambda y^iz^j\,;\,i,j\in\mathbb{N},\lambda\in\Bbbk^\times\}$. Prove that, if $\alpha\in\mathbb{Q}_+^\times$, there are distinct monomials having the same eigenvalue, and then their $\Bbbk$-linear combinations belong to $E(B)$. [Hint: consider $y^{2p}z^q$ and y^pz^{2q} for $p,q\in\mathbb{N}$, $\alpha=pq^{-1}$].

b) Prove that, if $\alpha\notin\mathbb{Q}$ or $\alpha\in\mathbb{Q}_+^\times$, then $\mathrm{Z_P}(B)=\Bbbk$. Prove that, if $\alpha\in\mathbb{Q}_-^\times$, then $\mathrm{Z_P}(B)\neq\Bbbk$. [Hint: consider y^pz^q for $\alpha=-pq^{-1}$, $p,q\in\mathbb{N}$]. Using example (1) p. 20, deduce that $\mathrm{Z_P}(\operatorname{Frac}B)=\operatorname{Frac}\mathrm{Z_P}(B)$ if and only if $\alpha\notin\mathbb{Q}$ or $\alpha\in\mathbb{Q}_+^\times$.

c) Prove that, if $\alpha\notin\mathbb{Q}$, then $\Lambda\simeq\{i+j\alpha\,;\,i,j\in\mathbb{N}\}$ which is a free abelian semigroup of rank 2, and Λ' is a free abelian group of rank 2. [Hint: show more explicitly that $\Lambda=\mu_1\mathbb{N}+\mu_2\mathbb{N}$ where $\mu_1=x^*$ and $\mu_2=\alpha x^*$ in $\mathfrak{g}^*$]. Prove that, if $\alpha=pq^{-1}$ with p,q relatively prime, then Λ' is free abelian of rank 1 generated by $q^{-1}x^*$.

d) We assume that $\alpha\notin\mathbb{Q}$. Observe that $\mathrm{SZ_P}(B)=\Bbbk[y,z]$, $\widehat{\mathfrak{g}}=\Bbbk y\oplus\Bbbk z$, $\widehat{B}=\Bbbk[y,z]$, and compare the expression $B=\Bbbk[y,z][x]_{0,\delta}$ with proposition 2.

e) We assume $\alpha\notin\mathbb{Q}$ and choose $e:=yz$ in $E(B)$. Prove that $B_e=\Bbbk[y^{\pm1},z^{\pm1}][x]_{0,\delta}$, $\mathrm{Z_P}(B_e)=\Bbbk$, $\mathrm{SZ_P}(B_e)=\Bbbk[y^{\pm1},z^{\pm1}]$, and compare with proposition 3.

(ii) Poisson-analogues of McConnell algebras. Let V be a $\mathbb{k}$ vector space of finite dimension n. Let ω a linear alternating bilinear form on V. Let G a free finitely generated subgroup of rank m in the additive group V^*. Denoting by $S(V)$ the symmetric algebra of V and by $\mathbb{k}[G]$ the group algebra of G, we have an isomorphism of commutative algebras:

$$S(V) \otimes_{\mathbb{k}} \mathbb{k}[G] \simeq \mathbb{k}[x_1, \ldots, x_n, y_1^{\pm 1}, \ldots, y_m^{\pm 1}]. \tag{49}$$

• There exists a unique Poisson structure on $S(V) \otimes_{\mathbb{k}} \mathbb{k}[G]$ such that:

$$\{v, w\} = \omega(v, w), \quad \{g, h\} = 0, \quad \{g, v\} = \lambda_g(v)g, \qquad \text{for all } v, w \in V, \ g, h \in G, \tag{50}$$

where the notation λ_g is for the linear form associated with g. The so-defined Poisson algebra is denoted by $\mathscr{B}_{\mathbb{k}}(V, \omega, G)$. It appears as a natural Poisson analogue of the noncommutative algebras introduced in the seminal work of J.C. Connell on the prime quotients of enveloping algebras of solvable Lie algebras (see chapter 14.8 of [25]).
Let $S_\omega(V)$ be the Poisson-subalgebra of $\mathscr{B}_{\mathbb{k}}(V, \omega, G)$ generated by V. Denoting by 2ℓ the rank of ω, and by V^ω the kernel of ω, we have clearly the isomorphism of Poisson algebras:

$$S_\omega(V) \simeq \mathbb{S}_\ell(\mathbb{k}) \otimes_{\mathbb{k}} S(V^\omega), \tag{51}$$

where $\mathbb{S}_\ell(\mathbb{k})$ is the Poisson-Weyl algebra defined in 1.1.1. In order to characterize the property for $\mathscr{B}_{\mathbb{k}}(V, \omega, G)$ to be Poisson-simple (i.e. to admit non nontrivial proper Poisson ideal, see exercise 3 of 1.1.2), we introduce the following notations. The orthogonal of G in V is $V^G := \bigcap_{g \in G} \ker \lambda_g$. For any $g \in G$, there exists a unique derivation D_{λ_g} in the polynomial algebra $S(V)$ such that $D_{\lambda_g}(v) = \lambda_g(v)$ for any $v \in V$. This derivation is locally nilpotent and we define the automorphism $\varphi_g = \exp D_{\lambda_g}$ of $S(V)$. Then it can be proved (see 7.5 in [36]) that the following conditions are equivalent:

(a) $\mathscr{B}_{\mathbb{k}}(V, \omega, G)$ is Poisson-simple ;
(b) the set of $r \in S(V^\omega)$ such that $\varphi_g(r) = r$ for all $g \in G$ reduces to $\mathbb{k}$;
(c) $V^G \cap V^\omega = (0)$.

In such a Poisson-simple $\mathscr{B}_{\mathbb{k}}(V, \omega, G)$, the Poisson-centralizer C of the "group part" $\mathbb{k}[G]$ is $S_\omega(V^G) \otimes_k \mathbb{k}[G]$ (see 7.6 in [36]), and so:

$$\boldsymbol{d}(\mathscr{B}_{\mathbb{k}}(V, \omega, G)) = \dim V + \operatorname{rk} G, \quad \boldsymbol{d}(\mathbb{k}[G]) = \operatorname{rk} G, \quad \boldsymbol{d}(C) = \dim V^G + \operatorname{rk} G,$$

where $\boldsymbol{d}$ is the Gel'fand-Kirillov dimension (see [19] or [25]), that is in the case of a commutative finitely generated domain A the transcendence degree of $\operatorname{Frac} A$. The simple algebras $\mathscr{B}_{\mathbb{k}}(V, \omega, G)$ are naturally connected with the Poisson-Weyl algebras and with the prime quotients $S(\mathfrak{g})/Q$, as shown by the following two observations.

Observation 1. *We suppose that $\mathscr{B}_{\mathbb{k}}(V, \omega, G)$ is Poisson-simple. We denote: $n = \dim V$, $2\ell = \operatorname{rk} \omega$, $t = \dim V^\omega = n - 2\ell$, and $m = \operatorname{rk} G$. Then there exists a Poisson-algebra embedding $\mathscr{B}_{\mathbb{k}}(V, \omega, G) \to \mathbb{S}_\ell(\mathbb{k}) \otimes_{\mathbb{k}} \mathbb{S}'_m(\mathbb{k})$.*

Proof. We recall that $\mathbb{S}_\ell(\mathbb{k})$ is the Poisson-Weyl algebra defined in 1.1.1 and $\mathbb{S}'_m(\mathbb{k})$ is its localized form defined in exercise 4 of 1.1.2. We denote $\mathbb{S}_\ell(\mathbb{k}) = \mathbb{k}[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$ and $S'_m(\mathbb{k}) = \mathbb{k}[w_1, \ldots, w_m, z_1^{\pm 1}, \ldots, z_m^{\pm 1}]$, with brackets:

$$\{x_i, y_j\} = \delta_{ij},\ \{x_i, x_j\} = \{y_i, y_j\} = 0,\ \{z_i, w_j\} = \delta_{ij} z_i,\ \{z_i, z_j\} = \{w_i, w_j\} = 0.$$

Let $(u_1, \dots, u_\ell, v_1, \dots, v_\ell, s_1, \dots, s_t)$ be a $\Bbbk$-basis of V adapted to ω, i.e. $(s_1, \dots, s_t)$ is a basis of V^ω and the bracket of the Poisson subalgebra $S_\omega(V)$ of $\mathscr{B}_\Bbbk(V, \omega, G)$ is:

$$\{u_i, v_j\} = \delta_{ij},\ \{u_i, u_j\} = \{v_i, v_j\} = 0 \ \text{ and } \ \{s_i, u_j\} = \{s_i, v_j\} = \{s_i, s_j\} = 0.$$

Let $(g_1, \dots, g_m)$ a basis of the $\mathbb{Z}$-module G. There exist scalars (λ_{ji}), (μ_{ji}) and (ν_{jk}) such that, for all $1 \le i \le \ell, 1 \le j \le m, 1 \le k \le t$:

$$\{g_j, u_i\} = \lambda_{ji} g_j,\ \{g_j, v_i\} = \mu_{ji} g_j,\ \{g_j, s_k\} = \nu_{jk} g_j.$$

Hence the $\Bbbk$-algebra homomorphism $\theta : \mathscr{B}_\Bbbk(V, \omega, G) \to \mathbb{S}_\ell(\Bbbk) \otimes \mathbb{S}'_m(\Bbbk)$ define from:

$$u_i \mapsto x_i + \textstyle\sum_{j=1}^{\ell} \lambda_{ji} w_j, \quad v_i \mapsto y_i + \sum_{j=1}^{\ell} \mu_{ji} w_j, \quad s_k \mapsto \sum_{j=1}^{m} \nu_{jk} w_j, \quad g_j \mapsto z_j$$

is a Poisson-morphism, which is injective by Poisson-simplicity of $\mathscr{B}_\Bbbk(V, \omega, G)$. □

Observation 2. *We suppose that $\mathscr{B}_\Bbbk(V, \omega, G)$ is Poisson-simple. We denote: $n = \dim V$, $2\ell = \operatorname{rk} \omega$, $t = \dim V^\omega = n - 2\ell$, and $m = \operatorname{rk} G$. Then there exists a solvable Lie algebra $\mathfrak{g}$ and a Poisson ideal Q in $S(\mathfrak{g})$ such that $\mathscr{B}_\Bbbk(V, \omega, G)$ is Poisson-isomorphic to the localization $(S(\mathfrak{g})/Q)_E$ of $S(\mathfrak{g})/Q$ by the subset E of its nonzero semi-invariants.*

Proof. With the same notations as in previous lemma, the Lie algebra $\mathfrak{g}$ of dimension $2\ell + t + m + 1$ defined adjoining a one-dimensional subspace $\Bbbk w$ to the space $V \oplus (\bigoplus_{j=1}^m \Bbbk g_j)$, for the Lie bracket (where $1 \le i, j \le \ell,\ 0 \le k, k' \le t,\ 1 \le p, q \le m$):

$$[u_i, u_j] = [v_i, v_j] = [s_k, u_i] = [s_k, v_j] = [s_k, s_{k'}] = 0,\ \ [u_i, v_j] = \delta_{ij} w,$$

$$[g_p, g_q] = 0, \quad [g_p, v] = \lambda_{g_p}(v) g_p \text{ for any } v \in V,$$

$$[w, u] = 0 \ \text{ for any } u \in V \oplus (\textstyle\bigoplus_{j=1}^m \Bbbk g_j).$$

Then $\mathfrak{g}$ is solvable, $Q := (w-1)S(\mathfrak{g})$ is a $\mathfrak{g}$-stable prime Poisson ideal of $S(\mathfrak{g})$, and the set E of nonzero semi-invariants of $S(\mathfrak{g})/Q$ is the multiplicative semi-group generated by the elements $\overline{g}_p$ $(1 \le p \le m)$. By Poisson-simplicity of $\mathscr{B}_\Bbbk(V, \omega, G)$ the localized algebra $(S(\mathfrak{g})/Q)_E$ is Poisson isomorphic to $\mathscr{B}_\Bbbk(V, \omega, G)$. □

• Conversely, a fundamental result of [36] consists in a description of the prime quotients $S(\mathfrak{g})/Q$ for any solvable $\mathfrak{g}$ in terms of Poisson-simple algebras $\mathscr{B}_\Bbbk(V, \omega, G)$. The strategy, similar to that developed in McConnell's work for enveloping algebras (see chapter 14 of [25]), is in two steps.

– In the case where $\mathfrak{g}$ is nilpotent, denoting by E the set of nonzero semi-invariant elements of $B = S(\mathfrak{g})/Q$ and by K the Poisson-center of $L = \operatorname{Frac} B$, it is proved in section 8 of [36]) that there exists an integer $n \ge 0$ such that the localized algebra B_E is Poisson isomorphic to $\mathbb{S}_n(K) = K \otimes_\Bbbk \mathbb{S}_n(\Bbbk)$.

This result is actually obtained as a corollary of the following more general key result (8.2 and 8.3 in [36]; also 14.6.8 and 14.6.9 in [25] for the noncommutative analogue), devoted to situations where the Poisson semi-center and center are similar:

Main lemma. *Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over $\Bbbk$, and $B = S(\mathfrak{g})/Q$ a quotient by a $\mathfrak{g}$-stable prime ideal of $S(\mathfrak{g})$ satisfying $\mathrm{SZ_P}(B) = \mathrm{Z_P}(B)$.*

Suppose also that $\mathfrak{g}$ *is an ideal of some solvable Lie algebra* $\mathfrak{h}$ *and that* $\mathfrak{s}$ *is a subalgebra of* $\mathfrak{h}$ *acting semisimply on* $\mathfrak{g}$*. Then:*

(i) *there exists a nonzero element* $e \in \mathrm{Z_P}(B)$ *such that* B_e *is Poisson isomorphic to* $\mathrm{Z_P}(B)_e \otimes_{\Bbbk} \mathbb{S}_n(\Bbbk)$ *for some integer* $n \geq 0$*;*

(ii) $\mathrm{Z_P}(B_e) = \mathrm{Z_P}(B)_e$ *is finitely generated over* $\Bbbk$*.*

Moreover, the following additional details hold (which will be useful in the case where $\mathfrak{g}$ is algebraic):

(iii) *e can be chosen to be an* $\mathfrak{h}$*-eigenvector;*

(iv) *generators* x_i, y_i *of* $S_n(\Bbbk)$ *in* B_e *satisfying relations* (3) *can be chosen to be* $\mathfrak{s}$*-eigenvectors.*

– In the case where $\mathfrak{g}$ is solvable with abelian maximum nilpotent ideal, it is proved in section 9 of [36]) that there exist suitable V, ω, G such that $\mathscr{B}_K(V, \omega, G)$ is Poisson-simple and isomorphic to B_E (as Poisson-algebras over K).

Combining these two particular situations with a decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$ where $\mathfrak{n}$ is the maximum nilpotent ideal of $\mathfrak{g}$ (see for instance section 1.4 of [17]) and $\mathfrak{a}$ a complementary subspace of $\mathfrak{n}$ in $\mathfrak{g}$, section 10 of [36] finally prove:

Theorem (P. Tauvel, R. Yu). *Let* $\mathfrak{g}$ *be a finite dimensional solvable Lie algebra over an algebraically closed field* $\Bbbk$ *of characteristic zero. Let* Q *be a* $\mathfrak{g}$*-stable prime ideal of* $S(\mathfrak{g})$*. Let us denote by* E *the set of nonzero semi-invariant elements of* $S(\mathfrak{g})/Q$ *and* K *the Poisson center of* $\mathrm{Frac}\,(S(\mathfrak{g})/Q)$*. Then, the localized Poisson algebra* $(S(\mathfrak{g})/Q)_E$ *is Poisson isomorphic over* K *to a simple Poisson algebra* $\mathscr{B}_K(V, \omega, G)$*.*

(iii) The case where $\mathfrak{g}$ ***is algebraic.*** The Lie algebra $\mathfrak{g}$ is algebraic when it is the Lie algebra of a linear algebraic group. We need here the following facts about solvable algebraic Lie algebras (see [36] 11.3, or [25] 7.2):

(a) $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{s}$ where $\mathfrak{n}$ is the maximum nilpotent ideal of $\mathfrak{g}$ and $\mathfrak{s}$ is an abelian Lie subalgebra of $\mathfrak{g}$ acting semisimply[7] on $\mathfrak{n}$;

(b) the rank of the $\mathbb{Z}$-submodule $\Gamma(\mathfrak{g})$ of $\mathfrak{g}^*$ spanned by the Jordan-Hölder weights of $\mathfrak{g}$ [see (48)] is the dimension of the $\Bbbk$-subspace $W(\mathfrak{g})$ of $\mathfrak{g}^*$ spanned by these weights.

Exercise 1. With the notations of point (b), prove that $\dim_{\Bbbk} W' = \mathrm{rk}_{\mathbb{Z}}\,\Gamma'$ for any $\Bbbk$-subspace W' and $\mathbb{Z}$-submodule Γ' of $\Gamma(\mathfrak{g})$ such that $\Bbbk\Gamma' = W'$. [Hint: choose simultaneous bases for Γ' and $\Gamma(\mathfrak{g})$ over $\mathbb{Z}$].

Exercise 2. For $\mathfrak{g}$ the solvable algebraic Lie algebra of lower triangular $n \times n$ matrices with entries in $\Bbbk$, prove that: the Lie subalgebra $\mathfrak{n}$ of strictly lower triangular matrices is nilpotent, the Lie subalgebra $\mathfrak{s}$ of diagonal matrices is abelian, $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{s}$, and the action of an element $s = \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \in \mathfrak{s}$ on each matrix e_{ij} of the canonical basis of $\mathfrak{n}$ is given by $s \cdot e_{ij} = (\alpha_j - \alpha_i)e_{ij}$ [hence each e_{ij} is

[7] The action of $\mathfrak{s}$ on $\mathfrak{n}$ is semisimple when $\mathfrak{n}$ is a semisimple $\mathfrak{s}$-module, i.e. a direct sum of simple $\mathfrak{s}$-modules; the field $\Bbbk$ being algebraically closed, this is equivalent here to the existence of a basis of $\mathfrak{s}$-eigenvectors in $\mathfrak{n}$ (see [17] 1.2.7, 1.2.9 and 1.3.13).

an $\mathfrak{s}$-eigenvector with eigenvalue $\lambda_{ij} : \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_j - \alpha_i$]. Deduce that $\dim_{\Bbbk} W(\mathfrak{g}) = \mathrm{rk}_{\mathbb{Z}} \Gamma(\mathfrak{g}) = n-1$.

EXERCISE 3. For $\mathfrak{g}$ the solvable algebraic Lie algebra defined in example (1) p. 20, prove that $\mathfrak{n} = \Bbbk y + \Bbbk z$ and $\mathfrak{s} = \Bbbk x$.

For algebraic solvable $\mathfrak{g}$, the following improvement of proposition 2 holds.

PROPOSITION 4. *We have:* $\mathrm{SZ}_{\mathrm{P}}(B) = \mathrm{SZ}_{\mathrm{P}}(\widehat{B}) = \mathrm{Z}_{\mathrm{P}}(\widehat{B})$.

PROOF. Let $(\mu_1, \ldots, \mu_\ell)$ be a basis of the $\mathbb{Z}$-submodule Λ' of $\mathfrak{g}^*$ (see p. 24). Using exercise 1, $(\mu_1, \ldots, \mu_\ell)$ is also a basis of the $\Bbbk$-subspace genetared by Λ'. By (47), we have $\widehat{\mathfrak{g}} = \bigcap_{i=1}^{\ell} \ker \mu_i$. Let us consider a decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{s}$ in the sense of point (a) above. For any $1 \leq i \leq \ell$, there exists a nonzero element $c \in L$ such that $x \cdot c = \mu_i(x) c$ for any $x \in \mathfrak{g}$; in particular, the action of an element $x \in \mathfrak{n}$ being locally nilpotent, we have $\mu_i(x) = 0$. Hence $\mathfrak{n} \subset \widehat{\mathfrak{g}}$. We consider $\mathfrak{t}$ a complement to $\widehat{g} \cap \mathfrak{s}$ in $\mathfrak{s}$, so that $\mathfrak{g} = \widehat{\mathfrak{g}} \oplus \mathfrak{t}$ where the abelian Lie subalgebra $\mathfrak{t}$ acts semisimply on $\widehat{\mathfrak{g}}$. Therefore $\mathfrak{t}$ acts semisimply on $\widehat{B}$. Hence any element of $\mathrm{Z}_{\mathrm{P}}(\widehat{B})$ is a sum of $\mathfrak{t}$-eigenvectors. Since $\widehat{\mathfrak{g}}$ acts trivially on $\mathrm{Z}_{\mathrm{P}}(\widehat{B})$, these $\mathfrak{t}$-eigenvectors are $\mathfrak{g}$-eigenvectors. This shows that $\mathrm{Z}_{\mathrm{P}}(\widehat{B}) \subseteq \mathrm{SZ}_{\mathrm{P}}(B)$. By proposition 2, the proof is complete. □

We recall the notations $\mathbb{S}_p(\Bbbk)$ for the Poisson-Weyl algebra (1.1.1), $\mathbb{S}'_\ell(\Bbbk)$ for its localized form (exercise 4 of 1.1.2), and $\mathbb{F}_n(\Bbbk)$ for the field $\mathrm{Frac}\, \mathbb{S}_n(\Bbbk) = \mathrm{Frac}\, \mathbb{S}'_n(\Bbbk)$ (see 1.2.1).

THEOREM (P. Tauvel, R. Yu). *Let $\mathfrak{g}$ be a finite dimensional algebraic solvable Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero. Let Q be a $\mathfrak{g}$-stable prime ideal of $S(\mathfrak{g})$. We denote by B the Poisson algebra $S(\mathfrak{g})/Q$ and by L its field of fractions. Then:*

(i) *there exist a nonzero semi-invariant element e of B and uniquely determined integers p, ℓ such that we have the following isomorphism of Poisson-algebras:*
$$B_e \simeq \mathrm{Z}_{\mathrm{P}}(B_e) \otimes \mathbb{S}_p(\Bbbk) \otimes \mathbb{S}'_\ell(\Bbbk);$$

(ii) *consequently, we have: $L \simeq \mathbb{F}_m(K)$ for $K = \mathrm{Z}_{\mathrm{P}}(L)$ and $m = p + \ell$.*

PROOF. We start with the decomposition $\mathfrak{g} = \widehat{\mathfrak{g}} \oplus \mathfrak{t}$ seen in the proof of proposition 4, where $\mathfrak{t}$ is an abelian Lie subalgebra acting semisimply on $\widehat{\mathfrak{g}}$. Applying the main lemma p. 26, there exists a nonzero element $e \in \mathrm{SZ}_{\mathrm{P}}(\widehat{B})$ such that we have the Poisson algebras isomorphism:

$$\widehat{B}_e \simeq \mathrm{Z}_{\mathrm{P}}(\widehat{B}_e) \otimes_{\Bbbk} \mathbb{S}_p(\Bbbk) \tag{52}$$

for some $p \geq 0$. Moreover, we can choose generators $x_1, y_1, \ldots, x_p, y_p$ of $S_p(\Bbbk)$ in $\widehat{B}_e$ satisfying relations (3) which are $\mathfrak{t}$-eigenvectors. Let us denote $\nu_1, \theta_1, \ldots, \nu_p, \theta_p$ the linear forms in $\mathfrak{t}^*$ defined by $\{t, x_i\} = \nu_i(t) x_i$ and $\{t, y_i\} = \theta_i(t) y_i$ for all $1 \leq i \leq p$ and $t \in \mathfrak{t}$. It follows from relations $\{x_i, y_i\} = 1$ that $\nu_i = -\theta_i$ for any $1 \leq i \leq p$. For any $t \in \mathfrak{t}$, the element $t' := t - \sum_{i=1}^{p} \nu_i(t) x_i y_i$ in $\widehat{B}_e$ satisfies: $\{t', x_i\} = \{t', y_i\} = 0$ for any $1 \leq i \leq p$, and $\{t', s\} = 0$ for any $s \in \mathfrak{t}$. Since B_e is generated by $\widehat{B}_e$ and the elements $t \in \mathfrak{t}$, or equivalently by $\widehat{B}_e$ and the element t' for $t \in \mathfrak{t}$, we deduce from proposition 2 that, for $(z_1, \ldots, z_r)$ is a basis of $\mathfrak{t}$, we have:

$$B_e \simeq \mathrm{Z}_{\mathrm{P}}(\widehat{B}_e)[z_1]_{0,\delta_1}[z_2]_{0,\delta_2} \ldots [z_r]_{0,\delta_r} \otimes_{\Bbbk} \mathbb{S}_p(\Bbbk). \tag{53}$$

By proposition 3, there exists a nonzero semi-invariant element $f \in B$ such that $\mathrm{SZ_P}(B_e) \simeq \mathrm{Z_P}(B_e)[w_1^{\pm 1}, \dots, w_\ell^{\pm 1}]$, where $w_1, \dots, w_\ell$ are nonzero semi-invariants elements in B_f algebraically independent over $\mathrm{Z_P}(B_f)$. Up to replace f by ef, we may suppose that we have:

$$\mathrm{SZ_P}(B_e) \simeq \mathrm{Z_P}(B_e)[w_1^{\pm 1}, \dots, w_\ell^{\pm 1}] \simeq \mathrm{Z_P}(B_e) \otimes_{\mathbb{k}} \mathbb{k}[w_1^{\pm 1}, \dots, w_\ell^{\pm 1}]. \tag{54}$$

More precisely, we have seen in the proof of proposition 4 that $\widehat{\mathfrak{g}} = \bigcap_{i=1}^{\ell} \ker \mu_i$ where $(\mu_1, \dots, \mu_\ell)$ is a basis of the $\mathbb{Z}$-submodule Λ' of $\mathfrak{g}^*$ (or equivalently of the $\mathbb{k}$-subspace generated by Λ'). The semi-invariants $w_1, \dots, w_\ell$ are of weight $\mu_1, \dots, \mu_\ell$ respectively (see proposition 3). For any λ in Λ, we have $\widehat{\mathfrak{g}} \subset \ker \lambda$ and so λ is uniquely the extension of an eigenvalue of $\mathfrak{t}$ on B. It follows that the $\mathbb{k}$-subspace of $\mathfrak{t}^*$ spanned by the restrictions to $\mathfrak{t}$ of $\mu_1, \dots, \mu_\ell$ has dimension ℓ. By an elementary argument of linear algebra[8], it must equal $\mathfrak{t}^*$. Thus the restrictions of $\mu_1, \dots, \mu_\ell$ form a basis for $\mathfrak{t}^*$. Denote by $(v_1, \dots, v_\ell)$ its dual basis in $\mathfrak{t}$, and set $t_i = w_i^{-1} v_i$ for any $1 \le i \le \ell$. For all $1 \le i, j \le \ell$, we calculate:

$$\{t_i, w_j\} = w_i^{-1}\{v_i, w_j\} = w_i^{-1}(v_i \cdot w_j) = w_i^{-1}\mu_j(v_i) w_j = \delta_{ij} w_i^{-1} w_j = \delta_{ij}.$$

We have $\{w_i, w_j\} = 0$ by propositions 1 and 3, and $\{v_i, v_j\} = 0$ because $\mathfrak{t}$ is abelian; consequently $\{t_i, t_j\} = 0$. Hence, by (54) and proposition 4, we have the isomorphism of Poisson algebras:

$$\mathrm{Z_P}(\widehat{B}_e)[z_1]_{0,\delta_1}[z_2]_{0,\delta_2} \dots [z_r]_{0,\delta_r} \simeq \mathrm{Z_P}(B_e) \otimes_{\mathbb{k}} \mathbb{k}[w_1^{\pm 1}, \dots, w_\ell^{\pm 1}][t_1]_{0,\partial_{w_1}}[t_2]_{0,\partial_{w_2}} \dots [t_\ell]_{0,\partial_{w_\ell}} \tag{55}$$

We conclude with (53), (55) and exercise 4 of 1.1.2 that $B_e \simeq \mathrm{Z_P}(B_e) \otimes_{\mathbb{k}} \mathbb{S}'_\ell(\mathbb{k}) \otimes_{\mathbb{k}} \mathbb{S}_p(\mathbb{k})$. The proof of (i) is complete. Point (ii) follows using (22) and example 3 p. 12. □

Last comment. By point (ii) of the theorem, the field $L(\mathfrak{g}) = \operatorname{Frac} S(\mathfrak{g})$ is Poisson isomorphic to a Poisson-Weyl field $\mathbb{F}_m(K)$ for $K = \mathrm{Z_P}(L(\mathfrak{g}))$. The answer to the problem p. 19 will be complete after proving that K is a purely transcendental extension of $\mathbb{k}$. This question (which is not addressed in [36]) can be solved at least in two ways: by repeating mutatis mutandis the proof of the similar result for the enveloping algebra[9], or by deducing it directly by symmetrization[10] from the similar result for the enveloping algebra. After this final step, we deduce:

Conclusion. *For any finite dimensional algebraic solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbb{k}$ of characteristic zero, the field $L(\mathfrak{g}) = \operatorname{Frac} S(\mathfrak{g})$ is Poisson isomorphic to a Poisson-Weyl field $\mathscr{F}_{m,t}(\mathbb{k})$ for some integers $m, t \ge 0$.*

[8] From [25] 14.1.20: let V be a finite dimensional $\mathbb{k}$-vector space and A a subspace of V^* such that no nonzero $v \in V$ is annihilated by all $a \in A$; then $A = V^*$.

[9] The center of the field of fractions of the enveloping algebra $U(\mathfrak{g})$ for solvable $\mathfrak{g}$ over an algebraically closed base field $\mathbb{k}$ of characteristic zero is a purely transcendental extension of $\mathbb{k}$ of degree $\le \dim \mathfrak{g}$ (proposition 4.4.8 in [17]). More generally, the field of fractions of any prime quotient $U(\mathfrak{g})/P$ is an extension of finite type of $\mathbb{k}$ (proposition 4.4.11 in [17]).

[10] For any finite dimensional Lie algebra $\mathfrak{g}$, there exists a canonical bijection ϕ (the symmetrization) from the symmetric algebra $S(\mathfrak{g})$ to the enveloping algebra $U(\mathfrak{g})$, which is a $\mathfrak{g}$-module isomorphism. We have then $\phi(\mathrm{Z_P}(S(\mathfrak{g}))) = Z(U(\mathfrak{g}))$ but the restriction $\phi_{|\mathrm{Z_P}(S(\mathfrak{g}))}$ is not in general an isomorphism of algebras (see 2.4.10, 2.4.11 and 4.9.6.b in [17]) from $\mathrm{Z_P}(S(\mathfrak{g}))$ to $Z(U(\mathfrak{g}))$. However, $\mathrm{Z_P}(S(\mathfrak{g})) \simeq Z(U(\mathfrak{g}))$ can be proved when $\mathfrak{g}$ is nilpotent (see 4.8.12 in [17]), more generally solvable (see 6.6.9 in [17]), or for arbitrary $\mathfrak{g}$ when $\mathbb{k}$ is algebraically closed (see 10.4.5 in [17]).

2.2 Rational equivalence for semiclassical limits of quantum algebras

The problem discussed in the previous section can also be formulated for Poisson algebras S arising from semiclassical limits of quantum polynomial algebras. In this context, the question becomes:

> PROBLEM. *Do we have a Poisson isomorphism* $\operatorname{Frac} S \simeq \mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ *for some integer* $n \geq 2$ *and some* $n \times n$ *antisymmetric matrix* $\boldsymbol{\lambda}$ *with entries in* $\Bbbk$ *?*

This question must be seen as a Poisson analogue of the noncommutative quantum version of Gel'fand-Kirillov problem (see [1], [32] or sections I.2.11 and II.10.4 of [13] for references). We present here an outline of the paper [20] which gives a complete answer for large classes of such Poisson-quantum algebras. The main tool is an adaptation to the case of iterated Poisson-Ore polynomial algebras of an algorithmic method invented by G. Cauchon (see [14]) for noncommutative iterated Ore extensions, which consists in deleting the "δ-part" by localization. Some specific assumptions are necessary for applying this process, which are satisfied for the algebras under consideration because they support some suitable rational action of a torus.

2.2.1 Poisson-derivations deleting method

We start with a Poisson-Ore extension $A = B[x]_{\sigma,\delta}$ in the sense of 1.1.2 over a $\Bbbk$-algebra B. The construction of the δ-deleting map, denoted by θ, proceeds in two steps (we refer to the paper [20] for the details of calculations which are only sketched in the following).

LEMMA. *Assume that* δ *is locally nilpotent, and that there exists* $s \in \Bbbk^\times$ *such that* $\sigma\delta = \delta(\sigma + s)$*. Let us define, for any* $b \in B$*:*

$$\theta(b) = \sum_{n\geq 0} \tfrac{1}{n!} \left(\tfrac{-1}{s}\right)^n \delta^n(b) x^{-n}.$$

Hence we have:

(i) θ *defines a* $\Bbbk$*-algebra homomorphism* $B \to B[x^{\pm 1}]$,

(ii) $\{x, \theta(b)\} = \theta\sigma(b)x$ *for any* $b \in B$,

(iii) θ *is a Poisson homomorphism from* B *to* $B[x^{\pm 1}]_{\sigma,\delta}$.

PROOF. The linearity of θ is clear and equality $\theta(a)\theta(b) = \theta(ab)$ for all $a, b \in B$ is easily deduced from Leibniz rule. Point (ii) follows from:

$$\begin{aligned}
\{x, \delta(b)\} &= \sum_{n\geq 0} \tfrac{1}{n!} \left(\tfrac{-1}{s}\right)^n \{x, \delta^n(b)\} x^{-n} = \sum_{n\geq 0} \tfrac{1}{n!} \left(\tfrac{-1}{s}\right)^n (\sigma\delta^n(b)x + \delta^{n+1}(b)) x^{-n} \\
&= \theta\delta(b) + \sum_{n\geq 0} \tfrac{1}{n!} \left(\tfrac{-1}{s}\right)^n \delta^n(\sigma + ns)(b) x^{1-n} \\
&= \theta\delta(b) + \theta\sigma(b)x - \sum_{n=1} \tfrac{1}{(n-1)!} \left(\tfrac{-1}{s}\right)^{n-1} \delta^n(b) x^{1-n} = \theta\delta(b) + \theta\sigma(b)x - \theta\delta(b) = \theta\sigma(b)x.
\end{aligned}$$

The nontrivial point is assertion (iii). We compute:

$$\begin{aligned}\{\theta(a),\theta(b)\} &= \sum_{l\geq 0}\tfrac{1}{l!}\left(\tfrac{-1}{s}\right)^l\{\delta^l(a)x^{-l},\theta(b)\} = \sum_{l\geq 0}\tfrac{1}{l!}\left(\tfrac{-1}{s}\right)^l(\{\delta^l(a),\theta(b)\}x^{-l} - l\delta^l(a)\{x,\theta(b)\}x^{-l-1})\\ &= \sum_{l\geq 0}\tfrac{1}{l!}\left(\tfrac{-1}{s}\right)^l(\{\delta^l(a),\theta(b)\} - l\delta^l(a)\theta\sigma(b))x^{-1}\\ &= \sum_{l,m\geq 0}\tfrac{1}{l!m!}\left(\tfrac{-1}{s}\right)^{l+m}(\{\delta^l(a),\delta^m(b)x^{-m}\} - l\delta^l(a)\delta^m\sigma(b)x^{-m})x^{-l}\\ &= \sum_{l,m\geq 0}\tfrac{1}{l!m!}\left(\tfrac{-1}{s}\right)^{l+m}(C_{lm}+D_{lm})x^{-l-m},\end{aligned}$$

with notations:

$$\begin{aligned}C_{lm} &= \{\delta^l(a),\delta^m(b)\} + m\delta^l\sigma(a)\delta^m(b) - l\delta^l(a)\delta^m\sigma(b),\\ D_{lm} &= lms\delta^l(a)\delta^m(b) + m\delta^{m+1}(a)\delta^m(b)x^{-1}.\end{aligned}$$

We check that $\sum\limits_{l,m\geq 0}\frac{1}{l!m!}\left(\frac{-1}{s}\right)^{l+m}D_{lm}x^{-l-m} = 0$ and use the auxiliary (nontrivial) calculation:

$$\delta^n(\{a,b\}) = \sum_{l+m=n}\binom{n}{l}(\{\delta^l(a),\delta^m(b)\} + m\delta^l\sigma(a)\delta^m(b) - l\delta^l(a)\delta^m\sigma(b))$$

to conclude that:

$$\begin{aligned}\{\theta(a),\theta(b)\} &= \sum_{l,m\geq 0}\tfrac{1}{l!m!}\left(\tfrac{-1}{s}\right)^{l+m}C_{lm}x^{-l-m} = \sum_{n\geq 0}\sum_{l+m=n}\tfrac{1}{l!m!}\left(\tfrac{-1}{s}\right)^n C_{lm}x^{-n}\\ &= \sum_{n\geq 0}\tfrac{1}{n!}\sum_{l+m=n}\tfrac{n!}{l!m!}\left(\tfrac{-1}{s}\right)^n C_{lm}x^{-n} = \sum_{n\geq 0}\tfrac{1}{n!}\left(\tfrac{-1}{s}\right)^n x^{-n}\sum_{l+m=n}\binom{n}{l}C_{lm}\\ &= \sum_{n\geq 0}\tfrac{1}{n!}\left(\tfrac{-1}{s}\right)^n\delta^n(\{a,b\})x^{-n} = \theta(\{a,b\}),\end{aligned}$$

and the proof is complete. □

Proposition. *Assume that δ is locally nilpotent, and that there exists $s \in \Bbbk^\times$ such that $\sigma\delta = \delta(\sigma + s)$. Then there exists a Poisson isomorphism $\theta : B[y^{\pm 1}]_{\sigma,0} \to B[x^{\pm 1}]_{\sigma,\delta}$ defined by:*

$$\theta(y) = x, \quad \text{and} \quad \theta(b) = \sum_{n\geq 0}\tfrac{1}{n!}\left(\tfrac{-1}{s}\right)^n\delta^n(b)x^{-n} \text{ for any } b\in B.$$

Proof. It is clear that the Poisson homomorphism $\theta : B \to B[x^{\pm 1}]$ of the lemma extends by setting $\theta(y) = x$ into an homomorphism of $\Bbbk$-algebras $B[y^{\pm 1}] \to B[x^{\pm 1}]$. Concerning the Poisson structure, the forms $\theta(\{-,-\})$ and $\{\theta(-),\theta(-)\}$ agree on pairs of elements of B by point (iii) of the lemma, and on pairs of elements of $B \cup \{y^{\pm 1}\}$ since we have by point (ii) of the lemma:

$$\theta(\{y,b\}) = \theta(\sigma(b)y) = \theta(\sigma(b))\theta(y) = \theta(\sigma(b))x = \{x,\theta(b)\} = \{\theta(y),\theta(b)\}.$$

From which we deduce that they agree on pairs of elements of $B[y^{\pm 1}]_{\sigma,0}$ par derivation et linearity. Hence θ is a Poisson morphism $B[y^{\pm 1}]_{\sigma,0} \to B[x^{\pm 1}]_{\sigma,\delta}$.

For surjectivity, we already have $\theta(y^{\pm 1}) = x^{\pm 1}$ and we just need to see that B is contained in the image of θ. Let b be a fixed nonzero element of B. Since δ is locally nilpotent, there exists an integer $l \geq 1$ such that $\delta^l(b) = 0$, and we proceed by induction on l. If $l = 1$ then $\delta(b) = 0$ and in this case $\theta(b) = b$. Now let $l > 1$ and write:

$$\theta(b) = b + \sum_{n=1}^{l-1} \tfrac{1}{n!} \left(\tfrac{-1}{s}\right)^n \delta^n(b) x^{-n}.$$

We have $\delta^{l-1}(\delta^n(b)) = \delta^{n+l-1}(b) = 0$ for $n = 1, \ldots, l-1$, then $\theta(\delta^{l-1}(b)) = \delta^{l-1}(b)$ and therefore $\delta^{l-1}(b) \in \mathrm{Im}\theta$. Since $\theta(\delta^{l-2}(b)) = \delta^{l-2}(b) - \frac{1}{s}\delta^{l-1}(b)x^{-1}$, we also have $\delta^{l-2}(b) \in \mathrm{Im}\theta$. By induction with respect to l, we deduce that $\delta^{l-1}(b), \ldots, \delta^1(b)$ lie in $\mathrm{Im}\theta$, then $\theta(b) - b$ lies in $\mathrm{Im}\theta$, and finally $b \in \mathrm{Im}\theta$.

For injectivity, we take a nonzero Laurent polynomial $p \in B[y^{\pm 1}]$ developed as $p = \sum_{i=l}^{m} b_i y^i$ with $b_i \in B$ and some integers $l \leq m$ such that $b_m \neq 0$. Each of the terms $\theta(b_i y^i)$ is a Laurent polynomial of the form $b_i y^i +$ [lower terms]. Hence $\theta(p) = b_m x^m +$ [lower terms]. Thus $\theta(p) \neq 0$ (because $b_m \neq 0$), and consequently θ is injective. □

We can now apply the method to iterated Poisson-Ore polynomial algebras in the sense of (11) in order to obtain the following Poisson version of theorem 6.1.1 of [14].

THEOREM (K. Goodearl, S. Launois). *Let $A = \Bbbk[x_1][x_2]_{\sigma_2,\delta_2}[x_3]_{\sigma_3,\delta_3} \ldots [x_n]_{\sigma_n,\delta_n}$ be an iterated Poisson-Ore polynomial algebra satisfying the three conditions:*

(a) *δ_i is locally nilpotent for any $2 \leq i \leq n$;*

(b) *there exists $s_i \in \Bbbk^\times$ such that $\sigma_i\delta_i = \delta_i(\sigma_i + s_i)$ for any $2 \leq i \leq n$;*

(c) *there exists $\lambda_{ij} \in \Bbbk$ such that $\sigma_i(x_j) = \lambda_{ij}x_j$ for all $1 \leq j < i \leq n$.*

Then Frac A *is Poisson-isomorphic to the Poisson-quantum field $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ for $\boldsymbol{\lambda}$ the $n \times n$ antisymmetric matrix whose entries below the diagonal agree with the scalars λ_{ij} in* (c)*.*

PROOF. We proceed by a double induction: first, with respect to the number d of indexes i for which $\delta_i \neq 0$, and second (downward) with respect to the maximum index t for which $\delta_t \neq 0$ (with convention $t = n+1$ if $d = 0$; there is nothing to prove is this case).

Case 1: $t = n$, i.e. $\delta_n \neq 0$. Set $B = k[x_1][x_2]_{\sigma_2,\delta_2} \cdots [x_{n-1}]_{\sigma_{n-1},\delta_{n-1}}$. By the proposition above, the localization $B[x_n^{-1}]_{\sigma_n,\delta_n}$ of $A = B[x_n]_{\sigma_n,\delta_n}$ is Poisson isomorphic $B[y^{\pm 1}]_{\sigma_n,0}$. Thus the iterated Poisson-Ore extension $A' := B[y]_{\sigma_n,0}$ is such that Frac A' and Frac A are Poisson isomorphic. Since the number of nonzero maps among $\delta_2, \ldots, \delta_{n-1}$ is $d-1$, we can apply the first induction to the algebra A' and the expected result for Frac A is proved in this case.

Case 2: $t < n$, i.e. $\delta_n = 0$.. Since $\{x_n, x_1\} = \lambda_{n1}x_1x_n$, we see that $\{x_n, \Bbbk[x_1]\} \subseteq \Bbbk[x_1]x_n$, and so $\Bbbk[x_1, x_n]$ is a Poisson-Ore algebra of the form $\Bbbk[x_1][x_n]_{\sigma'_n,0}$. For $i = 2, \ldots, n-1$, we have $\{x_i, \Bbbk[x_1, \ldots, x_{i-1}]\} \subseteq \Bbbk[x_1, \ldots, x_{i-1}]x_i + \Bbbk[x_1, \ldots, x_{i-1}]$ and $\{x_i, x_n\} = -\lambda_{ni}x_ix_n = \lambda_{in}x_nx_i$, from which it follows that $\{x_i, \Bbbk[x_1, \ldots, x_{i-1}, x_n]\} \subseteq \Bbbk[x_1, \ldots, x_{i-1}, x_n]x_i + \Bbbk[x_1, \ldots, x_{i-1}, x_n]$. Hence, we may rewrite A in the form:

$$A = \Bbbk[x_1][x_n]_{\sigma'_n,0}[x_2]_{\sigma'_2,\delta'_2}[x_{n-1}]_{\sigma'_{n-1},\delta'_{n-1}}$$

for suitable σ'_i and δ'_i, such that $\sigma'_i(x_j) = \lambda_{ij}x_j$ for $j < i$ and for $j = n$. Note that σ'_i and δ'_i restrict to σ_i and δ_i on $\Bbbk[x_1, \ldots, x_{i-1}]$, and that $\delta'_i(x_n) = 0$. It follows easily that δ'_i is locally nilpotent, and that $\sigma'_i\delta'_i = \delta'_i(\sigma'_i + s_i)$. Finally, the map δ'_t is nonzero because it restricts to δ_t, and it occurs in position $t+1$ in the list $0, 0, \delta'_2, \dots, \delta'_{n-1}$. Thus, the second induction yield the result. $\square$

Remark. The previous theorem is in fact a slightly simplified version of the result of [20], which establishes such a rational Poisson-isomorphism not only for the Poisson algebra A itself, but also for all its quotients by Poisson prime ideals.

2.2.2 Rational Poisson-actions of tori and applications

(i) A general argument. Our goal is to show (following [20]) that the existence of a suitable action of a torus on an iterated Poisson-Ore algebra implies condition (b) in the previous theorem. We recall some preliminary results([13] pp. 149-150, or [20]).

We fix a torus $H = (\Bbbk^\times)^r$, with $r \geq 1$ and denote by $\widehat{H}$ the group of all characters of H (i.e. all morphisms of groups $\chi : H \to \Bbbk^\times$).

• We suppose that H acts by automorphisms on a $\Bbbk$-algebra A, by $H \times A \to A, (h, a) \mapsto h.a$. An H-eigenvector is a nonzero element $a \in A$ such that $h.a \in \Bbbk a$ for all $h \in H$; we associate to a the character $\chi_a \in \widehat{H}$ defined by $h.a = \chi_a(h)a$ for all $h \in H$.

Conversely, for any $\chi \in \widehat{H}$, we define $A_\chi = \{a \in A\,;\, h.a = \chi(h)a \text{ for any } h \in H\}$. When $A_\chi \neq 0$, we call it the H-eigenspace associated with the eigenvalue χ.

• We suppose moreover that the action of H is rational; because H is a torus, that means that A is a direct sum of its H-eigenspace (semisimplicity) and the corresponding eigenvalues are rational characters (i.e. morphisms of algebraic varieties). Let us denote by $X(H)$ the group of rational character of H. The base field being infinite, $X(H)$ is free abelian of rank r with a basis consisting of the r projections $(\Bbbk^\times)^r \to \Bbbk^\times$. It could be useful to identify $X(H)$ with the additive group $\mathbb{Z}^r$ via the isomorphism $\varphi : \mathbb{Z}^r \to X(H)$ defined for any $z = (z_1, \dots, z_r)$ by $\chi := \varphi(z) : (h_1, \dots, h_r) \mapsto h_1^{z_1} \dots h_r^{z_r}$.

Denoting by $\mathfrak{h}$ the $\Bbbk$-vector space $\Bbbk^r$, we set:

$$\eta.a = (\eta|\chi)a \qquad \text{for all } \ \eta \in \mathfrak{h},\ \chi \in X(H),\ a \in A_\chi,$$

where $(\eta|\chi)$ is the ordinary scalar product $(\eta|\chi) = \eta_1 z_1 + \dots + \eta_r z_r$ of $\eta = (\eta_1, \dots, \eta_r)$ by $\varphi^{-1}(\chi) = (z_1, \dots, z_r)$. The action of H being rational thus semisimple, we can define:

$$\eta.a = \sum_{\chi \in X(H)} (\eta|\chi)a_\chi \qquad \text{for all } \eta \in \mathfrak{h},\ a = \sum_{\chi \in X(H)} a_\chi \in A, \text{ with } a_\chi \in A_\chi.$$

It is clear that the so-defined map $\mathfrak{h} \times A \to A, (\eta, a) \mapsto \eta.a$ is an action of the additive group $\mathfrak{h}$ (it could be viewed from a more theoretical point of view as the differential action of the Lie algebra $\mathfrak{h}$ deduced from the rational action of the algebraic group H). We claim that $\mathfrak{h}$ acts by derivations on A, i.e. $\eta.(ab) = (\eta.a)b + a(\eta.b)$ for all $\eta \in \mathfrak{h}, a, b \in A$.

> Take $\eta \in \mathfrak{h}$, a in some eigenspace A_χ and b in another one $A_{\chi'}$ where $\chi, \chi' \in X(H)$. Since H acts by automorphisms on A, we have $h.(ab) = (h.a)(h.b) = \chi(h)\chi'(h)ab$ for any $h \in H$, and then the element ab lies in the eigenspace $A_{\chi\chi'}$. Therefore, $\eta.(ab) = (\eta|\chi\chi')(ab) = \big((\eta|\chi) + (\eta|\chi')\big)ab = (\eta|\chi)ab + a(\eta|\chi')b = (\eta.a)b + a(\eta.b)$. The same identity holds for all $a, b \in A$ by semisimplicity, hence the claim is proved.

The differential $\mathfrak{h}$-action commutes with the rational H-action (an easy calculation shows that $\eta.(h.a) = h.(\eta.a)$ for all $\eta \in \mathfrak{h}, h \in H, a \in A$).

• We suppose moreover that A is a Poisson algebra and that the rational action of H is a Poisson action, i.e. $h.\{a,b\} = \{h.a, h.b\}$ for all $h \in H, a, b \in A$. We claim that $\mathfrak{h}$ acts by Poisson derivations on A, i.e. $\eta.\{a,b\} = \{\eta.a, b\} + \{a, \eta.b\}$ for all $\eta \in \mathfrak{h}, a, b \in A$.

> Take $\eta \in \mathfrak{h}$, a in some eigenspace A_χ and b in another one $A_{\chi'}$ where $\chi, \chi' \in X(H)$. Since $h.\{a,b\} = \{h.a, h.b\} = \{\chi(h)a, \chi'(h)b\} = \chi(h)\chi'(h)\{a,b\}$ for any $h \in H$, the element $\{a,b\}$ lies in the eigenspace $A_{\chi\chi'}$. Therefore, $\eta.\{a,b\} = (\eta|\chi\chi')\{a,b\} = \big((\eta|\chi) + (\eta|\chi')\big)\{a,b\} = \{(\eta|\chi)a, (\eta|\chi')b\} = \{\eta.a, b\} + \{a, \eta.b\}$. The same identity holds for all $a, b \in A$ by semisimplicity, hence the claim is proved.

Lemma. *Let $A = \Bbbk[x_1][x_2]_{\sigma_2,\delta_2}[x_3]_{\sigma_3,\delta_3} \dots [x_n]_{\sigma_n,\delta_n}$ be an iterated Poisson-Ore polynomial algebra supporting a rational action by a torus $H := (\Bbbk^\times)^r$ such that $x_1, \dots, x_n$ are H-eigenvectors. We assume that there exist $\eta_1, \dots, \eta_n \in \mathfrak{h}$ satisfying, for any $2 \le i \le n$:*

$$\eta_i.x_j = \sigma_i(x_j) \text{ for all } j < i, \quad \text{and} \quad \eta_i.x_i = s_i x_i \text{ for some } s_i \neq 0 \text{ in } \Bbbk.$$

Then $\sigma_i\delta_i = \delta_i(\sigma_i + s_i)$ for any $2 \le i \le n$.

Proof. Fix $2 \le i \le n$. There exists $\chi_i \in X(H)$ such that $x_i \in A_{\chi_i}$, hence $\eta_i.x_i = (\eta_i|\chi_i)x_i$, and therefore $(\eta_i|\chi_i) = s_i \neq 0$. By assumption, the derivations $a \mapsto \eta_i.a$ and σ_i agree on $x_1, \dots, x_{i-1}$ hence agree on the algebra $A_{i-1} = \Bbbk[x_1, \dots, x_{i-1}]$. Then for any element $f \in A_{i-1}$ taken in an H-eigenspace A_χ for some $\chi \in X(H)$, we have:

$$\{x_i, f\} = \sigma_i(f)x_i + \delta_i(f) = (\eta_i.f)x_i + \delta_i(f) = (\eta_i|\chi)fx_i + \delta_i(f)$$

As noted above, $x_i \in A_{\chi_i}$ and $f \in A_\chi$ imply $x_i f \in A_{\chi_i+\chi}$ and $\{x_i, f\} \in A_{\chi_i+\chi}$, and consequently $\delta_i(f) = \{x_i, f\} - (\eta_i|\chi)fx_i$ lies in $A_{\chi_i+\chi}$. We calculate:

$$\sigma_i\delta_i(f) = \eta_i.\delta_i(f) = (\eta_i \,|\, \chi_i + \chi)\delta_i(f) = \delta_i((\eta_i|\chi)f + (\eta_i|\chi_i)f) = \delta_i(\eta_i.f + s_i f) = \delta_i(\sigma_i + s_i)(f).$$

The result follows by semisimplicity of the H-action. □

(ii) Application to the semiclassical limit of quantum matrices. We consider here the Poisson algebra $A = \mathscr{O}(M_n(\Bbbk)))_{\boldsymbol{p},\lambda}$ defined in example 4 of 1.1.3. We suppose $\lambda \neq 0$. We have seen that, adjoining the $n \times n$ generators x_{ij} in lexicographical order, A can be described as an iterated Poisson-Ore extension (19). We claim that A satisfies the three assumptions (a), (b) and (c) of the main theorem of 2.2.1.

Condition (c) is clearly given by relations (20). We deduce immediately from (21) that

$$\delta_{lm}^2(x_{ij}) = 0 \quad \text{for all } (i,j) <_{\text{lex}} (l,m),$$

hence (a) is satisfied. We set $H = (\Bbbk^\times)^{2n}$ and define an action of H on A by automorphisms from:

$$h.x_{ij} = h_i h_{n+j} x_{ij} \qquad \text{for all } \; h = (h_1, \dots, h_{2n}) \in H, \; 1 \le i, j \le n. \tag{56}$$

For any monomial $\boldsymbol{x} = \prod_{1\le i,j\le n} \alpha_{ij} x_{ij}^{m_{ij}}$, where $\alpha_{ij} \in \mathbb{k}$ and m_{ij} nonnegative integers, the action of an element $h \in H$ on $\boldsymbol{x}$ is given by $h.\boldsymbol{x} = (\prod_{1\le i,j\le n} h_i^{m_{ij}} h_{n+j}^{m_{ij}})\boldsymbol{x}$, from which we deduce that $\boldsymbol{x}$ is an H-eigenvector and that the action is rational. We check moreover by direct calculations using (18) and (56) that this is a Poisson action.
By definition, the rational character $h \mapsto h_i h_{n+j}$ is an eigenvalue for x_{ij}, which corresponds in the identification of $X(H) \simeq \mathbb{Z}^r$ detailed in the previous paragraph (i) to the element $\epsilon_i + \epsilon_{n+j}$ of $\mathfrak{h} = \mathbb{k}^{2n}$, where $(\epsilon_i)_{1\le i\le 2n}$ denotes the canonical basis of $\mathfrak{h}$. Hence the $\mathfrak{h}$-action by Poisson derivations on A obtained as the differential of the rational H-action by Poisson automorphisms is defined by $\eta.x_{ij} = (\eta \,|\, \epsilon_i + \epsilon_{n+j})x_{ij}$, or equivalently:

$$\eta.x_{ij} = (\eta_i + \eta_{n+j})x_{ij} \qquad \text{for all} \;\; \eta = (\eta_1, \dots, \eta_{2n}) \in \mathfrak{h},\; 1 \le i,j \le n. \tag{57}$$

Now, using the parameters $\boldsymbol{p}$ and λ of the definition of A, we define for all $1 \le l,m \le n$ the following element in $\mathfrak{h}$:

$$\eta_{lm} = (\underbrace{p_{l1},\, \dots,\, p_{ln}}_{n},\, \underbrace{p_{1m},\, \dots,\, p_{m-1,m}}_{m-1},\, \lambda,\, \underbrace{\lambda + p_{m+1,m},\, \dots,\, \lambda + p_{nm}}_{n-m-1})$$

By construction, we have for all $1 \le l,m \le n$ from (20) and (57):

$$\eta_{lm}.x_{ij} = \sigma_{lm}(x_{ij}) \;\; \text{for } (i,j) <_{\text{lex}} (l,m), \qquad \text{and} \qquad \eta_{lm}.x_{lm} = \lambda x_{lm}.$$

Thus the lemma in previous paragraph (i) applies, we have $\delta_{lm}\sigma_{lm} = \delta_{lm}(\sigma_{lm} + \lambda)$ for any $(1,1) <_{\text{lex}} (l,m) \le_{\text{lex}} (n,n)$, and the assumption (b) in theorem 2.2.1 is satisfied.

So we have proved that, for any $\boldsymbol{p}$ and $\lambda \ne 0$, this theorem hold for the Poisson algebra $\mathscr{O}(M_n(\mathbb{k}))_{\boldsymbol{p},\lambda}$. Since the case $\lambda = 0$ corresponds to the situation where $\mathscr{O}(M_n(\mathbb{k}))_{\boldsymbol{p},\lambda}$ is simply a Poisson-quantum space, we conclude:

Theorem (K. Goodearl, S. Launois). *Let $\boldsymbol{p} = (p_{ij})$ a $n \times n$ antisymmetric matrix with entries in $\mathbb{k}$, and $\lambda \in \mathbb{k}$. Then there exists a $n^2 \times n^2$ antisymmetric matrix $\boldsymbol{\lambda}$ with entries in $\mathbb{k}$ such that the field of fractions of $\mathscr{O}(M_n(\mathbb{k}))_{\boldsymbol{p},\lambda}$ is Poisson-isomorphic to the Poisson-quantum field $\mathbb{Q}^{\boldsymbol{\lambda}}_{n^2}(\mathbb{k})$.*

More precisely, $\operatorname{Frac}\mathscr{O}(M_n(\mathbb{k}))_{\boldsymbol{p},\lambda} \simeq \mathbb{k}(y_{ij})_{1\le i,j\le n}$ *with Poisson bracket:*

$$\{y_{lm}, y_{ij}\} = \begin{cases} (p_{li} + p_{jm})y_{ij}y_{lm} & \text{if } l \ge i \text{ and } m > j, \\ (\lambda + p_{li} + p_{jm})y_{ij}y_{lm} & \text{if } l > i \text{ and } m \le j. \end{cases} \tag{58}$$

(iii) Application to other semiclassical limits of quantum algebras. The same method holds for other significant classes of Poisson algebras obtained by semiclassical limits of quantum symplectic spaces, quantum euclidian spaces, quantum symmetric matrices, quantum antisymmetric matrices. Complete proofs can be found in [20], including the following example (introduced in [28]) covering the cases of quantum symplectic and even-dimensional euclidean spaces:

EXERCISE 1. For $\Gamma = (\gamma_{ij})$ an antisymmetric matrix in $M_n(k)$, $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_n)$ two vectors in $\mathbb{k}^n$ such that $p_i \neq q_i$ for any $1 \leq i \leq n$, $A := A_{n,\Gamma}^{P,Q}(\mathbb{k})$ is the commutative polynomial algebra in $2n$ indeterminates $\mathbb{k}[x_1, y_1 \dots, x_n, y_n]$ with Poisson bracket defined by:

$$\begin{array}{llll} \{y_i, y_j\} = \gamma_{ij} y_i y_j & (\text{all } i, j), & \{x_i, y_j\} = (p_j + \gamma_{ji}) x_i y_j & (i < j), \\ \{x_i, x_j\} = (q_i - p_j + \gamma_{ij}) x_i x_j & (i < j), & \{x_i, y_j\} = (q_j + \gamma_{ji}) x_i y_j & (i > j), \end{array}$$

$$\{x_i, y_i\} = q_i x_i y_i + \sum_{l<i} (q_l - p_l) x_l y_l (\text{all } i).$$

1) Prove that A is the iterated Poisson-Ore extension:

$$A = k[x_1][y_1]_{\sigma_1, \delta_1}[x_2]_{\sigma'_2, 0}[y_2]_{\sigma_2, \delta_2} \dots [x_n]_{\sigma'_n, 0}[y_n]_{\sigma_n, \delta_n},$$

where the $\sigma'_j, \sigma_j, \delta_j$ are defined for all $j > i$ by:

$$\begin{array}{lll} \sigma_j(x_i) = (\gamma_{ij} - p_j) x_i & \sigma'_j(x_i) = (p_j - q_i + \gamma_{ji}) x_i & \delta_j(x_i) = 0 \\ \sigma_j(y_i) = \gamma_{ji} y_i & \sigma'_j(y_i) = (q_i + \gamma_{ij}) y_i & \delta_j(y_i) = 0 \\ \sigma_j(x_j) = -q_j x_j & & \delta_j(x_j) = \sum_{l<i} (p_l - q_l) x_l y_l \end{array}$$

Deduce that conditions (a) and (c) of theorem 2.2.1 are satisfied.

2) Prove that $H := (\mathbb{k}^\times)^{n+1}$ acts rationaly on A by Poisson automorphisms from:

$$h.x_i = h_i x_i \text{ and } h.y_i = h_1 h_{n+1} h_i^{-1} y_i \quad \text{for any } h = (h_1, \dots h_n, h_{n+1}) \in H,$$

and that the associated action of $\mathfrak{h} := \mathbb{k}^{n+1}$ by Poisson derivations on A is:

$$\eta.x_i = \eta_i x_i \text{ and } \eta.y_i = (\eta_1 + \eta_{n+1} - \eta_i) y_i \quad \text{for any } \eta = (\eta_1, \dots \eta_n, \eta_{n+1}) \in \mathfrak{h}.$$

3) Define $\eta_j, \eta'_j \in \mathfrak{h}$ as follows:

$$\begin{array}{ll} \eta_1 = (-q_1, 0, 0, \dots, 0, 1), & \\ \eta_j = (-p_j + \gamma_{1j}, \dots, -p_j + \gamma_{j-1,j}, -q_j, 0, \dots, 0, \gamma_{j1}) & (j > 1), \\ \eta'_j = (-q_1 + p_j + \gamma_{j1}, \dots, -q_n + p_j + \gamma_{jn}, q_1 + \gamma_{1j}) & (j > 1). \end{array}$$

Prove that $\eta_1.x_1 = \sigma_1(x_1)$ and $\eta_1.y_1 = y_1$, and for any $j > 1$:

$$\begin{array}{lll} \eta_j.x_i = \sigma_j(x_i) \text{ if } i \leq j, & \eta_j(y_i) = \sigma_j(y_i) \text{ if } i < j, & \eta_j.y_j = (q_j - p_j) y_j, \\ \eta'_j.x_i = \sigma'_j(x_i) \text{ if } i \leq j, & \eta'_j(y_i) = \sigma'_j(y_i) \text{ if } i < j, & \eta'_j.x_j = (p_j - q_j) x_j. \end{array}$$

4) Conclude that $\operatorname{Frac} A_{n,\Gamma}^{P,Q}(\mathbb{k})$ is Poisson-isomorphic to the Poisson-quantum field $\mathbb{Q}_{2n}^{\boldsymbol{\lambda}}(\mathbb{k})$ for some suitable antisymmetric matrix $\boldsymbol{\lambda}$ in $M_{2n}(\mathbb{k})$, and more precisely to $\mathbb{k}(v_1, \dots, v_n, w_1, \dots, w_n)$ with Poisson brackets:

$$\begin{array}{llll} \{w_i, w_j\} = \gamma_{ij} w_i w_j & \text{for all } i, j, & \{v_i, w_j\} = (p_j + \gamma_{ji}) v_i w_j & \text{if } i < j, \\ \{v_i, v_j\} = (q_i - p_j + \gamma_{ij}) v_i v_j & \text{if } i < j, & \{v_i, w_j\} = (q_j + \gamma_{ji}) v_i w_j & \text{if } i \geq j. \end{array}$$

A LAST COMMENT. We emphasized in this section on the two main cases (Poisson-Weyl fields and Poisson-quantum fields) to the extent that they play a crucial role in many significant situations. But we can easily construct fields of rational functions whose Poisson-structure mixes the two types of brackets. The most simple example is the following (which can be viewed as a semiclassical limit of the noncommutative algebras of eulerian operators on quantum spaces studied in [32]).

EXERCICE 2.

1) Prove that there exists on the polynomial algebra $S = \Bbbk[x, y, z]$ a Poisson bracket defined by $\{x, y\} = 1$, $\{y, z\} = \lambda yz$ and $\{x, z\} = -\lambda xz$, where $\lambda \in \Bbbk^\times$. (Hint: use exercise 1 of 1.1.1).

2) Set $t = xy$ and prove that $F := \operatorname{Frac} A = \Bbbk(t, y, z)$ with Poisson brackets $\{t, z\} = 0$, $\{y, t\} = -y$ and $\{y, z\} = \lambda zy$. Deduce that $F = \Bbbk(t, z)(y)_{\sigma,0}$ for $\sigma = \lambda z\partial_z - \partial_t$ and that the Poisson center of F is $Z_P(F) = \ker \sigma = \Bbbk$. (Hint: use lemma 1.2.1).

3) Deduce from question 2) that F cannot be Poisson-isomorphic to a Poisson-Weyl field $\mathscr{F}_{n,t}(\Bbbk)$ (Hint: use (32) p. 13).

4) Prove that F cannot be Poisson-isomorphic to a Poisson-quantum field $\mathbb{Q}_n^{\boldsymbol{\lambda}}(\Bbbk)$ (Hint: use proposition 2 of 1.2.2).

3 Poisson analogue of Noether's problem

3.1 Commutative rational invariants

Let S be a commutative ring. For any subgroup G of $\operatorname{Aut} S$, we denote by S^G the invariant subring $\{a \in S\,;\, g(a) = a \text{ for any } g \in G\}$. Assume that S is a domain and consider $F = \operatorname{Frac} S$ the field of fractions of S. Any automorphism of S extends into an automorphism of F and it's obvious that, for any subgroup G of $\operatorname{Aut} S$, we have $\operatorname{Frac} S^G \subseteq F^G$. For finite G, the converse is true:

PROPOSITION. *If G is a finite subgroup of automorphisms of a commutative domain S with field of fractions F, then we have:* $\operatorname{Frac} S^G = F^G$.

PROOF. For any $x \in F^G$, there exist $a, b \in S$, $b \neq 0$, such that $x = \frac{a}{b}$. Define $b' = \prod_{g \in G, g \neq \mathrm{id}_S} g(b)$. Then $bb' \in S^G$ and $x = \frac{ab'}{bb'}$, with $ab' = x(bb') \in F^G \cap S = S^G$. □

This applies in particular to a polynomial algebra $S = \Bbbk[x_1, \ldots, x_n]$ and its field of rational functions $F = \Bbbk(x_1, \ldots, x_n)$, and in this case the following problem is classically formulated about the structure of F^G.

3.1.1 Noether's problem

Let $\Bbbk$ be commutative field of characteristic zero. Let G be a finite subgroup of $\mathrm{GL}_n(\Bbbk)$ acting canonically by linear automorphisms on $S = \Bbbk[x_1, \ldots, x_n]$, and then on $F = \operatorname{Frac} S = \Bbbk(x_1, \ldots, x_n)$. We consider the subfield $F^G = \operatorname{Frac} S^G$ of F.

Remark 1. It's well known (by Artin's lemma) that $[F : F^G] = |G|$, and then $\mathrm{trdeg}_{\Bbbk} F^G = \mathrm{trdeg}_{\Bbbk} F = n$.

Remark 2. We know from classical invariant theory that S^G is finitely generated (say by m elements) as a $\Bbbk$-algebra. Thus F^G is finitely generated (say by p elements) as a field extension of $\Bbbk$, with $p \leq m$. We can have $p < m$; example: $S = \Bbbk(x, y)$ and $G = \langle g \rangle$ for $g : x \mapsto -x, y \mapsto -y$, then $S^G = \Bbbk[x^2, y^2, xy] = \Bbbk[X, Y, Z]/(Z^2 - XY)$ and $F^G = \Bbbk(xy, x^{-1}y)$.

Remark 3. Suppose that S^G is not only finitely generated, but isomorphic to a polynomial algebra $\Bbbk[y_1, \ldots, y_m]$, with $y_1, \ldots, y_m$ algebraically independent over $\Bbbk$. Then we have $F^G = \Bbbk(y_1, \ldots, y_m)$. Thus $m = n$ by remark 1.

Now we can consider the main question:

Problem (Noether's problem): *is F^G a purely transcendental extension of $\Bbbk$?*

An abundant literature has been devoted (and is still devoted) to this question and it's out of the question to give here a comprehensive presentation of it. The first counterexamples (Swan 1969, Lenstra 1974) were for $\Bbbk = \mathbb{Q}$ (and G the cyclic group of order n in S_n for $n = 47$ and $n = 8$ respectively) and D. Saltman produced in 1984 the first counterexample for $\Bbbk$ algebraically closed (see [22], [33]). Here we just point out the following elementary facts.

- The answer is positive if S^G is a polynomial algebra. By remark 3, we have then $S^G = \Bbbk[x_1, \ldots, x_n]$ and $F^G = \Bbbk(x_1, \ldots, x_n)$. This is in particular the case when G is the symmetric group S_n acting by permutation of the x_j's, or more generally when Shephard-Todd and Chevalley theorem applies.

- The answer is positive if $n = 1$. This is an obvious consequence of Lüroth's theorem: if $F = \Bbbk(x)$ is a purely transcendental extension of degree 1 of $\Bbbk$, then for any intermediate subfield $\Bbbk \subsetneq L \subset F$, there exists some $v \in F$ transcendental over $\Bbbk$ such that $F = \Bbbk(v)$.

- The answer is positive if $n = 2$. This is an obvious consequence of Castelnuovo's theorem: if $F = \Bbbk(x, y)$ is a purely transcendental extension of degree 2 of $\Bbbk$, then for any intermediate subfield $\Bbbk \subsetneq L \subset F$ such that $[F : L] < +\infty$, there exists some $v, w \in F$ such that $F = \Bbbk(v, w)$ is purely transcendental of degree 2.

- The answer is positive for all $n \geq 1$ when G is abelian and $\Bbbk$ is algebraically closed. This is a classical theorem by E. Fischer (1915), see corollary 2 in 3.1.2 below.

3.1.2 Miyata's theorem

The following result (see [22] or [26]) concerns invariants under actions on rational functions resulting from actions on polynomials. Observe that the group G is not necessarily finite.

Theorem (T. Miyata). *Let K be a commutative field, $S = K[x]$ the commutative ring of polynomials in one variable over K, and $F = K(x)$ the field of fractions of S. Let G be a group of ring automorphisms of S such that $g(K) \subseteq K$ for any $g \in G$.*

(i) *if* $S^G \subseteq K$, *then* $F^G = S^G = K^G$.

(ii) *if* $S^G \not\subseteq K$, *then for any* $u \in S^G, u \notin K$ *of degree* $m = \min\{\deg_x y\,;\, y \in S^G, y \notin K\}$, *we have* $S^G = K^G[u]$ *and* $F^G = K^G(u)$.

Proof. We simply denote here deg for $\deg_x$. Take $g \in G$ and $n = \deg g(x)$; the assumption $g(K) \subseteq K$ implies $\deg g(s) \in \mathbb{N} \cup \{-\infty\}$ for all $s \in S$ and so $n = 1$ since g is surjective. We deduce:

$$\deg g(s) = \deg s \text{ for all } g \in G \text{ and } s \in S. \tag{*}$$

If $S^G \subset K$, then $S^G = K^G$. If $S^G \not\subseteq K$, let us choose in $\{s \in S^G\,;\, \deg s \geq 1\}$ an element u of minimal degree m. The inclusion that $K^G[u] \subseteq S^G$ is clear. For the converse, let us fix $s \in S^G$. There exist q_1 and r_1 unique in S such that $s = q_1 u + r_1$ and $\deg r_1 < \deg u$. For any $g \in G$, we have then: $s = g(s) = g(q_1)g(u) + g(r_1) = g(q_1)u + g(r_1)$. Since $\deg g(r_1) = \deg r_1 < \deg u$ by (*), it follows from the uniqueness of q_1 and r_1 that $g(q_1) = q_1$ and $g(r_1) = r_1$. So $r_1 \in S^G$; since $\deg r_1 < \deg u$ and $\deg u$ is minimal, we deduce that $r_1 \in K^G$. Moreover, $q_1 \in S^G$, and $\deg q_1 < \deg s$ because $\deg u \geq 1$. To sum up, we obtain $s = q_1 u + r_1$ with $r_1 \in K^G$ and $q_1 \in S^G$ such that $\deg q_1 < \deg s$. We decompose similarly q_1 into $q_1 = q_2 u + r_2$ with $r_2 \in K^G$ and $q_2 \in S^G$ such that $\deg q_2 < \deg q_1$. We obtain $s = q_2 u^2 + r_2 u + r_1$. By iteration, it follows that $s \in K^G[u]$.

In both cases (i) and (ii), the inclusion $\mathrm{Frac}\,(S^G) \subseteq F^G$ is clear. For the converse (which follows from the first proposition of 3.1 in the particular case where G is finite), we have to prove that:

$$\text{for any } a \text{ and } b \text{ non-zero in } S,\ ab^{-1} \in F^G \text{ implies } ab^{-1} \in \mathrm{Frac}\,(S^G). \tag{**}$$

Let a and b be two nonzero relatively prime elements in S such that $t := ab^{-1} \in F^G$. If $a \in K$ or $b \in K$ and the result is clear. We suppose now $\deg a > 0$ and $\deg b > 0$. Up to replace t by t^{-1}, we can without any restriction suppose that $\deg b \leq \deg a$. The assumption $g(t) = t$ for any $g \in G$ implies $g(a)b = g(b)a$, hence a is a divisor of $g(a)$ and b is a divisor of $g(b)$ in S. By (*), it follows that $g(a) = k_g a$ and $g(b) = k_g b$ for some nonzero $k_g \in K$. Moreover, there exist $q, r \in S$ uniquely determined such that:

$$a = qb + r \qquad \text{with } \deg r < \deg b \leq \deg a. \tag{***}$$

Applying any $g \in G$, we have $k_g a = k_g b g(q) + g(r)$. Since $\deg g(r) = \deg r$, the uniqueness of (q, r) implies that $g(q) = q$ and $g(r) = k_g r$. Hence $q \in S^G$ and $ab^{-1} = (qb + r)b^{-1} = q + rb^{-1}$ with $q \in S^G$ et $rb^{-1} \in F^G$ such that $\deg(r) + \deg(b) < 2\deg(b) \leq \deg(a) + \deg(b)$. We conclude by induction on $\deg a + \deg b$. □

Corollary 1 (W. Burnside). *The answer to Noether's problem is positive if* $n = 3$.

Proof. Let G be a finite subgroup of $\mathrm{GL}_3(\Bbbk)$ acting linearly on $S = \Bbbk[x, y, z]$. We introduce in $F = \Bbbk(x, y, z)$ the subalgebra $S_1 = \Bbbk(\frac{y}{x}, \frac{z}{x})[x]$, which satisfies $\mathrm{Frac}\, S_1 = F$. Let $g \in G$. We have:

$$g(x) = \alpha x + \beta y + \gamma z, \quad g(y) = \alpha' x + \beta' y + \gamma' z, \quad g(z) = \alpha'' x + \beta'' y + \gamma'' z.$$

Thus:

$$g(\frac{y}{x}) = \frac{\alpha' + \beta' \frac{y}{x} + \gamma' \frac{z}{x}}{\alpha + \beta \frac{y}{x} + \gamma \frac{z}{x}} \text{ and } g(\frac{z}{x}) = \frac{\alpha'' + \beta'' \frac{y}{x} + \gamma'' \frac{z}{x}}{\alpha + \beta \frac{y}{x} + \gamma \frac{z}{x}}.$$

It follows that the subfield $K = \Bbbk(\frac{z}{x}, \frac{y}{x})$ is stable under the action of G, and we can apply the theorem to the algebra $S_1 = K[x]$. The finiteness of G implies that $[F : F^G]$ is finite and so $S_1^G \not\subset K$. Thus we are in the second case of the theorem. There exists $u \in S_1^G$ of minimal degree ≥ 1 such that $S_1^G = K^G[u]$ and $F^G = K^G(u)$. By Castelnuovo's theorem (see in 3.1.1 above), $K^G = \Bbbk(v; w)$ is purely transcendental of degree two, and then $F^G = \Bbbk(v, w)(u) = \Bbbk(u, v, w)$. $\square$

Of course, we can prove similarly that the answer to Noether's problem is positive if $n = 2$ using Lüroth's theorem instead of Castelnuovo's theorem.

Corollary 2 (E. Fischer). *If $\Bbbk$ is algebraically closed, the answer to Noether's problem is positive for G abelian.*

Proof. We assume that G is a finite abelian subgroup of $\mathrm{GL}_n(\Bbbk)$. By total reducibility[11] and Schur's lemma[12], we can suppose up to conjugation that there exist complex characters $\chi_1, \ldots, \chi_n$ of G such that $g(x_j) = \chi_j(g)x_j$ for all $1 \leq j \leq n$ and all $g \in G$. In particular, G acts on $S_1 = \Bbbk(x_2, \ldots, x_n)[x_1]$ stabilizing $K_1 = \Bbbk(x_2, \ldots, x_n)$; thus $\Bbbk(x_1, \ldots, x_n)^G = K_1^G(u_1)$ for some $u_1 \in S_1^G$. We apply then Miyata's theorem inductively to conclude. $\square$

Another application due to E. B. Vinberg concerns the rational finite dimensional representations of solvable connected linear algebraic groups and uses Lie-Kolchin theorem about triangulability of such representations in order to apply inductively Miyata's theorem (see [39] for more details).

3.2 Invariants for rational extensions of Poisson polynomial automorphisms

3.2.1 Problem

Let $S = \Bbbk[x_1, \ldots, x_n]$ be a commutative polynomial algebra over a base field $\Bbbk$, and $F = \Bbbk(x_1, \ldots, x_n)$ its field of fractions. We suppose moreover that S is equipped with a Poisson structure and we consider a group G of Poisson automorphisms of S. The Poisson bracket extends canonically to F, and the action of G extends canonically into an action by Poisson automorphisms on F. We can consider F^G, from one hand as a field extension of $\Bbbk$, and from the other hand as a Poisson $\Bbbk$-algebra. Hence the problem of recognition of the Poisson structure on F^G adds to the initial transcendence question in classical Noether's problem. The most natural formulation is for G finite; since in this case $\operatorname{Frac} S^G = F^G$, the question is then for the Poisson algebras S^G and S to be rationally equivalent or not:

Problem 1. *Let G a finite group of Poisson automorphisms of S, do we have a Poisson isomorphism $F^G \simeq F$?*

[11] Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group G with V a finite dimensional $\Bbbk$-vector space. If the order of G order doesn't divide the characteristic of $\Bbbk$, then $V = V_1 \oplus \cdots \oplus V_m$, where each V_i is G-stable and irreducible (i.e. V_i doesn't admit proper and non zero G-stable subspace).

[12] If moreover $\Bbbk$ is algebraically closed and G is abelian, then any finite dimensional irreducible representation of G is of dimension one.

Depending on the specific Poisson structure chosen on S, more precise formulations arise. In particular following the philosophy of a Poisson analogue of the Gel'fand-Kirillov problem, the following question appears as a relevant formulation of Noether's problem for Poisson-Weyl algebras.

PROBLEM 2. *Let G a subgroup (finite or not) of the symplectic group $\mathrm{Sp}_{2n}(\Bbbk)$ acting linearly by Poisson automorphisms on the Poisson-Weyl algebra $\mathbb{S}_n(\Bbbk)$. Do we have a Poisson isomorphism $\mathbb{F}_n(\Bbbk)^G \simeq \mathscr{F}_{m,t}$ for some nonegative integers m, t ?*

Considering the transcendence degree, a positive answer to the question is possible only for $2m + t \leq 2n$ (with equality when G is finite).

Similar formulations of the problem can be given for the Poisson-quantum spaces (see further 3.2.4).

A useful general argument in the study of these problems is the following improvement of Miyata's theorem involving a Poisson-Ore structure on the polynomial algebra (the particular case where $\sigma = 0$ was underlying with a different formulation in [7]).

THEOREM. *Let K be a commutative field with Poisson structure over $\Bbbk$, σ a Poisson derivation and δ a Poisson σ-derivation of K. We consider the Poisson-Ore polynomial ring $S = K[x]_{\sigma,\delta}$ and its field of fractions $F = \mathrm{Frac}\, S = K(x)_{\sigma,\delta}$. Let G be a group of Poisson automorphisms of S such that $g(K) \subseteq K$ for any $g \in G$.*

(i) *if $S^G \subseteq K$, then $F^G = S^G = K^G$ are Poisson subalgebras of K.*

(ii) *if $S^G \not\subseteq K$, then for any $u \in S^G, u \notin K$ of degree $m = \min\{\deg_x y\,;\, y \in S^G, y \notin K\}$, there exist a Poisson derivation σ' of K^G and a Poisson σ'-derivation δ' of K^G such that $S^G = K^G[u]_{\sigma',\delta'}$ and $F^G = K^G(u)_{\sigma',\delta'}$.*

PROOF. Point (i) obviously follows from point (i) of Miyata's theorem 3.1.2 and remark 3 of 1.1.1. If $S^G \not\subseteq K$, let us choose an element $u \in S^G, u \notin K$ of degree $m = \min\{\deg_x y\,;\, y \in S^G, y \notin K\}$. We know by point (ii) of Miyata's theorem that $S^G = K^G[u]$ and $F^G = \mathrm{Frac}\,(S^G) = K^G(u)$. Because the elements of G are Poisson automorphisms of S, we have $\{u, a\} \in S^G$ for any $a \in K^G$. We fix such an element $a \in K^G$ and develop $\{u, a\}$ as a polynomial into the variable u with coefficient in K^G. Let us denote $n = \deg_u\{u, a\}$. We have then $\deg_x\{u, a\} = nm$. From the other hand, with notation $u = \sum_{0\leq k\leq m} a_i x^k, a_k \in K, a_m \neq 0, m \geq 1$, we compute

$$\{u, a\} = \sum_{0\leq k\leq m} (\{a_k, a\}x^k + a_k\{x, a\}kx^{k-1}) = \sum_{0\leq k\leq m} ([\{a_k, a\} + ka_k\sigma(a)]x^k + ka_k\delta(a)x^{k-1})$$

in order to observe that $\deg_x\{u, a\} \leq m$. Hence $mn \leq m$ with $m \geq 1$. Therefore $n \leq 1$. In other words, for any $a \in K^G$, there exists $\sigma'(a) \in K^G$ and $\delta'(a) \in K^G$ such that $\{u, a\} = \sigma'(a)u+\delta'(a)$. Since S^G is a Poisson algebra by remark 3 of 1.1.1, it follows from proposition 1.1.2 that σ' is a Poisson derivation of K^G, δ' is a Poisson σ-derivation of K^G, and $S^G = K^G[u]_{\sigma',\delta'}$. The equality $F^G = K^G(u)_{\sigma',\delta'}$ is then clear. $\square$

3.2.2 The case of the Kleinian surfaces

We fix $\Bbbk = \mathbb{C}$. The group $\mathrm{SL}_2(\mathbb{C})$ (briefly denoted by SL_2 if there is no doubt about the base field) acts linearly by Poisson automorphisms on the first Poisson-Weyl algebra $\mathbb{S}_1(\Bbbk) = \mathbb{C}[x, y]$ with Poisson bracket $\{x, y\} = 1$:

$$g.x = \alpha x + \beta y \;\text{ and }\; g.y = \gamma x + \delta y, \quad \text{for any } g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2. \tag{59}$$

Let us recall that finite subgroups of SL_2 are classified up to conjugation in five types, two infinite families parameterized by the positive integers (the type A_{n-1} corresponding of the cyclic group of order n and the type D_n corresponding to the binary dihedral group of order $4n$) and three groups E_6, E_7, E_8 of respective orders 24, 48, 120 (see for instance [34] or [6]). Since any finite subgroup G of SL_2 is conjugate in SL_2 to a subgroup G' of these types (then $\mathbb{S}_1^G$ is Poisson isomorphic to $\mathbb{S}_1^{G'}$), we can suppose without restriction in the determination of the Poisson algebra $\mathbb{S}_1^G$ for G finite subgroup of SL_2 that G is of type A_{n-1}, D_n, E_6, E_7 or E_8.
Concerning the $\Bbbk$-algebra structure, the description of the algebras $\mathbb{C}[x, y]^G$ is a classical topic in algebraic and geometric invariant theory. In each case, one can compute (see [34]) a system of three generators f_1, f_2, f_3 of the algebra $\mathbb{S}_1^G$.

type	generators of $\mathbb{C}[x, y]^G$	equation of $\mathscr{F}$
A_{n-1}	$f_1 = xy, \quad f_2 = x^n, \quad f_3 = y^n$	$X^n - YZ = 0$
D_n	$f_1 = x^2y^2, \quad f_2 = x^{2n} + (-1)^n y^{2n}$, $f_3 = x^{2n+1}y - (-1)^n xy^{2n+1}$	$X^{n+1} + XY^2 + Z^2 = 0$
E_6	$f_1 = xy^5 - x^5y, \quad f_2 = x^8 + 14x^4y^4 + y^8$, $f_3 = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$	$X^4 + Y^3 + Z^2 = 0$
E_7	$f_1 = x^8 + 14x^4y^4 + y^8, \; f_2 = x^{10}y^2 - 2x^6y^6 + x^2y^{10}$ $f_3 = x^{17}y - 34x^{13}y^5 + 34x^5y^{13} - xy^{17}$	$X^3Y + Y^3 + Z^2 = 0$
E_8	$f_1 = x^{11}y + 11x^6y^6 - xy^{11}$, $f_2 = x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20}$, $f_3 = x^{30} + 522x^{25}y^5 - 10005x^{20}y^{10} - 10005x^{10}y^{20} - 522x^5y^{25} + y^{30}$	$X^5 + Y^3 + Z^2 = 0$

Moreover the algebra $\mathbb{C}[x, y]^G = \mathbb{C}[f_1, f_2, f_3]$ appears as the factor of the polynomial algebra $\mathbb{C}[X, Y, Z]$ in three variables by the ideal generated by one irreducible polynomial F (of degree n, $n+1$, 4, 4, 5 respectively). The corresponding surfaces $\mathscr{F}$ of $\mathbb{C}^3$ are the Kleinian surfaces, which are the subject of many geometric, algebraic and homological studies. It is proved in [34] that, for G and G' two groups among the types $A_{n-1}, D_n, E_6, E_7, E_8$, the algebras $\mathbb{C}[x, y]^G$ and $\mathbb{C}[x, y]^{G'}$ are isomorphic if and only if $G = G'$.

Concerning the Poisson structure of $\mathbb{S}_1^G$, the link with a Poisson structure on the three dimensional space via the Kleinian surfaces is specified by the following proposition. Let us consider on $\mathbb{C}[X, Y, Z]$ the jacobian Poisson bracket associated with F, in the sense of example 3 of 1.1.1. For any polynomials $P \in \mathbb{C}[X, Y, Z]$ and $QF \in (F)$, we have

$\{P, QF\} = \{P, Q\}F + \{P, F\}Q = \{P, Q\}F + \mathrm{Jac}(P, F, F)Q = \{P, Q\}F + 0 \in (F)$. Then (F) is a Poisson ideal and we can take the induced Poisson structure on $\mathbb{C}[X, Y, Z]/(F)$.

PROPOSITION. *There exists a Poisson isomorphism between* $\mathbb{S}_1^G$ *and* $\mathbb{C}[X, Y, Z]/(F)$ *for the jacobian Poisson structure associated with* F.

PROOF. With the notations above, the surjective morphism of algebras $\phi : \mathbb{C}[X, Y, Z] \to \mathbb{C}[x, y]^G$ defined by $X \mapsto f_1, Y \mapsto f_2, Z \mapsto f_3$ induces a surjective morphism $\Phi : \mathbb{C}[X, Y, Z]/(F) \to \mathbb{C}[x, y]^G$ because $\ker \phi \supset (F)$. From classical ringtheoretical results, the Krull dimension of $\mathbb{C}[x, y]^G$ is 2, and the irreducibility of F implies that $\mathbb{C}[X, Y, Z]/(F)$ is also of Krull dimension 2. We conclude that Φ is an algebra isomorphism. The strategy to deduce from Φ a Poisson isomorphism consists in the calculation of three constants $a_1, a_2, a_3 \in \mathbb{Q}$ such that the polynomials $h_1 = a_1 f_1$, $h_2 = a_2 f_2$ and $h_3 = a_3 f_3$ in $\mathbb{C}[x, y]^G$ satisfy the relations:

$$\{h_1, h_2\} = F_3'(h_1, h_2, h_3), \quad \{h_2, h_3\} = F_1'(h_1, h_2, h_3), \quad \{h_3, h_1\} = F_2'(h_1, h_2, h_3) \qquad (\star)$$

with $F(h_1, h_2, h_3) = 0$, so that the isomorphism $\Psi : \mathbb{C}[X, Y, Z]/(F) \to \mathbb{C}[x, y]^G$ deduced from the map $X \mapsto h_1, Y \mapsto h_2, Z \mapsto h_3$ becomes a Poisson isomorphism.

The determination of a_1, a_2, a_3 is case by case. For instance, for G of type A_{n-1}, we have $f_1 = xy$, $f_2 = x^n$ and $f_3 = y^n$, with $F = X^n - YZ$ so $F_1' = nX^{n-1}$, $F_2' = -Z$ and $F_3' = -Y$. We compute $\{f_1, f_2\} = -nf_2$, $\{f_2, f_3\} = n^2 f_1^{n-1}$ and $\{f_3, f_1\} = -nf_3$. Setting $h_1 = a_1 f_1$, $h_2 = a_2 f_2$ and $h_3 = a_3 f_3$ and identifying in the above relations $(\star)$, we obtain $a_1 = \frac{1}{n}$ and $a_2 a_3 = \frac{1}{n^n}$. Similar (but more complicated) calculations are detailed for each case in [6]. $\square$

COMMENT. We deduce from this proposition an interesting link between the Poisson structure of $\mathbb{S}_1^G \simeq \mathbb{C}[X, Y, Z]/(F)$ and some geometrical invariant of the hypersurface $\mathscr{F}$ defined by F in the three dimensional affine space. From one hand, there exists for any Poisson $\mathbb{C}$-algebra A a notion of Poisson homology ; the first term of it is just the $\mathbb{C}$-vector space: $\mathrm{HP}_0(A) = A/\{A, A\}$, where $\{A, A\}$ is the subspace generated by all $\{a, b\}$ for $a, b \in A$. From the other hand, the Milnor number of the surface $\mathscr{F}$ is defined as the codimension of the jacobian ideal (i.e. the ideal generated by the derivate polynomials F_1', F_2', F_3'), that is: $\mu(\mathscr{F}) = \dim_{\mathbb{C}} \mathbb{C}[X, Y, Z]/(F_1', F_2', F_3')$. The main theorem of [6] proves that, for any finite subgroup of G, we have in the Poisson algebra isomorphism $\mathbb{S}_1^G \simeq \mathbb{C}[X, Y, Z]/(F)$ the equality : $\dim \mathrm{HP}_0(\mathbb{C}[x, y]^G) = \mu(\mathscr{F})$. The direct calculation of $\mu(\mathscr{F})$ for each of the five types allows to conclude that:

type	A_{n-1}	D_n	E_6	E_7	E_8
$\dim_{\mathbb{C}} \mathrm{HP}_0(\mathbb{S}_1(\mathbb{C})^G)$	$n-1$	$n+2$	6	7	8

These values coincide with the dimensions of $\mathrm{HH}_0(A_1(\mathbb{C})^G)$, the first group in the Hochschild homology of $A_1(\mathbb{C})^G$, where G acts by linear automorphisms on the (noncommutative) Weyl algebra $A_1(\mathbb{C})$. Hence the deformation process of $\mathbb{S}_1(\mathbb{C})$ to $A_1(\mathbb{C})$ not only induces a deformation of $\mathbb{S}_1(\mathbb{C})^G$ to $A_1(\mathbb{C})^G$, but can also be interpreted as a deformation at the level of the homological trace groups (see [6]).

We come back now to our main motivation which concerns the separation up to Poisson-isomorphism of the field of fractions.

Preliminary examples. Following the notation of (23) and (24), we introduce in $\mathbb{F}_1(\mathbb{C}) = \operatorname{Frac}\mathbb{S}_1(\mathbb{C})$ the element $w = xy$. For any $m \geq 1$, we denote by Q_m the subfield generated by w and x^m. Since

$$\{x^m, w\} = mx^m \quad \text{for any } m \geq 1, \tag{60}$$

each $Q_m = \mathbb{C}(w)(x^m)_{m\partial_w,0}$ is a Poisson subfield. Hence the element $z_m := \frac{1}{m}x^{-m}w = \frac{1}{m}yx^{1-m}$ satisfies $\{z_m, w\} = -mz_m$ and we deduce:

$$Q_m = \mathbb{C}(x^m)(z_m)_{0,\partial_{x^m}}, \quad \text{with } \{z_m, x^m\} = 1 \text{ for any } m \geq 1. \tag{61}$$

So each Q_m is Poisson-isomorphic to $\mathbb{F}_1(\mathbb{C})$. We also need the element $v := x^{-1}y = 2z_2$; because $wv^{-1} = x^2$, we have

$$Q_2 = \mathbb{C}(w)(x^2)_{2\partial_w,0} = \mathbb{C}(v)(w)_{0,2v\partial_v}, \quad \text{with } \{w, v\} = 2v \text{ for any } m \geq 1. \tag{62}$$

• Example (type A_{n-1}). Let G be the cyclic subgroup of order n in SL_2 generated by the automorphism g_n acting on $\mathbb{S}_1 = \mathbb{C}[y][x]_{0,\partial_y}$ by $g_n.x = \zeta_n x$ and $g_n.y = \zeta_n^{-1}y$ for ζ_n a n-th primitive root of one. Then $g_n(w) = w$. The algebra $S := \mathbb{C}(w)[x]$ is such that $\operatorname{Frac} S = \mathbb{F}_1(\mathbb{C})$ and g_n acts on S fixing w and multiplying x by ζ_n. Thus it is clear that $S^G = \mathbb{C}(w)[x^n]$ and it follows directly from theorem 3.2.1 that $\mathbb{F}_1(\mathbb{C})^G = \mathbb{C}(w)(x^n)_{n\partial_w,0} = Q_n$. By (61) we have proved that:

$$\mathbb{F}_1(\mathbb{C})^G = \mathbb{C}(y_n)(x_n)_{0,\partial_{y_n}} \quad \text{with } x_n = -\tfrac{1}{n}yx^{1-n} \text{ and } y_n = x^n.$$

• Example (type D_n). Let G be the binary dihedral subgroup of order $4n$ in SL_2 generated by the automorphism g_{2n} acting on $\mathbb{S}_1(\mathbb{C})$ by $g_{2n}.x = \zeta_{2n}x$ and $g_{2n}.y = \zeta_{2n}^{-1}y$ (for ζ_{2n} a $2n$-th primitive root of one), and by the automorphism μ defined by $\mu.x = iy$ and $\mu.y = ix$ (see [34]). We have $\mathbb{F}_1(\mathbb{C})^G = (\mathbb{F}_1(\mathbb{C})^{g_{2n}})^\mu = Q_{2n}^\mu$. Since $x^2 = wv^{-1}$, we have $x^{2n} = w^n v^{-n}$; thus $Q_{2n} = \mathbb{C}(w)(x^{2n})_{2n\partial_w,0} = \mathbb{C}(w)(v^n)_{-2n\partial_w,0}$, with $\{v^n, w\} = -2nv^n$. The action of μ on Q_{2n} is given by $\mu(w) = -w$ and $\mu(v^n) = v^{-n}$. The element $s_n := \frac{1}{2n}(v^{-n} - v^n)w$ satisfies $\mu(s_n) = s_n$ and $\{s_n, v^n\} = 1 - v^{2n}$, then $Q_{2n} = \mathbb{C}(v^n)(s_n)_{0,(1-v^{2n})\partial_{v^n}}$. By a last change of variable $t_n := (v^n + 1)(v^n - 1)^{-1}$, we deduce that $\mathbb{C}(v^n) = \mathbb{C}(t_n)$ by Lüroth's theorem, and the action of μ reduces to $\mu(t_n) = -t_n$. Because $\mu(s_n) = s_n$, we have $\mathbb{C}(s_n, v_n)^\mu = \mathbb{C}(s_n, t_n)^\mu = \mathbb{C}(s_n, t_n^2)$. We compute:

$$\begin{aligned}\{s_n, t_n\} &= \big(\{s_n, v^n\}(v^n - 1) - (v^n + 1)\{s_n, v^n\}\big)(v^n - 1)^{-2} \\ &= -2(1 - v^{2n})(1 - v^n)^{-2} = 2t_n,\end{aligned}$$

and then $\{s_n, t_n^2\} = 2t_n\{s_n, t_n\} = 4t_n^2$. It follows that $Q_{2n}^\mu = \mathbb{C}(t_n^2)(s_n)_{0,4t_n^2\partial_{t_n^2}}$. Denoting finally $x_n := (2t_n)^{-2}s_n$ and $y_n := t_n^2$ to obtain $\{x_n, y_n\} = 1$, we conclude that $Q_{2n}^\mu = \mathbb{C}(y_n)(x_n)_{0,\partial_{y_n}}$,

We have proved that:

$$\mathbb{F}_1(\mathbb{C})^G = \mathbb{C}(y_n)(x_n)_{0,\partial_{y_n}}$$

with: $x_n = \frac{1}{8n}\big((x^{-1}y)^{-n} - (x^{-1}y)^n\big)\left(\frac{(x^{-1}y)^n - 1}{(x^{-1}y)^n + 1}\right)^2 xy$, and $y_n = \left(\frac{(x^{-1}y)^n + 1}{(x^{-1}y)^n - 1}\right)^2$.

The positive answer to problem 2 p. 41 given by explicit calculation of generators in these both cases is a particular case of the following theorem (see [7]):

Theorem (J. Baudry). *For any finite subgroup G of SL_2 acting linearly on $\mathbb{S}_1(\mathbb{C})$, the field $\mathbb{F}_1(\mathbb{C})^G$ is Poisson isomorphic to $\mathbb{F}_1(\mathbb{C})$.*

Proof. The proof is somewhat formally similar to the noncommutative case in [3]. Let G any finite subgroup of SL_2. The cyclic case being solved in the first above example, we can suppose that the type of G is D_n, E_6, E_7 or E_8. In these four cases, G contains the involution e defined by $e.x = -x$ et $e.y = -y$ (see [34]). As seen for the type A_1, we have $\mathbb{F}_1(\mathbb{C})^e = Q_2$ with notation (62). Take any $g \in G$. There exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\alpha\delta - \beta\gamma = 1$ such that $g.x = \alpha x + \beta y$ et $g.y = \gamma x + \delta y$. Recall that $w := xy$ and $v := x^{-1}y$. From $g.x = x(\alpha + \beta v)$ et $g.y = x(\gamma + \delta v)$, we obtain:

$$g(v) = \tfrac{\gamma+\delta v}{\alpha+\beta v} \in k(v). \qquad (\dagger)$$

Moreover, $g.w = \alpha\gamma x^2 + \beta\delta y^2 + \alpha\delta xy + \beta\gamma yx$ and then:

$$g.w = \left(\tfrac{\beta\delta v^2+(\alpha\delta+\beta\gamma)v+\alpha\gamma}{v}\right) w. \qquad (\ddagger)$$

It follows from $(\dagger)$ and $(\ddagger)$ that the restrictions to the algebra $S = \mathbb{C}(v)[w]$ of the extensions to $\mathbb{F}_1(\mathbb{C})$ of the elements of G determine a subgroup $G' \simeq G/(e)$ of $\mathrm{Aut}_{\mathbb{C}} S$. Because $e \in G$ and $\mathbb{F}_1^e = Q_2 = \mathrm{Frac}\, S$, we deduce that $\mathbb{F}_1(\mathbb{C})^G = Q_2^{G'}$.

Denoting $K = \mathbb{C}(v)$, assertion $(\dagger)$ allows to apply theorem 3.1.2 with $S = K[w]$ and $Q_2 = \mathrm{Frac}\, S = K(w)_{0,2v\partial_v}$. Since $S^{G'} \nsubseteq K$ because $[Q_2 : Q_2^{G'}] = |G'| < +\infty$, there exists $u \in S^{G'}$ of degree $w \geq 1$ minimal among the degrees of all elements $S^{G'} \setminus K^{G'}$ such that $S^{G'} = K^{G'}[u]_{\sigma',\delta'}$ and $Q_2^{G'} = K^{G'}(u)_{\sigma',\delta'}$ for suitable σ' and δ'. Denote $u = a_m(v)w^m + a_{m-1}(v)w^{m-1} + \cdots + a_1(v)w + a_0(v)$, with $a_i(v) \in K$ for any $0 \leq i \leq m$ and $a_m(v) \neq 0$. For any $h(v) \in K$, we have $\{a_i(v)w^i, h(v)\} = a_i(v)\{w^i, h(v)\} = a_i(v)\{w, h(v)\}w^{i-1} = 2v\partial_v(h(v))a_i(v)w^{i-1}$. Therefore:

$$\{u, h(v)\} = 2mva_m(v)\partial_v(h(v))w^{m-1} + \cdots \text{ for any } h(v) \in K$$

In particular, if $h(v) \in K^{G'}$, then $\{u, h(v)\} \in S^{G'}$ because $u \in S^{G'}$ and $S^{G'}$ is a Poisson algebra. By minimality of the degree m of u among degrees (related to w) of elements in $S^{G'} \setminus K^{G'}$, it is impossible that $m - 1 \geq 1$ when $\partial_v(h(v)) \neq 0$. So we have proved:

$$\text{if } h(v) \in K^{G'} \text{ with } h(v) \notin \mathbb{C}, \text{ then } \{u, h(v)\} \in K.$$

By Lüroth's theorem, $\mathbb{C}(v)^{G'}$ is a purely transcendental extension $\mathbb{C}(z)$ of $\mathbb{C}$. Since $z \in K$ and $z \notin \mathbb{C}$, it follows from previous calculations that $m = 1$ and $\{u, z\} = 2va_1(v)\partial_v(z(v)) \neq 0$. We introduce $t := \{u, z\}^{-1}u$ in order to obtain $\{t, z\} = 1$, we deduce $Q_2^{G'} = K^{G'}(u)_{\sigma',\delta'} = \mathbb{C}(z)(u)_{\sigma',\delta'} = \mathbb{C}(z)(t)_{0,\partial_z}$, and the proof is complete. $\square$

We complete this section by a short application of the previous result:

EXERCISE ([7]). Let G be the Weyl group B_2 over $\mathbb{C}$, of order 8, generated by three elements $\epsilon, \epsilon', \tau$ described by their action on $\mathbb{C}[x_1, y_1, x_2, y_2]$ by:

$$\begin{array}{lllll} \epsilon: & x_1 \mapsto -x_1, & y_1 \mapsto -y_1, & x_2 \mapsto x_2, & y_2 \mapsto y_2 \\ \epsilon': & x_1 \mapsto x_1, & y_1 \mapsto y_1, & x_2 \mapsto -x_2, & y_2 \mapsto -y_2 \\ \tau: & x_1 \mapsto x_2, & y_1 \mapsto y_2, & x_2 \mapsto x_1, & y_2 \mapsto y_1 \end{array}$$

Check that G acts by Poisson automorphism on $\mathbb{S}_2(\mathbb{C})$. Denoting by V the subgroup generated by ϵ and ϵ', prove that $\mathbb{C}(x_1, y_1, x_2, y_2)^V = \mathbb{C}(u_1, v_1, u_2, v_2)$, where $u_i = x_i^2$ and $v_i = \frac{1}{2} y_i x_i^{-1}$ satisfy $\{u_i, v_i\} = 1$ and the other brackets are zero. Introduce $p_1 = \frac{1}{2}(u_1+u_2)$, $q_1 = v_1+v_2$, $a = \frac{1}{2}(u_1-u_2)$ and $b = v_1-v_2$. Prove that $\mathbb{C}(u_1, v_1, u_2, v_2) = \mathbb{C}(p_1, q_1, a, b)$ and τ fixes p_1 and q_1, and maps $a \mapsto -a, b \mapsto -b$. Denote $p_2 = a^2$ and $q_2 = \frac{1}{2} b a^{-1}$. Deduce that $\mathbb{C}(x_1, y_1, x_2, y_2)^G = \mathbb{C}(u_1, v_1, u_2, v_2)^\tau = \mathbb{C}(p_1, q_1, p_2, q_2)$ with $\{p_i, q_i\} = 1$ and the other brackets are zero.

Conclude that $\operatorname{Frac} \mathbb{S}_2(\mathbb{C})^G = \mathbb{F}_2(\mathbb{C})^G$ is Poisson isomorphic to $\mathbb{F}_2(\mathbb{C})$.

3.2.3 The case of finite abelian groups of linear Poisson automorphisms

We start with a group G (finite or not) and an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{k}$. Let us denote by $(e_1, \ldots, e_n)$ a $\mathbb{k}$-basis of V and by $(x_1, \ldots, x_n)$ its dual basis on V^*. The canonical action of G by automorphisms on V defined by

$$g.v = \rho(g)(v) \qquad \text{for all } \; g \in G, \; v \in V \tag{63}$$

extends into an action by automorphisms on $\mathbb{k}[V] \simeq S(V^*) \simeq \mathbb{k}[x_1, \ldots, x_n]$ by:

$$(g.f)(v) = f(g^{-1}.v) \qquad \text{for all } \; g \in G, \; f \in \mathbb{k}[V], \; v \in V \tag{64}$$

whose restriction to V^* corresponds to the standard dual representation. Hence combining (63) and (64), we obtain an action on $W = V^* \oplus V$:

$$g.(v, f) = (g.v, g.f) \qquad \text{for all } \; g \in G, \; v \in V, \; f \in V^* \tag{65}$$

whose dualization allows to define an action of $\mathbb{k}[W] \simeq S(W^*) \simeq \mathbb{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, with $(y_1, \ldots, y_n)$ the dual basis of $(x_1, \ldots, x_n)$. In particular, this action satisfies, for any $g \in G$:

$$g.x_i \in \mathbb{k}[x_1, \ldots, x_n], \quad g.y_i \in \mathbb{k}[y_1, \ldots, y_n], \qquad \text{for all } g \in G, \; 1 \leq i \leq n, \tag{66}$$

Denoting by $(\beta_{i,j})$ the matrix of g in the basis $(y_1, \ldots, y_n)$ and $(\gamma_{i,j})$ the matrix of g^{-1} in the basis $(x_1, \ldots, x_n)$, we can calculate:

$$\beta_{j,i} = \Big(\sum_{m=1}^{n} \beta_{m,i} y_m\Big)(x_j) = (g.y_i)(x_j) = y_i(g^{-1}.x_j) = y_i\Big(\sum_{m=1}^{n} \gamma_{m,j} x_m\Big) = \gamma_{i,j}.$$

It follows that, if we consider the Poisson-Weyl bracket defined on $\mathbb{k}[W]$ by standard relations (3), then $\{g^{-1}.x_j, y_i\} = \{\sum_{m=1}^n \gamma_{m,j} x_m, y_i\} = \gamma_{i,j} = \beta_{j,i} = \{x_j, \sum_{m=1}^n \beta_{m,i} y_m\} =$

$\{x_j, g.y_i\}$. We deduce by linearity that $\{g(x_j), g(y_i)\} = g(\{x_j, y_i\})$ for all $g \in G$ and finally that g acts by Poisson automorphisms on $\mathbb{k}[W]$. We summarize this construction in the following proposition.

Proposition. *Any representation of dimension n over $\mathbb{k}$ of a group G defines canonically an action of G by Poisson linear automorphisms on the Poisson-Weyl algebra $\mathbb{S}_n(\mathbb{k}) = [y_1, \ldots, y_n, x_1, \ldots, x_n]$ such that:*

$$\{g.x_i, y_j\} = \{x_i, g^{-1}.y_j\} \qquad \text{for all } g \in G,\ 1 \leq i, j \leq n, \tag{67}$$

or equivalently

$$g.x_i = \sum_{j=1}^{n} \{x_i, g^{-1}.y_j\}\, x_j \qquad \text{for all } g \in G,\ 1 \leq i \leq n. \tag{68}$$

Example 1 (diagonal action). The most simple situation is for G acting as a subgroup of the torus $(\mathbb{k}^\times)^n$ by Poisson automorphisms on $\mathbb{S}_n(\mathbb{k})$ by:

$$g.y_i = \alpha_i y_i, \quad g.x_i = \alpha_i^{-1} x_i, \quad \text{with } g = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{k}^\times)^n. \tag{69}$$

(i) Suppose that $G = (\mathbb{k}^\times)^n$. Any monomial $u = y_1^{j_1} \ldots y_n^{j_n} x_1^{i_1} \ldots x_n^{i_n}$ is an eigenvector under the action, and any element of $\mathbb{S}_n(\mathbb{k})^G$ is a $\mathbb{k}$-linear combination of invariant monomials. For $g = (\lambda_1, 1, \ldots, 1)$ with λ_1 of infinite order in $\mathbb{k}^*$, the relation $g.u = u$ implies $i_1 = j_1$. Proceeding on the same way for all diagonal entries, we obtain:

$$\text{if } \ G = (\mathbb{k}^\times)^n, \text{ then } \mathbb{S}_n(\mathbb{k})^G = \mathbb{k}[x_1y_1, x_2y_2, \ldots, x_ny_n],$$

with trivial Poisson structure. In particular, $\operatorname{Frac} \mathbb{S}_n(\mathbb{k})^G$ is Poisson isomorphic to $\mathscr{F}_{0,n}(\mathbb{k})$.

(ii) If G is now a finite subgroup of $(\mathbb{k}^\times)^n$, the invariant algebra $\mathbb{S}_n(\mathbb{k})^G$ is finitely generated over $\mathbb{k}$ (by classical Noether's theorem). Since every monomial in the y_i's and x_i's is an eigenvector under the action of G, it's clear that we can find a finite family of $\mathbb{k}$-algebra generators of $\mathbb{S}_n(\mathbb{k})^G$ constituted by invariant monomials.
If $n = 1$, denoting by p the order of the cyclic group G, we have $\mathbb{S}_1(\mathbb{k})^G = \mathbb{k}[x^p, y^p, xy]$, with Poisson structure defined from $\{x^p, xy\} = px^p$, $\{y^p, xy\} = -py^p$ and $\{x^p, y^p\} = p^2(xy)^{p-1}$. This is just the first example p. 44, so $\operatorname{Frac} \mathbb{S}_1(\mathbb{k})^G$ is Poisson isomorphic to $\mathbb{F}_1(\mathbb{k}) = \mathscr{F}_{1,0}(\mathbb{k})$.
For $n > 1$, the determination of such a family becomes an arithmetical and combinatorial question depending on the mixing between the actions on the various subalgebras $\mathbb{k}[y_i, x_i] \simeq \mathbb{S}_1(\mathbb{k})$ in $\mathbb{S}_n(\mathbb{k})$. We shall solve it completely at the level of the rational functions at the end of this this paragraph.

> Exercise. We consider $G = \langle g \rangle$ cyclic of order 6 and $G' = \langle g' \rangle$ cyclic of order 2 acting as Poisson automorphisms on $\mathbb{S}_2(\mathbb{C}) = \mathbb{C}[x_1, x_2, y_1, y_2]$ by:
>
> $$g: \ x_1 \mapsto -x_1, \quad y_1 \mapsto -y_1, \quad x_2 \mapsto jx_2, \quad y_2 \mapsto j^2 y_2,$$

$$g' : \ x_1 \mapsto -x_1, \quad y_1 \mapsto -y_1, \quad x_2 \mapsto -x_2, \quad y_2 \mapsto -y_2,$$

Prove that: $\mathbb{S}_2(\mathbb{C})^G$ is generated by $x_1^2, x_1y_1, y_1^2, x_2^3, x_2y_2, y_2^3$,

$\mathbb{S}_2(\mathbb{C})^{G'}$ is generated by $x_1^2, x_1y_1, x_1x_2, x_1y_2, y_1^2, y_1x_2, y_1y_2, x_2^2, x_2y_2, y_2^2$,

and that $\operatorname{Frac}\mathbb{S}_2(\mathbb{C})^G$ and $\operatorname{Frac}\mathbb{S}_2(\mathbb{C})^{G'}$ are Poisson isomorphic to $\mathbb{F}_2(\mathbb{C}) = \mathscr{F}_{2,0}(\mathbb{C})$.

Example 2. We take $\Bbbk = \mathbb{C}$ and $n = 2$; any finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$ acts by Poisson automorphisms on the Poisson-Weyl algebra $\mathbb{S}_2(\mathbb{C}) = \mathbb{C}[y_1, y_2, x_1, x_2]$ by:

$$\begin{cases} g.y_1 = \alpha y_1 + \beta y_2, & g.x_1 = \delta x_1 - \gamma x_2, \\ g.y_2 = \gamma y_1 + \delta y_2 & g.x_2 = -\beta x_1 + \alpha x_2, \end{cases} \quad \text{for any } g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C}). \tag{70}$$

Using the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$, one can obtain a positive answer to problem 2 p. 41 in this case proving that $\operatorname{Frac}\mathbb{S}_2(\mathbb{C})^G$ is Poisson isomorphic to $\mathbb{F}_2(\mathbb{C})$. The proof (which is a Poisson analogue of the rational equivalence of differential operator algebras over Kleinian surfaces studied in [4]) is somewhat long and technical and will not be developed here.

We concentrate now on the case of abelian groups actions, obtaining positive answers to problem 2 p. 41 as consequences of the more general following result.

Theorem. *We suppose that $\Bbbk$ is of characteristic zero. We consider a representation of a group G (non necessarily finite) which is a direct summand of n representations of dimension one, and the associated action of G by Poisson automorphisms on the Poisson-Weyl algebra $\mathbb{S}_n(\Bbbk)$. Then there exists a unique integer $0 \leq s \leq n$ such that $\operatorname{Frac}\mathbb{S}_n(\Bbbk)^G$ is Poisson isomorphic to $\mathscr{F}_{n-s,s}(\Bbbk)$.*

Proof. By (32), the integer s is no more than the transcendence degree over $\Bbbk$ of the Poisson-center of $\mathscr{F}_{n-s,s}(\Bbbk)$ and so is unique. We proceed by induction on n to prove the existence of s.

1) For $n = 1$. Then G acts on $S_1(\Bbbk) = \Bbbk[y][x]_{0,\partial_y}$ by Poisson automorphisms of the form:

$$g.y = \chi(g)y, \quad g.x = \chi(g)^{-1}x, \quad \text{for all } g \in G$$

where χ is a character $G \to \Bbbk^\times$. The element $w = xy$ is invariant under G. Using (24), we can consider in $\mathbb{F}_1(\Bbbk) = \Bbbk(w)(x)_{\partial_w,0}$ the subalgebra $S = \Bbbk(w)[x]_{\partial_w,0}$. We have $\operatorname{Frac} S \simeq \mathbb{F}_1(\Bbbk)$. Any $g \in G$ fixes the elements of $\Bbbk(w)$ and acts on x by $g.x = \chi(g)x$. We apply the theorem of 3.2.1. If $S^G \subseteq \Bbbk(w)$, then $\mathbb{F}_1(\Bbbk)^G = S^G = \Bbbk(w)^G = \Bbbk(w)$; thus $F_1(\Bbbk)^G \simeq \mathscr{F}_{0,1} \simeq \mathscr{F}_{1-s,s}(\Bbbk)$ with $s = 1$. If $S^G \not\subseteq \Bbbk(w)$, then S^G is a Poisson-Ore extension $\Bbbk(w)[u]_{\sigma',\delta'}$ for some Poisson derivation σ' and some Poisson σ'-derivation δ' of $\Bbbk(w)$, with $u \in \Bbbk(w)[x], u \notin \Bbbk(w)$ such that $g.u = u$ for all $g \in G$ and of minimal nonzero degree. Because of the form of the action of G on x, we can choose without any restriction $u = x^m$ for an integer $m \geq 1$, and so $\sigma' = m\partial_w$ and $\delta' = 0$. To sum up, $\mathbb{F}_1(\Bbbk)^G = \operatorname{Frac} S^G = \Bbbk(w)(x^m)_{m\partial_w,0}$. This field is also generated over $\Bbbk$ by $x' = x^m$ and $y' = m^{-1}x^{-m}w$ which satisfy $\{x', y'\} = 1$. We conclude: $\mathbb{F}_1(\Bbbk)^G \simeq \mathbb{F}_1(\Bbbk) = \mathscr{F}_{1-s,s}(\Bbbk)$ for $s = 0$.

2) Now suppose that the theorem is true for any direct summand of $n-1$ representations of dimension one of any group over any field of characteristic zero. Let us consider a direct summand of n representations of dimension one of a group G over $\Bbbk$. Then G acts on $\mathbb{S}_n(\Bbbk)$ by Poisson automorphisms of the form:

$$g.y_i = \chi_i(g)y_i, \quad g.x_i = \chi_i(g)^{-1}x_i, \quad \text{for all } g \in G \text{ and } 1 \leq i \leq n,$$

where $\chi_1, \chi_2, \ldots, \chi_n$ are characters $G \to \Bbbk^\times$. Thus, recalling notation $w_i = p_i q_i$ of (30), we have:

$$g.w_i = w_i, \quad \text{for any } g \in G \text{ and any } 1 \leq i \leq n.$$

In $\mathbb{F}_n(\Bbbk) = \Bbbk(w_1, w_2, \ldots, w_n)(x_1)_{\partial_{w_1},0}(x_2)_{\partial_{w_2},0} \ldots (x_n)_{\partial_{w_n},0}$, let us consider the Poisson subfields:

$$\begin{array}{rcl} L & = & \Bbbk(w_n), \\ K & = & \Bbbk(w_1, w_2, \ldots, w_n)(x_1)_{\partial_{w_1},0}(x_2)_{\partial_{w_2},0} \cdots (x_{n-1})_{\partial_{w_{n-1}},0} \\ & = & \Bbbk(w_n)(w_1, w_2, \ldots, w_{n-1})(x_1)_{\partial_{w_1},0}(x_2)_{\partial_{w_2},0} \cdots (x_{n-1})_{\partial_{w_{n-1}},0} \quad \simeq \quad \mathbb{F}_{n-1}(L), \end{array}$$

and the Poisson subalgebra $S = K[x_n]_{\partial_{w_n},0}$, which satisfies $\mathrm{Frac}\, S = \mathbb{F}_n(\Bbbk)$. Applying the induction hypothesis to the restriction of the action of G by Poisson L-automorphisms on $\mathbb{S}_{n-1}(L)$, there exists an integer $0 \leq s \leq n-1$ such that: $\mathbb{F}_{n-1}(L)^G \simeq \mathscr{F}_{n-1-s,s}(L) \simeq \mathscr{F}_{n-(s+1),s+1}(\Bbbk)$. Since K is stable under the action of G, we can apply the theorem of 3.2.1 to the ring $S = K[x_n]_{\partial_{w_n},0}$. Two cases are possible.

First case: $S^G = K^G$. Then we obtain:

$$\mathbb{F}_n(\Bbbk)^G = \mathrm{Frac}\,(S^G) = K^G \simeq \mathbb{F}_{n-1}(L)^G \simeq \mathscr{F}_{n-(s+1),s+1}(\Bbbk).$$

Second case: there exists a polynomial $u \in S$ with $\deg_{x_n} u \geq 1$ such that $g(u) = u$ for all $g \in G$. Choosing u such that $\deg_{x_n} u$ is minimal, there exist an Poisson derivation σ' and a Poisson σ'-derivation δ' of K^G such that $S^G = K^G[u]_{\sigma',\delta'}$ and $\mathbb{F}_n(\Bbbk)^G = \mathrm{Frac}\, S^G = K^G(u)_{\sigma',\delta'}$.
Let us develop $u = f_m x_n^m + \cdots + f_1 x_n + f_0$ with $m \geq 1$ and $f_i \in K^G$ for all $0 \leq i \leq m$. In view of the action of G on x_n, it's clear that the monomial $f_m x_n^m$ is then invariant under G. Using the embedding in Poisson field of Laurent series (see the proof of the lemma in 1.2.1), we can develop f_m in:

$$\overline{K} = \Bbbk(w_1, w_2, \ldots, w_n)(\!(x_1)\!)_{\partial_{w_1},0}(\!(x_2)\!)_{\partial_{w_2},0} \cdots (\!(x_{n-1})\!)_{\partial_{w_{n-1}},0}.$$

The action of G extends to $\overline{K}$ acting diagonally on the x_i's and fixing w_i's. Therefore we can choose without any restriction a monomial u:

$$u = x_1^{a_1} \ldots x_n^{a_n} \text{ with } (a_1, \ldots, a_n) \in \mathbb{Z}^n, \text{ et } a_n \geq 1.$$

By (29), we have $\{u, w_j\} = a_j u$ for any $1 \leq j \leq n$. Let us introduce the invariant elements:

$$w'_1 = w_1 - a_n^{-1} a_1 w_n, \quad w'_2 = a_n w_2 - a_n^{-1} a_2 w_n, \quad \ldots, \quad w'_{n-1} = a_n w_{n-1} - a_n^{-1} a_{n-1} w_n.$$

They satisfy $\{u, w'_j\} = 0$ and $\{x_i, w'_j\} = \{x_i, w_j\} = \delta_{i,j} x_i$ for all $1 \leq i, j \leq n-1$. They generate in K the Poisson subfield

$$F = \Bbbk(w'_1, w'_2, \ldots, w'_{n-1})(x_1)_{\partial_{w'_1},0}(x_2)_{\partial_{w'_2},0} \cdots (x_{n-1})_{\partial_{w'_{n-1}},0},$$

which is Poisson isomorphic to $\mathbb{F}_{n-1}(\Bbbk)$. More precisely, F is the field of fractions of

$$A = \Bbbk[y'_1, \ldots, y'_{n-1}][x_1]_{0,\partial_{y'_1}} \cdots [x_{n-1}]_{0,\partial_{y'_{n-1}}},$$

where $y'_i = w_i x_i^{-1}$ for any $1 \leq i \leq n-1$, which is Poisson isomorphic to the Poisson-Weyl algebra $\mathbb{S}_{n-1}(\Bbbk)$. The group G acts on A by $g.x_i = \chi_i(g)^{-1}x_i, g.y_i = \chi_i(g)y_i$ for all $g \in G, 1 \leq i \leq n-1$. We apply the induction hypothesis: there exists $0 \leq s \leq n-1$ such that $F^G \simeq \mathscr{F}_{n-1-s,s}(\Bbbk)$. It's clear by definition of the w'_j's that $\Bbbk(w_n)(w'_1, w'_2, \ldots, w'_{n-1}) = \Bbbk(w_n)(w_1, w_2, \ldots, w_{n-1})$; since $\{w_n, z\} = 0$ for all $z \in F$, we deduce that $K = F(w_n)_{0,0}$. The Poisson algebra $S^G = K^G[u]_{\sigma',\delta'}$ can then be written $S^G = F^G(w_n)[u]_{\sigma',\delta'}$. The generator u satisfies $\{u, w'_j\} = 0$ for any $0 \leq j \leq n-1$ as we have seen above, $\{u, x_i\} = 0$ for any $1 \leq i \leq n$ by definition, and $\{u, w_n\} = a_n u$. Therefore the change of variables $w'_n = a_n^{-1} w_n$ implies: $S^G = F^G(w'_n)[u]_{\partial_{w'_n}}$. It follows that: $\operatorname{Frac} S^G \simeq \mathbb{F}_1(F^G) \simeq \mathbb{F}_1(\mathscr{F}_{n-1-s,s}(\Bbbk)) \simeq \mathscr{F}_{n-s,s}(\Bbbk)$. □

Corollary (Application to finite abelian groups). *We suppose here that $\Bbbk$ is algebraically closed of characteristic zero. Then, for any finite dimensional representation of a finite abelian group G, we have the Poisson isomorphism $\mathbb{F}_n(\Bbbk)^G \simeq \mathbb{F}_n(\Bbbk)$.*

Proof. By total reducibility and Schur's lemma (see the notes p. 40), any finite representation of G is a direct summand of one dimensional representations. Therefore the previous theorem applies. The finiteness of G implies that, at each step of the proof, we are in the second case of application of theorem 3.2.1. In the initialization of the induction, we have $\mathbb{F}_1(\Bbbk)^G \simeq \mathbb{F}_1(\Bbbk)$. A the end of the proof, the Poisson isomorphism $\operatorname{Frac} S^G \simeq \mathbb{F}_1(F^G)$ is for $F \simeq \mathbb{F}_{n-1}(\Bbbk)$, and then $F^G \simeq \mathbb{F}_{n-1}(\Bbbk)$ by induction hypothesis. Hence $\operatorname{Frac} S^G \simeq \mathbb{F}_n(\Bbbk)$. □

The last corollary proves in particular that for non necessarily finite groups G, all possible values of s can be obtained in the previous theorem.

Corollary (Application to the canonical action of subgroups of a torus). *We suppose that $\Bbbk$ is of characteristic zero. For an integer $n \geq 1$, we consider the diagonal action of the torus $(\Bbbk^\times)^n$ by Poisson automorphisms on $\mathbb{S}_n(\Bbbk)$. Then:*

(i) for any subgroup G of $(\Bbbk^\times)^n$, there exists a unique integer $0 \leq s \leq n$ such that we have the Poisson isomorphism $\mathbb{F}_n(\Bbbk)^G \simeq \mathscr{F}_{n-s,s}(\Bbbk)$;

(ii) for any integer $0 \leq s \leq n$, there exists at least one subgroup G of $(\Bbbk^\times)^n$ such that we have the Poisson isomorphism $\mathbb{F}_n(\Bbbk)^G \simeq \mathscr{F}_{n-s,s}(\Bbbk)$;

(iii) in particular $s = n$ if $G = (\Bbbk^\times)^n$, and $s = 0$ if G is finite.

Proof. Point (i) is just the application of the previous theorem. For (ii), let us fix an integer $0 \leq s \leq n$ and consider in $(\Bbbk^\times)^n$ the subgroup:

$$G = \{\operatorname{Diag}(\alpha_1, \ldots, \alpha_s, 1, \ldots, 1) \ ; \ (\alpha_1, \ldots, \alpha_s) \in (\Bbbk^\times)^s\} \simeq (\Bbbk^\times)^s,$$

acting by Poisson automorphisms on $\mathbb{S}_n(\Bbbk)$:

$$\begin{array}{lll} y_i \mapsto \alpha_i y_i, & x_i \mapsto \alpha_i^{-1} x_i, & \text{pour tout } \ 1 \leq i \leq s, \\ y_i \mapsto y_i, & x_i \mapsto x_i, & \text{pour tout } \ s+1 \leq i \leq n. \end{array}$$

In the Poisson field $\mathbb{F}_n(\Bbbk) = \Bbbk(w_1, w_2, \ldots, w_n)(x_1)_{\partial_{w_1}}(x_2)_{\partial_{w_2}} \cdots (x_n)_{\partial_{w_n}}$, we introduce the subfield $K = \Bbbk(w_1, w_2, \ldots, w_n)(x_{s+1})_{\partial_{w_{s+1}}}(x_{s+2})_{\partial_{w_{s+2}}} \cdots (x_n)_{\partial_{w_n}}$. Then the Poisson subalgebra $S = K[x_1]_{\partial_{w_1}} \cdots [x_s]_{\partial_{w_s}}$ satisfies $\operatorname{Frac} S = \mathbb{F}_n(\Bbbk)$. By construction, K is invariant under the action of G. If $S^G \not\subset K$, we can find in particular in S^G a monomial:

$$u = v x_1^{d_1} x_2^{d_2} \cdots x_s^{d_s}, \quad v \in K,\ v \neq 0,\ d_1, \ldots, d_s \in \mathbb{N},\ (d_1, \ldots, d_s) \neq (0, \ldots, 0),$$

then $\alpha_1^{d_1} \alpha_2^{d_2} \cdots \alpha_s^{d_s} = 1$ for all $(\alpha_1, \alpha_2, \ldots, \alpha_s) \in (\Bbbk^\times)^s$, and so a contradiction. We conclude with theorem 3.2.1 that $(\operatorname{Frac} S)^G = S^G = K^G$, and so $\mathbb{F}_n(\Bbbk)^G = K$. It's clear that $K \simeq \mathscr{F}_{n-s,s}(\Bbbk)$; this achieves the proof of point (ii). The first assertion of point (iii) is just the case (i) of example 1 p. 47, and the second one follows then from the previous corollary. $\square$

3.2.4 The case of the multiplicative Poisson structure on the plane

We summarize here some elementary observations about a formulation of problem 2 p. 41 in the case of the polynomial algebra $\mathbb{C}[x, y]$ and Laurent polynomial algebra $\mathbb{C}[x^{-1}, y^{-1}]$ for the multiplicative bracket (5) parametrized by some $\lambda \in \mathbb{C}$, i.e. :

$$\{x^a y^b, x^c y^d\} = (ad - bc)\lambda x^{a+c} y^{b+d} \qquad \text{for all } a, b, c, d \in \mathbb{Z}. \tag{71}$$

We recall that the corresponding Poisson polynomial algebra is the Poisson-quantum plane denoted by $\mathbb{P}_2^\lambda(\mathbb{C})$, its localization into a Poisson algebra of Laurent polynomials is the Poisson-quantum torus denoted by $\mathbb{T}_2^\lambda(\mathbb{C})$, and their common field of fractions is the Poisson-quantum field $\mathbb{Q}_2^\lambda(\mathbb{C})$.

Theorem. *Let λ be a nonzero complex number.*

(i) *The group of Poisson automorphisms of $\mathbb{P}_2^\lambda(\mathbb{C})$ reduces to 2-dimensional torus $(\mathbb{C}^\times)^2$ acting by $x \mapsto \alpha x, y \mapsto \beta y$ for $\alpha, \beta \in \mathbb{C}^\times$.*

(ii) *For any finite subgroup G of Poisson automorphisms of $\mathbb{P}_2^\lambda(\mathbb{C})$, the field $\mathbb{Q}_2^\lambda(\mathbb{C})^G$ is Poisson isomorphic to $\mathbb{Q}_2^{\lambda'}(\mathbb{C})$ for $\lambda' = \lambda.|G|$.*

Proof. An element $z \in \mathbb{P}_2(\mathbb{C})^\lambda$ is said to be normal if the principal ideal $z\mathbb{P}_2(\mathbb{C})^\lambda$ is stable under the Poisson bracket. It is clear by (71) that any monomial $x^a y^b$ si normal. Conversely, let z be a normal element: there exist $u, v \in \mathbb{P}_2^\lambda$ such that $\{z, y\} = uz$ and $\{z, x\} = vz$. Denoting $z = \sum_m f_m(y) x^m$, we have $\{z, y\} = \sum_m \lambda m f_m(y) y x^m$ and the first equality implies that $u \in \mathbb{C}[y]$. More precisely $\sum_m f_m(y)[m\lambda y - u(y)]x^m = 0$. Hence $u(y) = m\lambda y$ for any m in the support of z. By assumption $\lambda \neq 0$, we deduce that z is a monomial $z = f_i(y) x^i$ for some nonnegative integer i. From the second equality $\{f_i(y)x^i, x\} = vz$, it is easy to deduce that $f_i(y) = \alpha y^j$ for some nonnegative integer j and $\alpha \in \mathbb{C}$. This proves that the normal elements of the Poisson algebra $\mathbb{P}_2^\lambda(\mathbb{C})$ are the monomials $\alpha y^j x^i$.

Let g be an Poisson $\mathbb{C}$-automorphism of $\mathbb{P}_2^\lambda(\mathbb{C})$. It preserves the set of nonzero normal elements. Hence we have $g(x) = \alpha y^j x^i$ and $g(y) = \beta y^k x^h$ with $\alpha, \beta \in \mathbb{C}^\times$ and j, i, k, h nonnegative integers. Because $\lambda \neq 0$, the relation $\{g(x), g(y)\} = \lambda g(y) g(x)$ implies by (71)that $ik - hj = 1$. Writing similar formulas for g^{-1} and identifying the exponents in $g^{-1}(g(x)) = x$ and $g^{-1}(g(y)) = y$, we obtain easily $j = h = 0$ and $i = k = 1$. This achieves the proof of point (i).

Let G a finite group of Poisson automorphisms of $\mathbb{P}_2^\lambda(\mathbb{C})$. There exists for any $g \in G$ a pair $(\alpha_g, \beta_g) \in \mathbb{C}^\times \times \mathbb{C}^\times$ such that $g(y) = \alpha_g y$ and $g(x) = \beta_g x$. Denote by m and m' the orders of the cyclic groups $\{\alpha_g \,;\, g \in G\}$ and $\{\beta_g \,;\, g \in G\}$ of $\mathbb{C}^\times$ respectively. In particular, $\mathbb{C}(y)^G = \mathbb{C}(y^m)$. We can apply theorem 3.2.1 to the extension $S = \mathbb{C}(y)[x]_\sigma$ of $\mathbb{P}_2^\lambda(\mathbb{C}) = \mathbb{C}[y][x]_\sigma$, where $\sigma = \lambda y \partial_y$. We have $S^G \neq \mathbb{C}(y)^G$ because $x^{m'} \in S^G$. Let n be the minimal degree related to x of the elements of S^G of positive degree. For any $u \in S^G$ of degree n, there exist σ' a Poisson derivation and δ' a Poisson σ'-derivation of $\mathbb{C}(y)$ such that $S^G = \mathbb{C}(y^m)[u]_{\sigma',\delta'}$. We develop

$$u = a_n(y)x^n + \cdots + a_1(y)x + a_0(y), \text{ with } n \geq 1,\ a_i(y) \in \mathbb{C}(y) \text{ for all } 0 \leq i \leq n \text{ et } a_n(y) \neq 0.$$

Identifying the terms of greater and lower degree (related to x) in the two members of equality $\{u, y^m\} = \sigma'(y^m)u + \delta'(y^m)$, we deduce that $\delta' = 0$ and $\sigma' = \lambda mn \partial_{y^m}$. Hence $\mathbb{Q}_2^\lambda(\mathbb{C})^G = \operatorname{Frac} S^G = \mathbb{C}(y^m)(u)_{\lambda mn\partial_{y^m}}$ is Poisson isomorphic to $\mathbb{Q}_2^{\lambda'}(\mathbb{C})$ for $\lambda' = mn\lambda$. We just have to check that $mn = |G|$. Denote by $p \in \mathbb{Z}$ the valuation (related to y) of $a_n(y)$ in the extension $\mathbb{C}(\!(y)\!)$ of $\mathbb{C}(y)$. The action of G being diagonal on $\mathbb{C}x \oplus \mathbb{C}y$, we deduce from $u = a_n(y)x^n + \cdots \in S^G$ that the monomial $y^p x^n$ lies in S^G. So we obtain $\mathbb{Q}_2^\lambda(\mathbb{C})^G = \mathbb{C}(y^m)(y^p x^n)_{\lambda mn \partial_{y^m}}$. We remind Artin's lemma $[\mathbb{C}(x,y) : \mathbb{C}(x,y)^G] = |G|$ and the extensions

$$\mathbb{C}(x,y)^G = \mathbb{C}(y^m)(y^p x^n) \ \subseteq\ L = \mathbb{C}(y)(y^p x^n) = \mathbb{C}(y)(x^n) \subseteq \mathbb{C}(y)(x) = \mathbb{C}(x,y),$$

which satisfy $[\mathbb{C}(x,y) : L] = n$ and $[L : \mathbb{C}(x,y)^G] = m$ to conclude that $|G| = mn$. $\square$

Lemma. *Let λ be a nonzero complex number.*

(i) *The group $\mathrm{SL}_2(\mathbb{Z})$ acts by Poisson automorphisms on $\mathbb{T}_2^\lambda(\mathbb{C})$ by:*

$$g.x = x^a y^c \quad \text{and} \quad g.y = x^b y^d \qquad \text{for } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \tag{72}$$

or more generally for any $m, n \in \mathbb{Z}$,

$$g.(x^m y^n) = x^{am+bn} y^{cm+dn} \qquad \text{for } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}). \tag{73}$$

(ii) *The group of Poisson automorphisms of $\mathbb{T}_2^\lambda(\mathbb{C})$ is $(\mathbb{C}^\times)^2 \ltimes \mathrm{SL}_2(\mathbb{Z})$, where the torus $(\mathbb{C}^\times)^2$ acts by: $x \mapsto \alpha x$ and $y \mapsto \beta y$ for all $\alpha, \beta \in \mathbb{C}^\times$.*

Proof. Point (i) is a straightforward verification (see exercise 3 of 1.1.1). Point (ii) follows from the fact that any ring automorphism of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ stabilizes the group of invertible elements. $\square$

The classification up to conjugation of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ is well known; the description of the twelve types (classically denoted $\mathscr{G}_1$ to $\mathscr{G}_{12}$) can be found in [24]. In particular the finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ correspond to the four (all cyclic) cases:

$$\mathscr{G}_7 = \langle \mathrm{x} \rangle \simeq \mathrm{C}_6, \quad \mathscr{G}_8 = \langle \mathrm{ds} \rangle \simeq C_4, \quad \mathscr{G}_9 = \langle \mathrm{x}^2 \rangle \simeq \mathrm{C}_3, \quad \mathscr{G}_{10} = \langle \mathrm{x}^3 \rangle \simeq \mathrm{C}_2,$$

where $\mathrm{x} = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathrm{d} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathrm{s} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are the three basic matrices used in the description of any finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$. Explicitly:

$$\mathrm{x} = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \quad \mathrm{ds} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \quad \mathrm{x}^2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right), \quad \mathrm{x}^3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

Remark. It is easy to verify that the property for any finite subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ to be conjugated in $\mathrm{GL}_2(\mathbb{Z})$ to a $\mathscr{G}_i$ $(i = 7, 8, 9, 10)$ implies that G is conjugated to $\mathscr{G}_i$ in $\mathrm{SL}_2(\mathbb{Z})$ (see [7] p. 73). Denoting by h an element in $\mathrm{SL}_2(\mathbb{Z})$ such that $G = h^{-1}\mathscr{G}_i h$, the assignment $P \mapsto h.P$ defines a Poisson isomorphism $\mathbb{T}_2^\lambda(\mathbb{C})^G \to \mathbb{T}_2^\lambda(\mathbb{C})^{\mathscr{G}_i}$. In conclusion, in the study of Poisson invariants of $\mathbb{T}_2^\lambda(\mathbb{C})$ under the action of a finite subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ we can suppose without restriction thet G is one of the $\mathscr{G}_7, \mathscr{G}_8, \mathscr{G}_9, \mathscr{G}_{10}$.

Just like for the Kleinian surfaces for the case of finite groups of $\mathrm{SL}_2(\mathbb{C})$ acting on $\mathbb{S}_1(\mathbb{C})$, the invariant subalgebra $\mathbb{T}_2^\lambda(\mathbb{C})^G$ for each type of finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is generated (as a commutative algebra) by three elements with one relation. From [24], we have:

G	generators of $\mathbb{T}_2^\lambda(\mathbb{C})^G$ and relation
$\mathscr{G}_{10} \simeq C_2$	$\xi_1 = x + x^{-1}, \quad \xi_2 = y + y^{-1}, \quad \theta = xy + x^{-1}y^{-1}$
	$\theta\xi_1\xi_2 = \theta^2 + \xi_1^2 + \xi_2^2 - 4$
$\mathscr{G}_9 \simeq C_3$	$\eta_+ = x + y + x^{-1}y^{-1}, \quad \eta_- = x^{-1} + y^{-1} + xy, \quad \varphi = xy^2 + x^{-2}y^{-1} + xy^{-1} + 6$
	$\varphi\eta_+\eta_- = \eta_+^3 + \eta_-^3 + \varphi^2 - 9\varphi + 27$
$\mathscr{G}_8 \simeq C_4$	$\sigma_1 = \xi_1 + \xi_2, \quad \sigma_2 = \xi_1\xi_2, \quad \rho = xy^2 + x^{-1}y^{-2} + x^2y^{-1} + x^{-2}y + 3\sigma_1$
	$\rho^2 = \rho\sigma_1(\sigma_2 + 4) + 4\sigma_1^2\sigma_2 - \sigma_1^4 - \sigma_2(\sigma_2 + 4)^2$
$\mathscr{G}_7 \simeq C_6$	$\tau_1 = \eta_+ + \eta_-, \quad \tau_2 = \eta_+\eta_-, \quad \sigma = \eta_+\varphi + \eta_-(x^{-1}y^{-2} + x^2y + x^{-1}y + 6)$
	$\sigma^2 = \tau_1(\tau_2 + 9)\sigma - \tau_2(\tau_2 + 9)^2 + (\tau_1^2 - 4\tau_2)(3\tau_1\tau_2 - \tau_1^3 - 27)$

The surfaces in the 3-dimensional affine space corresponding to the algebraic relation between the three generators in each case are studied in [7] (in particular the type of the isolated singularities are determined).

Comment. These first results open the way for a wide program of systematic study of the Poisson algebras $\mathbb{T}_2^\lambda(\mathbb{C})^G$ in parallel with the Kleinian surfaces. This program is greatly initiated in [7], concerning in particular the finiteness of the underlying Lie algebra structure of the Poisson algebras $\mathbb{T}_2^\lambda(\mathbb{C})$ and $\mathbb{T}_2^\lambda(\mathbb{C})^G$, and the comparison of the dimension of $\mathrm{HP}_0(\mathbb{T}_2^\lambda(\mathbb{C})^G)$ with the dimension of $\mathrm{HH}_0(\mathbb{C}_\lambda[x^\pm, y^\pm]^G)$ for the (noncommutative) quantum torus $\mathbb{C}_\lambda[x^\pm, y^\pm]$.

Back to our original motivation for the rational equivalence and the Poisson-Noether problem, we can give here as an exploratory result (see [7]) the case of the type $\mathscr{G}_{10}$:

Proposition. *Let λ be a nonzero complex number. For the subgroup $\mathscr{G}_{10}$ of $\mathrm{SL}_2(\mathbb{Z})$ acting multiplicatively on $\mathbb{T}_2^\lambda(\mathbb{C})$, the field $\mathbb{Q}_2^\lambda(\mathbb{C})^G$ is Poisson isomorphic to $\mathbb{Q}_2^\lambda(\mathbb{C})$.*

Proof. Here G is just $\{I_2, e\}$ where $e := -I_2$ acts by (72), that is $e.x = x^{-1}$ and $e.y = y^{-1}$. It is known that $\mathbb{T}_2^\lambda(\mathbb{C})^G$ is generated by $\xi_1 = x + x^{-1}$, $\xi_2 = y + y^{-1}$ and $\theta = xy + x^{-1}y^{-1}$, submitted to the relation $\theta\xi_1\xi_2 - \theta^2 - \xi_1^2 - \xi_2^2 + 4 = 0$.

Step 1: field generators. In $\mathbb{C}(x,y)^G = \mathbb{C}(\xi_1,\xi_2,\theta)$, this algebraic dependence relation rewrites into:

$$(2\theta-\xi_1\xi_2)^2 = \xi_1^2\xi_2^2 - 4(\xi_1^2+\xi_2^2-4) \;\Leftrightarrow\; (2\theta-\xi_1\xi_2)^2 = (\xi_1^2-4)(\xi_2^2-4)$$
$$\Leftrightarrow\; \Big(\frac{2\theta-\xi_1\xi_2}{\xi_2-2}\Big)^2 = (\xi_1^2-4)\,\frac{\xi_2+2}{\xi_2-2}$$

Let us introduce $\eta := \dfrac{2\theta-\xi_1\xi_2}{\xi_2-2} \in F^G$ and $\mu := \dfrac{\eta^2}{\xi_1^2-4} = \dfrac{\xi_2+2}{\xi_2-2} \in \mathbb{C}(\eta,\xi_1)$.

We have: $\xi_2 = \dfrac{2(\mu+1)}{\mu-1} \in \mathbb{C}(\eta,\xi_1)$ and then $\theta = \frac{1}{2}(\eta(\xi_2-2)+\xi_1\xi_2) \in \mathbb{C}(\eta,\xi_1)$.

We conclude that $F^G = \mathbb{C}(\eta,\xi_1)$.

Step 2: Poisson structure. We compute:

$$\{\xi_1,\xi_2\} = \lambda(2\theta-\xi_1\xi_2), \quad \{\xi_2,\theta\} = \lambda(2\xi_1-\theta\xi_2) \quad \text{and} \quad \{\theta,\xi_1\} = \lambda(2\xi_2-\theta\xi_1).$$

Thus:

$$\begin{aligned}
\{\eta,\xi_1\} &= \{\frac{2\theta-\xi_1\xi_2}{\xi_2-2}\,,\,\xi_1\} = \frac{-2\theta+\xi_1\xi_2}{(\xi_2-2)^2}\{\xi_2-2\,,\,\xi_1\} + \frac{1}{\xi_2-2}\{2\theta-\xi_1\xi_2\,,\,\xi_1\} \\
&= \lambda\Big(\frac{2\theta-\xi_1\xi_2}{\xi_2-2}\Big)^2 + \frac{\lambda}{\xi_2-2}\big(2(2\xi_2-\theta\xi_1)+\xi_1(2\theta-\xi_1\xi_2)\big) \\
&= \lambda\eta^2 + \frac{\lambda\xi_2}{\xi_2-2}(4-\xi_1^2) = \lambda\eta^2 + \frac{1}{2}\lambda(\mu+1)(4-\xi_1^2) = \frac{1}{2}\lambda(\eta^2-\xi_1^2+4).
\end{aligned}$$

Hence: $\{\eta\,,\,\eta^2-\xi_1^2+4\} = -\lambda\xi_1(\eta^2-\xi_1^2+4)$ and $\{\xi_1\,,\,\eta^2-\xi_1^2+4\} = -\lambda\eta(\eta^2-\xi_1^2+4)$.

Therefore: $\{\eta+\xi_1\,,\,\eta^2-\xi_1^2+4\} = -\lambda(\eta+\xi_1)(\eta^2-\xi_1^2+4)$ and then:

$$\{\tfrac{1}{\eta+\xi_1}\,,\,\eta^2-\xi_1^2+4\} = \tfrac{\lambda}{\eta+\xi_1}(\eta^2-\xi_1^2+4).$$

Step 3: conclusion. We define $x' := \frac{1}{\eta+\xi_1}$ and $y' := \eta^2-\xi_1^2+4$. From the first step, we have $\mathbb{C}(x,y)^G = \mathbb{C}(\eta+\xi_1,\eta-\xi_1) = \mathbb{C}(x'^{-1},x'(y'-4)) = \mathbb{C}(x',y')$. From the second step, $\{x',y'\} = \lambda x'y'$, and the proof is complete. □

3.3 Invariants for rational Poisson automorphisms

We can naturally take in consideration in the problem 1 of p. 40 not only Poisson automorphisms of some polynomial (or Laurent polynomial) Poisson algebra S extended to the field of fractions $K = \operatorname{Frac} S$, but also Poisson automorphisms of K himself. We concentrate in the following on the first Poisson-quantum field.

3.3.1 The case of rational triangular actions

We are interested here in Poisson automorphisms of $\mathbb{Q}_2^\lambda(\mathbb{C}) = \mathbb{C}(x,y)$ with $\{x,y\} = \lambda xy$ which preserve the embedding $\mathbb{C}(y) \subset \mathbb{C}(x,y)$, that is which stabilize the subfield $\mathbb{C}(y)$. We fix $\lambda \in \mathbb{C}^\times$.

Proposition. *The subgroup of Poisson automorphisms of $\mathbb{C}(x,y) = \mathbb{Q}_2^\lambda(\mathbb{C})$ stabilizing $\mathbb{C}(y)$ is $\langle\sigma\rangle \ltimes \big(\mathbb{C}^\times \ltimes \mathbb{C}(y)^\times\big)$, where σ is the standard quadratic transformation $y \mapsto \frac{1}{y}, x \mapsto \frac{1}{x}$, and $\mathbb{C}^\times \ltimes \mathbb{C}(y)^\times$ acts by: $y \mapsto \alpha y, x \mapsto f(y)x$ for $\alpha \in \mathbb{C}^\times, f \in \mathbb{C}(y)^\times$.*

Proof. Let θ be a Poisson automorphism of $\mathbb{Q}_2^\lambda$ whose restriction to $\mathbb{C}(y)$ is an automorphism of $\mathbb{C}(y)$. Hence there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\alpha\delta - \beta\gamma \neq 0$ such that $\theta(y) = \frac{\alpha y+\beta}{\gamma y+\delta}$. Using the development $\theta(x) = \sum_{n\geq m} f_n(y)x^n$ in the extension $\mathbb{C}(y)(\!(x)\!)$ of $\mathbb{C}(x,y)$, with $m \in \mathbb{Z}$, $f_n(y) \in \mathbb{C}(y)$ for any $n \geq m$ and $f_m(y) \neq 0$, we compute:

$$\{\theta(x), \theta(y)\} = \sum_{n\geq m} f_n(y)\{x^n, \theta(y)\} = \sum_{n\geq m} f_n(y)\{x, \tfrac{\alpha y+\beta}{\gamma y+\delta}\}x^{n-1} = \lambda \sum_{n\geq m} n\tfrac{\alpha\delta-\beta\gamma}{(\gamma y+\delta)^2} f_n(y)yx^n,$$

and identify it with $\lambda\theta(x)\theta(y) = \lambda\sum_{n\geq m}\frac{\alpha y+\beta}{\gamma y+\delta}f_n(y)x^n$. For any n in the support $\operatorname{Supp}\theta(x)$, we have $n(\alpha\delta-\beta\gamma) = \alpha\gamma y^2 + (\alpha\delta+\beta\gamma)y + \delta\beta$. Therefore $\alpha\gamma = \beta\delta = 0$, with $\alpha\delta - \beta\gamma \neq 0$. Only two cases can occur. In the first case, $\beta = \gamma = 0$ then $n\alpha\delta y = \alpha\delta y$; hence $\operatorname{Supp}\theta(x) = \{1\}$ and $\theta(x) = f_1(y)x$ with $\theta(y) = \alpha\beta^{-1}y$. In the second case, $\alpha = \delta = 0$ then $-n\beta\gamma y = \beta\gamma y$; hence $\operatorname{Supp}\theta(x) = \{-1\}$ and $\theta(x) = f_{-1}(y)x^{-1}$ with $\theta(y) = \beta\gamma^{-1}y^{-1}$.

For any $\alpha \in \mathbb{C}^\times, f(y) \in \mathbb{C}(y)^\times$, let us denote by $\theta_{\alpha,f}$ the Poisson automorphism of $\mathbb{Q}_2^\lambda(\mathbb{C})$ defined by $\theta(x) = \alpha x$ and $\theta(y) = f(y)x$. Such automorphisms constitute a subgroup $\mathscr{G}$, isomorphic to the semidirect product $\mathbb{C}^\times \ltimes \mathbb{C}(y)^\times$ since $\theta_{\alpha,f}\theta_{\alpha',f'} = f_{\alpha\alpha',f\theta(f')}$. It follows from the first part of the proof that the subgroup of Poisson automorphisms of $\mathbb{Q}_2^\lambda(\mathbb{C})$ preserving the embedding $\mathbb{C}(y) \subset \mathbb{C}(x,y)$ is precisely the semidirect product $\langle\sigma\rangle \ltimes \mathscr{G}$. $\square$

Theorem. *For any finite subgroup G of Poisson automorphisms of $\mathbb{Q}_2^\lambda(\mathbb{C})$ stabilizing $\mathbb{C}(y)$, the invariant field $\mathbb{Q}_2^\lambda(\mathbb{C})^G$ is Poisson isomorphic to $\mathbb{Q}_2^{\lambda'}(\mathbb{C})$ for some $\lambda' \in \mathbb{C}^\times$.*

Proof. In $\mathbb{Q}_2^\lambda(\mathbb{C}) = \mathbb{C}(x,y)$, we consider the Poisson subalgebra $S = \mathbb{C}(y)[x]_{\sigma,0}$ with $\sigma = \lambda y\partial_y$, see (26), which satisfies $\operatorname{Frac} S = \mathbb{Q}_1^\lambda(\mathbb{C})$. The automorphisms of the form $\theta_{\alpha,f} : x \mapsto \alpha x, y \mapsto f(y)x$ with $\alpha \in \mathbb{C}^\times, f \in \mathbb{C}(y)^\times$, which constitute the group denoted by $\mathscr{G}$ in the proof of previous proposition, act as Poisson automorphisms on the subalgebra S. For G a finite subgroup of Poisson automorphisms of $\mathbb{Q}_2^\lambda(\mathbb{C})$ stabilizing $\mathbb{C}(y)$, we set $G^+ = G \cap \mathscr{G}$.

First step. We apply to G^+ the theorem of 3.2.1. Since $[\mathbb{C}(x,y) : \mathbb{C}(x,y)]^{G^+} = |G^+|$ we are necessarily in the second case of application of the theorem: there exists $u \in S^{G^+}$ of minimal nonzero degree $m \geq 1$ such that $S^{G^+} = \mathbb{C}(y)^{G^+}[u]_{\sigma',\delta'}$ and $\mathbb{Q}_2^\lambda(\mathbb{C})^{G^+} = \mathbb{C}(y)^{G^+}(u)_{\sigma',\delta'}$ for suitable σ', δ'. Because of the form of the action of G^+ on y and x, we can suppose without loss of generality that u is a monomial $u = h(y)x^m$, with $h(y) \in \mathbb{C}(y)$. The group of restrictions to $\mathbb{C}(y)$ of the elements of G^+ is isomorphic to a finite subgroup of $\mathbb{C}^\times$, hence is cyclic. Denoting by k its order, we have $\mathbb{C}(y)^{G^+} = \mathbb{C}(y^k)$. We calculate $\{u, y^k\} = h(y)\{x^m, y^k\} = \lambda mk\, uy^k$. Then $S^{G^+} = \mathbb{C}(y^k)[u]_{\sigma',0}$ with $\sigma' = \lambda mky^k\partial_{y^k}$. Consequently, $\mathbb{Q}_2^\lambda(\mathbb{C})^{G^+}$ is Poisson isomorphic to $\mathbb{Q}_2^{\lambda'}(\mathbb{C})$ for $\lambda' = \lambda mk$. Observe that $mk = |G^+|$ by Artin's lemma since:

$$\begin{aligned}[\mathbb{C}(x,y) : \mathbb{C}(u,y^k)] &= [\mathbb{C}(x,y) : \mathbb{C}(u,y)][\mathbb{C}(u,y) : \mathbb{C}(u,y^k)] \\ &= [\mathbb{C}(x,y) : \mathbb{C}(x^m,y)][\mathbb{C}(u,y) : \mathbb{C}(u,y^k)].\end{aligned}$$

When $G = G^+$, the proof is complete.

Second step. We suppose that $G^- := G \setminus G^+ \neq \emptyset$. We fix some automorphism θ in G^-. By the preliminary proposition, it is of the form: $y \mapsto \epsilon_\theta y^{-1}, x \mapsto g_\theta x^{-1}$ with $\epsilon_\theta \in \mathbb{C}^\times, g_\theta \in \mathbb{C}(y)^\times$. Its action on u is given by $\theta(u) = \theta(hx^m) = \theta(h)g_\theta^m x^{-m} = g_\theta^m \theta(h)hu^{-1}$. We set $a_\theta := g_\theta^m\theta(h)h$ to obtain $\theta : u \mapsto a_\theta u^{-1}$. We claim that $a_\theta \in \mathbb{C}(y)^{G^+}$. Let γ be an automorphism in G^+. Define $\theta' = \theta^{-1}\gamma$, which lies in G^-. The identity $u = \gamma(u)$ implies that $u = \theta(a_{\theta'}u^{-1}) = \theta(a_{\theta'})a_\theta^{-1}u$, then $\theta(a_{\theta'}) = a_\theta$. Similarly, $\theta'(a_\theta) = a_{\theta'}$. Finally $\gamma(a_\theta) = \theta(\theta'(a_\theta)) = \theta(a_{\theta'}) = a_{\theta'}$, and the claim is proved. We conclude that:

$$\theta \text{ acts on } \mathbb{Q}_2^\lambda(\mathbb{C})^{G^+} = \mathbb{C}(y^k)(u)_{\sigma',0} \text{ by: } y^k \mapsto \epsilon_\theta^k \tfrac{1}{y^k},\ u \mapsto a_\theta \tfrac{1}{u},\ \text{ with } \epsilon_\theta \in \mathbb{C}^\times,\ a_\theta \in \mathbb{C}(y^k)^\times.$$

The element $z := \mu y^k$ where μ is a square root of ϵ_θ^{-k} satisfies $\mathbb{C}(y^k) = \mathbb{C}(z)$, $\theta(z) = \frac{1}{z}$ and $\{u, z\} = \mu\{u, y^k\} = \mu\lambda' u y^k = \lambda' uz$. In particular, the element a_θ is a rational function $q(z)$ into the variable z. We have $\theta^2 \in G^+$, hence $\theta^2(u) = u$, then $\theta(a_\theta)a_\theta^{-1} = 1$; in other words $q(z) = q(\frac{1}{z})$ in $\mathbb{C}(z)^\times$. Similarly $q(z)^{-1} = q(\frac{1}{z})^{-1}$ and by a classical argument (comparing the zeros and poles, or transforming the question by the change of variables $z \mapsto t := \frac{z+1}{z-1}$ into the question of writing any even rational function $b(t)$ as a product $c(t)c(-t)$), there exists $p \in \mathbb{C}(z)^\times$ such that $q(z)^{-1} = p(z)p(\frac{1}{z})$. We set $v := p(z)u$ which satisfies $\mathbb{C}(z, u) = \mathbb{C}(z, v)$, $\theta(v) = p(\frac{1}{z})q(z)\frac{1}{u} = \frac{1}{v}$ and $\{v, z\} = \{p(z)u, z\} = p(z)\{z, u\} = p(z)\lambda' uz = \lambda' vz$. We conclude:

$$\theta \text{ acts on } \mathbb{Q}_2^\lambda(\mathbb{C})^{G^+} = \mathbb{C}(z)(v)_{\sigma',0}, \quad \text{where } \sigma' = \lambda' z\partial_z, \quad \text{by:} \quad z \mapsto \frac{1}{z},\ v \mapsto \frac{1}{v}. \tag{74}$$

We denote to simplify $Q := \mathbb{Q}_2^\lambda(\mathbb{C})^{G^+}$. From one hand, it is clear that $Q^\theta = Q^{\theta'}$ for any $\theta' \in G^-$ since $\theta\theta' \in G^+$; therefore $Q^\theta = \mathbb{Q}_2^\lambda(\mathbb{C})^G$. From the other hand, it follows from (74) and the last proposition of 3.2.4 that Q^θ is Poisson isomorphic to $\mathbb{Q}_2^{\lambda'}(\mathbb{C})$. $\square$

3.3.2 The Poisson Cremona group

(i) We recall the following subgroups of the Cremona group $\operatorname{Aut}\mathbb{C}(x, y)$.

• The fractional linear transformations are the automorphisms:

$$x \mapsto \frac{\alpha x + \beta y + \gamma}{\alpha'' x + \beta'' y + \gamma''},\ y \mapsto \frac{\alpha' x + \beta' y + \gamma'}{\alpha'' x + \beta'' y + \gamma''}, \quad \text{for } \begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{pmatrix} \in \mathrm{PGL}(3, \mathbb{C}).$$

The subgroup of such automorphisms is isomorphic to $\mathrm{PGL}(3, \mathbb{C})$ and denoted by $\mathbf{A}$.

• The Jonquières automorphisms are the automorphisms:

$$x \mapsto \frac{a(y)x + b(y)}{c(y)x + d(y)},\ y \mapsto \frac{\alpha y + \beta}{\gamma y + \delta}, \quad \text{for } \begin{pmatrix} a(y) & b(y) \\ c(y) & d(y) \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{C}(y)),\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{C}),$$

The subgroup of such automorphisms, which preserve the embedding $\mathbb{C}(y) \subset \mathbb{C}(x, y)$, is isomorphic to the semi-direct product $\mathrm{PGL}(2, \mathbb{C}(y)) \rtimes \mathrm{PGL}(2, \mathbb{C})$ and denoted by $\mathbf{J}$.

• Defining the transposition $\tau \in \mathbf{A}$ and the standard quadratic transformation $\sigma \in \mathbf{J}$:

$$\tau : x \mapsto y,\ y \mapsto x, \quad \text{and} \quad \sigma : x \mapsto \frac{1}{x},\ y \mapsto \frac{1}{y},$$

the classical Noether-Castelnuovo theorem asserts that $\operatorname{Aut}\mathbb{C}(x,y)$ is generated by $\mathbf{A}$ and σ. It was later proved by Iskovskikh in [21] that $\operatorname{Aut}\mathbb{C}(x,y)$ is generated by $\mathbf{J}$ and τ. Recently, Blanc proved in [10] that $\operatorname{Aut}\mathbb{C}(x,y)$ is the amalgamated product of $\mathbf{A}$ and $\mathbf{J}$ along their intersection, divided by the unique relation $\sigma\tau=\tau\sigma$.

(ii) We consider now on $\mathbb{C}(x,y)$ the Poisson structure defined from the bracket

$$\{x,y\}=xy. \tag{75}$$

As observed previously in 3.2.4, the groups $(\mathbb{C}^\times)^2$ and $\mathrm{SL}(2,\mathbb{Z})$ act by Poisson automorphisms on $\mathbb{T}_2^1(\mathbb{C})=\mathbb{C}[x^\pm,y^\pm]$ and then on $\mathbb{Q}_2^1(\mathbb{C})=\mathbb{C}(x,y)$ by:

$$x\mapsto \eta x^a y^b,\ y\mapsto \mu x^c y^d, \qquad \text{with } (\eta,\mu)\in(\mathbb{C}^\times)^2,\ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}(2,\mathbb{Z}). \tag{76}$$

We denote by $\mathbf{H}\simeq(\mathbb{C}^\times)^2\rtimes\mathrm{SL}(2,\mathbb{Z})$ the subgroup of such automorphisms in $\operatorname{Aut}\mathbb{C}(x,y)$ (a systematic study of the embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into the Cremona group can be found in [12]). We introduce the rational automorphism π defined (see [9], [11], [37]) by :

$$\pi:\ x\mapsto y,\ y\mapsto \frac{y+1}{x}. \tag{77}$$

The somewhat surprising following property is related to the mutation in the cluster algebra of rank 2 associated with the Dynkin diagram A_2.

Lemma. *π is a Poisson automorphism of $\mathbb{C}(x,y)$, of order 5.*

Proof. We observe that $\pi=\mu\gamma\in\operatorname{Aut}\mathbb{C}(x,y)$ with $\mu: x\mapsto (y+1)^{-1}x,\ y\mapsto y$ in the subgroup $\mathbf{J}$ and and $\gamma: x\mapsto y,\ y\mapsto x^{-1}$ of order 4 in the subgroup $\mathbf{H}$. We have:

$$\{\pi(x),\pi(y)\}=\{y,x^{-1}(y+1)\}=\{y,x^{-1}\}(y+1)=-x^{-2}\{y,x\}(y+1)=x^{-1}y(y+1),$$

hence $\{\pi(x),\pi(y)\}=\pi(\{x,y\})$. Now we compute:

$\pi(y)=x^{-1}(y+1)$,

$\pi^2(y)=y^{-1}\big(x^{-1}(y+1)+1\big)=x^{-1}+y^{-1}x^{-1}+y^{-1}$,

$\pi^3(y)=y^{-1}+(y+1)^{-1}xy^{-1}+(y+1)^{-1}x=y^{-1}(y+1)^{-1}(y+1+x+xy)=y^{-1}(1+x)$

$\pi^4(y)=(y+1)^{-1}x(1+y)=(y+1)^{-1}(1+y)x=x.$

$\pi^5(y)=\pi(x)=y$, and $\pi^5(x)=\pi^4(y)=x$. $\square$

The key role of this automorphism appears in the following important theorem.

Theorem (J. Blanc). *The subgroup of Poisson automorphisms in $\operatorname{Aut}\mathbb{C}(x,y)$ for the Poisson bracket $\{x,y\}=xy$ is generated by $\mathbf{H}$ and π.*

We refer to the paper [11] for the proof, which solve a conjecture formulated in [37] and gives moreover an explicit presentation (3 generators and 5 relations) of the subgroup of Poisson automorphisms generated by $\mathrm{SL}(2,\mathbb{Z})$ and π.

(iii) Concerning our motivation about the invariant field $\mathbb{C}(x,y)^G$ and its Poisson structure for finite subgroups G of Poisson automorphisms of $\mathbb{C}(x,y)$, we don't go back to the case of finite subgroups of $\mathbf{H}$ (studied in 3.2.4) and consider the case where G is the cyclic group of order 5 generated by the rational Poisson automorphism π. Reformulating in terms of affine coordinates the last remark of [9], we decompose as a sequence of changes of variables the conjugation of π with the canonical transformation $X \mapsto \zeta X, Y \mapsto \zeta^{-1}Y$ where $\zeta = \exp 2i\pi/5$.

Step 1. We consider in $K = \mathbb{C}(x, y)$ the linear automorphism:

$$\gamma_1 : \; x \mapsto -x, \quad y \mapsto y + 1. \tag{78}$$

The elements $u := \gamma_1(x)$ and $v := \gamma_1(y)$ satisfy $K = \mathbb{C}(u, v)$ and the expression of π is:

$$\pi(u) = 1 - v \;\text{ and }\; \pi(v) = 1 - \frac{v}{u}. \tag{79}$$

Step 2. We consider in $K = \mathbb{C}(u, v)$ the endomorphism:

$$\gamma_2 : \; u \mapsto \frac{u - \omega v}{u - v}.\frac{v - 1}{\omega^2 v - 1}, \quad v \mapsto \omega^{-1}\frac{u - \omega v}{u - v}.\frac{1 - u}{1 - \omega u}, \tag{80}$$

where $\omega = \frac{1}{2}(1 + \sqrt{5})$. We check by straightforward calculations that γ_2 is an involution, and therefore an automorphism of K. The elements $s := \gamma_2(u)$ and $t := \gamma_2(v)$ satisfy $K = \mathbb{C}(s, t)$ and the expression of π is:

$$\pi(s) = 1 - \frac{\omega^{-2}}{t} \;\text{ and }\; \pi(t) = 1 - \frac{\omega^{-1}s}{t}, \tag{81}$$

which corresponds to the linear fractional transformation with matrix $L = \begin{pmatrix} 0 & -\omega^{-1} & 0 \\ 1 & 1 & 1 \\ -\omega^{-2} & 0 & 0 \end{pmatrix}$. This matrix is diagonalizable with eigenvalues $\lambda_1 = \omega^{-1}\zeta$, $\lambda_2 = \omega^{-1}\zeta^{-1}$, $\lambda_3 = \omega^{-1}$ and associated eigenvectors $e_1 = (-\omega\zeta, \omega\zeta^2, 1)$, $e_2 = (-\omega\zeta^{-1}, \omega\zeta^{-2}, 1)$, $e_3 = (1, -1, -\omega^{-1})$, where $\zeta = \exp 2i\pi/5$.

Step 3. We consider in $K = \mathbb{C}(s, t)$ the linear fractional automorphism (deduced from the above diagonalization):

$$\gamma_3 : \; s \mapsto \frac{-\omega\zeta s + \omega\zeta^2 t + 1}{s - t - \omega^{-1}}, \quad t \mapsto \frac{-\omega\zeta^{-1}s + \omega\zeta^{-2}t + 1}{s - t - \omega^{-1}}, \tag{82}$$

The elements $x' := \gamma_3(s)$ and $y' := \gamma_3(t)$ satisfy $K = \mathbb{C}(x', y')$ and the expression of π is:

$$\pi(x') = \zeta x' \;\text{ and }\; \pi(y') = \zeta^{-1}y'. \tag{83}$$

We conclude that the invariant field $\mathbb{C}(x, y)$ under the group of order 5 generated by π is $\mathbb{C}(x, y)^{\pi} = \mathbb{C}(x'y', x'^5)$. The automorphisms $\gamma_1, \gamma_2, \gamma_3$ realizing the conjugation of π with the automorphism $x \mapsto \zeta x, y \mapsto \zeta^{-1}y$ are not Poisson automorphisms and, to the best of our knowledge, our main question remains open in this case:

Problem. *Can we find X and Y such that $\mathbb{C}(x,y)^\pi = \mathbb{C}(X,Y)$ and $\{X,Y\} = \lambda XY$ for some $\lambda \in \mathbb{C}^\times$?*

(iv) We consider now the first quantum Weyl skewfield $D_1^q(\mathbb{C}) = \mathbb{C}_q(x,y)$, with noncommutative product deduced from relation

$$xy = qyx, \tag{84}$$

for $q \in \mathbb{C}^\times$ fixed not a root of one. Some partial information about the quantum Cremona group $\mathrm{Aut}\, D_1^q(\mathbb{C})$ can be found in [2], for instance a description of analogues for $\mathrm{Aut}\, D_1^q(\mathbb{C})$ of the subgroups $\mathbf{A}$ and $\mathbf{J}$ of the classical Cremona group, but without general theorem describing $\mathrm{Aut}\, D_1^q(\mathbb{C})$ as generated by this subgroups.

From the deformation point of view, the theorem above is enlightening: from one hand relation (76) also defines an action of $(\mathbb{C}^\times)^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ as a group of automorphisms of $D_1^q(\mathbb{C})$ still denoted by $\mathbf{H}$ (see [2]), and from the other hand it is easy to construct a quantum version of π (already mentioned in [37]), that is the automorphism π_q of order 5 in $D_1^q(\mathbb{C})$, defined by:

$$\pi_q : \ x \mapsto y, \quad y \mapsto x^{-1}(y + q^{-1}). \tag{85}$$

Then at least two questions naturally arise:

(Q1) is $\mathrm{Aut}\, D_1^q(\mathbb{C})$ generated by $\mathbf{H}$ and π_q (up to the specifically noncommutative role of the inner automorphism subgroup) ?

(Q2) is the invariant skewfield $D_1^q(\mathbb{C})^{\pi_q}$ isomorphic to some $D_1^{q'}(\mathbb{C})$?

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