CIMPA-UNESCO-ARGENTINA
"Homological methods and representations of non-commutative algebras",
Mar del Plata, Argentina March 6 - 17, 2006.

An introduction to noncommutative polynomial invariants

François Dumas

Université Blaise Pascal (Clermont-Ferrand, France)
Laboratoire de Mathématiques (UMR 6620 du CNRS)

Contents

Foreword. This course attempts to reach two apparently contradictory goals: to be a basic introduction with minimal prerequisites, and to introduce to some recent aspects of the current research in the area. This motivates our main orientations: to start with an overview on some topics from the heart of classical invariant theory, to be self-contained for a beginner (and so to remind if necessary some well known results), to give a particular emphasis to concrete examples and explicit calculations, to follow as a main thread some significant mathematical objects in various contexts (for instance the finite subgroups of SL_2), to select some recent subjects without any other criterion that the subjective interest of the author and, more seriously, their capacity to illustrate interesting general noncommutative methods and to lead to relevant current topics. These notes have been written in some rush; so the author apologizes in advance for all misprints, mistakes and misspells in this draft version.

1. Commutative polynomial invariants: some classical results

1.1. Polynomial invariants for linear actions.

1.1.1. *Polynomial functions.* We fixe $\mathbb{k}$ a infinite commutative field. Let V be a $\mathbb{k}$-vector space of finite dimension $n \geq 1$. We denote by $\mathbb{k}[V]$ the ring of polynomial (or regular) functions on V. Let us recall that $f : V \to \mathbb{k}$ is an element of $\mathbb{k}[V]$ means that, for any $\mathbb{k}$-basis $(e_1, \dots, e_n)$ of V, there exists some polynomial $\varphi \in \mathbb{k}[x_1, \dots, x_n]$ such that $f(\alpha_1 e_1 + \cdots + \alpha_n e_n) = \varphi(\alpha_1, \dots, \alpha_n)$ for all $(\alpha_1, \dots, \alpha_n) \in \mathbb{k}^n$. In other words, f is a polynomial into the elements $x_1 = e_1^*, \dots, x_n = e_n^*$ of the dual basis. It is clear that:

$$\mathbb{k}[V] \simeq S(V^*) \simeq \mathbb{k}[x_1, \dots, x_n]. \tag{1}$$

1.1.2. *Linear actions.* Let G be a group. For any representation $\rho : G \to \mathrm{GL}(V)$ of G on V, the corresponding left action of G on V defined by:

$$\forall\ g \in G,\ \forall\ v \in V,\ g.v = \rho(g)(v), \tag{2}$$

can be canonically extended into an left action of G on $\mathbb{k}[V]$ by:

$$\forall\ g \in G,\ \forall\ f \in \mathbb{k}[V],\ \forall\ v \in V,\ (g.f)(v) = f(g^{-1}.v) = f(\rho(g^{-1})(v)). \tag{3}$$

The function f being polynomial and $\rho(g^{-1})$ being linear, it is clear that $f(\rho(g^{-1})(v))$ is polynomial in the coordinates of v with respect of any basis of V and so $g.f \in \mathbb{k}[V]$; then it is trivial to check that $g.(g'.f) = gg'.f$ and $1.f = f$.

• Remark 1. By definition, the dual representation of ρ is $\rho^* : G \to \mathrm{GL}(V^*)$ such that, for any $f \in V^*$, the linear form $\rho^*(g)(f)$ is given by $v \mapsto f(\rho(g^{-1})(v))$. Then formula (3) is just the extension of the action associated to the dual representation from V^* to $\mathbb{k}[V]$.

• Remark 2. Let $(\alpha_{i,j})_{1\leq i,j\leq n}$ be the matrix of $\rho(g)$ relative to the basis $(e_1, \dots, e_n)$ of V, and $(\beta_{i,j})_{1\leq i,j\leq n}$ its inverse in $\mathrm{GL}_n(\mathbb{k})$. Denoting by $(x_1, \dots, x_n)$ the dual basis, the linear form $g.x_j$ (for any $1 \leq j \leq n$) is defined from (3) by $(g.x_j)(e_i) = x_j(\rho(g^{-1})(e_i)) = x_j\big(\sum_{k=1}^n \beta_{i,k} e_k\big) = \beta_{i,j}$ for all $1 \leq i \leq n$. Then $g.x_j = \sum_{i=1}^n \beta_{i,j} x_i$. Finally, G acts by $\mathbb{k}$-algebra automorphisms on $\mathbb{k}[x_1, \dots, x_n]$:

$$g.\big(\sum_j \alpha_j x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}\big) = \sum_j \alpha_j \big(\sum_{i=1}^n \beta_{i,1} x_i\big)^{j_1} \big(\sum_{i=1}^n \beta_{i,2} x_i\big)^{j_2} \dots \big(\sum_{i=1}^n \beta_{i,n} x_i\big)^{j_n}.$$

1.1.3. *Invariants.* Let G be a group and $\rho : G \to \mathrm{GL}(V)$ a representation of G on V. A polynomial function $f \in \mathbb{k}[V]$ is invariant under the action of G if $g.f = f$ for all $g \in G$. It is clear that the invariants form a subalgebra of $\mathbb{k}[V]$ called the invariant algebra and denoted by $\mathbb{k}[V]^G$. So we have:

$$\begin{aligned} \mathbb{k}[V]^G &= \{f \in \mathbb{k}[V],\ g.f = f,\ \forall\ g \in G\} \\ &= \{f \in \mathbb{k}[V],\ f(g.v) = f(v),\ \forall\ g \in G,\ \forall\ v \in V\}. \end{aligned}$$

In other words a polynomial function $f \in \mathbb{k}[V]$ is invariant if and only if it is constant on all orbits $Gv = \{g.v\,;\ g \in G\}$ of elements $v \in V$ under the action of G.

• Remarks.

(i) Any H subgroup of G acts on $\mathbb{k}[V]$ and $\mathbb{k}[V]^G \subset \mathbb{k}[V]^H$

(ii) If H is normal in G, then G/H acts on $\mathbb{k}[V]^H$ (via $\overline{g}.f = g.f$ for any $f \in \mathbb{k}[V]^H$) and we have $(\mathbb{k}[V]^H)^{G/H} = \mathbb{k}[V]^G$.

(iii) Let H be a subgroup of G and $g \in G$. For any $f \in \mathbb{k}[V]$, we have $f \in \mathbb{k}[V]^H$ if and only if $g.f \in \mathbb{k}[V]^{gHg^{-1}}$.

(iv) In particular, if H and K are conjugate in G, then $\mathbb{k}[V]^H$ is isomorphic to $k[V]^K$.

1.1.4. *Grading.* For any integer $d \geq 0$, a polynomial function $f \in \Bbbk[V]$ is said to be homogeneous of degree d if $f(\alpha v) = \alpha^d f(v)$ for any $\alpha \in \Bbbk$ and $v \in V$. We denote by $\Bbbk[V]_d$ the subspace of $\Bbbk[V]$ of homogeneous functions of degree d. In particular $\Bbbk[V]_0 = \Bbbk$ and $\Bbbk[V]_1 = V^*$, and $\Bbbk[V]_d$ is canonically identified in the first isomorphism of (1) with the d-th symmetric power $S^d(V^*)$. In the second isomorphism of (1), $\Bbbk[V]_d$ is identified with the subspace of $\Bbbk[x_1, \ldots, x_n]$ generated by monomials $x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}$ such that $d_1 + d_2 \cdots + d_n = d$. We deduce in particular from a classical combinatorial result that $\dim \Bbbk[V]_d = \binom{n+d-1}{d}$.
It is clear that the family $(\Bbbk[V]_d)_{d \geq 0}$ is a grading of the algebra $\Bbbk[V]$, i.e.

$$\Bbbk[V] = \bigoplus_{d \geq 0} \Bbbk[V]_d \quad \text{and} \quad \Bbbk[V]_d \, \Bbbk[V]_{d'} \subset \Bbbk[V]_{d+d'}. \tag{4}$$

Moreover $\Bbbk[V]_d$ is stable under the action (3) because any $g \in G$ acts as a degree one function. With the natural notation:

$$\Bbbk[V]_d^G = \Bbbk[V]_d \cap \Bbbk[V]^G, \tag{5}$$

we obtain the following grading of the algebra of invariants:

$$\Bbbk[V]^G = \bigoplus_{d \geq 0} \Bbbk[V]_d^G \quad \text{with} \quad \Bbbk[V]_d^G \, \Bbbk[V]_{d'}^G \subset \Bbbk[V]_{d+d'}^G. \tag{6}$$

1.2. **First example: symmetric polynomials.** We begin with the following well known and historical situation. We fix an integer $n \geq 1$ and consider the symmetric group $G = \mathcal{S}_n$ on n letters. In the canonical representation of $\mathcal{S}_n$ on a n-dimensional vector space $V = \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n$, a permutation g acts by $g(e_i) = e_{g(i)}$. The associated action on $\Bbbk[V] \simeq \Bbbk[x_1, \ldots, x_n]$ is defined by:

$$\forall\, g \in \mathcal{S}_n, \ \forall\, f \in \Bbbk[x_1, \ldots, x_n], \quad g.f(x_1, \ldots, x_n) = f(x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)}). \tag{7}$$

The elements of the invariant algebra $\Bbbk[x_1, \ldots, x_n]^{\mathcal{S}_n}$ are no more than the usual symmetric polynomials:

$$\Bbbk[x_1, \ldots, x_n]^{\mathcal{S}_n} = \{ f \in \Bbbk[x_1, \ldots, x_n] \ ; \ f(x_{g(1)}, \ldots, x_{g(n)}) = f(x_1, \ldots, x_n), \ \forall\, g \in \mathcal{S}_n \}. \tag{8}$$

In particular, the following so called elementary symmetric polynomials are invariant:

$$\begin{aligned}
&\sigma_1 = x_1 + x_2 + \cdots + x_n, \\
&\sigma_2 = x_1 x_2 + x_1 x_3 + \cdots + x_1 x_n + x_2 x_3 + \cdots + x_2 x_n + \cdots + x_{n-1} x_n, \\
&\ldots \\
&\sigma_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k} \qquad (\text{sum of } \tbinom{n}{k} \text{ terms}), \\
&\ldots \\
&\sigma_n = x_1 x_2 \ldots x_n.
\end{aligned}$$

They satisfy in $\Bbbk[x_1, x_2, \ldots, x_n, z]$ the relation:

$$\textstyle\prod_{1 \leq i \leq n} (z - x_i) = z^n - \sigma_1 z^{n-1} + \sigma_2 z^{n-2} - \cdots + (-1)^{n-1} \sigma_{n-1} z + (-1)^n \sigma_n. \tag{9}$$

The following classical theorem gives then a very precise description of the invariant algebra as an algebra of polynomials.

1.2.1. Theorem. *The elementary symmetric polynomials $\sigma_1, \dots, \sigma_n$ are algebraically independent and generate the algebra of symmetric polynomials*

$$\Bbbk[x_1, x_2, \dots, x_n]^{\mathcal{S}_n} = \Bbbk[\sigma_1, \sigma_2, \dots, \sigma_n].$$

Proof ([35]). We proceed by induction on n. We assume in the following that the theorem is true for the elementary symmetric polynomials $\sigma'_1, \dots, \sigma'_{n-1}$ in the variables $x_1, x_2, \dots, x_{n-1}$. We have:

$$\sigma_1 = \sigma'_1 + x_n, \quad \sigma_2 = \sigma'_2 + x_n\sigma'_1, \ \dots, \ \sigma_{n-1} = \sigma'_{n-1} + x_n\sigma'_{n-2}, \quad \sigma_n = x_n\sigma'_{n-1}.$$

We prove first that $\sigma_1, \dots, \sigma_n$ are algebraically independent. Suppose that there exists some algebraic relation $P(\sigma_1, \dots, \sigma_n) = 0$ of minimal degree, with $P \in \Bbbk[t_1, \dots, t_n]$. Set $P = Qt_n + R$ with $Q \in \Bbbk[t_1, \dots, t_n]$ and $R \in \Bbbk[t_1, \dots, t_{n-1}]$. From the above relations, it is clear that $R(\sigma_1, \dots, \sigma_{n-1})$ equals $R(\sigma'_1, \dots, \sigma'_{n-1})$ modulo x_n in $\Bbbk[x_1, \dots, x_n]$. Then $P(\sigma_1, \dots, \sigma_n) = 0$ implies that $R(\sigma'_1, \dots, \sigma'_{n-1})$ is divisible by x_n in $\Bbbk[x_1, \dots, x_n]$. Because $R(\sigma'_1, \dots, \sigma'_{n-1})$ lies in $\Bbbk[x_1, \dots, x_{n-1}]$, we deduce $R(\sigma'_1, \dots, \sigma'_{n-1}) = 0$. By assumption, $\sigma'_1, \dots, \sigma'_n$ are algebraically independent and then $R = 0$ in $\Bbbk[t_1, \dots, t_{n-1}]$. Hence P is divisible by t_n in $\Bbbk[t_1, \dots, t_n]$, which contradicts the minimality.

We prove now that any symmetric polynomial $f \in \Bbbk[x_1, \dots, x_n]$ lies in $\Bbbk[\sigma_1, \dots, \sigma_n]$. According to (6), we can suppose that f is homogeneous of some degree d. Writing $f = \sum_{i=0}^m f_i(x_1, \dots, x_{n-1})x_n^i$, we have $f_i \in \Bbbk[x_1, \dots, x_{n-1}]^{\mathcal{S}_{n-1}}$ (consider the permutations in $\mathcal{S}_n$ fixing the letter n). By assumption $\Bbbk[x_1, \dots, x_{n-1}]^{\mathcal{S}_{n-1}} = \Bbbk[\sigma'_1, \dots, \sigma'_{n-1}]$. From the above relations, we deduce that $f \in \Bbbk[\sigma_1, \dots, \sigma_{n-1}, x_n]$. Thus f has the form $f = p(\sigma_1, \dots, \sigma_{n-1}) + x_n h(\sigma_1, \dots, \sigma_{n-1}, x_n)$ with two polynomials $p \in \Bbbk[t_1, \dots, t_{n-1}]$ and $h \in \Bbbk[t_1, \dots, t_n]$. Again we can assume that, in the algebra $\Bbbk[x_1, \dots, x_n]$, $p(\sigma_1, \dots, \sigma_{n-1})$ is homogeneous of degree d and $h(\sigma_1, \dots, \sigma_{n-1}, x_n)$ is homogeneous of degree $d-1$. It follows that $f - p(\sigma_1, \dots, \sigma_{n-1})$ is again homogeneous of degree d and is divisible by x_n. Since it is clearly symmetric, it is also divisible by $x_1, \dots, x_{n-1}$, and then by the product $x_1x_2 \dots x_n = \sigma_n$. So $f - p = \sigma_n f'$ with a symmetric polynomial f' of degree at most $d - n$. We achieve the proof by induction on d. □

1.2.2. *Remark.* Among other useful examples of symmetric polynomials, we must mention:

- the Newton functions: $s_k = x_1^k + x_2^k + \dots + x_n^k$, for any integer $k \geq 1$,
- the Wronski polynomials: $w_k = \sum_{i_1+i_2+\dots+i_n=k} x_1^{i_1}x_2^{i_2} \dots x_n^{i_n}$, for any integer $k \geq 1$,
- the discriminant: $\delta = \prod_{1\leq i<j\leq n}(x_i - x_j)^2$.

In particular, it is easy to see that:

$$\begin{array}{ll} s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \dots + (-1)^{k-1}\sigma_{k-1}s_1 + (-1)^k k\sigma_k = 0, & \text{for all } 1 \leq k \leq n, \\ s_\ell - \sigma_1 s_{\ell-1} + \sigma_2 s_{\ell-2} + \dots + (-1)^n\sigma_n s_{\ell-n} = 0, & \text{for all } \ell > n. \end{array}$$

We deduce:

$$s_1 = \sigma_1, \quad s_2 = s_1\sigma_1 - 2\sigma_2 = \sigma_1^2 - 2\sigma_2, \quad s_3 = s_2\sigma_1 - s_1\sigma_2 + 3\sigma_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3, \quad \cdots$$

and $s_k \in \Bbbk[\sigma_1, \dots, \sigma_k]$ for all $1 \leq k \leq n$. If moreover $\Bbbk$ is of characteristic zero, we also have:

$$\sigma_1 = s_1, \qquad \sigma_2 = \tfrac{1}{2}s_1^2 - \tfrac{1}{2}s_2, \qquad \sigma_3 = \tfrac{1}{6}s_1^3 - \tfrac{1}{2}s_1s_2 + \tfrac{1}{3}s_3, \quad \cdots$$

In this case, the Newton function $s_1, \dots, s_n$ are algebraically independent and generate the algebra of symmetric polynomials

$$\Bbbk[x_1, x_2, \dots, x_n]^{\mathcal{S}_n} = \Bbbk[s_1, s_2, \dots, s_n].$$

En particular, we observe that:

$$\forall\ \ell > n, \quad s_\ell \in \Bbbk[s_1, \dots, s_n]. \tag{10}$$

1.3. **Second example: actions of** SL_2**.** We consider here $G = \mathrm{SL}_2(\mathbb{C})$ (briefly denoted by SL_2 if there is no doubt about the base field), $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $\mathbb{C}[V] \simeq \mathbb{C}[x,y]$. The natural action corresponds to the trivial two dimensional representation $\mathrm{SL}_2 \to \mathrm{GL}(V)$ and is defined by:

$$\forall\ g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2, \quad g.e_1 = \alpha e_1 + \gamma e_2 \ \text{ and } \ g.e_2 = \beta e_1 + \delta e_2.$$

Following remark 2 of 1.1.2, the associated action on $\mathbb{C}[x,y]$ is the left action defined from:

$$\forall\ g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2, \quad g.x = \delta x - \beta y \ \text{ and } \ g.y = -\gamma x + \alpha y \tag{11}$$

and extended by algebra automorphism to any polynomial.

1.3.1. *Some examples of calculations.* We compute the algebra of invariants under this action for some subgroups of SL_2.

1. For $T = \{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\ ;\ \alpha \in \mathbb{C}^\times\}$, we have $\mathbb{C}[x,y]^T = \mathbb{C}[xy]$.

   Proof. Choose $\delta \in \mathbb{C}^\times$ of infinite order and denote by g the automorphism $x \mapsto \delta x$ and $y \mapsto \delta^{-1}y$. For any monomial $\lambda_{i,j}x^iy^j$ with $\lambda_{i,j} \in \mathbb{C}$, we have $g(\lambda_{i,j}x^iy^j) = \lambda_{i,j}\delta^{i-j}x^iy^j$. Then a polynomial $\sum_{i,j}\lambda_{i,j}x^iy^j$ lies in $\mathbb{C}[x,y]^T$ if and only if $\lambda_{i,j} = 0$ for $i \neq j$. □

2. For $U = \{\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)\ ;\ \beta \in \mathbb{C}\}$, we have $\mathbb{C}[x,y]^U = \mathbb{C}[y]$.

   Proof. Choose $\beta \in \mathbb{C}^\times$ and denote by g the automorphism $x \mapsto x - \beta y$ and $y \mapsto y$. Any polynomial $f \in \mathbb{C}[x,y]$ can be written $f = h_m(y)x^m + h_{m-1}(y)x^{m-1} + \cdots + h_0(y)$ with $h_i(y) \in \mathbb{C}[y]$. Then $g(f) = h_m(y)(x^m - m\beta y x^{m-1} + \cdots) + h_{m-1}(y)(x^{m-1} + \cdots) + \cdots + h_0(y)$. Supposing $g(f) = f$, we observe by a trivial identification that $m\beta y h_m(y) = 0$. We conclude that $f = h_0(y) \in \mathbb{C}[y]$. □

3. We deduce in particular that $\mathbb{C}[x,y]^{\mathrm{SL}_2} = \mathbb{C}$.

4. We fix an integer $n \geq 1$. The subgroup $C_n = \{\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}\ ;\ \zeta^n = 1\}$ of SL_2 is cyclic of order n. We have $\mathbb{C}[x,y]^{C_n} = \mathbb{C}[x^n, xy, y^n]$.

   Proof. Choose ζ a primitive n-th root of one and denote by g the automorphism $x \mapsto \zeta x$ and $y \mapsto \zeta^{-1}y$. Each monomial being an eigenvector for the action, $\mathbb{C}[x,y]^{C_n}$ is generated by invariant monomials. Let $f = x^iy^j$ be a invariant monomial. If $i = j$, then $f \in \mathbb{C}[xy]$. If $i > j$, then $f = (xy)^jx^{i-j}$; the identity $g(f) = f$ implies $i - j = kn$ for some integer $n \geq 1$ and we conclude that $f \in \mathbb{C}[xy, x^n]$. In the same way we obtain $f \in \mathbb{C}[xy, y^n]$ when $j > i$. □

5. We fix an integer $n \geq 1$. The binary dihedral group is the subgroup D_n of SL_2 generated by C_{2n} and the matrix $\mu = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. Its order is $4n$; C_{2n} is normal in D_n; any element of D_n can be written $c\mu^\ell$ with $c \in C_{2n}$ and $\ell = 0$ or 1. We have:

$$\mathbb{C}[x,y]^{D_n} = \mathbb{C}[x^2y^2\,,\ x^{2n} + (-1)^n y^{2n}\,,\ x^{2n+1}y - (-1)^n xy^{2n+1}].$$

   Proof. We put $X = x^{2n}$, $Y = y^{2n}$ and $Z = xy$. Then $\mathbb{C}[x,y]^{C_{2n}} = \mathbb{C}[X,Y,Z]$ with $XY = Z^{2n}$. The automorphism $g : x \mapsto iy, y \mapsto ix$ of $\mathbb{C}[x,y]$ associated to μ acts on $\mathbb{C}[x,y]^{C_{2n}}$ by $g(X) = (-1)^nY$, $g(Y) = (-1)^nX$ and $g(Z) = -Z$. Then $\mathbb{C}[x,y]^{D_{2n}} = \mathbb{C}[X,Y,Z]^g$. We have:

$$\mathbb{C}[X,Y,Z] = \bigoplus_{d\geq 1}\mathbb{C}[Z^2]ZX^d \oplus \bigoplus_{d\geq 1}\mathbb{C}[Z^2]X^d \oplus \mathbb{C}[Z] \oplus \bigoplus_{d\geq 1}\mathbb{C}[Z^2]Y^d \oplus \bigoplus_{d\geq 1}\mathbb{C}[Z^2]ZY^d.$$

   For even n, we deduce:

$$\mathbb{C}[X,Y,Z]^g = \bigoplus_{d\geq 1}\mathbb{C}[Z^2]Z(X^d - Y^d) \oplus \bigoplus_{d\geq 1}\mathbb{C}[Z^2](X^d + Y^d) \oplus \mathbb{C}[Z^2].$$

Using relation $XY = Z^{2n}$ and the binomial formula, it is easy to prove by induction on d that $X^d + Y^d$ belongs to the algebra generated over $\mathbb{C}$ by Z^2 and $X+Y$, and that $X^d - Y^d$ is the product of $X - Y$ by an element of the algebra generated over $\mathbb{C}$ by Z^2 and $X+Y$. We conclude in this case that $\mathbb{C}[X,Y,Z]^g = \mathbb{C}[Z^2, X+Y, Z(X-Y)]$. The proof for odd n is similar. $\square$

1.3.2. *First additional comment: Kleinian surfaces.* The finite subgroups of SL_2 are classified up to conjugation in five types, two infinite families parameterized by the positive integers (the type A_{n-1} corresponding of the cyclic group of order n and the type D_n corresponding to the binary dihedral group of order $4n$) and three groups E_6, E_7, E_8 of respective orders 24, 48, 120. This groups can be explicitly described in the following way.
Let us denote $\zeta_n = \exp(2i\pi/n) \in \mathbb{C}$ for any integer $n \geq 1$ and consider in SL_2 the matrices:

$$\theta_n = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \nu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} -\zeta_5^3 & 0 \\ 0 & -\zeta_5^2 \end{pmatrix},$$

$$\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^5 & \zeta_8 \end{pmatrix}, \qquad \psi = \frac{1}{\zeta_5^2 - \zeta_5^{-2}} \begin{pmatrix} \zeta_5 + \zeta_5^{-1} & 1 \\ 1 & -(\zeta_5 + \zeta_5^{-1}) \end{pmatrix}.$$

We define the following subgroups of SL_2:

- type A_{n-1} : the cyclic group C_n, of order n, generated by θ_n,
- type D_n : the binary dihedral group D_n, of order $4n$, generated by θ_{2n} and μ,
- type E_6 : the binary tetrahedral group T, of order 24, generated by θ_4, μ and η,
- type E_7 : the binary octahedral group O, of order 48, generated by θ_8, μ and η,
- type E_8 : the binary icosahedral group I, of order 120, generated by φ, ν and ψ.

Since any finite subgroup G of SL_2 is conjugate to one of these types, it follows from remark (iv) of 1.1.3 that we can suppose without restriction in the determination of the algebra of invariants $\mathbb{C}[x,y]^G$ for the natural action (11) that G is C_n, D_n, T, O or I. In each case, one can compute (see [51]) a system of three generators f_1, f_2, f_3 of the algebra of invariants $\mathbb{C}[x,y]^G$ for the natural action. Observe that the two first cases are no more than examples 4 and 5 above.

type	generators	equation
A_{n-1}	$f_1 = xy, \quad f_2 = x^n, \quad f_3 = y^n$	$X^n + YZ = 0$
D_n	$f_1 = x^2y^2, \quad f_2 = x^{2n} + (-1)^n y^{2n}$, $f_3 = x^{2n+1}y - (-1)^n xy^{2n+1}$	$X^{n+1} + XY^2 + Z^2 = 0$
E_6	$f_1 = xy^5 - x^5y, \quad f_2 = x^8 + 14x^4y^4 + y^8$, $f_3 = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$	$X^4 + Y^3 + Z^2 = 0$
E_7	$f_1 = x^8 + 14x^4y^4 + y^8, \; f_2 = x^{10}y^2 - 2x^6y^6 + x^2y^{10}$ $f_3 = x^{17}y - 34x^{13}y^5 + 34x^5y^{13} - xy^{17}$	$X^3Y + Y^3 + Z^2 = 0$
E_8	$f_1 = x^{11}y + 11x^6y^6 - xy^{11}$, $f_2 = x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20}$, $f_3 = x^{30} + 522x^{25}y^5 - 10005x^{20}y^{10} - 10005x^{10}y^{20} - 522x^5y^{25} + y^{30}$	$X^5 + Y^3 + Z^2 = 0$

In all cases, the algebra $\mathbb{C}[x,y]^G = \mathbb{C}[f_1, f_2, f_3]$ appears as the factor of the polynomial algebra $\mathbb{C}[X,Y,Z]$ in three variables by the ideal generated by one relation (of degree n, $n+1$, 4, 4, 5 respectively). The corresponding surfaces of $\mathbb{C}^3$ are the Kleinian surfaces, which are the subject of many geometric, algebraic and homological studies. It is proved in [51] that, for G and G' two groups among the types $A_{n-1}, D_n, E_6, E_7, E_8$, the algebras $\mathbb{C}[x,y]^G$ and $\mathbb{C}[x,y]^{G'}$ are isomorphic if and only if $G = G'$.

1.3.3. *Second additional comment: irreducible representations of* SL_2. For any integer $d \geq 0$, we denote here by W_d the vector space $\mathbb{C}[x,y]_d$ of homogeneous polynomials of degree d. In the terminology of classical invariant theory, the elements of W_d are called the binary forms of degree d. A $\mathbb{C}$-basis of W_d is $(e_i)_{0\leq i\leq d}$ where $e_i = x^i y^{d-i}$. As we have seen in 1.1.4, any space W_d is stable under the natural action of SL_2 on $\mathbb{C}[x,y]$ defined by (11). Then we obtain a representation $\rho_d : \mathrm{SL}_2 \to \mathrm{GL}(W_d)$, satisfying:

$$\forall\ g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2,\ \forall\ 0 \leq i \leq d,\quad \rho_d(g)(e_i) = (\delta x - \beta y)^i(-\gamma x + \alpha y)^{d-i}. \tag{12}$$

It is not difficult to verify that the representation ρ_d is irreducible (i.e. there is no proper and non trivial subspace W' of W_d such that $\rho_d(g)(W') \subset W'$ for any $g \in \mathrm{SL}_2$). A more profound theorem asserts that any irreducible rational representation of finite dimension $d+1$ of SL_2 (the notion of rational representation is connected with the structure of algebraic group of SL_2) is equivalent to ρ_d (see for instance [51]).

1.4. **Third example: duality double.** The following situation will turn to be important in further considerations about actions on Weyl algebras. We return to the general situation of a representation $\rho : G \to \mathrm{GL}(V)$ of a group G on a n-dimensional $\Bbbk$-vector space V. We put $W = V \oplus V^*$. Any element of W can be written uniquely $w = v + x$ with $v \in V$ and $x \in V^*$. Then we denote $w = (v,x)$. Combining the action (2) of G on V and the associated action (3) on V^*, we define the action:

$$\forall\ g \in G,\ \forall\ w = (v,x) \in W = V \oplus V^*,\ \ g.w = (g.v, g.x). \tag{13}$$

Here $g.v = \rho(g)(v) \in V$ and $g.x \in V^*$ is defined by $(g.x)(u) = x(\rho(g^{-1})(u))$ for all $u \in V$. We define the following bilinear form $q : W \to \Bbbk$:

$$\forall\ (v,x) \in W,\ \ q(v,x) = x(v). \tag{14}$$

1.4.1. Lemma. *Let $(e_1,\ldots,e_n)$ a $\Bbbk$-basis of V and $(x_1,\ldots,x_n)$ its dual basis in V^*; we consider the basis $(e_1,\ldots,e_n,x_1,\ldots,x_n)$ of W and its dual basis $(x_1,\ldots,x_n,\zeta_1,\ldots,\zeta_n)$ in W^*. Then:*

$$q = x_1\zeta_1 + \cdots + x_n\zeta_n.$$

Proof. By definition of the x_i's and ζ_i's, we have $x_i(v,x) = x_i(v,0) = x_i(v)$ and $\zeta_i(v,x) = \zeta_i(0,x) = \zeta_i(x)$ for all $(v,x) \in W$. It follows that the polynomial function $q' = x_1\zeta_1 + \cdots + x_n\zeta_n$ is a bilinear form $W \to \Bbbk$. For any $1 \leq i,j \leq n$, we have: $q'(e_i,x_j) = \sum_{k=1}^n x_k(e_i,x_j)\zeta_k(e_i,x_j)$. Since $x_k(e_i,x_j) = \delta_{i,k}$ and $\zeta_k(e_i,x_j) = \delta_{j,k}$, we obtain $q'(e_i,x_j) = \delta_{i,j} = x_j(e_i) = q(e_i,x_j)$. Using the bilinearity of q and q', this proves that $q' = q$. □

1.4.2. Proposition. *For any $\rho : G \to \mathrm{GL}(V)$, we have: $q \in \Bbbk[W]^G$.*

Proof. It's clear from previous lemma that $q \in \Bbbk[W]$. Moreover, for any $g \in G$ and $(v,x) \in W$, we have $q(g.(v,x)) = (g.x)(g.v) = x\big(\rho(g^{-1})(\rho(g)(v))\big) = x(v) = q(v,x)$. □

1.4.3. Proposition. *For the natural representation of* $\mathrm{GL}(V)$ *on* V*, we have:* $\Bbbk[W]^{\mathrm{GL(V)}} = \Bbbk[q]$.

Proof. First we observe that the subset $W_q = \{w \in W\,;\, q(w) \neq 0\}$ is Zariski-dense on W (i.e. every function $f \in \Bbbk[W]$ which vanishes on W_q is the zero function). Indeed, if f vanishes on W_q, then fq vanishes on W by definition of W_q, hence $fq = 0$. Since q is nonzero this implies in the domain $\Bbbk[W]$ that $f = 0$.
Now fix some vector $w_0 = (v_0, x_0) \in W_q$ such that $x_0(v_0) = 1$. By standard arguments of linear algebra, one can check that, for any $w = (v, x) \in W_q$, there exists $g \in \mathrm{GL}(V)$ such that $g.w = (v_0, \lambda x_0)$ where $\lambda = x(v) \in \Bbbk^\times$. Take $f \in \Bbbk[W]^{\mathrm{GL}(V)}$. We can write $f = f_0 + f_1 + \cdots + f_d$ with any f_j homogeneous of degree j related to the component in V^* (i.e. $f_j(v, \alpha x) = \alpha^j f_j(v, x)$ for any $v \in V$, $x \in V^*$ and $\alpha \in \Bbbk^\times$). Let $p(t)$ be the polynomial in $\Bbbk[t]$ defined by: $p(t) = \sum_{j=0}^d f_j(v_0, x_0)t^j$. Then, considering any $w \in W_q$ with $g \in \mathrm{GL}(V)$ satisfying $g.w = (v_0, \lambda x_0)$ with $\lambda = q(w) \in \Bbbk^\times$, we obtain:

$$f(w) = f(g.w) = f(v_0, \lambda x_0) = \textstyle\sum_{j=0}^d f_j(v_0, \lambda x_0) = \sum_{j=0}^d \lambda^j f_j(v_0, x_0) = p(\lambda) = p(q(w)) = p(q)(w).$$

Then the polynomial functions f and $p(q) = \sum_{j=0}^d f_j(v_0, x_0)q^j$ are equal on W_q. Because W_q is Zariski-dense in W, we conclude that $f = p(q)$. □

1.4.4. Proposition. *We recall the notations of 1.4.1 and denote by* T_n *the subgroup of linear automorphisms* $g \in \mathrm{GL}(V)$ *with diagonal matrices with respect of the basis* $(e_1, \ldots, e_n)$*. Then:*

$$\Bbbk[W]^{\mathrm{T}_n} = \Bbbk[x_1\zeta_1, x_2\zeta_2, \ldots, x_n\zeta_n].$$

Proof. An element $g \in \mathrm{T}_n$, with matrix $M_g = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, acts by $g.x_j = \lambda_j^{-1}x_j$ and $g.\zeta_j = \lambda_j\zeta_j$. Therefore, any monomial $y = x_1^{j_1} \ldots x_n^{j_n}\zeta_1^{i_1} \ldots \zeta_n^{i_n}$ is an eigenvector under the action, and any element of $\Bbbk[W]^{\mathrm{T}_n}$ is a $\Bbbk$-linear combination of invariant monomials. If we choose $M_g = (\lambda_1, 1, \ldots, 1)$ with λ_1 of infinite order in $\Bbbk^\times$, the relation $g.y = y$ implies $i_1 = j_1$. Proceeding on the same way for all diagonal entries, we obtain $y = (x_1\zeta_1)^{i_1}(x_2\zeta_2)^{i_2} \ldots (x_n\zeta_n)^{i_n}$. The result follows. □

1.5. **Finiteness theorems.** We start with the following well known result, which is one of the most simple about invariant under finite group actions.

1.5.1. Theorem (E. Noether). *Assume that* $\Bbbk$ *is of characteristic zero. For any representation* $\rho : G \to \mathrm{GL}(V)$ *of a finite group* G*, the invariant algebra* $\Bbbk[V]^G$ *is generated by the homogeneous invariants functions of degree less than or equal to the order of* G*.*

Proof. We can find many different proofs in the literature (see for instance [48],[50],...). The following is particulary enlightening and proceeds from [35].
• We choose a basis of V and identify $\Bbbk[V] = \Bbbk[x_1, \ldots, x_n]$, where $n = \dim V$. For any integer $j \geq 0$, we consider the polynomial p_j in $\Bbbk[x_1, \ldots, x_n, t_1, \ldots, t_n]$ defined by:

$$p_j(x_1, \ldots, x_n, t_1, \ldots, t_n) = \sum_{g\in G} \big[(g.x_1)t_1 + (g.x_2)t_2 + \ldots + (g.x_n)t_n\big]^j.$$

Denoting $X_g = (g.x_1)t_1 + (g.x_2)t_2 + \ldots + (g.x_n)t_n$ for any $g \in G$, and $G = \{g_1, g_2, \ldots, g_d\}$ with d the order of G, we can observe that $p_j = X_{g_1}^j + X_{g_2}^j + \cdots + X_{g_d}^j$ is the j-th Newton function on the d variables X_{g_i}. It follows from (10) that $p_j \in \Bbbk[p_1, \ldots, p_d]$ for any j.
• For any n-tuple $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ of non-negative integers, define the following polynomial:

$$h_\mu = \sum_{g\in G} g.(x_1^{\mu_1}x_2^{\mu_2} \ldots x_n^{\mu_n}).$$

Then $h_\mu \in \Bbbk[x_1, \ldots, x_n]^G$ and h_μ is homogeneous of degree $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n$. By definition of the p_j's and h_μ's, we have:

$$p_j = \sum_{|\mu|=j} \tfrac{j!}{\mu_1!\mu_2!\ldots\mu_n!}\, h_\mu\, t_1^{\mu_1}t_2^{\mu_2} \ldots t_n^{\mu_n}.$$

Because each p_j with $j > d$ can be expressed as a polynomial in the p_i's with $i \leq d$, we deduce from this relation that the invariants h_μ for $|\mu| > d$ can be written as polynomials in the h_η where $|\eta| \leq d$.

• Finally consider any invariant polynomial function $f = \sum_\mu \lambda_\mu x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n}$, where $\lambda_\mu \in \Bbbk$. We have $f = \frac{1}{d}\sum_{g\in G} g.f = \frac{1}{d}\sum_\mu \sum_{g\in G} \lambda_\mu g.(x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n}) = \frac{1}{d}\sum_\mu \lambda_\mu h_\mu$. This proves that any element of $\Bbbk[x_1,\dots,x_n]^G$ is a polynomial in the h_μ's, and then (by the second step of the proof) is a polynomial in the h_η such that $|\eta| \le d$. □

1.5.2. *Remark.* For a given finite group G, the minimal number m such that for every representation $\rho : G \to \mathrm{GL}(V)$ the invariant algebra $\Bbbk[V]^G$ can be generated by invariants of degree less or equal to m is denoted by $\beta(G)$. Noether's theorem asserts that $\beta(G) \le |G|$. It is possible to prove that we always have $\beta(G) < |G|$ unless for cyclic G (the part "G cyclic implies $\beta(G) = |G|$" is obvious considering a one dimensional faithful representation), and to compute $\beta(G)$ for some small groups (see [48]).

1.5.3. *More about finiteness of invariants for finite groups.* If we only consider the question on the finite generation of invariants (independently of the research of some bound on the degree of generators), we can obtain results in more general contexts. Observe in particular that the following theorem doesn't apply only to linear actions, but to any finite group of automorphisms (see further 1.6 for comments on this point).

Theorem (E. Noether). *Let A be a commutative noetherian ring, R a commutative finitely generated A-algebra, and G a finite group of A-algebra automorphisms of R. Then:*

(i) *R^G is a finitely generated A-algebra,* (ii) *R is finitely generated R^G-module.*

Proof. Let $r_1, r_2, \dots, r_n$ be generators of R over A. We denote $R = A[r_1, r_2, \dots, r_n]$. For any $p \in R$, we consider in $R[x]$ the monic polynomial $q(x) = \prod_{g\in G}(x - g.p)$. It is clear from formula (9) that $q \in R^G[x]$. Since $q(p) = 0$, we deduce that R is integral over R^G. Each generator r_i ($1 \le i \le n$) satisfies a monic polynomial relation:

$$r_i^{d_i} + \sum_{j=0}^{d_i-1} \alpha_{i,j} r_i^j = 0, \quad \text{with } \alpha_{i,j} \in R^G \text{ for all } 1 \le i \le n \text{ and } 1 \le j \le d_i.$$

Denoting simply $\{a_1, \dots, a_\ell\}$ the finite set of all coefficients $\alpha_{i,j}$, we introduce the algebra $B = A[a_1, \dots, a_\ell]$ generated over A by the a_j's. We have $B \subset R^G$. From Hilbert's basis theorem, B is noetherian (as a factor of a polynomial algebra with coefficients in a noetherian ring). The n monic relations above imply that any monomial in the r_i's is a linear combination with coefficients in B of monomials $r_1^{j_1} r_2^{j_2} \dots r_n^{j_n}$ with $j_i \le d_i - 1$. Thus R is a finitely generated B-module. Because B is noetherian, it follows that any B-submodule of R is itself finitely generated. In particular R^G (which is obviously a B submodule of R since $B \subset R^G \subset R$) is a finitely generated B-module. As B is a finitely generated A-algebra, point (i) of the theorem is proved. Finally, R finitely generated as B-module and $B \subset R^G$ trivially imply point (ii). □

1.5.4. *Finiteness results for reductive groups.* We must mention to finish the following important theorem which is related to linear actions of non necessarily finite groups.

Theorem (D. Hilbert) *Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional representation of a group G. Assume that the representation of G on the polynomial functions algebra $\Bbbk[V]$ is completely reducible. Then the invariant algebra $\Bbbk[V]^G$ is finitely generated as a $\Bbbk$-algebra.*

The proof (that we don't develop here) uses two main arguments: the Hilbert's basis theorem (as in the previous theorem) and the existence under the hypothesis of a linear projection $R : \Bbbk[V] \to \Bbbk[V]^G$ which is $\Bbbk[V]^G$-linear (i.e. $R(hf) = hR(f)$ for all $h \in \Bbbk[V]^G, f \in \Bbbk[V]$) and equivariant (i.e. $R(g.f) = R(f)$ for all $g \in G, f \in \Bbbk[V]$). Such an R is called a Reynolds operator.

The theorem applies in particular to the class of reductive groups, including finite groups (in this case $R(f) = \frac{1}{|G|}\sum_{g\in G} g.f$), but also tori, linear and special linear groups, orthogonal and special orthogonal groups, symplectic groups,...

This theorem is the starting point of a wide literature around Hilbert's 14-th problem. We just enumerate some points of reference and refer the reader to [50], [51], [52], [35]

1. The condition "G reductive" is sufficient but not necessary for $\Bbbk[V]^G$ to be finitely generated.
2. There exist non reductive groups G with linear finite dimensional actions such that $\Bbbk[V]^G$ is not finitely generated (the first counter example is by Nagata for V of dimension 18).
3. It is possible to characterize the groups G such that, for any finite dimensional representation on V, the algebra $\Bbbk[V]^G$ is not only finitely generated, but also a polynomial algebra. The Shephard-Todd and Chevalley theorem asserts that this is the case if and only if G is generated by pseudo-reflections.

1.6. **Non linear actions and polynomial automorphisms.** As we have seen in 1.5.3, many problems on polynomial invariants make sense for non necessarily linear actions. We recall here some basic facts about the automorphism groups of commutative polynomial algebra (and refer to [52] for more details).

1.6.1. *Linear automorphisms, triangular automorphisms.* Consider the polynomial algebra $R = \Bbbk[x_1, \dots, x_n]$, and denote by $\operatorname{Aut} R$ the group of $\Bbbk$-algebra automorphisms of R.

• An element $g \in \operatorname{Aut} R$ is said to be linear if it stabilizes the vector space $\Bbbk x_1 \oplus \Bbbk x_2 \oplus \cdots \oplus \Bbbk x_n$. So the subgroup $\mathrm{GL}(R)$ of linear automorphisms of R is just (up to isomorphism) $\mathrm{GL}_n(\Bbbk)$ acting as we have seen in the previous sections. In particular g is said to be diagonal if it stabilizes $\Bbbk x_i$ for all $1 \leq i \leq n$. Then up to isomorphism the subgroup of linear automorphisms of R is just $\mathrm{GL}_n(\Bbbk)$ acting as we have seen in the previous sections. An element $g \in \operatorname{Aut} R$ is said to be affine if it acts on the x_i's by:

$$g(x_i) = \sum_{j=1}^{n} \alpha_{i,j} x_j + \beta_i, \quad \text{with } (\alpha_{i,j})_{1\leq i,j\leq n} \in \mathrm{GL}_n(\Bbbk) \text{ and } (\beta_1, \dots, \beta_n) \in \Bbbk^n.$$

We denote by $\mathrm{Aff}(R)$ the subgroup of affine automorphisms of R.

• An element $g \in \operatorname{Aut} R$ is said to be triangular if it acts on the x_i's by:

$$\begin{array}{rcl} g(x_1) & = & \lambda_1 x_1 + f_1(x_2, x_3, \dots, x_n) \\ g(x_2) & = & \lambda_2 x_2 + f_2(x_3, x_4, \dots, x_n) \\ \dots & & \\ g(x_{n-1}) & = & \lambda_{n-1} x_{n-1} + f_{n-1}(x_n) \\ g(x_n) & = & \lambda_n x_n + f_n \end{array} \qquad \begin{array}{l} \text{with } \lambda_i \in \Bbbk^\times \text{ and } f_i \in \Bbbk[x_{i+1}, \dots, x_n] \\ \text{for any } 1 \leq i \leq n. \end{array}$$

The subgroup of $\operatorname{Aut} R$ consisting of all triangular automorphisms is traditionally denoted by $\mathrm{J}(R)$ (from de Jonquières). The following proposition (see [4]) is elementary but gives useful informations about the case of a finite subgroup of $\mathrm{J}(R)$.

Proposition.

(i) *Any finite subgroup of triangular automorphisms of $R = \Bbbk[x_1, x_2, \dots, x_n]$ is conjugated in $\operatorname{Aut} R$ to a subgroup of diagonal automorphisms.*

(ii) *Any finite subgroup of affine automorphisms of $R = \Bbbk[x_1, x_2, \dots, x_n]$ is conjugated in $\operatorname{Aut} R$ to a subgroup of linear automorphisms.*

Proof. Let G be a finite subgroup of triangular automorphisms of R. It acts on $\mathbb{k}x_n \oplus \mathbb{k}$ fixing $\mathbb{k}$. By semi-simplicity of G (see the lemma below), there exists $x'_n \in \mathbb{k}x_n \oplus \mathbb{k}$ such that G stabilizes $\mathbb{k}x'_n$ and $\mathbb{k}x_n \oplus \mathbb{k} = \mathbb{k}x'_n \oplus \mathbb{k}$. Then G acts on $\mathbb{k}x_{n-1} \oplus \mathbb{k}[x_n] = \mathbb{k}x_{n-1} \oplus \mathbb{k}[x'_n]$ stabilizing $\mathbb{k}[x'_n]$. By semi-simplicity of G, there exists $x'_{n-1} \in \mathbb{k}x_{n-1} \oplus \mathbb{k}[x'_n]$ such that G stabilizes $\mathbb{k}x'_{n-1}$ and $\mathbb{k}x_{n-1} \oplus \mathbb{k}[x'_n] = \mathbb{k}x'_{n-1} \oplus \mathbb{k}[x'_n]$. In the third step, G acts on $\mathbb{k}x_{n-2} \oplus \mathbb{k}[x_{n-1}, x_n] = \mathbb{k}x_{n-2} \oplus \mathbb{k}[x'_{n-1}, x'_n]$ stabilizing $\mathbb{k}[x'_{n-1}, x'_n]$. By semi-simplicity of G, there exists $x'_{n-2} \in \mathbb{k}x_{n-2} \oplus \mathbb{k}[x'_{n-1}, x'_n]$ such that G stabilizes $\mathbb{k}x'_{n-2}$ and $\mathbb{k}x_{n-2} \oplus \mathbb{k}[x'_{n-1}, x'_n] = \mathbb{k}x'_{n-2} \oplus \mathbb{k}[x'_{n-1}, x'_n]$. We construct so inductively a family $x'_n, x'_{n-1}, \ldots, x'_1$ such that, for any $1 \leq i \leq n$, we have $x'_i \in \mathbb{k}x_i \oplus \mathbb{k}[x_{i+1}, \ldots, x_n] = \mathbb{k}x_i \oplus \mathbb{k}[x'_{i+1}, \ldots, x'_n]$, $x'_i \notin \mathbb{k}[x_{i+1}, \ldots, x_n]$, and $\mathbb{k}x'_i$ is stable under the action of G. Denoting by h the triangular automorphism defined by $h(x_i) = x'_i$ for any $1 \leq i \leq n$, we conclude that $h^{-1}Gh$ acts diagonally on the x_i's. This proves point (i).
For (ii), let G be a finite subgroup of affine automorphisms of R. It acts on $\mathbb{k}x_1 \oplus \mathbb{k}x_2 \oplus \cdots \oplus \mathbb{k}x_n \oplus \mathbb{k}$ fixing $\mathbb{k}$. By semi-simplicity of G, there exists $\mathbb{k}$-linearly independents elements $x'_1, x'_2, \ldots, x'_n$ of $\mathbb{k}x_1 \oplus \mathbb{k}x_2 \oplus \cdots \oplus \mathbb{k}x_n \oplus \mathbb{k}$ such that G stabilizes $\mathbb{k}x'_1 \oplus \mathbb{k}x'_2 \oplus \cdots \oplus \mathbb{k}x'_n$ and $\mathbb{k}x_1 \oplus \mathbb{k}x_2 \oplus \cdots \oplus \mathbb{k}x_n \oplus \mathbb{k} = \mathbb{k}x'_1 \oplus \mathbb{k}x'_2 \oplus \cdots \oplus \mathbb{k}x'_n \oplus \mathbb{k}$. Denoting by h the affine automorphism defined by $h(x_i) = x'_i$ for any $1 \leq i \leq n$, we conclude that $h^{-1}Gh$ acts linearly on the x_i's. $\square$

In order to be complete, we recall in the following lemma the semi-simplicity argument used in the proof of the proposition.

Lemma (Maschke). *Let $\rho : G \to \mathrm{GL}(V)$ a representation of a finite group G whose order doesn't divide the characteristic of $\mathbb{k}$, with V a non necessarily finite dimensional vector space. Suppose that $V = W \oplus W_1$ with W and W_1 subspaces such that W is G-stable. Then there exists a G-stable subspace $W_2 \simeq W_1$ such that $V = W \oplus W_2$.*

Proof. Denote by p_1 the canonical projection $p_1 : V \to W$ and define $q : V \to V$ by $q(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)(p_1(\rho(g)^{-1}(v)))$. Because W is G-stable, we have $q(v) \in W$ for all $v \in V$ and $q(w) = w$ for all $w \in W$. Then $\operatorname{Im} q = W$. An easy calculation shows that $q(\rho(h)(v)) = \rho(h)(q(v))$ for any $h \in G$ and $v \in V$. It follows that $\operatorname{Ker} q$ is G-stable. Then the lemma is proved with $W_2 = \operatorname{Ker} q$. $\square$

1.6.2. *The case $n = 2$.* For $R = \mathbb{k}[x_1, x_2]$ the structure of the group $\operatorname{Aut} R$ is very explicitly known. Papers by Jung, Van der Kulk, Rentschler, Makar-Limanov (see [52] for more complete references) led to prove that $\operatorname{Aut} R$ is generated by the subgroup $\mathrm{Aff}(R)$ of linear automorphisms and the subgroup $\mathrm{J}(R)$ of triangular automorphisms. More precisely, $\operatorname{Aut} R$ is the amalgamated free product of $\mathrm{Aff}(R)$ and $\mathrm{J}(R)$ over their intersection (i.e. if $g_i \in \mathrm{J}(R) \setminus \mathrm{Aff}(R)$ and $h_i \in \mathrm{Aff}(R) \setminus \mathrm{J}(R)$, then $g_1 h_1 g_2 h_2 \ldots g_n h_n g_{n+1} \notin \mathrm{Aff}(R)$). It follows by a theorem of Serre (see [49], théorème 8 p. 53) that any finite subgroup G of $\operatorname{Aut} R$ is conjugate either to a subgroup of $\mathrm{Aff}(R)$ or to a subgroup of $\mathrm{J}(R)$. Applying proposition 1.6.1, we finally conclude that:

Corollary. *If $R = \mathbb{k}[x_1, x_2]$, any finite subgroup of* $\operatorname{Aut} R$ *is conjugate to a subgroup of linear automorphisms.*

For $\mathbb{k} = \mathbb{C}$, the finite subgroups of GL_2 are classified up to isomorphism and the corresponding invariant algebras determined in [45] similarly to the description given in 1.3.2 for SL_2.

1.6.3. *Comments.* For any $n \geq 1$, the subgroup of $\operatorname{Aut} R$ generated by $\mathrm{Aff}(R)$ and $\mathrm{J}(R)$ is called the group of tame automorphisms and is denoted by $\mathrm{T}(R)$. The results of 1.6.2 are no more right for $n > 2$. Firstly, it is easy to observe that, if $n \geq 3$, then $\mathrm{T}(R)$ is not the amalgamated free product of $\mathrm{Aff}(R)$ and $\mathrm{J}(R)$ over their intersection (define g the automorphism of R exchanging x_2 and x_3, and h the automorphism $x_1 \mapsto x_1 + x_2^2, x_2 \mapsto x_2, x_3 \mapsto x_3$; the automorphism $t = ghg^{-1}$ and h belong to $\mathrm{J}(R) \setminus \mathrm{Aff}(R)$, however $t^{-1}gh = g \in \mathrm{Aff}(R)$). More profoundly, Nagata conjectured in 1972 that $\mathrm{T}(R) \neq \mathrm{Aut}\,(R)$ for $n = 3$, and proposed as a possible counterexample the automorphism $x_1 \mapsto x_1 - 2x_2(x_3x_1 + x_2^2) - x_3(x_3x_1 + x_2^2)^2$, $x_2 \mapsto x_2 + x_3(x_3x_1 + x_2)^2$, $x_3 \mapsto x_3$.

This conjecture has been solved (by the affirmative) only in 2001 by Chestakhov and Urmibaev. A canonical way to obtain automorphisms of R consists in considering the exponential of a locally nilpotent derivation D of R, and in particular of the product of a triangular derivation d by an element of $\operatorname{Ker} d$ (for instance Nagata's automorphism is $\exp D$ for $D = (x_3x_1 + x_2^2)d$, where $d = -2x_2\partial_{x_1} + x_3\partial_{x_2}$). Fixing a locally nilpotent derivation D of R and denoting $g_t = \exp(tD)$ for any $t \in \Bbbk$, the subgroup $E_D = \{g_t\,;\, t \in \Bbbk\}$ is a subgroup of $\operatorname{Aut} R$ isomorphic to the additive group $\mathbb{G}_a = (\Bbbk, +)$ and it is easy to observe that conversely any action of the algebraic group $\mathbb{G}_a$ on the affine space $\Bbbk^n$ arises in this way. Many questions about the $\mathbb{G}_a$-actions (triangulability, fixed point freeness, cancellation problem, finite generation of the invariants,...) reduce to algebraic problems on locally nilpotent derivations of R (in particular about their kernels) and conjugation of subgroups E_D in $\operatorname{Aut} R$. We refer the reader interested by this wealthy research area to [52].

2. Actions on noncommutative polynomial algebras

2.1. **Invariants of noetherian rings under finite groups actions.**

2.1.1. *Noncommutative noetherian rings.* Let R be a ring (non necessarily commutative). A left R-module M is said to be noetherian if M satisfies the ascending chain condition on left submodule, or equivalently if every left submodule of M is finitely generated. The ring R himself is a left noetherian ring if it is noetherian as left R-module. There is of course a similar definition for right modules, and a ring R is said to be noetherian if it is left noetherian and right noetherian (i.e. if every left ideal is finitely generated and every right ideal is finitely generated). It is classical and easy to prove that any finite direct sum of noetherian modules is noetherian, and that, for any submodule N of a module M, we have: M noetherian if and only if N and M/N are noetherian. These properties imply in particular the following useful observation: if R a left noetherian ring, then all finitely generated left R-modules are left noetherian.

2.1.2. *Skew group rings.* Let R be a ring and G a subgroup of the group $\operatorname{Aut} R$ of ring automorphisms of R. The skew group ring (or trivial crossed product) $R\#G$ is defined as the free left R-module with elements of G as a basis and with multiplication defined from relation:

$$\forall\, r, s \in R,\ \forall\, g, h \in G,\ (rg)(sh) = rg^{-1}(s)gh$$

Every element of $R\#G$ as a unique expression as $\sum_{g\in G} r_g g$ with $r_g \in R$ for any $g \in G$ and $r_g = 0$ for all but finitely many g. It is clear that R is a subring of $R\#G$ (identifying r with $r1_G$), and that $R\#G$ is also a right R-module. Using the last observation of 2.1.1, we deduce immediately that:

(15) if G is finite and R is left noetherian, then $R\#G$ is left noetherian.

Note that the noetherianity of $R\#G$ can be proved in the more general context where G is polycyclic by finite, see [39]. The skew group ring $R\#G$ is closely related to the invariant ring R^G, as shows for instance the following lemma.

Lemma. *Let R be a ring, G a finite subgroup of* $\operatorname{Aut} R$*, and $S = R\#G$. Suppose that $|G|$ is invertible in R and consider in S the element $e = \frac{1}{|G|}\sum_{g\in G} g$. Then we have:*

$$\text{(i)}\ \ e^2 = e, \qquad \text{(ii)}\ \ eS = eR, \qquad \text{(iii)}\ \ eSe = eR^G \simeq R^G.$$

Proof. We have $eg = e$ for all $g \in G$. Relation (i) is then obvious. For any $x = \sum_{g\in G} r_g g \in S$, we have $ex = \sum_{g\in G} er_g g$. Since $rg = gg(r)$ for all $g \in G$ and $r \in R$ by definition of the multiplication in S, we obtain $ex = \sum_{g\in G} egg(r_g)$. As $eg = e$, it follows that $ex = \sum_{g\in G} eg(r_g) = e\sum_{g\in G} g(r_g) \in eR$. We conclude that $eS \subset eR$. The converse is clear and so equality (ii) holds. It follows from point (ii) that $eSe = eRe$. For $r \in R$, we compute:

$$\begin{aligned} ere = \tfrac{e}{|G|}\textstyle\sum_{g\in G} rg = \tfrac{e}{|G|}\sum_{g\in G} gg(r) &= \tfrac{1}{|G|}\textstyle\sum_{g\in G} egg(r) \\ &= \tfrac{1}{|G|}\textstyle\sum_{g\in G} eg(r) = \tfrac{e}{|G|}\sum_{g\in G} g(r) = \tfrac{e}{|G|}\tau(r) = e\tau(\tfrac{r}{|G|}), \end{aligned}$$

where $\tau : R \to R^G$ is the trace of G on R. This proves that $eSe = e\tau(R)$. The assumption $|G|$ invertible in R implies that any $r \in R^G$ can be written $r = \tau(\frac{1}{|G|}r)$, so $R^G \subset \tau(R)$, and finally $R^G = \tau(R)$. Hence $eSe = eR^G$. As $er = re$ for any $r \in R^G$, the map $r \mapsto er$ defines a ring isomorphism $R^G \to eR^G$. □

2.1.3. *A finiteness theorem.* The following theorem is due to S. Montgomery and L. W. Small (see [42]) and can be viewed as a noncommutative analogue of Noether's theorem 1.5.3.

Theorem. *Let A be a commutative noetherian ring, R a non necessarily commutative ring such that A is a central subring of R and R is a finitely generated A-algebra, and G a finite group of A-algebra automorphisms of R such that $|G|$ is invertible in R. If R is left noetherian, then R^G is a finitely generated A-algebra.*

Proof. Let us introduce $S = R\#G$. As we have observed in 2.1.2, S is left noetherian. It is clear from the hypothesis that A is a central subring of S and that S is finitely generated as A-algebra (if $\{q_1, \dots, q_m\}$ generate R over A and $G = \{g_1, \dots, g_d\}$, then $\{q_1, \dots, q_m, g_1, \dots, g_d\}$ generate S over A).

As in 2.1.2, consider in S the element $e = \frac{1}{|G|}\sum_{g\in G} g$ which satisfies $e^2 = e$. In particular, eSe is a subring of S, eS is a left eSe-module, and SeS is a two-sided ideal of S. Observe firstly that eS is a finitely generated left eSe-module.

> Because S is left noetherian, SeS is finitely generated as a left ideal of S. Say that $SeS = \sum_i Sx_i$, and write $x_i = \sum_j v_{ij}ew_{ij}$ with $v_{ij} \in S$ and $w_{ij} \in S$ for all j. Choose $r \in S$. Then $er = eeer \in e(SeS)$, and so $er = e(\sum_i s_ix_i) = \sum es_iv_{ij}ew_{ij} = \sum es_iv_{ij}e^2w_{ij}$. Thus the finite set $\{ew_{ij}\}$ generates eS as a left eSe-module.

Denote more briefly $eS = \sum_{i=1}^n eSex_i$ with $x_i \in S$, and take $t_1, t_2, \dots, t_m$ generators of S as a A-algebra. Now write $et_j = \sum_{i=1}^n ey_{ij}ex_i$ and $ex_kt_j = \sum_{i=1}^n ez_{ijk}ex_i$ with $y_{ij} \in S$ and $z_{ijk} \in S$ for all $1 \le j \le m$ and $1 \le k \le n$. Consider the finite set $E = \{ex_ie,\ ey_{ij}e,\ ez_{ijk}e\}_{1\le i,k\le n,\,1\le j\le m}$. We compute:

$$et_1t_2e = (\sum_{i=1}^n ey_{i1}ex_i)t_2e = \sum_{i=1}^n ey_{i1}e(ex_it_2)e = \sum_{i=1}^n ey_{i1}e(\sum_{\ell=1}^n ez_{\ell 2i}ex_\ell)e = \sum_{i=1}^n ey_{i1}e(\sum_{\ell=1}^n ez_{\ell 2i}eex_\ell e),$$

and prove so inductively that any monomial $et_{j_1}t_{j_2}\dots t_{j_k}e$ with $1 \le j_1, j_2, \dots, j_k \le m$ can be expressed by a finite sum of products of elements of E. As any element of eSe is a linear combination of such monomials with coefficients in A, we conclude that E generates eSe as a A-algebra. By lemma 2.1.2, this achieves the proof. □

This theorem will apply in particular to the iterated Ore extensions (see further 2.3).

2.2. **Invariants of simple rings under finite groups actions.**

2.2.1. *Definitions.* Recall that a ring R is simple when (0) and R are the only two-sided ideals of R. An automorphism $g \in \operatorname{Aut} R$ is said to be inner if there exists $a \in R$ invertible in R such that $g(x) = axa^{-1}$ for all $x \in R$, and is said to be outer if it is not inner. A subgroup G of $\operatorname{Aut} R$ is outer when the identity map is the only inner automorphism in G.

2.2.2. *Simplicity of the invariants.* We start with the following lemma about simplicity of crossed products.

Lemma. *Let R be a simple ring and G a finite outer subgroup of* $\operatorname{Aut} G$. *Then the ring $R\#G$ is simple.*

Proof. For any nonzero element $x = \sum_{g\in G} r_g g$ in S, define the length of x as the cardinal of the support $\{g \in G\,;\, r_g \neq 0\}$ of x. Let I be a two-sided nonzero ideal of $S = R\#G$ and ℓ be the minimal length of nonzero elements of I. Because I is a two-sided ideal and ℓ is minimal, it is clear that the set J consisting of 0 and of all nonzero elements of I of length ℓ is a two-sided ideal of S. Thus the set K of all elements $r \in R$ appearing as a coefficient in the decomposition of some element of J is a two-sided ideal of R. Since R is simple, we have $1 \in K$. So there exists in I some element with decomposition $1.g_0 + \sum_{g\in G, g\neq g_0} r_g.g$. Multiplying at the right by g_0^{-1}, we deduce that I contains an element $x = 1.1_G + \sum_{g\in G, g\neq 1_G} r_g.g$ of length ℓ.

If $x = 1.1_G$ (i.e. $\ell = 1$), then $I = S$ and we are done. Assume that $r_h \neq 0$ for some $h \in G, h \neq 1_G$. For any $r \in R$, the bracket $rx - xr = \sum_{g\in G, g\neq 1_G}(rr_g - r_g g^{-1}(r)).g$ lies in I and has shorter length than x. Since ℓ is minimal, it follows that $rx - xr = 0$. In particular: $rr_h - r_h h^{-1}(r) = 0$ for all $r \in R$. Therefore $r_h R = R r_h$ is a two-sided ideal of R. The simplicity of R implies that $1 \in r_h R$, and so r_h is invertible in R. Hence $h^{-1}(r) = r_h^{-1} r r_h$ for all $r \in R$, which says that h^{-1} is an inner automorphism of R, which is impossible since G is outer and $h \neq 1_G$. □

We need now a brief account on the notion of Morita equivalence. Two rings S and T are Morita equivalent when their categories of modules are equivalent. There exist several methods to characterize such an equivalence. None is obvious and we refer for instance to [14] or [39] for a serious presentation of this classical subject. In the limited frame of this notes, our basis will be the following concrete criterion (see [39], proposition 3.5.6): S and T are Morita equivalent if and only if there exist an integer n and an idempotent element $e \in M_n(S)$ such that $T \simeq eM_n(S)e$ and $M_n(S)eM_n(S) = M_n(S)$.

Proposition. *Let R be a simple ring and G a finite outer subgroup of* $\operatorname{Aut} G$ *such that $|G|$ is invertible in R. Then:*

(i) *R^G and $R\#G$ are Morita equivalent,*

(ii) *the ring R^G is simple.*

Proof. Set $S = R\#G$. By the lemma of 2.1.2, the element $e = \frac{1}{|G|}\sum_{g\in G} g$ of S satisfies $e^2 = e$ and we have a ring isomorphism $eSe \simeq R^G$. It is clear that SeS is a two-sided ideal of S. Thus $SeS = S$ since S is simple by the previous lemma. We just apply the above Morita equivalence criterion (with $n = 1$) to conclude that S and R^G are Morita equivalent. The simplicity being a Morita invariant, R^G is then simple. □

This proposition is a fundamental argument in all homological studies of invariants of Weyl algebras (see further).

2.3. **Iterated Ore extensions.**

2.3.1. *Definitions.* Let A a non necessarily commutative ring. For any $\sigma \in \operatorname{Aut} A$, a σ-derivation of A is an additive map $\delta : A \to A$ such that $\delta(\alpha\beta) = \sigma(\alpha)\delta(\beta) + \delta(\alpha)\beta$ for all $\alpha, \beta \in A$.

For any automorphism σ of A and any σ-derivation δ of A, it is a technical elementary exercise to verify that there exists a ring R containing A as a subring and an element $x \in R$ such that R is a free left A-module with basis $\{x^n\,, n \geq 0\}$ and:

$$x\alpha = \sigma(\alpha)x + \delta(\alpha) \quad \text{for any} \quad \alpha \in A. \tag{16}$$

The ring R is called the Ore extension of R defined by σ and δ, and is denoted by $R = A[x\,;\,\sigma,\delta]$. Any element can be written uniquely as a finite sum $y = \sum_i \alpha_i x^i$ with $\alpha_i \in A$. The addition in R is the ordinary addition of polynomials, and the noncommutative multiplication in R is defined inductively from the commutation law (16). For $y \neq 0$, the nonnegative integer $n = \max\{i\,,\alpha_i \neq 0\}$ is called the degree of y and denoted by $\deg_x y$, and the corresponding α_n is the leading coefficient of y. By convention 0 has degree $-\infty$ and leading coefficient 0. When A is a domain, it is clear that, if y, z are two non zero elements of R of respective degrees n, m and leading coefficients α, β, then yz has degree $n+m$ and leading coefficient $\alpha\sigma^n(\beta)$. We deduce:

$$\text{if } A \text{ is a domain, then } A[x\,;\,\sigma,\delta] \text{ is a domain.}$$

In the particular case where $\delta = 0$, we simply denote $R = A[x\,;\,\sigma]$. The commutation relation becomes:

$$x\alpha = \sigma(\alpha)x \quad \text{for any} \quad \alpha \in A. \tag{17}$$

In the particular case where $\sigma = \mathrm{id}_A$, the map δ is an ordinary derivation of A and we simply denote $R = A[x\,;\,\delta]$. The commutation relation becomes:

$$x\alpha = \alpha x + \delta(\alpha) \quad \text{for any} \quad \alpha \in A. \tag{18}$$

When the coefficient ring A is a field, we have as in the commutative case an euclidian algorithm in $A[x\,;\,\sigma,\delta]$. The proofs of the following two results are left to the reader (see for instance [20]).

Proposition. *Let $R = K[x\,;\,\sigma,\delta]$ where K is a non necessarily commutative field, σ is an automorphism of K, and δ is a σ-derivation of K. For any $a, b \in R$, with $b \neq 0$, there exist $q, r \in R$ unique such that $a = qb + r$ with $\deg_x r < \deg_x b$, and there exist $q', r' \in R$ unique such that $a = bq' + r'$ with $\deg_x r' < \deg_x b$.*

Corollary. *For K a non necessarily commutative field, all right ideal and all left ideals of $R = K[x\,;\,\sigma,\delta]$ are principal.*

2.3.2. *Examples.* Take $A = \mathbb{k}[y]$ the commutative polynomial ring in one variable over a commutative field $\mathbb{k}$.

(i) For $\delta = \partial_y$ the usual derivative, $\mathbb{k}[y][x\,;\,\partial_y]$ is the first Weyl algebra $A_1(\mathbb{k})$, with commutation law $xy - yx = 1$.

(ii) For $\delta = y\partial_y$, $\mathbb{k}[y][x\,;\,y\partial_y]$ is the enveloping algebra $U_1(\mathbb{k})$ of the non abelian two dimensional Lie algebra, with commutation law $xy - yx = y$. Note that $yx = (x-1)y$ and then $U_1(\mathbb{k})$ can also be viewed as $\mathbb{k}[x][y\,;\,\sigma]$ for σ the $\mathbb{k}$-automorphism of $\mathbb{k}[x]$ defined by $x \mapsto x - 1$.

(iii) For $\delta = y^2\partial_y$, $\mathbb{k}[y][x\,;\,y^2\partial_y]$ is the Jordanian plane, with homogeneous commutation law $xy - yx = y^2$.

(iv) For σ the $\mathbb{k}$-automorphism of $\mathbb{k}[y]$ defined by $y \mapsto qy$ for some fixed scalar $q \in \mathbb{k}^\times$, $\mathbb{k}[y][x\,;\,\sigma]$ is the quantum plane, denoted by $\mathbb{k}_q[x, y]$, with commutation law $xy = qyx$.

(v) Consider again σ the $\mathbb{k}$-automorphism of $\mathbb{k}[y]$ defined by $y \mapsto qy$ for some fixed scalar $q \in \mathbb{k}^\times, q \neq 1$. The Jackson derivative is the additive map $\delta : \mathbb{k}[y] \to \mathbb{k}[y]$ defined by $\delta(f) = \frac{f(qy)-f(y)}{qy-y}$; it is a σ-derivation. The algebra $\mathbb{k}[y][x\,;\,\sigma,\delta]$ is then the first Weyl algebra, denoted by A_1^q, with commutation law $xy - qyx = 1$.

2.3.3. *Iterated Ore extension.* Starting with a commutative field $\mathbb{k}$ and the commutative polynomial ring $R_1 = \mathbb{k}[x_1]$, and considering an automorphism σ_2 and a σ_2-derivation δ_2 of R_1, we can build the Ore extension $R_2 = R_1[x_2\,;\,\sigma_2,\delta_2]$. Taking an automorphism σ_3 and a σ_3-derivation

δ_3 of R_2, we consider then $R_3 = R_2[x_3\,;\,\sigma_3,\delta_3]$. Iterating this process, we obtain a so called iterated Ore extension:

(19) $$R_m = \Bbbk[x_1][x_2\,;\,\sigma_2,\delta_2][x_3\,;\,\sigma_3,\delta_3]\cdots[x_m\,;\,\sigma_m,\delta_m].$$

It is clear from the construction that $\{x_1^{i_1}x_2^{i_2}\dots x_m^{i_m}\}_{(i_1,i_2,\dots,i_m)\in\mathbb{N}^m}$ is a left $\Bbbk$-basis of R_m, and that R_m is a domain. We give here some elementary examples (see also 2.4.1 below).

1. The Lie algebra $\mathfrak{sl}_2(\Bbbk)$ is $\Bbbk e\oplus\Bbbk f\oplus\Bbbk h$ with Lie brackets $[h,e]=2e$, $[h,f]=-2f$ and $[e,f]=h$. By Poincaré-Birkhoff-Witt's theorem, its enveloping algebra $U(\mathfrak{sl}_2)$ admits $(h^ie^jf^k)_{i,j,k\in\mathbb{N}}$ as a left $\Bbbk$-basis. Then $U(\mathfrak{sl}_2)=\Bbbk[h][e\,;\,\sigma'][f\,;\,\sigma,\delta]$, where σ' is the $\Bbbk$-automorphism of $\Bbbk[h]$ defined by $h\mapsto h-2$, σ is the $\Bbbk$-automorphism of $\Bbbk[h][e\,;\,\sigma']$ defined by $h\mapsto h+2, e\mapsto e$, and δ is the σ-derivation of $\Bbbk[h][e\,;\,\sigma']$ defined by $\delta(h)=0$ and $\delta(e)=-h$.
2. The Heisenberg Lie algebra $\mathfrak{sl}_3^+(\Bbbk)$ is $\Bbbk x\oplus\Bbbk y\oplus\Bbbk z$ with Lie brackets $[x,z]=[y,z]=0$ and $[x,y]=z$. Then $U(\mathfrak{sl}_3^+)=\Bbbk[z][y][x\,;\,\delta]$ for $\delta=z\partial_y$. It can be proved much more generally that the enveloping algebra of any nilpotent Lie algebra of dimension n is an iterated Ore extension on n variables (with $\sigma_1=\mathrm{id}$ for all i's in the formula 19).
3. Let $Q=(q_{ij})$ a $m\times m$ matrix with entries in $\Bbbk^\times$ such that $q_{ii}=1$ and $q_{ij}=q_{ji}^{-1}$ for all i,j's. The quantum m-dimensional affine space parameterized by Q is the algebra $\Bbbk_Q[x_1,\dots,x_m]$ generated over $\Bbbk$ by m generators $x_1,\dots,x_m$ satisfying the commutation relations $x_ix_j=q_{ij}x_jx_i$. It is the iterated Ore extension:
$$\Bbbk_Q[x_1,\dots,x_m]=\Bbbk[x_1][x_2\,;\,\sigma_2][x_3\,;\,\sigma_3]\cdots[x_m\,;\,\sigma_m]$$
with σ_i the $\Bbbk$-automorphism of $\Bbbk[x_1][x_2\,;\,\sigma_2,]\cdots[x_{i-1}\,;\,\sigma_{i-1}]$ defined by $\sigma_i(x_j)=q_{ij}x_j$ for any $1\le j\le i-1$.

2.3.4. *Noetherianity of Ore extension.* The following theorem can be viewed as a noncommutative version of Hilbert's basis theorem (see in particular the historical note of [30] p. 20).

THEOREM. *Let A a non necessarily commutative ring, σ an automorphism and δ a σ-derivation of A. If A is right (resp. left) noetherian, then $A[x\,;\,\sigma,\delta]$ is right (resp. left) noetherian.*

Proof. Assume that A is right noetherian. Let J be a non zero right ideal of $R=A[x\,;\,\sigma,\delta]$. We claim that the set L of leading coefficients of elements of J is a right ideal of A.

> Take $\alpha,\beta\in L$. If $\alpha+\beta=0$, we have $\alpha+\beta\in L$ obviously. So we assume $\alpha+\beta\neq 0$. Let $y,z\in J$ of respective degrees $m,n\in\mathbb{N}$ with respective leading coefficients α,β. In other words, $y=\alpha x^m+\cdots$ and $z=\beta x^n+\cdots$. If $n\ge m$, then $yx^{n-m}+z=(\alpha+\beta)x^n+\cdots$ lies in J, thus $\alpha+\beta\in L$. If $m>n$, then $y+zx^{m-n}=(\alpha+\beta)x^m+\cdots$ lies in J and $\alpha+\beta\in L$. Now take $\gamma\in A$ such that $\alpha\gamma\neq 0$. We have $y\sigma^{-m}(\gamma)=\alpha\gamma x^m+\cdots$. As $y\sigma^{-m}(\gamma)\in J$, it follows that $\alpha\gamma\in L$. We conclude that L is a right ideal of A.

A being right noetherian, introduce nonzero generators $\alpha_1,\dots,\alpha_k$ of L as a right ideal of A. For any $1\le i\le k$, let y_i be an element of J with leading coefficient α_i. Denote n_i the degree of y_i and $n=\max\{n_1,\dots,n_k\}$. Each y_i can be replaced by $y_ix^{n-n_i}$. Hence there is no loss of generality in assuming that $y_1,\dots,y_k$ all have the same degree n. Set N the left A-submodule of R generated by $1,x,x^2,\dots,x^n$ (i.e. the set of elements of R whose degree is lower or equal than n). Using the commutation law $\alpha x=x\sigma^{-1}(\alpha)-\delta(\sigma^{-1}(\alpha))$ for any $\alpha\in A$, we observe that N is also the right A-submodule of R generated by $1,x,x^2,\dots,x^n$. So N is a noetherian right A-module (any right module finitely generated over a right noetherian ring is right noetherian, see the last observation of 2.1.1). It follows that the right A-submodule $J\cap N$ of N is finitely generated, say generated by $z_1,\dots,z_t$. Thus we have $J\cap N=z_1A+z_2A+\cdots+z_tA$. Set $I=y_1R+y_2R+\cdots+y_kR+z_1R+z_2R+\cdots+z_tR$. We will show that $J=I$.

The inclusion $I \subset J$ is trivial (all y_i and z_j are in the right ideal J of R). For the converse inclusion observe first that, A being a subring of R, we have: $J \cap N = z_1A + z_2A + \cdots + z_tA \subset z_1R + z_2R + \cdots + z_tR \subset I$. Thus I contains all elements of J with degree less than n. We will prove by induction on m that, for any integer $m \geq n$, we have: $\{p \in J\,;\, \deg_x p \leq m\} \subset I$.

> The assertion is right for $m = n$. Assume that it is satisfied up to a rank $m - 1 \geq n$. Take $p \in J$ with degree m and leading coefficient α. We have $\alpha \in L$, then there exist $\beta_1, \ldots, \beta_k \in A$ such that $\alpha = \alpha_1\beta_1 + \cdots + \alpha_k\beta_k$. Set $q = \big[y_1\sigma^{-n}(\beta_1) + y_2\sigma^{-n}(\beta_2) + \cdots + y_k\sigma^{-n}(\beta_k)\big]x^{m-n}$, which lies in I by definition of I. Each y_i being of degree n and leading coefficient α_i, the degree of q is m and its leading coefficient is $\alpha_1\beta_1 + \cdots + \alpha_k\beta_k = \alpha$. It follows that $p - q$ is of degree less than m. We have $p \in J$ and $q \in I \subset J$, thus $p - q \in J$ and we can apply the induction assumption to deduce that $p - q \in I$, and then $p \in I$.

So we have proved that $J = I$. Since J was any right ideal of R and I is finitely generated as a right ideal of R, we conclude that R is right noetherian.
Now if A is left noetherian, the opposite ring A^{op} is right noetherian. It is easy to observe that $A[x\,;\,\sigma, \delta]^{\text{op}}$ is isomorphic to $A^{\text{op}}[x\,;\,\sigma^{-1}, -\delta\sigma^{-1}]$. Then the left noetherianity of R follows from the first part of the proof. □

Corollary. *Every iterated Ore extension over a commutative field $\mathbb{k}$ is a noetherian domain.*

Proof. We have seen in 2.3.1 that $A[x\,;\,\sigma, \delta]$ is a domain when A is a domain. We apply this argument and the previous theorem inductively starting from $\mathbb{k}$. □

2.3.5. *Invariants of iterated Ore extension under finite groups.* From the previous corollary and theorem 2.1.3, we deduce immediately the following practical result:

Theorem. *Let R be an iterated Ore extension over a commutative field $\mathbb{k}$. Let G be a finite group of $\mathbb{k}$-automorphisms of R. We suppose that the order of G is prime with the characteristic of $\mathbb{k}$. Then R^G is a finitely generated $\mathbb{k}$-algebra.*

2.4. **Actions on Weyl algebras.**

2.4.1. *Definition and first properties of the Weyl algebras.* We fix an integer $n \geq 1$ and a commutative base field $\mathbb{k}$. Let $S = \mathbb{k}[q_1, q_2, \ldots, q_n]$ be the commutative polynomial algebra in n variables. We denote by $\text{End}_k S$ the $\mathbb{k}$-algebra of $\mathbb{k}$-linear endomorphisms of S. The canonical embedding $\mu : S \to \text{End}_k S$ consisting in the identification of any polynomial f with the multiplication μ_f by f in S is a morphism of algebras. We consider in $\text{End}_k S$ the k-vector space $\text{Der}_k S$ consisting of the $\mathbb{k}$-derivations of S. It is a S-module with basis $(\partial_{q_1}, \partial_{q_2}, \ldots, \partial_{q_n})$, where ∂_{q_i} is the usual derivative related to q_i. Then the algebra $\text{Diff}\, S$ of differential operators on S is the subalgebra of $\text{End}_k S$ generated by $\mu_{q_1}, \ldots, \mu_{q_n}, \partial_{q_1}, \ldots, \partial_{q_n}$. This algebra $\text{Diff}\, S = \text{Diff}\, \mathbb{k}[q_1, \ldots, q_n]$ is called the n-th Weyl algebra over $\mathbb{k}$, and is denoted by $A_n(\mathbb{k})$. For all $d \in \text{Der}_k S$ and $f, h \in S$, the ordinary rule $d(fh) = d(f)h + fd(h)$ can be written $d\mu_f = \mu_f d + \mu_{d(f)}$ in $\text{End}_k S$ or, up to the identification mentioned above, $df - fd = d(f)$. Denoting by p_i the derivative ∂_{q_i}, we obtain the following formal definition of $A_n(\mathbb{k})$:

Definition. The Weyl algebra $A_n(\mathbb{k})$ is the algebra generated over $\mathbb{k}$ by $2n$ generators $q_1, \ldots, q_n$, $p_1, \ldots, p_n$ with relations:

$$[p_i, q_i] = 1, \quad [p_i, q_j] = [p_i, p_j] = [q_i, q_j] = 0 \ \text{ for } \ i \neq j, \tag{20}$$

where $[\,.\,,\,.\,]$ is the canonical commutation bracket (i.e. $[a,b]=ab-ba$ for all $a,b\in A_n(\Bbbk)$). The monomials $(q_1^{i_1}\dots q_n^{i_n}p_1^{j_1}\dots p_n^{j_n})_{(i_1,\dots,i_n,j_1,\dots,j_n)\in\mathbb{N}^{2n}}$ are a $\Bbbk$-left basis of the algebra $A_n(\Bbbk)$, which can be viewed as the iterated Ore extensions:

$$A_n(\Bbbk)=A_{n-1}(\Bbbk)[q_n][p_n\,;\,\partial_{q_n}], \tag{21}$$

$$A_n(\Bbbk)=\Bbbk[q_1,q_2,\dots,q_n][p_1\,;\,\partial_{q_1}][p_2\,;\,\partial_{q_2}]\dots[p_n\,;\,\partial_{q_n}]. \tag{22}$$

It follows in particular that the invertible elements of $A_n(\Bbbk)$ are only the nonzero scalar in $\Bbbk^\times$, and so that any nontrivial automorphism of $A_n(\Bbbk)$ is outer.

Proposition. *If $\Bbbk$ is of characteristic zero, $A_n(\Bbbk)$ is a simple noetherian domain of center $\Bbbk$.*

Proof. By 2.3.4, $A_n(\Bbbk)$ is a noetherian domain independently of the characteristic. Let $a=\sum_{i,j}a_{i,j}q_n^ip_n^j$ be any element of $A_n(\Bbbk)$, with $a_{i,j}\in A_{n-1}(\Bbbk)$. We have:

$$[p_n,a]=\sum_{i,j}i\,a_{i,j}\,q_n^{i-1}p_n^j \qquad\text{and}\qquad [a,q_n]=\sum_{i,j}j\,a_{i,j}\,q_n^ip_n^{j-1} \tag{23}$$

If a is central in $A_n(\Bbbk)$, we have $[p_n,a]=[a,q_n]=0$. Since $\Bbbk$ is of characteristic zero, we deduce from (23) that a reduces to $a_{0,0}$, and then $a\in A_{n-1}(\Bbbk)$. As a must be central in $A_{n-1}(\Bbbk)$, it follows by induction that $a\in\Bbbk$. Now consider a two-sided ideal I of $A_n(\Bbbk)$ and suppose that a is non zero in I. We must have $aq_n\in I$ and $q_na\in I$, thus $[a,q_n]\in I$. Similarly, $[p_n,a]\in I$. Applying (23), we deduce after a finite number of steps that $a_{0,0}\in I$. We repeat the process with the element $a_{0,0}$ in $A_{n-1}(\Bbbk)$, and then inductively up to obtain $1\in I$. This proves that the only two-sided ideals of $A_n(\Bbbk)$ are (0) and $A_n(\Bbbk)$. □

Proposition. *If $\Bbbk$ is of characteristic zero, then $A_n(\Bbbk)^G$ is a simple noetherian domain of center $\Bbbk$ and a finitely generated $\Bbbk$-algebra, for any finite subgroup G of* $\operatorname{Aut}A_n(\Bbbk)$.

Proof. $A_n(\Bbbk)^G$ is simple by point (ii) of proposition 2.2.2. $A_n(\Bbbk)^G$ is noetherian by point (i) of proposition 2.2.2 and observation (15) of 2.1.2. $A_n(\Bbbk)^G$ is a finitely generatd $\Bbbk$-algebra by theorem 2.3.5. Any nonzero central element of $A_n(\Bbbk)^G$ generates a two-sided principal ideal in $A_n(\Bbbk)^G$, so is invertible since $A_n(\Bbbk)^G$ is simple, and then belongs to $\Bbbk$. □

Proposition. *For any nonnegative integer m, denote by F_m the $\Bbbk$-vector space generated in $A_n(\Bbbk)$ by monomials $q_1^{i_1}\dots q_n^{i_n}p_1^{j_1}\dots p_n^{j_n}$ such that $i_1+\cdots+i_n+j_1+\cdots+j_n\le m$. Then:*

(i) *$\mathcal{B}=(F_m)_{m\in\mathbb{N}}$ is a filtration of $A_n(\Bbbk)$, called the Bernstein filtration;*
(ii) *the associated graded algebra $\operatorname{gr}(A_n(\Bbbk))$ is the commutative polynomial algebra in $2n$ variables over $\Bbbk$:*
(iii) *for any finite subgroup of G of linear automorphisms of $A_n(\Bbbk)$, the action of G induces an action on $\operatorname{gr}(A_n(\Bbbk))$, the filtration $\mathcal{B}$ induces a filtration of $A_n(\mathbb{C})^G$, and we have: $\operatorname{gr}(A_n(\Bbbk)^G)\simeq\operatorname{gr}(A_n(\Bbbk))^G$.*

Proof. It is clear that $A_n(\Bbbk)=\bigcup_{i\in\mathbb{N}}F_i$, $F_i\subset F_j$ for $i\le j$, and $F_iF_j\subset F_{i+j}$. By definition, the associated graded algebra is $T=\bigoplus_{i\ge0}T_i$ for $T_0=\Bbbk$ and $T_i=F_i/F_{i-1}$; then, by a straightforward verification, the $\overline{p}_i$'s and $\overline{q}_i$'s in T_1 generate T as a $\Bbbk$-algebra and are algebraically independent (see [22] for a detailed proof). Point (iii) is left to the reader. □

2.4.2. *Action of* SL_2 *on the Weyl algebra* $A_1(\mathbb{C})$. Here we take $n = 1$ and $\Bbbk = \mathbb{C}$. We denote simply p for p_1 and q for q_1. Thus, $A_1(\mathbb{C})$ is the algebra generated over $\mathbb{C}$ by p, q with the only relation $[p, q] = 1$.

$$A_1(\mathbb{C}) = \mathbb{C}[q][p\,;\,\partial_q] = \mathbb{C}[p][q\,;\,-\partial_p]. \tag{24}$$

Any element of $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{C})$ gives rise to a linear algebra automorphism on $A_1(\mathbb{C})$ defined by:

$$\forall\ g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2,\quad g(p) = \alpha p + \beta q\ \text{ and }\ g(q) = \gamma p + \delta q. \tag{25}$$

A subgroup of $\mathrm{Aut}\,A_1(\mathbb{C})$ is said to be linear admissible if it is the image by the canonical injection $\iota : \mathrm{SL}_2 \hookrightarrow \mathrm{Aut}\,A_1(\mathbb{C})$ of one of the five types A_{n-1}, D_n, E_6, E_7, E_8 defined in 1.3.2. We can now formulate:

Theorem.

(i) *Any finite subgroup of* $\mathrm{Aut}\,A_1(\mathbb{C})$ *is conjugate to a linear admissible subgroup.*

(ii) *If G and G' are two linear admissible subgroups of* $\mathrm{Aut}\,A_1(\mathbb{C})$, *then* $A_1(\mathbb{C})^G \simeq A_1(\mathbb{C})^{G'}$ *if and only if* $G = G'$.

Proof. It is not possible to give here a complete self contained proof of this theorem, which is based on many non trivial theorems from various papers. We indicate the structure of the main arguments and refer the interested reader to the original articles for further details. First we can naturally introduce two kinds of automorphisms of $A_1(\mathbb{C})$. The linear ones (preserving the vector space $\mathbb{C}p \oplus \mathbb{C}q$) correspond to the action (25) of SL_2. The triangular ones are of the form: $p \mapsto \alpha p + \beta, q \mapsto \alpha^{-1}q + f(p)$ with $\alpha \in \mathbb{C}^\times, \beta \in \mathbb{C}, f(p) \in \mathbb{C}[p]$, and form a subgroup denoted by J. It is proved in [24] that $\mathrm{Aut}\,A_1(\mathbb{C})$ is generated by the subgroups J and SL_2 (in fact the image of SL_2 by the canonical injection ι). More precisely, it is shown in [1] that $\mathrm{Aut}\,A_1(\mathbb{C})$ is the amalgamated free products of SL_2 and J over their intersection. Exactly as in 1.6.2, it follows by the theorem of Serre that any finite subgroup of $\mathrm{Aut}\,A_1(\mathbb{C})$ is conjugate either to a subgroup of SL_2 or to a subgroup of J. But on the same way that in proposition 1.6.1, a semi-simplicity argument proves that any finite subgroup of J is conjugate to a subgroup of linear automorphisms (see further lemma 2.4.3). Since the finite subgroups of SL_2 are classified up to conjugation in the five types $A_{n-1}, D_n, E_6, E_7, E_8$ (see 1.3.2), the point (i) follows. The separation (ii), which cannot be obtained by the standard dimensional invariants, was first proved in [9] by an original method of "reduction modulo p" (see the second additional comment below for other arguments). □

• *First additional comment: finite generation of* $A_1(\mathbb{C})^G$. By theorem 2.3.5, $A_1(\mathbb{C})^G$ is a finitely generated $\mathbb{C}$-algebra, and we can ask for explicit generators of $A_1(\mathbb{C})^G$ for any type of admissible G, similarly to the commutative case in 1.3.2.

Example: consider the action $p \mapsto \zeta p,\ q \mapsto \zeta^{-1}q$ of the cyclic group C_n on $A_1(\mathbb{C})$, with ζ a primitive n-th root of unity in $\mathbb{C}$. As in example 4 of 1.3.1, it is clear that $A_1(\mathbb{C})^{C_n}$ is generated by invariants monomial p^iq^j. For $j \geq i$, write $p^iq^j = (p^iq^i)q^{j-i}$, and observe that p^iq^i is invariant to deduce that $j-i = kn$ for some $k \geq 1$, and then $p^iq^j = (p^iq^i)q^{kn}$. Similarly, $p^iq^j = p^{kn}(p^jq^j)$ if $i > j$. We conclude with the formula:

$$p^jq^j = pq(pq+1)(pq+2)\dots(pq+j-1)$$

that $A_1(\mathbb{C})^{C_n}$ is generated by q^n, p^n and pq. This result is formally similar to the one of example 4 of 1.3.1, but we must of course take care that the generators don't commute here. More precisely we have: $pqp^n = p^n(pq-n),\ \ q^npq = (pq-n)q^n$, and

$$p^nq^n - q^np^n = \textstyle\prod_{i=1}^n(pq+i-1) - (-1)^n\prod_{i=1}^n(-pq+i).$$

We refer to [18] for the calculation of generators for each of the five types of admissible G.

• *Second additional comment: $A_1(\mathbb{C})^G$ as a deformation of the kleinian surfaces.* The linear action of the finite group G on the noncommutative algebra $A_1(\mathbb{C})$ induces canonically a linear action on the commutative graded algebra $S = \mathrm{gr}(A_1(\mathbb{C})) = \mathbb{C}[x, y]$ associated to the Bernstein filtration, which is the standard action considered in 1.3.2. We have then $\mathrm{gr}(A_1(\mathbb{C})^G) = S^G$, what allows to see the invariant algebra $A_1(\mathbb{C})^G$ as a noncommutative deformation of the kleinian surface S^G. The next step would be to see the Hochschild homology of $A_1(\mathbb{C})^G$ as a deformation of the Poisson homology of S^G (the commutative algebra S^G inherits in a natural way a Poisson algebra structure whose bracket defined from relation $\{x, y\} = 1$ is induced by the commutator bracket of $A_1(\mathbb{C})^G$). This program is initiated in [10], which proves that:

$$\dim HH_0(A_1(\mathbb{C})^G) = s(G) - 1 = \dim HP_0(S^G),$$

where $s(G)$ is the number of conjugacy classes in G. (Recall that: (1) $HH_0(A) = A/[A, A]$ where $[A, A]$ denote the $\mathbb{C}$-vector space generated by all brackets $[a, b] = ab - ba$ with $a, b \in A$); (2) $HP_0(S) = S/\{S, S\}$ where $\{S, S\}$ is the $\mathbb{C}$-vector space generated by all $\{a, b\}$ with $a, b \in S$).

2.4.3. *Action of* Sp_{2n} *on the Weyl algebra A_n.* An automorphism g of $A_n(\Bbbk)$ is linear if the $\Bbbk$-vector subspace $W = \Bbbk q_1 \oplus \cdots \oplus \Bbbk q_n \oplus \Bbbk p_1 \oplus \cdots \oplus \Bbbk p_n$ is stable under g. The restriction to W of the commutation bracket in $A_n(\mathbb{C})$ defines an alternated bilinear form and the relations (20) mean that $\mathcal{B} = (p_1, q_1, p_2, q_2 \ldots, p_n, q_n)$ is a symplectic basis of W. Then it is clear that the group of linear automorphisms of $A_n(\Bbbk)$ is isomorphic to the symplectic group $\mathrm{Sp}_{2n} = \mathrm{Sp}_{2n}(\Bbbk)$. The previous example 2.4.2 is just the case $n = 1$. For finite abelian groups of linear automorphisms and for $\Bbbk = \mathbb{C}$, the following result (from [6]) simplifies the situation in a way which is used as a key argument by many studies of this kind of actions (see [12], [11], [7], and further 3.4.2).

Proposition. *Any finite abelian subgroup of linear automorphisms of $A_n(\mathbb{C})$ is conjugated in Sp_{2n} to a subgroup of diagonal automorphisms.*

More precisely, with the above notations, for any finite abelian subgroup G of Sp_{2n}, there exist a symplectic basis $\mathcal{C} = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ of W and complex characters $\chi_1, \chi_2, \ldots, \chi_n$ of G such that:

$$g(x_j) = \chi_j(g)x_j \quad \text{and} \quad g(y_j) = \chi_j(g)^{-1}y_j, \quad \text{for all } g \in G.$$

Proof. By Schur's lemma and total reducibility, there exists a basis $\mathcal{U} = (u_1, u_2, \ldots, u_{2n})$ of W and complex characters $\varphi_1, \varphi_2, \ldots, \varphi_n$ of G such that $g(u_j) = \varphi_j(g)u_j$ for any $1 \leq j \leq 2n$. Set $\omega_{i,j} = [u_i, u_j]$ for all $1 \leq i, j \leq 2n$. Up to permutate the u_i's, one can suppose that $\omega_{1,2} \neq 0$. For any $3 \leq j \leq 2n$, let us define:

$$v_j = \omega_{1,2}u_j - \omega_{j,2}u_1 + \omega_{j,1}u_2.$$

Denote $x_1 = u_1$ and $y_1 = \omega_{1,2}^{-1}u_2$. Then $(x_1, y_1, v_3, v_4, \ldots, v_{2n})$ is a basis of W satisfying $[x_1, y_1] = 1$ and $[x_1, v_j] = [y_1, v_j] = 0$ for any $3 \leq j \leq 2n$. The action of G on this new basis can be described on the following way. It is clear that $g(x_1) = \varphi_1(g)x_1$ and $g(y_1) = \varphi_2(g)y_1$ for any $g \in G$. Since $\omega_{1,2} \neq 0$, we have $\varphi_2(g) = \varphi_1(g)^{-1}$. For $3 \leq j \leq 2n$, it follows from the definition of v_j that:

$$g(v_j) = \varphi_j(g)v_j + \omega_{j,2}\big(\varphi_j(g) - \varphi_1(g)\big)u_1 - \omega_{j,1}\big(\varphi_j(g) - \varphi_2(g)\big)u_2.$$

If $\omega_{j,2} \neq 0$, then $\varphi_j(g) = \varphi_2(g)^{-1} = \varphi_1(g)$. Similarly $\omega_{j,1} \neq 0$ implies $\varphi_j(g) = \varphi_2(g)$. Hence $g(v_j) = \varphi_j(g)v_j$ for any $3 \leq j \leq 2n$. Finally we conclude that the basis $(x_1, y_1, v_3, v_4, \ldots, v_{2n})$ of W satisfies $[x_1, y_1] = 1$ and $[x_1, v_j] = [y_1, v_j] = 0$ for any $3 \leq j \leq 2n$, and that G acts by:

$$g(x_1) = \varphi_1(g)x_1, \quad g(y_1) = \varphi_1(g)^{-1}y_1, \quad g(v_j) = \varphi_j(g)v_j \ \text{ for } 3 \leq j \leq 2n.$$

We repeat the process with the subspace generated by $v_3, \dots, v_{2n}$. As W doesn't contain any totally isotropic subspace of dimension $\geq n+1$, we can iterate this construction n times to obtain the basis $\mathcal{C}$ and the characters $\chi_1 = \varphi_1, \chi_2 = \varphi_3, \dots, \chi_n = \varphi_{2n-1}$ of the proposition. □

In order to be complete, we recall in the following lemma two classical arguments on representation theory used at the beginning of the proof.

Lemma.
(i) (Total reducibility). *Let $\rho : G \to \mathrm{GL}(V)$ a representation of a finite group G whose order doesn't divide the characteristic of $\Bbbk$, with V a finite dimensional vector space. Then $V = V_1 \oplus \dots \oplus V_m$ with V_i G-stable and irreducible (i.e. V_i doesn't admit proper and non zero G-stable subspace) for any $1 \leq i \leq m$.*
(ii) (Schur's lemma). *If $\Bbbk$ is algebraically closed and G is abelian, then any finite dimensional irreducible representation of G is of dimension one.*

Proof. Because V is finite dimensional, (i) just follows from Maschke's lemma (see 1.6.1). For (ii), consider a finite dimensional irreducible representation $\rho : G \to \mathrm{GL}(V)$ of an abelian group G. Fix $s \in G$ and set $t = \rho(s)$. For any $g \in G$, $gs = sg$ implies $\rho(g)t = t\rho(g)$. Let $\lambda \in \Bbbk^\times$ be a eigenvalue of t and denote $W = \{v \in V\,;\, t(v) = \lambda v\} \neq (0)$. For any $v \in W$, we have: $t(\rho(g)(v)) = \rho(g)(t(v)) = \rho(g)(\lambda v) = \lambda(\rho(g)(v))$ so $\rho(g)(v) \in W$. Hence W is G-stable and then $W = V$. We have proved: for all $s \in G$, there exists $\lambda \in \Bbbk^\times$ such that $\rho(s) = \lambda\, \mathrm{id}_V$. In particular any one-dimensional subspace of V in G-stable. Since V is irreducible, we conclude that V is of dimension one. □

This proposition applies in particular to the subgroup generated by one automorphism of finite order. Under this form, it appears in [12] and [11] as an ingredient for the homological study of $A_n(\mathbb{C})^G$ when G is finite not necessarily abelian (another fundamental ingredient is the Morita equivalence between $A_n(\mathbb{C})^G$ and $A_n(\mathbb{C})\#G$ by proposition 2.2.2, as $A_n(\mathbb{C})$ doesn't admit non-trivial inner automorphisms). We cannot develop here the elaborate proofs of these papers leading in particular to the following theorem, which describes very precisely the Hochschild (co)homology and Poincaré duality: *for any finite subgroup of linear automorphisms of $A_n(\mathbb{C})$, we have for all nonnegative integer j:*

$$\dim_{\mathbb{C}} HH_j(A_n(\mathbb{C})^G) = \dim_{\mathbb{C}} HH^{2n-j}(A_n(\mathbb{C})^G)) = a_j(G)$$

where $a_j(G)$ is the number of conjugacy classes of elements of G which admit the eigenvalue 1 with multiplicity j.

• *Additional comment: finite triangular automorphism groups.* Let g be an automorphism of $A_n(\Bbbk)$ and suppose that g is triangular with respect of the iterated Ore extension:

$$A_n(\Bbbk) = \Bbbk[q_1][p_1\,;\,\partial_{q_1}][q_2][p_2\,;\,\partial_{q_2}]\dots[q_n][p_n\,;\,\partial_{q_n}]. \tag{26}$$

By straightforward calculations from relations (20), we can check that g stabilizes in fact any subalgebra $\Bbbk[q_i][p_i\,;\,\partial_{q_i}] \simeq A_1(\Bbbk)$, for $1 \leq i \leq n$, acting on the generators by:

$$g(q_i) = \alpha_i q_i + \gamma_i, \quad g(p_i) = \alpha_i^{-1} p_i + f_i(q_i), \quad \text{with } \alpha_i \in \Bbbk^\times,\ \gamma_i \in \Bbbk,\ f_i \in \Bbbk[q_i]. \tag{27}$$

So, similarly to proposition 1.6.1, we have:

Lemma. *Any finite subgroup of triangular automorphisms of $A_n(\Bbbk)$ is conjugated in* $\mathrm{Aut}\,(A_n(\Bbbk))$ *to a finite abelian subgroup of diagonal automorphisms.*

Proof. Let G be a finite subgroup of triangular automorphisms of $A_n(\Bbbk)$. In each subalgebra $\Bbbk[q_i][p_i\,;\,\partial_{q_i}]$, $1 \leq i \leq n$, consider the $\Bbbk$-vector spaces $F_i = \Bbbk \oplus \Bbbk q_i$ and $E_i = \Bbbk[q_i] \oplus \Bbbk p_i$. By (27), G acts on F_i fixing $\Bbbk$ and on E_i stabilizing $\Bbbk[q_i]$. By the semi-simplicity lemma 1.6.1, there exist $y_i \in F_i$ with $F_i = \Bbbk \oplus \Bbbk y_i$

and $x_i \in E_i$ with $E_i = \Bbbk[q_i] \oplus \Bbbk x_i$ such that $\Bbbk y_i$ and $\Bbbk x_i$ are G-stable. By construction, $y_i = \alpha_i q_i + \gamma_i$ where $\alpha_i \in \Bbbk^\times$ and $\gamma_i \in \Bbbk$. Up to multiply by a nonzero scalar, we can suppose that $x_i = \alpha_i^{-1} p_i + f_i(q_i)$ with $f_i \in \Bbbk[q_i]$. Let h be the triangular automorphism of $A_n(\Bbbk)$ defined by $h(q_i) = y_i$ and $h(p_i) = x_i$ for all $1 \leq i \leq n$. Then $h^{-1}Gh$ acts diagonally on the vectors of the basis $q_1, p_1, \dots, q_n, p_n$. □

• *Remark.* As seen by previous results, some favorable situations reduce to diagonal actions, i.e. actions of subgroups of the torus $(\Bbbk^\times)^n$ by $g(q_i) = \alpha_i q_i$ and $g(p_i) = \alpha_i^{-1} p_i$ with $\alpha_i \in \Bbbk^\times$. This is the most simple case of the following construction.

2.4.4. *Dual action of* GL_n *on the Weyl algebra* A_n. We consider here the case of a linear action on $A_n(\Bbbk)$ which extends an action on the polynomial functions by the duality process of 1.4.

We start with a vector space V of finite dimension n over $\Bbbk$, $(q_1, \dots, q_n)$ a $\Bbbk$-basis of the dual V^*, $S := \Bbbk[V] \simeq S(V^*) \simeq \Bbbk[q_1, \dots, q_n]$. As in 2.4.1, we denotes by $\mathrm{End}_k S$ the $\Bbbk$-algebra of $\Bbbk$-linear endomorphisms of S, $\mu : S \to \mathrm{End}_k S$ the canonical embedding defined by the multiplication, $\mathrm{Der}_k S$ the subspace of $\mathrm{End}_k S$ consisting of the $\Bbbk$-derivations of S, and $A_n(\Bbbk) = \mathrm{Diff}\, S$ the subalgebra of $\mathrm{End}_k S$ generated by $\mu_{q_1}, \dots, \mu_{q_n}, \partial_{q_1}, \dots, \partial_{q_n}$.
Let G be a subgroup of $\mathrm{GL}_n(\Bbbk)$ acting by linear automorphisms on V, via the natural representation $\rho : G \to \mathrm{GL}(V)$. By 1.1.2, this action extends canonically in an action by automorphisms on S. Recall that the restriction of this action to the subspace $V^* = \Bbbk q_1 \oplus \Bbbk q_2 \oplus \cdots \Bbbk q_n$ just corresponds to the dual representation of ρ. Let us define the application:

$$G \times \mathrm{End}_k S \to \mathrm{End}_k S, \quad (g, \varphi) \mapsto g.\varphi := g \varphi g^{-1}. \tag{28}$$

For any $f \in S$, we have $g.\mu_f = \mu_{g(f)}$. So we obtain an action of G on $\mathrm{End}_k S$ which extends the action on S making covariant the morphism μ. We observe easily that the subspace $\mathrm{Der}_k S$ is stable under this action. We conclude that the restriction to $\mathrm{Diff}\, S$ of the action of G determines an action of G on the Weyl algebra. We claim that the restriction of this action to the vector space $U = \Bbbk\partial_{q_1} \oplus \Bbbk\partial_{q_2} \oplus \cdots \Bbbk\partial_{q_n}$ corresponds to the initial representation ρ.

Proof. For all $1 \leq i, j \leq n$ and $g \in G$, we compute

$$(g.\partial_{q_i})(q_j) = g\partial_{q_i} g^{-1}(q_j) = g\partial_{q_i}\big(\sum_{m=1}^{n} \beta_{m,j} q_m\big) = \beta_{i,j} = \partial_{q_i}(g^{-1}(q_j)) = \partial_{q_i}(g^{-1}.q_j),$$

where $(\beta_{i,j})$ denotes the matrix of g^{-1} in the basis $(q_1, \dots, q_n)$ of V^*. By (3), it follows that the action on U is dual to the action on V^*, which is itself dual of the initial action on V. □

In other words, the so-defined action of G on $A_n(\Bbbk)$ is obtained from the linear action of G on S applying the duality process exposed in 1.4. In particular, lemma 1.4.1 applies. We summarize this results in the following proposition (with the notation $p_i = \partial_{q_i}$).

Proposition. *For any subgroup G of $\mathrm{GL}_n(\Bbbk)$, the action of G by linear automorphisms on $S = \Bbbk[q_1, \dots, q_n]$ extends in an action by linear automorphisms on the Weyl algebra $A_n(\Bbbk)$ by:*

$$[g(p_i), q_j] = [p_i, g^{-1}(q_j)] \quad \text{for all } g \in G,\ 1 \leq i, j \leq n, \tag{29}$$

or equivalently

$$g(p_i) = \sum_{j=1}^{n} \partial_{q_i}(g^{-1}(q_j)) p_j \quad \text{for all } g \in G,\ 1 \leq i \leq n. \tag{30}$$

In this action, the element $w = q_1 p_1 + q_2 p_2 + \cdots + q_n p_n$ lies in $A_n(\Bbbk)^G$ for any choice of G.

• *First example: diagonal action.* The most simple situation (but interesting as we have seen before) is when G acts as a diagonal subgroup of $\mathrm{GL}_n(\Bbbk)$, and then acts on $A_n(\Bbbk)$ as a subgroup of the torus $(\Bbbk^\times)^n$ by:

$$(31) \qquad g(q_i) = \alpha_i q_i, \quad g(p_i) = \alpha_i^{-1} p_i, \quad \text{with } g = (\alpha_1, \ldots, \alpha_n) \in (\Bbbk^\times)^n.$$

Applying proposition 1.4.4, we have in particular:

$$\text{if } G = (\Bbbk^\times)^n, \text{ then } \ A_n(\Bbbk)^G = \Bbbk[q_1p_1, q_2p_2, \ldots, q_np_n].$$

If G is a finite subgroup of $(\Bbbk^\times)^n$ acting so, the invariant algebra $A_n(\Bbbk)^G$ is finitely generated over $\Bbbk$ (by theorem 2.3.5). Since every monomial in the q_i's and p_i's is an eigenvector under the action of G, it's clear that we can find a finite family of $\Bbbk$-algebra generators of $A_n(\Bbbk)^G$ constituted by invariant monomials. The case where $n = 1$ is detailed in the example of the first additional comment of 2.4.2. For $n > 1$, the determination of such a family becomes an arithmetical and combinatorial question depending on the mixing between the actions on the various copies of $A_1(\Bbbk)$ in $A_n(\Bbbk)$. We shall solve it completely at the level of the rational functions in the next section (see 3.4.2). For the moment, we only give the two following toy illustrations:

Example. For $G = \langle g \rangle$ the cyclic group of order 6 acting on $A_2(\mathbb{C})$ by:

$$g: \ p_1 \mapsto -p_1, \quad q_1 \mapsto -q_1, \quad p_2 \mapsto jp_2, \quad q_2 \mapsto j^2 q_2,$$

$A_2(\mathbb{C})^G$ is generated by $p_1^2, p_1q_1, q_1^2, p_2^3, p_2q_2, q_2^3$.

Example. For $G = \langle h \rangle$ the cyclic group of order 2 acting on $A_2(\mathbb{C})$ by:

$$h: \ p_1 \mapsto -p_1, \quad q_1 \mapsto -q_1, \quad p_2 \mapsto -p_2, \quad q_2 \mapsto -q_2,$$

$A_2(\mathbb{C})^G$ is generated by $p_1^2, p_1q_1, p_1p_2, p_1q_2, q_1^2, q_1p_2, q_1q_2, p_2^2, p_2q_2, q_2^2$.

• *Second example: differential operators over Kleinian surfaces.* We take $\Bbbk = \mathbb{C}$, $n = 1$, G a finite subgroup of SL_2 acting on $A_2(\mathbb{C})$ by:

$$(32) \qquad \forall\, g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2, \quad \begin{cases} g(q_1) = \alpha q_1 + \beta q_2, & g(p_1) = \delta p_1 - \gamma p_2, \\ g(q_2) = \gamma q_1 + \delta q_2 & g(p_2) = -\beta p_1 + \alpha p_2. \end{cases}$$

This action is the extension, following the process described at the beginning of this paragraph, of the canonical action (11) on $\mathbb{C}[q_1, q_2]$ (don't mistake with (25) corresponding to the action on $A_1(\mathbb{C})$ described in 2.4.2). Applying theorem 5 from [37] (since G doesn't contain non trivial pseudo-reflections), we have $\mathrm{Diff}(S)^G = A_2(\mathbb{C})^G \simeq \mathrm{Diff}(S^G)$, the differential operator algebra over the Kleinian surface associated to G. As an application of the main results of part 3, we will prove further in 3.4.3 that $A_2(\mathbb{C})^G$ is rationally equivalent to $A_2(\mathbb{C})$.

• *Third example: dual action of the Weyl group on a Cartan subalgebra of a semi-simple complex Lie algebra.* Let $\mathfrak{g}$ a semi-simple Lie algebra of rank ℓ over $\mathbb{C}$ and $\mathfrak{h}$ a Cartan subalgebra. The Weyl group acts by linear automorphisms on $\mathbb{C}[\mathfrak{h}^*] \simeq S(\mathfrak{h})$, and then on $\mathrm{Diff}(\mathfrak{h}^*) \simeq A_\ell(\mathbb{C})$ following the process that we described above. The interested reader could find in [12] homological results and calculations concerning this action.

2.5. **Non linear actions and polynomial automorphisms.** Of course, the questions discussed in 1.6 about invariants under subgroups of non necessarily linear automorphisms of a commutative polynomial algebra make sense for noncommutative polynomial algebras. It is not possible to give here a complete survey of the many papers devoted to the determination of such automorphism groups (see for instance the bibliographies of [1], [2], [3], [5], [24], [29],...). With the results of part 3 in mind, we focus here on the iterated Ore extension in two variables over $\mathbb{C}$, for which we have a complete answer.

2.5.1. *Examples of automorphism groups.* We have already recalled in 1.6.2 and 2.4.2 the description of the automorphism groups of the commutative ring $\mathbb{C}[x,y]$ and of the Weyl algebra $A_1(\mathbb{C})$. In both cases, the group is "rich", generated by linear and by triangular automorphisms. This is not the case for the quantum plane $\mathbb{C}_q[x,y]$ (with commutation rule $xy = qyx$, see example (iv) of 2.3.2), and for the quantum Weyl algebra $A_1^q(\mathbb{C})$ (with commutation rule $xy - qyx = 1$, see example (v) of 2.3.2), as the following proposition shows.

Proposition. *Suppose that $q \in \mathbb{C}^\times$ is not a root of one.*

(i) *The automorphism group of the quantum plane $\mathbb{C}_q[x,y]$ is isomorphic to the torus $(\mathbb{C}^\times)^2$ acting by $(\alpha,\beta) : x \mapsto \alpha x,\ y \mapsto \beta y$.*

(ii) *The automorphism group of the quantum Weyl algebra $A_1^q(\mathbb{C})$ is isomorphic to the torus $(\mathbb{C}^\times)$ acting by $\alpha : x \mapsto \alpha x,\ y \mapsto \alpha^{-1} y$.*

Proof. Assertion (i) first appeared in [2], as a particular case of more general results. We give here a short independent proof. Recall that an element is normal in $\mathbb{C}_q[x,y]$ if it generates a two-sided ideal. Let z be a normal element of $\mathbb{C}_q[x,y]$. We have in particular $zy = uz$ and $zx = vz$ for some $u, v \in \mathbb{C}_q[x,y]$. Considering $\deg_x$ in the first equality, we have $u \in \mathbb{C}[y]$ and it follows by straightforward identifications that $z = f(y)x^i$ for some nonnegative integer i and some $f \in \mathbb{C}[y]$. From the second equality $f(y)x^{i+1} = xv$, it is easy to deduce that $z = \alpha y^j x^i$ for some nonnegative integer j and some $\alpha \in \mathbb{C}$. This proves that the normal elements of $\mathbb{C}_q[x,y]$ are the monomials. Let g be an $\mathbb{C}$-automorphism of $\mathbb{C}_q[x,y]$. It preserves the set of nonzero normal elements. Hence we have $g(x) = \alpha y^j x^i$ and $g(y) = \beta y^k x^h$ with $\alpha, \beta \in \mathbb{C}^\times$ and j, i, k, h nonnegative integers satisfying $ik - hj = 1$ (traducing the relation $xy = qyx$). Writing similar formulas for g^{-1} and identifying the exponents in $g^{-1}(g(x)) = x$ and $g^{-1}(g(y)) = y$, we obtain $j = h = 0$ and $i = k = 1$.
Assertion (ii) can be proved by somewhat similar arguments (see [3] for details). □

Proposition. *Suppose that δ is an ordinary derivation of $\mathbb{C}[y]$ satisfying $\delta(y) \notin \mathbb{C}$. Let p be the non constant polynomial in $\mathbb{C}[y]$ such that $\delta = p\partial_y$. Any automorphism of $R = \mathbb{C}[y][x\,;\,\delta]$ is triangular, of the form:*

$$y \mapsto \alpha y + \beta, \quad x \mapsto \lambda x + f,$$

with $f \in \mathbb{C}[y]$, and $\alpha \in \mathbb{C}^\times$, $\lambda \in \mathbb{C}^\times$, $\beta \in \mathbb{C}$ satisfying $p(\alpha y + \beta) = \alpha\lambda p(y)$.

Proof. For any $u \in \mathbb{C}[y]$, we have $xu = ux + p\partial_y(u)$, and then $xp = p.(x + \partial_y(p))$. Thus p is normal in R. It follows that the two-sided ideal I generated by the commutators $[r,s] = rs - sr$ with $r, s \in R$ is the principal ideal generated by $p = [x,y]$. For any automorphism $g \in \operatorname{Aut} R$, the element $g(p)$ generates I. So there exists $\varepsilon \in \mathbb{C}^\times$ such that $g(p) = \varepsilon p \in \mathbb{C}[y]$. As $\deg_x g(p) = n \deg_x g(y)$ where $n = \deg_x p \geq 1$ (by assumption), we deduce that $\deg_x g(y) = 0$, therefore $g(y) \in \mathbb{C}[y]$. Hence $g(\mathbb{C}[y]) \subset \mathbb{C}[y]$, and it's clear that there exists $\alpha \in \mathbb{C}^\times, \beta \in \mathbb{C}$ such that $g(y) = \alpha y + \beta$. Then, the surjectivity of g implies that $\deg_x(g(x)) = 1$. So there exist $\lambda \in \mathbb{C}^\times, f \in \mathbb{C}[y]$ such that $g(x) = \lambda x + f$. We have $p(\alpha y + \beta) = g(p) = [g(x), g(y)] = \alpha\lambda p(y)$. □

2.5.2. Classification lemma. *Let σ be a $\mathbb{C}$-automorphism of $\mathbb{C}[y]$ and δ a σ-derivation of $\mathbb{C}[y]$. Set $R = \mathbb{C}[y][x\,;\sigma,\delta]$. Up to $\mathbb{C}$-isomorphism, we have one and only one of the following five cases.*

(i) *$R = \mathbb{C}[x, y]$ is commutative;*
(ii) *there exists some $q \in \mathbb{C}^\times$, $q \neq 1$, such that $R = \mathbb{C}_q[x, y]$;*
(iii) *there exists some $q \in \mathbb{C}^\times$, $q \neq 1$, such that $R = A_1^q(\mathbb{C})$;*
(iv) *δ is an ordinary k-derivation such that $\delta(y) \notin \mathbb{C}$ and $R = \mathbb{C}[y][x\,;\delta]$;*
(v) *$R = A_1(\mathbb{C})$.*

Hence, by the results of 2.5.1, the group $\operatorname{Aut} R$ *is explicitly known in all cases.*

Proof. There exists $q \in \mathbb{C}^\times$ and $s \in \mathbb{C}$ such that $\sigma(y) = qy+s$. If $q \neq 1$ we set $y' = y+s(q-1)^{-1}$ and obtain $R = \mathbb{C}[y'][x\,;\sigma,\delta]$ with $\sigma(y') = qy'$ and $\delta(y') = \delta(y) \in \mathbb{C}[y]$. In $\mathbb{C}[y']$ write $\delta(y') = \phi(y')(1-q)y' + r$ with $\phi(y') \in \mathbb{C}[y']$ and $r \in \mathbb{C}$. It follows that $x' = x - \phi(y')$ satisfies $x'y' - qy'x' = r$. Hence $R = \mathbb{C}[y'][x'\,;\sigma,\delta']$ with $\delta'(y') = r \in k$. If $r = 0$, then $R = \mathbb{C}_q[x', y']$. If $r \neq 0$, we set $x'' = r^{-1}x'$ and conclude that $R = A_1^q(\mathbb{C})$. Assume now that $q = 1$. If $s = 0$ then $\sigma = \mathrm{id}$ and $R = \mathbb{C}[y][x\,;\delta]$; we are in case (i) when $\delta = 0$, in case (v) when $\delta(y) \in \mathbb{C}^\times$, and in case (iv) when $\delta \neq 0$. If $s \neq 0$, we set first $y' = s^{-1}y$ to reduce to $R = k[y'][x\,;\sigma,\delta]$ with $\sigma(y') = y'+1$ and $\delta(y') = s^{-1}\delta(y)$. Then we denote $x' = x + \delta(y')$, which satisfies $x'y' = (y'+1)x'$, so that $R = \mathbb{C}[y'][x'\,;\sigma]$ is the enveloping algebra $U_1(\mathbb{C})$ introduced in example (v) of 2.3.2. We write $U_1(\mathbb{C}) = \mathbb{C}[x'][y'\,;-x'\partial_{x'}]$ and are then in case (iv). $\square$

3. Actions on rational functions

3.1. A survey on some commutative results.

3.1.1. *Extension of an action to the field of fractions.* Let S be a commutative ring. Assume that S is a domain and consider $F = \operatorname{Frac} S$ the field of fractions of S. Any automorphism of S extends into an automorphism of F and it's obvious that, for any subgroup G of $\operatorname{Aut} S$, we have $\operatorname{Frac} S^G \subseteq F^G$. For finite G, the converse is true:

Proposition. *If G is a finite subgroup of automorphisms of a commutative domain S with field of fractions F, then we have:* $\operatorname{Frac} S^G = F^G$.

Proof. For any $x \in F^G$, there exist $a, b \in S$, $b \neq 0$, such that $x = \frac{a}{b}$. Define $b' = \prod_{g\in G, g\neq \mathrm{id}_S} g(b)$. Then $bb' \in S^G$ and $x = \frac{ab'}{bb'}$, with $ab' = x(bb') \in F^G \cap S = S^G$. $\square$

This applies in particular to a polynomial algebra $S = \Bbbk[x_1, \dots, x_n]$ and its field of rational functions $F = \Bbbk(x_1, \dots, x_n)$, and we formulate in this case the following problem about the structure of F^G.

3.1.2. *Noether's problem.* Let $\Bbbk$ be commutative field of characteristic zero. Let G be a finite subgroup of $\mathrm{GL}_n(\Bbbk)$ acting by linear automorphisms on $S = \Bbbk[x_1, \dots, x_n]$ (in the sense of 1.1.2), and then on $F = \operatorname{Frac} S = \Bbbk(x_1, \dots, x_n)$. We consider the subfield $F^G = \operatorname{Frac} S^G$ of F.

Remark 1. It's well known (by Artin's lemma, see for instance [36] page 194) that $[F : F^G] = |G|$, and then $\mathrm{trdeg}_\Bbbk F^G = \mathrm{trdeg}_\Bbbk F = n$.

Remark 2. We know by theorem 1.5.3 that S^G is finitely generated (say by m elements) as a $\Bbbk$-algebra. Thus F^G is finitely generated (say by p elements) as a field extension of $\Bbbk$, with $p \leq m$. We can have $p < m$; example: $S = \Bbbk(x, y)$ and $G = \langle g \rangle$ for $g : x \mapsto -x, y \mapsto -y$, then $S^G = \Bbbk[x^2, y^2, xy] = \Bbbk[X, Y, Z]/(Z^2 - XY)$ and $F^G = \Bbbk(xy, x^{-1}y)$.

Remark 3. Suppose that S^G is not only finitely generated, but isomorphic to a polynomial algebra $\Bbbk[y_1, \ldots, y_m]$, with $y_1, \ldots, y_m$ algebraically independent over $\Bbbk$. Then we have $F^G = \Bbbk(y_1, \ldots, y_m)$. Thus $m = n$ by remark 1.

Now we can consider the main question:

Problem (Noether's problem) : is F^G a purely transcendental extension of $\Bbbk$?

An abundant literature has been devoted (and is still devoted) to this question and it's out of the question to give here a comprehensive presentation of it. We just point out the following facts.

• The answer is positive if S^G is a polynomial algebra. By remark 3, we have then $S^G = \Bbbk[x_1, \ldots, x_n]$ and $F^G = \Bbbk(x_1, \ldots, x_n)$. This is in particular the case when G is the symmetric group S_n acting by permutation of the x_j's (by theorem 1.2.1), or more generally when Shephard-Todd and Chevalley theorem applies (see the last comment after theorem 1.5.4).

• The answer is positive if $n = 1$. This is an obvious consequence of Lüroth's theorem (see [32] p. 520): if $F = \Bbbk(x)$ is a purely transcendental extension of degree 1 of $\Bbbk$, then for any intermediate subfield $\Bbbk \subsetneq L \subset F$, there exists some $v \in F$ transcendental over $\Bbbk$ such that $F = \Bbbk(v)$.

• The answer is positive if $n = 2$. This is an obvious consequence of Castelnuovo's theorem (see [32] p. 523): if $F = \Bbbk(x, y)$ is a purely transcendental extension of degree 2 of $\Bbbk$, then for any intermediate subfield $\Bbbk \subsetneq L \subset F$ such that $[F : L] < +\infty$, there exists some $v, w \in F$ such that $F = \Bbbk(v, w)$ is purely transcendental of degree 2.

• The answer is positive for all $n \geq 1$ when G is abelian and $\Bbbk$ is algebraically closed. This is a classical theorem by E. Fisher (1915), see [17] for a proof, or corollary 2 in 3.1.3 below.

Among other cases of positive results, we can cite the cases where G is any subgroup of S_n for $1 \leq n \leq 4$, the case where $G = A_5$ for $n = 5$ by Sheperd-Barron or Maeda (see [34] and [38]), the case where G is the cyclic group of order n in S_n for $1 \leq n \leq 7$ and $n = 11$.
The first counterexamples (Swan 1969, Lenstra 1974) were for $\Bbbk = \mathbb{Q}$ (and G the cyclic group of order n in S_n for $n = 47$ and $n = 8$ respectively). D. Saltmann produced in 1984 the first counter-example for $\Bbbk$ algebraically closed (see [34], [46], [47]).

3.1.3. *Miyata's theorem.* The following result concerns invariants under actions on rational functions resulting from an action on polynomials.

Theorem (T. Miyata). *Let K be a commutative field, $S = K[x]$ the commutative ring of polynomials in one variable over K, and $F = K(x)$ the field of fractions of S. Let G be a subgroup of ring automorphisms of S such that $g(K) \subseteq K$ for any $g \in G$.*

(i) *if $S^G \subseteq K$, then $F^G = S^G = K^G$.*

(ii) *if $S^G \not\subseteq K$, then for any $u \in S^G, u \notin K$ of degree $m = \min\{\deg_x y\,;\, y \in S^G, y \notin K\}$ we have $S^G = K[u]$ and $F^G = K(u)$.*

We don't give a proof of this theorem here, because we will prove it further (see 3.3) in the more general context of Ore extensions; for a self-contained proof on the commutative case, we refer the reader to [34] or [40]. Observe that the group G is not necessarily finite.

Corollary 1 (W. Burnside). *The answer to Noether's problem is positive if $n = 3$.*

Proof. Let G be a finite subgroup of $\mathrm{GL}_3(\Bbbk)$ acting linearly on $S = \Bbbk[x, y, z]$. We introduce in $F = \Bbbk(x, y, z)$ the subalgebra $S_1 = \Bbbk(\frac{y}{x}, \frac{z}{x})[x]$, which satisfies $\operatorname{Frac} S_1 = F$. Let $g \in G$. We have:

$$g(x) = \alpha x + \beta y + \gamma z, \quad g(y) = \alpha' x + \beta' y + \gamma' z, \quad g(z) = \alpha'' x + \beta'' y + \gamma'' z.$$

Thus:

$$g(\tfrac{y}{x}) = \frac{\alpha' + \beta' \frac{y}{x} + \gamma' \frac{z}{x}}{\alpha + \beta \frac{y}{x} + \gamma \frac{z}{x}} \quad \text{and} \quad g(\tfrac{z}{x}) = \frac{\alpha'' + \beta'' \frac{y}{x} + \gamma'' \frac{z}{x}}{\alpha + \beta \frac{y}{x} + \gamma \frac{z}{x}}.$$

It follows that the subfield $K = \Bbbk(\frac{z}{x}, \frac{y}{x})$ is stable under the action of G, and we can apply the theorem to the algebra $S_1 = K[x]$. The finiteness of G implies that $[F : F^G]$ is finite and so $S_1^G \not\subset K$. Thus we are in the second case of the theorem. There exists $u \in S_1^G$ of minimal degree ≥ 1 such that $S_1^G = K^G[u]$ and $F^G = K^G(u)$. By Castelnuovo's theorem (see in 3.1.2 above), $K^G = \Bbbk(v; w)$ is purely transcendental of degree two, and then $F^G = \Bbbk(v, w)(u) = \Bbbk(u, v, w)$. □

Of course, we can prove similarly that the answer to Noether's problem is positive if $n = 2$ using Lüroth's theorem instead of Castelnuovo's theorem.

Corollary 2 (E. Fischer). *If $\Bbbk$ is agebraically closed, the answer to Noether's problem is positive for G abelian.*

Proof. Here we assume that G is a finite abelian subgroup of $\mathrm{GL}_n(\Bbbk)$. By total reducibility and Schur's lemma (see 2.4.3) we can suppose up to conjugation that there exist complex characters $\chi_1, \ldots, \chi_n$ of G such that $g(x_j) = \chi_j(g)x_j$ for all $1 \leq j \leq n$ and all $g \in G$. In particular, G acts on $S_1 = \Bbbk(x_2, \ldots, x_n)[x_1]$ stabilizing $K_1 = \Bbbk(x_2, \ldots, x_n)$; thus $\Bbbk(x_1, \ldots, x_n)^G = K_1^G(u_1)$ for some $u_1 \in S_1^G$. We apply then Miyata's theorem inductively to conclude. □

Another application due to E. B. Vinberg concerns the rational finite dimensional representations of solvable connected linear algebraic groups and uses Lie-Kolchin theorem about triangulability of such representations in order to apply inductively Miyata's theorem (see [53] for more details).

3.2. **Noncommutative rational functions.**

3.2.1. *A survey on skewfields of fractions for noncommutative noetherian domains.* Let A be a ring (non necessarily commutative). Assume that A is a domain; then the set $S = \{a \in A\,; a \neq 0\}$ is multiplicative. We say that S is a (left and right) Ore set if it satisfies the two properties:

$$[\forall\ (a, s) \in A \times S,\ \exists\ (b, t) \in A \times S,\ at = sb] \ \text{ and } \ [\forall\ (a, s) \in A \times S,\ \exists\ (b', t') \in A \times S,\ t'a = b's].$$

In this case, we define an equivalence on $A \times S$ by $(a, s) \sim (b, t)$ if there exist $c, d \in A$ such that $ac = bd$ and $sc = td$. The factor set $D = (A \times S)/\sim$ is canonically equipped with a structure of skewfield (or noncommutative division ring), which is the smallest skewfield containing A. We name D the skewfield of fractions of A, denoted by $\mathrm{Frac}\, A$. Concretely, we have:

(33) $\quad \forall\ q \in \mathrm{Frac}\, A,\ [\exists\ (a, s) \in A \times S,\ q = as^{-1}]$ and $[\exists\ (b, t) \in A \times S,\ q = t^{-1}b]$,

and more generally:

(34)

$$\forall\ q_1, \ldots, q_k \in \mathrm{Frac}\, A,\ \exists\ a_1, \ldots, a_k, b_1, \ldots b_k \in A,\ \exists\ s, t \in S,\ \forall\ i \in \{1, \ldots, k\},\ q_i = a_i s^{-1} = t^{-1} b_i.$$

We refer the reader to [25], [30], [39] for more details on this standard construction. An important point is that noetherianity is a sufficient condition for A to admit such a skewfield of fractions.

Lemma. *Any noetherian domain admits a skewfield of fractions.*

Proof. Let $(a, s) \in A \times S$, $a \neq 0$, where S is the set of nonzero elements of A. For any integer $n \geq 0$, denote by I_n the left ideal generated by $a, as, as^2, \ldots, as^n$. We have $I_n \subseteq I_{n+1}$ for all $n \geq 0$. Since A is noetherian, there exists some $m \geq 0$ such that $I_m = I_{m+1}$. In particular, $as^{m+1} = c_0a + c_1as + \cdots + c_mas^m$ for some $c_0, c_1, \ldots, c_m \in A$. Denote by k the smallest index such that $c_k \neq 0$. Because A is a domain, we can simplify by s^k and write $as^{m+1-k} = c_ka + c_{k+1}as + \cdots + c_mas^{m-k}$. With $t' = c_k \in S$ and

$b' = as^{m-k} - c_{k+1}a - \cdots - c_m as^{m-k-1}$, we conclude that $t'a = b's$. So S is a left Ore set; the proof is similar on the right. □

Remark 1. Many results which are very simple for commutative fields of fractions become more difficult for skewfields. This is the case for instance of the following noncommutative analogue of proposition 3.1.1: *let R be a domain satisfying the left and right Ore conditions, let F be the skewfield of fractions of R, let G be a finite subgroup of automorphisms of R such that $|G|$ is invertible in R, then R^G satisfies the left and right Ore conditions and we have* $\operatorname{Frac} R^G = F^G$.

Sketch of the proof. We start with a preliminary observation. Let I and J be two nonzero left ideals of R. Take $a \in I, a \neq 0, s \in J, s \neq 0$. Since R satisfies the left Ore condition, there exist b', t' nonzero in R such that $t'a = b's$. This element is nonzero (since R is a domain) and lies in $I \cap J$. By induction, we prove similarly that: the intersection of any family of nonzero left ideals of R is a nonzero left ideal of R.

Now fix a nonzero element $x \in F^G$. By (33), there exist nonzero elements $b, t \in R$ such that $x = t^{-1}b$. It's clear that $I = \bigcap_{g \in G} g(Rt)$ is a left ideal of R which is stable under the action of G. Then we can apply Bergman's and Isaacs' theorem (see corollary 1.5 in [41] or original paper [15] for a proof of this nontrivial result) to deduce that I contains a nontrivial fixed point. In other words, there exists a nonzero element v in $R^G \cap I$. In particular $v \in Rt$ can be written $v = dt$ for some nonzero $d \in R$, and so $x = t^{-1}b = t^{-1}d^{-1}db = v^{-1}db$. Since $x \in F^G$ and $v \in R^G$, we have $db = vx \in F^G \cap R = R^G$. Denoting $u = db$, we have proved that: any nonzero $x \in F^G$ can be written $x = v^{-1}u$ with v and u nonzero elements of R^G.

Finally, let a, s be two nonzero elements of R^G. Then $x = st^{-1} \in F^G$. By the second step, there exist $u, v \in R^G$ such that $st^{-1} = v^{-1}u$, and then $vs = ut$. This proves that R^G satisfies the left Ore condition. The proof is similar on the right. Therefore R^G admits a skewfield of fractions and the equality $\operatorname{Frac} R^G = F^G$ is clear from the second step of the proof. □

Remark 2. There exists a noncommutative analogue of Galois theory. We cannot develop it here, but just mention the following version of Artin's lemma (see remark 1 of 3.1.2): *Let D be a skewfield and G a finite group of automorphisms of D. Then $[D : D^G] \leq |G|$. If moreover G doesn't contain any non trivial inner automorphism, then $[D : D^G] = |G|$.*

We refer the reader to [21] (theorem 3.3.7) or [41] (lemma 2.18). □

3.2.2. *Noncommutative rational functions.* Let A a ring, σ an automorphism of A, δ a σ-derivation of A, and $R = A[x\,;\,\sigma, \delta]$ the associated Ore extension. We have seen in 2.3.1 that R is a domain when A is a domain, and in 2.3.4 that R is noetherian when A is noetherian. So we conclude by the lemma of 3.2.1 that, if A is a noetherian domain, then the Ore extension $R = A[x\,;\,\sigma, \delta]$ admits a skewfield of fractions.
Denoting $K = \operatorname{Frac} A$, it's easy to check that σ and δ extend uniquely into an automorphism and a σ-derivation of K, and we can then consider the Ore extension $S = K[x\,;\,\sigma, \delta]$. It follows from (34) that any polynomial $f \in S$ can be written $f = gs^{-1} = t^{-1}h$ with s, t nonzero in A and $g, h \in R$. We deduce that $\operatorname{Frac} R = \operatorname{Frac} S$. This skewfield is denoted by $K(x\,;\,\sigma, \delta)$.

(35) If $\operatorname{Frac} A = K,\ R = A[x\,;\,\sigma, \delta],\ S = K[x\,;\,\sigma, \delta]$, then: $D = \operatorname{Frac} R = \operatorname{Frac} S = K(x\,;\,\sigma, \delta)$.

In the case of an iterated Ore extension (19) over a commutative base field $\Bbbk$, we have by induction:

if $R_m = \Bbbk[x_1][x_2\,;\,\sigma_2, \delta_2] \cdots [x_m\,;\,\sigma_m, \delta_m]$, then $\operatorname{Frac} R_m = \Bbbk(x_1)(x_2\,;\,\sigma_2, \delta_2) \cdots (x_m\,;\,\sigma_m, \delta_m)$.

We simply denote $D = K(x\,;\,\sigma)$ when $\delta = 0$ and $D = K(x\,;\,\delta)$ when $\sigma = \mathrm{id}_A$.

Remark. It's useful in many circumstances to observe (see proposition 8.7.1 of [20]) that $K(x\,;\,\sigma,\delta)$ can be embedded into the skewfield $F=K(\!(x^{-1}\,;\,\sigma^{-1},-\delta\sigma^{-1})\!)$ whose elements are the Laurent series $\sum_{j\geq m}\alpha_j x^{-j}$ with $m\in\mathbb{Z}$ and $\alpha_j\in K$, with the commutation law:

$$x^{-1}\alpha=\sum_{n\geq 1}\sigma^{-1}(-\delta\sigma^{-1})^{n-1}(\alpha)x^{-n}=\sigma^{-1}(\alpha)x^{-1}-x^{-1}\delta\sigma^{-1}(\alpha)x^{-1}\quad\text{for all }\alpha\in K.$$

Indeed, multiplying on the left and on the right by x, we obtain the commutation law of $S=K[x\,;\,\sigma,\delta]$; then S appears as a subring of F, and so D is a subfield of F.
In particular, for $\delta=0$, we denote $F=K(\!(x^{-1}\,;\,\sigma^{-1})\!)$ and just have: $x^{-1}\alpha=\sigma^{-1}(\alpha)x^{-1}$.
If $\sigma=\mathrm{id}_K$, then $F=K(\!(x^{-1}\,;\,-\delta)\!)$ is a pseudo-differential operator skewfield, with commutation law:

$$x^{-1}\alpha=\alpha x^{-1}-\delta(\alpha)x^{-2}+\cdots+(-1)^n\delta^n(\alpha)x^{-n-1}+\cdots=\alpha x^{-1}-x^{-1}\delta(\alpha)x^{-1}.$$

It follows from the embedding of D into $K(\!(x^{-1}\,;\,\sigma^{-1},-\delta\sigma^{-1})\!)$ that D is canonically equipped with the discrete valuation $v_{x^{-1}}$, or more simply v, satisfying $v(s)=-\deg s$ for all $s\in S$.

Lemma. *Let K be a skewfield, with center $Z(K)$.*

(i) *Let σ be an automorphism of K. Assume that, for all $n\geq 1$, the automorphism σ^n is not inner. Then the center $Z(D)$ of $D=K(x\,;\,\sigma)$ is the subfield $Z(K)\cap K^\sigma$, where $K^\sigma=\{a\in K\,;\,\sigma(a)=a\}$.*

(ii) *Let δ be a derivation of K. Assume that K is of characteristic zero and δ is not inner. Then the center $Z(D)$ of $D=K(x\,;\,\delta)$ is $Z(K)\cap K_\delta$, where $K_\delta=\{a\in K\,;\,\delta(a)=0\}$.*

Proof. In the embedding of $D=K(x\,;\,\sigma)$ in $F=K(\!(x^{-1}\,;\,\sigma^{-1})\!)$, any element $f\in D$ can be written $f=\sum_{j\geq m}\alpha_j x^{-j}$ with $m\in\mathbb{Z}$ and $\alpha_j\in K$ for all $j\geq m$. Assume that f is central. Then $xf=fx$ and $\alpha f=f\alpha$ for any $\alpha\in K$. This is equivalent to $\alpha_j\in K^\sigma$ and $\alpha\alpha_j=\alpha_j\sigma^{-j}(\alpha)$ for all $j\geq m$. Since σ^j is not inner, we necessarily have $\alpha_j=0$ for $j\neq 0$. This achieve the proof of (i). Under the assumptions of point (ii), let us consider now an element $f\in D=K(x\,;\,\delta)\subseteq F=K(\!(x^{-1}\,;\,-\delta)\!)$. From the relation $\alpha f=f\alpha$ for any $\alpha\in K$, we deduce using the fact that δ is not inner that $f\in K$, and so $f\in Z(K)$. Then $f\in K_\delta$ follows from the relation $fx=xf$. □

3.2.3. *Weyl skewfields.* We fix a commutative base field $\Bbbk$.

• We consider firstly as in (24) the first Weyl algebra $A_1(\Bbbk)=\Bbbk[q][p\,;\,\partial_q]=\Bbbk[p][q\,;\,-\partial_p]$. Its skewfield of fractions is named the first Weyl skewfield and is classically denoted by $D_1(\Bbbk)$:

(36) $$D_1(\Bbbk)=\operatorname{Frac}A_1(\Bbbk)=\Bbbk(q)(p\,;\,\partial_q)=\Bbbk(p)(q\,;\,-\partial_p).$$

It would be useful in many circumstances to give another presentation of $D_1(\Bbbk)$. Set $w=pq$; it follows from relation $pq-qp=1$ that $wq=qw+q$ and $pw=(w+1)p$. Thus the subalgebra of $A_1(\Bbbk)$ generated by q and w, and the subalgebra of $A_1(\Bbbk)$ generated by p and w are both isomorphic to the enveloping algebra $U_1(\Bbbk)$ defined in example (ii) of 2.3.2. It's clear that $\operatorname{Frac}A_1(\Bbbk)=\operatorname{Frac}U_1(\Bbbk)$. We conclude:

(37) $$D_1(\Bbbk)=\Bbbk(q)(w\,;\,d),\quad\text{with } d=q\partial_q \text{ the Euler derivation in } \Bbbk(q),$$

(38) $$D_1(\Bbbk)=\Bbbk(w)(p\,;\,\sigma),\quad\text{with } \sigma\in\operatorname{Aut}\Bbbk(w)\text{ defined by }\sigma(w)=w+1.$$

Applying the last lemma in 3.2.2, we obtain:

(39) $$\text{if }\ \Bbbk\text{ is of characteristic zero, then }\ Z(D_1(\Bbbk))=\Bbbk.$$

The situation where $\Bbbk$ is of characteristic $\ell > 0$ is quite different, and out of our main interest here, since $D_1(\Bbbk)$ is then of finite dimension ℓ^2 over its center $\Bbbk(p^\ell, q^\ell)$.

• We defined similarly the n-th Weyl skewfield $D_n(\Bbbk) = \operatorname{Frac} A_n(\Bbbk)$. Using (22,26), we write:

(40) $$D_n(\Bbbk) = \Bbbk(q_1, q_2, \dots, q_n)(p_1\,;\,\partial_{q_1})(p_2\,;\,\partial_{q_2})\dots(p_n\,;\,\partial_{q_n}),$$

(41) $$D_n(\Bbbk) = \Bbbk(q_1)(p_1\,;\,\partial_{q_1})(q_2)(p_2\,;\,\partial_{q_2})\dots(q_n)(p_n\,;\,\partial_{q_n}).$$

Reasoning as above on the products $w_i = p_i q_i$ for all $1 \le i \le n$, which satisfy the relations

(42) $$p_i w_i - w_i p_i = p_i, \qquad w_i q_i - q_i w_i = q_i, \qquad [p_i, w_j] = [q_i, w_j] = [w_i, w_j] = 0 \text{ si } j \neq i,$$

we obtain the alternative presentations:

(43) $$D_n(\Bbbk) = \Bbbk(q_1, q_2, \dots, q_n)(w_1\,;\,d_1)(w_2\,;\,d_2)\dots(w_n\,;\,d_n),$$

with d_i the Euler derivative $d_i = q_i\partial_{q_i}$ for all $1 \le i \le n$, and:

(44) $$D_n(\Bbbk) = \Bbbk(w_1, w_2, \dots, w_n)(p_1\,;\,\sigma_1)(p_2\,;\,\sigma_2)\cdots(p_n\,;\,\sigma_n),$$

where each automorphism σ_i is defined on $\Bbbk(w_1, w_2, \dots, w_n)$ by $\sigma_i(w_j) = w_j + \delta_{i,j}$, and fixes the p_j's for $j < i$.

• If we replace $\Bbbk$ by a purely transcendental extension $K = \Bbbk(z_1, z_2, \dots, z_t)$ of degree t of $\Bbbk$, the skewfield $D_n(K)$ is denoted by $\mathcal{D}_{n,t}(\Bbbk)$. By convention, we set $\mathcal{D}_{0,t}(\Bbbk) = K$. To sum up:

(45) $$\mathcal{D}_{n,t}(\Bbbk) = D_n(\Bbbk(z_1, \dots, z_t)) \quad \text{for all} \;\; t \ge 0, n \ge 0.$$

One can prove using inductively the last lemma of 3.2.2 (see also [31] or [7]) that:

(46) $$\text{if } \Bbbk \text{ is of characteristic zero, then } \; Z(\mathcal{D}_{n,t}(\Bbbk)) = \Bbbk(z_1, \dots, z_t).$$

The skewfields $\mathcal{D}_{n,t}$ play a fundamental role in Lie theory and are in the center of the important conjecture (the Gelfand-Kirillov conjecture) on rational equivalence of enveloping algebras (see [31], I.2.11 of [16], [13], [7], [8], [44], and 3.4 below).

• Finally, for any $q \in \Bbbk^\times$, the skewfield of fractions the quantum plane $\Bbbk_q[x, y]$ defined in example (iv) of 2.3.2 is sometimes called the first quantum Weyl skewfield, and is denoted by:

(47) $$D_1^q(\Bbbk) = \operatorname{Frac} \Bbbk_q[x, y] = \Bbbk_q(x, y) = \Bbbk(y)(x\,;\,\sigma) \quad \text{with } \; \sigma \in \operatorname{Aut} \Bbbk(y) \text{ defined by } \sigma(y) = qy.$$

These skewfields (or more generally their n-dimensional versions as in example 3 of 2.3.3) play for the quantum algebras a role similar to the one of Weyl skewfields in classical Lie theory (see II.10.4 of [16], [7], [44]). It follows from the last lemma in 3.2.2, that:

(48) $$\text{if } \; q \text{ is not a root of one in } \Bbbk, \;\; \text{then } Z(D_1^q(\Bbbk)) = \Bbbk.$$

The situation where q is of finite order $\ell > 0$ on $\Bbbk^\times$ is quite different, and out of our main interest here, since $D_1^q(\Bbbk)$ is then of finite dimension ℓ^2 over its center $\Bbbk(p^\ell, q^\ell)$.

Let us recall that the first quantum Weyl algebra (see example (v) of 2.3.2) is the algebra $A_1^q(\Bbbk)$ generated by x and y with commutation law $xy - qyx = 1$. We observe that the element $z = xy - yx = (q-1)yx + 1$ satisfies the relation $zy = qyz$. Since $x = (q-1)^{-1}y^{-1}(z-1)$, $\operatorname{Frac} A_1^q(\Bbbk)$ is equal to the subfield generated by z and y, which is clearly isomorphic to $D_1^q(\Bbbk)$. Thus we have proved that:

(49) $$\operatorname{Frac} A_1^q(\Bbbk) \simeq D_1^q(\Bbbk).$$

3.3. **Noncommutative analogue of Miyata's theorem.**

3.3.1. *The main result.* We can now formulate for Ore extensions an analogue of theorem 3.1.3. We start with a technical lemma.

Lemma. *Let K be a non necessarily commutative field, σ an automorphism and δ a σ-derivation of K. We consider the Ore extension $S = K[x\,;\,\sigma,\delta]$. Take $u \in S$ such that $\deg_x(u) \geq 1$.*

(i) *For any non necessarily commutative subfield L of K, the family $\mathcal{U} = \{u^i\,;\, i \in \mathbb{N}\}$ is right and left free over L.*

(ii) *If the left free L-module T generated by $\mathcal{U}$ is a subring of S, then there exist an ring endomorphism σ' and a σ'-derivation δ' of L such that $T = L[u\,;\,\sigma',\delta']$. If moreover T is equal to the right free L-module T' generated by $\mathcal{U}$, then σ' is an automorphism de L.*

(iii) *In the particular case where K is commutative, then σ' is the restriction of σ^m to L, with $m = \deg_x(u)$.*

Proof. Point (i) is straightforward considering the term of highest degree in left L-linear sum of a finite number of elements of $\mathcal{U}$. Consider now $\alpha \in L \subseteq T$. We have $\deg_x(u\alpha) = \deg_x u$ and $u\alpha \in T$; thus there exist uniquely determined $\alpha_0, \alpha_1 \in L$ such that $u\alpha = \alpha_0 + \alpha_1 u$. So we define two $L \to L$ maps $\sigma' : \alpha \longmapsto \alpha_1$ and $\delta' : \alpha \longmapsto \alpha_0$ satisfying $u\alpha = \sigma'(\alpha)u + \delta'(\alpha)$ for all $\alpha \in L$. Denoting $u = \lambda_m x^m + \cdots + \lambda_1 x + \lambda_0$ with $m \geq 1$, $\lambda_i \in K$ for any $0 \leq i \leq m$ and $\lambda_m \neq 0$, then $\lambda_m \sigma^m(\alpha) = \sigma'(\alpha)\lambda_m$ for all $\alpha \in L$. We deduce that σ' is a ring endomorphism of L, and proofs also point (iii). The associativity and distributivity in the ring T imply that δ' is a σ'-dérivation. When $T' = T$, there exists for all $\beta \in L$ two elements β_1 and β_0 in L such that $\beta u = u\beta_1 + \beta_0 = \sigma'(\beta_1)u + \delta'(\beta_1) + \beta_0$. Thus $\beta = \sigma'(\beta_1)$ and σ' is surjective. □

Theorem ([5]). *Let K be a non necessarily commutative field, σ an automorphism and δ a σ-derivation of K. We consider the Ore extension $S = K[x,;\,\sigma,\delta]$ and its skewfield of fractions $D = \operatorname{Frac} S = K(x\,;\,\sigma,\delta)$. Let G be a subgroup of ring automorphisms of S such that $g(K) \subseteq K$ for any $g \in G$.*

(i) *if $S^G \subseteq K$, then $D^G = S^G = K^G$.*

(ii) *if $S^G \not\subset K$, then for any $u \in S^G, u \notin K$ of degree $m = \min\{\deg_x y\,;\, y \in S^G, y \notin K\}$, there exist an automorphism σ' and a σ'-derivation δ' of K^G such that $S^G = K^G[u\,;\,\sigma',\delta']$ and $D^G = \operatorname{Frac}(S^G) = K^G(u\,;\,\sigma',\delta')$.*

Proof. We simply denote here deg for $\deg_x$. Take $g \in G$ and $n = \deg g(x)$; the assumption $g(K) \subseteq K$ implies $\deg g(s) \in n\mathbb{N} \cup \{-\infty\}$ for all $s \in S$ and so $n = 1$ since g is surjective. We deduce:

$$\deg g(s) = \deg s \text{ for all } g \in G \text{ and } s \in S. \tag{*}$$

If $S^G \subset K$, then $S^G = K^G$. If $S^G \not\subseteq K$, let us choose in $\{s \in S^G\,;\, \deg s \geq 1\}$ an element u of minimal degree m. In order to apply the previous lemma for $L = K^G$, we check that the free left K^G-module T generated by the powers of u is equal to the subring S^G of S. The inclusion $T \subseteq S^G$ is clear. For the converse, let us fixe $s \in S^G$. By the proposition in 2.3.1, there exist q_1 and r_1 unique in S such that $s = q_1 u + r_1$ and $\deg r_1 < \deg u$. For any $g \in G$, we have then: $s = g(s) = g(q_1)g(u) + g(r_1) = g(q_1)u + g(r_1)$. Since $\deg g(r_1) = \deg r_1 < \deg u$ by (*), it follows from the uniqueness of q_1 and r_1 that $g(q_1) = q_1$ and $g(r_1) = r_1$. So $r_1 \in S^G$; since $\deg r_1 < \deg u$ and $\deg u$ is minimal, we deduce that $r_1 \in K^G$. Moreover, $q_1 \in S^G$, and $\deg q_1 < \deg s$ because $\deg u \geq 1$. To sum up, we obtain $s = q_1 u + r_1$ with $r_1 \in K^G$ and $q_1 \in S^G$ such that $\deg q_1 < \deg s$. We decompose similarly q_1 into $q_1 = q_2 u + r_2$ with $r_2 \in K^G$ and $q_2 \in S^G$ such that $\deg q_2 < \deg q_1$. We obtain $s = q_2 u^2 + r_2 u + r_1$. By iteration, it follows that $s \in T$. The same process using the right euclidian division in S proves that S^G is also the right free L-module T' generated by the powers of u. Then we deduce from point (ii) of the previous lemma that there exist an automorphism σ' of K^G and a σ'-derivation δ' of de K^G such that $S^G = K^G[u\,;\,\sigma',\delta']$.

In both cases (i) and (ii), the inclusion $\mathrm{Frac}\,(S^G) \subseteq D^G$ is clear. For the converse (which follows from remark 1 of 3.2.1 in the particular case where G is finite), we have to prove that:

$$\text{for any } a \text{ and } b \text{ non-zero in } S,\ ab^{-1} \in D^G \text{ implies } ab^{-1} \in \mathrm{Frac}\,(S^G). \qquad (**)$$

We proceed by induction on $\deg a + \deg b$. If $\deg a + \deg b = 0$, then $a \in K$, $b \in K$. Thus $ab^{-1} \in D^G$ is equivalent to $ab^{-1} \in K^G \subseteq S^G$; the result follows. Assume now that (**) is satisfied for all (a, b) such that $\deg a + \deg b \leq n$, for a fixed integer $n \geq 0$. Suppose that a et b non-zero in S with $ab^{-1} \in D^G$ and $\deg a + \deg b = n + 1$. Up to replace ab^{-1} by its inverse, we can without any restriction suppose that $\deg b \leq \deg a$. By the proposition of 2.3.1, there exist $q, r \in S$ uniquely determined such that:

$$a = qb + r \quad \text{ with } \deg r < \deg b \leq \deg a. \qquad (***)$$

For all $g \in G$, we have $g(ab^{-1}) = ab^{-1}$ and we can so introduce the element $c = a^{-1}g(a) = b^{-1}g(b)$ in D. Denoting by val the discrete valuation $v_{x^{-1}}$ in D (see the remark in 3.2.2), it follows from (*) that $\mathrm{val}\, c = 0$. Applying g to (***), we have $g(a) = g(q)g(b)+g(r)$; in other words, $qbc+rc = ac = g(q)bc+g(r)$, or equivalently: $(g(q) - q)bc = rc - g(r)$. The valuation of the left member is $\mathrm{val}\,(g(q) - q) + \mathrm{val}\, b$. For the right member, we have $\mathrm{val}\, g(r) = -\deg g(r) = -\deg r = \mathrm{val}\, r = \mathrm{val}\, rc$, thus $\mathrm{val}\,(rc - g(r)) \geq \mathrm{val}\, r$. Since $g(q) - q$, b et r are in S, we conclude that: $\deg(g(q) - q) + \deg(b) \leq \deg(r)$. The inequality $\deg b \leq \deg r$ being incompatible with (***), it follows that $g(q) = q$, and then $g(r) = rc$. Therefore we have $g(rb^{-1}) = rc(bc)^{-1} = rb^{-1}$. So we have proved that $ab^{-1} = (qb + r)b^{-1} = q + rb^{-1}$ with $q \in S^G$ et $rb^{-1} \in D^G$ such that $\deg(r)+\deg(b) < 2\deg(b) \leq \deg(a)+\deg(b) = n+1$. If $r = 0$, then $ab^{-1} = q \in S^G$. If not, we apply the inductive assumption to rb^{-1}: there exist r_1 and b_1 non zero in S^G such that que $rb^{-1} = r_1 b_1^{-1}$, and so $ab^{-1} = (qb_1 + r_1)b_1^{-1} \in \mathrm{Frac}\,(S^G)$. □

3.3.2. *Application to the rational invariants of the first Weyl algebra.* We consider here the action of finite subgroups of automorphisms of the Weyl algebra $A_1(\mathbb{C})$ on its skewfield of fractions $D_1(\mathbb{C})$. We know from theorem 2.4.2 that the algebras $A_1(\mathbb{C})^G$ and $A_1(\mathbb{C})^{G'}$ are not isomorphic when the finite subgroups G and G' are not isomorphic. However, these algebras are always rationally equivalent, as proved by the following theorem from [5].

Theorem. *For any finite subgroup G of* $\mathrm{Aut}\, A_1(\mathbb{C})$, *we have:* $D_1(\mathbb{C})^G \simeq D_1(\mathbb{C})$.

Proof. With the notations of 2.4.2 and 3.2.3, we have $R = A_1(\mathbb{C})$ generated by p and q with $pq - qp = 1$ and $D = D_1(\mathbb{C}) = \mathrm{Frac}\, R$. The element $w = pq$ of R satisfies $p^m w - wp^m = mp^m$ for all $m \geq 1$. The field of fractions of the subalgebra U_m of R generated by p^m and w is $Q_m = \mathbb{C}(w)(p^m\,;\,\sigma^m)$, where σ is the $\mathbb{C}$-automorphism of $\mathbb{C}(w)$ defined by $\sigma(w) = w + 1$. In particular, $Q_1 = \mathbb{C}(w)(p\,;\,\sigma) = D$. It's clear $Q_m \simeq D$ for all $m \geq 1$. Let us define $v = p^{-1}q$, which satisfies $wv - vw = 2v$. Since $wv^{-1} = p^2$, we have $Q_2 = \mathbb{C}(w)(p^2\,;\,\sigma^2) = \mathbb{C}(v)(w\,;\,2v\partial_v)$. We denote by S the subalgebra $\mathbb{C}(v)[w\,;\,2v\partial_v]$.
Let G be a finite subgroup of $\mathrm{Aut}\, R$. From theorem 2.4.2, we can suppose without any restriction that G is linear admissible. In the cyclic case of order n, the group G is generated by the automorphism $g_n : p \mapsto \zeta_n p, q \mapsto \zeta_n^{-1} q$ for ζ_n a primitive n-th root of one. Then we have: $g_n(w) = w$, therefore $D^G = D^{g_n} = Q_1^{g_n} = Q_n \simeq D$. Assume now that we are in one of the cases D_n, E_6, E_7, E_8. Thus G necessarily contains the involution $e : p \mapsto -p, q \mapsto -q$ (because $\mu^2 = \nu^2 = \theta_2$ with the notations of 1.3.2), which satisfies $D^e = Q_2$. Let g be any element of G. By (25), there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ satisfying $\alpha\delta - \beta\gamma = 1$ such that $g(p) = \alpha p + \beta q$ et $g(q) = \gamma p + \delta q$. Thus $g(p) = p(\alpha + \beta v)$ and $g(q) = p(\gamma + \delta v)$, and so:

$$g(v) = \frac{\gamma + \delta v}{\alpha + \beta v} \in \mathbb{C}(v). \qquad (\dagger)$$

Moreover, $g(w) = \alpha\gamma p^2 + \beta\delta q^2 + \alpha\delta pq + \beta\gamma qp$. From relations $qp = pq - 1$, $p^2 = wv^{-1} = v^{-1}w - 2v^{-1}$ and $q^2 = v + vw = wv - v$, it follows that:

$$g(w) = \Big(\frac{\beta\delta v^2 + (\alpha\delta + \beta\gamma)v + \alpha\gamma}{v}\Big)w + \Big(\frac{\beta\delta v^2 - \beta\gamma v - 2\alpha\gamma}{v}\Big). \qquad (\ddagger)$$

We deduce from (†) and (‡) that the restrictions to the algebra $S = \mathbb{C}(v)[w\,;\,2v\partial_v]$ of the extensions to D of the elements of G determine a subgroup $G' \simeq G/(e)$ of $\operatorname{Aut} S$. Since $e \in G$ and $D^e = Q_2 = \operatorname{Frac} S$, we have $D^G = Q_2^{G'}$.
Assertion (†) allows to apply theorem 3.3.1 for $K = \mathbb{C}(v)$, $d = 2v\partial_v$ and $S = K[w\,;\,d]$. By remark 2 of 3.2.1, we have: $[Q_2 : Q_2^{G'}] \leq |G'| < +\infty$, therefore $S^{G'} \not\subseteq K$. From the theorem of 3.3.1 and point (iii) of the lemma of 3.3.1, there exists $u \in S^{G'}$ of positive degree (related to w) and d' a derivation of $\mathbb{C}(v)^{G'}$ such that $S^{G'} = \mathbb{C}(v)^{G'}[u\,;\,d']$ and $Q_2^{G'} = \mathbb{C}(v)^{G'}(u\,;\,d')$. By Lüroth theorem (see 3.1.2), $\mathbb{C}(v)^{G'}$ is a purely transcendental extension $\mathbb{C}(z)$ de $\mathbb{C}$. If d' vanishes on $\mathbb{C}(z)$, then the subfield $Q_2^{G'}$ of Q_2 would be $\mathbb{C}(z, u)$ with transcendence degree > 1 over $\mathbb{C}$, which is impossible since $Q_2 \simeq D_1(\mathbb{C})$ (it's a well known but not trivial result that $D_1(\mathbb{C})$ does'nt contain commutative subfields of transcendence degree > 1 ; see [39], corollary 6.6.18). Therefore $d'(z) \neq 0$; defining $t = d'(z)^{-1}u$, we obtain $Q_2^{G'} = \mathbb{C}(z)(t\,;\,\partial_z) \simeq D_1(\mathbb{C})$. □

Example 1. In the case where $G = C_n$ is cyclic of order n, we have seen in the proof that $D^G = Q_n$ is generated by $w = pq$ and p^n; then a pair (p_n, q_n) of generators of $D_1(\mathbb{C})^{C_n}$ satisfying $[p_n, q_n] = 1$ is $p_n = p^n$ et $q_n = (np^n)^{-1}pq$.

Example 2. In the case where $G = D_n$ is binary dihedral of order $4n$ (see 1.3.2), the interested reader could find in [5] the calculation of the following pair (p_n, q_n) of generators of $D_1(\mathbb{C})^{D_n}$ satisfying $[p_n, q_n] = 1$:

$$p_n = \frac{1}{16n}\big((p^{-1}q)^{-n} - (p^{-1}q)^n\big)\Big(\frac{(p^{-1}q)^n - 1}{(p^{-1}q)^n + 1}\Big)^2(2pq - 1), \qquad q_n = \Big(\frac{(p^{-1}q)^n + 1}{(p^{-1}q)^n - 1}\Big)^2.$$

3.3.3. *Application to the rational invariants of polynomial functions in two variables.* We consider $R = \mathbb{C}[y][x\,;\,\sigma, \delta]$, for σ a $\mathbb{C}$-automorphism and δ a σ-derivation of $\mathbb{C}[y]$.

Lemma. *If $R = \mathbb{C}[y][x\,;\,\delta]$, with δ an ordinary derivation of $\mathbb{C}[y]$ such that $\delta(y) \notin \mathbb{C}$, then* $\operatorname{Frac} R^G \simeq D_1(\mathbb{C})$ *for any finite subgroup of* $\operatorname{Aut} R$.

Proof. Let us denote $K = \mathbb{C}(y)$ and $D = \operatorname{Frac} R = \mathbb{C}(y)(x\,;\,\delta)$. Replacing x by $x' = \delta(y)^{-1}x$, we have $D = \mathbb{C}(y)(x'\,;\,\partial_y)$,and so $D \simeq D_1(\mathbb{C})$. Since $\delta(y) \notin \mathbb{C}$, the second proposition of 2.5.1 implies that any $g \in \operatorname{Aut} R$ satisfies $g(K) \subseteq K$ for $K = \mathbb{C}(y)$, and the restriction of g to $S = \mathbb{C}(y)[x\,;\,\delta]$ of the extension to $D = \operatorname{Frac} S$ determines an automorphism of S. For G a finite subgroup of $\operatorname{Aut} R$ we can apply the theorem of 3.3.1 and point (iii) of the lemma of 3.3.1: there exist $u \in S^G$ of positive degree and δ' a derivation of $\mathbb{C}(y)^G$ such that $S^G = \mathbb{C}(y)^G[u\,;\,\delta']$ and $D^G = \mathbb{C}(y)^G(u\,;\,\delta')$. Then we achieve the proof as in the proof of the previous theorem. □

Lemma. *If R is the quantum plane $\mathbb{C}_q[x, y]$ for $q \in \mathbb{C}^\times$ not a root of one, then* $\operatorname{Frac} R^G \simeq D_1^{q'}(\mathbb{C})$ *with $q' = q^{|G|}$ for any finite subgroup G of* $\operatorname{Aut} R$.

Proof. Let G a finite group of $\operatorname{Aut} R$ where $R = \mathbb{C}_q[x, y]$. By point (i) of the first proposition of 2.5.1, there exists for any $g \in G$ a pair $(\alpha_g, \beta_g) \in \mathbb{C}^\times \times \mathbb{C}^\times$ such that $g(y) = \alpha_g y$ and $g(x) = \beta_g x$. Denote by m and m' the orders of the cyclic groups $\{\alpha_g\,;\,g \in G\}$ and $\{\beta_g\,;\,g \in G\}$ of $\mathbb{C}^\times$ respectively. In particular, $\mathbb{C}(y)^G = \mathbb{C}(y^m)$. We can apply the theorem of 3.3.1 to the extension $S = \mathbb{C}(y)[x\,;\,\sigma]$ of $R = \mathbb{C}[y][x\,;\,\sigma]$, where $\sigma(y) = qy$. We have $S^G \neq \mathbb{C}(y)^G$ because $x^{m'} \in S^G$. Let n be the minimal degree related to x of the elements of S^G of positive degree. For any $u \in S^G$ of degree n, there exist σ' and δ' such that $S^G = \mathbb{C}(y^m)[u\,;\,\sigma', \delta']$. By assertion (iii) of the lemma of 3.3.1, the automorphism σ' of $\mathbb{C}(y^m)$ is the restriction of σ^n to $\mathbb{C}(y^m)$. We show firstly that we can choose u monomial. We develop $u = a_n(y)x^n + \cdots + a_1(y)x + a_0(y)$ with $n \geq 1$, $a_i(y) \in \mathbb{C}(y)$ for all $0 \leq i \leq n$ et $a_n(y) \neq 0$. Denote by $p \in \mathbb{Z}$ the valuation (related to y) of $a_n(y)$ in the extension $\mathbb{C}(\!(y)\!)$ of $\mathbb{C}(y)$. The action of G being diagonal on $\mathbb{C}x \oplus \mathbb{C}y$, the monomial $v = y^p x^n$ lies in S^G. So we obtain $S^G = \mathbb{C}(y^m)[v\,;\,\sigma^n]$ and $D^G = \mathbb{C}(y^m)(v\,;\,\sigma^n) \simeq D_1^{q'}$ for $q' = q^{mn}$. We have to check that $mn = |G|$. Let $g \in G$ determining an inner automorphism of $D = \operatorname{Frac} R = \operatorname{Frac} S$; there exists non-zero $r \in D$ such that $g(s) = rsr^{-1}$ of all $s \in D$. Denoting by d the order of g in G, we have then r^d central in D, and so $r^d \in \mathbb{C}$ by (48). Embedding $D = \mathbb{C}(y)(x\,;\sigma)$ in $\mathbb{C}(y)(\!(x^{-1}\,;\sigma^{-1})\!)$, see remark in 3.2.2, we deduce that $r \in \mathbb{C}$ and so $g = id_R$. We have proved that any nontrivial automorphism in G is outer. Applying remark 2 of 3.2.1, it follows that $[D : D^G] = |G|$. We have:

$$D^G = \mathbb{C}(y^m)(y^p x^n\,;\sigma^n) \ \subseteq\ L = \mathbb{C}(y)(y^p x^n\,;\sigma^n) = \mathbb{C}(y)(x^n\,;\sigma^n) \subseteq\ D = \mathbb{C}(y)(x\,;\sigma).$$

Thus $[D : L] = n$ and $[L : D^G] = m$. We conclude $|G| = [D : D^G] = mn$. ☐

Lemma. *If R is the quantum Weyl algebra $A_1^q(\mathbb{C})$ for $q \in \mathbb{C}^\times$ not a root of one, then* $\operatorname{Frac} R^G \simeq D_1^{q'}(\mathbb{C})$ *with $q' = q^{|G|}$ for any finite subgroup of* $\operatorname{Aut} R$.

Proof. The proof is easier than in the case of the quantum plane and left to the reader as an exercise (use assertion (49) and point (ii) of the first proposition of 2.5.1); see proposition 3.5 of [5] for details. ☐

We are now in position to summarize in the following theorem the results on rational invariants for Ore extensions in two variables.

Theorem ([5]). *Let $R = \mathbb{C}[y][x\,;\sigma,\delta]$ with σ an automorphism and δ a σ-derivation of $\mathbb{C}[y]$. Let $D = \operatorname{Frac} R$ with center $\mathbb{C}$. Then we are in one of the two following cases:*

(i) *$D \simeq D_1(\mathbb{C})$, and $D^G \simeq D_1(\mathbb{C})$ for any finite subgroup G of* $\operatorname{Aut} R$ *;*

(ii) *there exists $q \in \mathbb{C}^\times$ not a root of one such that $D \simeq D_1^q(\mathbb{C})$, and $D^G \simeq D_1^{q^{|G|}}(\mathbb{C})$ for any finite subgroup G of* $\operatorname{Aut} R$.

Proof. We just combine the classification lemma 2.5.2 with the assertions (39) and (48) on the centers, the main theorem of 3.3.2, and the three previous lemmas. ☐

Remark. It could be relevant to underline here that previous results only concern actions on $\operatorname{Frac} R$ which extend actions on R. The question of determining D^G for other types of subgroups G of $\operatorname{Aut} D$ is another problem, and the structure of the groups $\operatorname{Aut} D_1(\mathbb{C})$ and $\operatorname{Aut} D_1^q(\mathbb{C})$ remains unknown (see [4]). In particular, we can define a notion of rational triangular automorphism related to one of the presentations (36) or (38) of the Weyl skewfield $D_1(\mathbb{C})$; the three following results are proved in [6].

1. The automorphisms of $D_1(\mathbb{C}) = \mathbb{C}(q)(p\,;\partial_q)$ which stabilizing $\mathbb{C}(q)$ are of the form:
$$\theta:\ q \mapsto \theta(q) = \tfrac{\alpha q+\beta}{\gamma q+\delta},\quad p \mapsto \theta(p) = \tfrac{1}{\partial_q(\theta(q))}p + f(q),$$
for $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ and $f(q) \in \mathbb{C}(q)$.
2. The automorphisms of $D_1(\mathbb{C}) = \mathbb{C}(pq)(p\,;\sigma)$ stabilizing $\mathbb{C}(pq)$ are of the form:
$$\theta:\ pq \mapsto \theta(pq) = pq + \alpha,\quad p \mapsto \theta(p) = f(pq)p,$$
for $\alpha \in \mathbb{C}$ and $f(pq) \in \mathbb{C}(pq)$, or are the product of such an automorphism by the involution $pq \mapsto -pq,\ p \mapsto p^{-1}$.
3. For any finite subgroup of $\operatorname{Aut} D_1(\mathbb{C})$ stabilizing one of the three subfields $\mathbb{C}(p)$, $\mathbb{C}(q)$ or $\mathbb{C}(pq)$, we have $D_1(\mathbb{C})^G \simeq D_1(\mathbb{C})$.

3.4. **Noncommutative Noether's problem and the Gelfand-Kirillov conjecture.**

3.4.1. *Formulation of the problem.* Let $\mathbb{k}$ be a field of characteristic zero. We have seen in 2.4.4 that any representation of dimension n of a group G gives rise to an action of G on the commutative polynomial algebra $S = \mathbb{k}[q_1, \dots, q_n]$, which extends canonically into an action by automorphisms on the Weyl algebra $A_n(\mathbb{k})$ defined from relations (29) or (30), and then to the Weyl skewfield $D_n(\mathbb{k})$. Following the philosophy of the Gelfand-Kirillov problem by considering the Weyl skewfields $\mathcal{D}_{n,t}(\mathbb{k})$ as significant classical noncommutative analogues of the purely transcendental extensions of $\mathbb{k}$, the following question appears as a relevant noncommutative formulation of Noether's problem.

Question: do we have $D_n(\mathbb{k})^G \simeq \mathcal{D}_{m,t}(\mathbb{k})$ for some nonnegative integers m and t ?

By somewhat specialized considerations on various noncommutative versions of the transcendence degree (which cannot be developed here), we can give the following two precisions (see [8] for the proofs):

1. if we have a positive answer to the above question, then m and t satisfy $2m + t \leq 2n$;
2. if we have a positive answer to the above question for a finite group G, then $m = n$ and $t = 0$, and so $D_n(\mathbb{k})^G \simeq D_n(\mathbb{k})$.

3.4.2. *The case of a direct summand of representations of dimension one.* The main result is the following.

Theorem. *For a representation of a group G (non necessarily finite) which is a direct summand of n representations of dimension one, there exists a unique integer $0 \leq s \leq n$ such that $D_n(\mathbb{k})^G \simeq \mathcal{D}_{n-s,s}(\mathbb{k})$.*

Proof. By (46), the integer s is no more than the transcendence degree over $\mathbb{k}$ of the center of $\mathcal{D}_{n-s,s}(\mathbb{k})$ and so is unique. Now we proceed by induction on n to prove the existence of s.

1) Assume first that $n = 1$. Then G acts on $A_1(\mathbb{k}) = \mathbb{k}[q_1][p_1\,;\,\partial_{q_1}]$ by automorphisms of the form:

$$g(q_1) = \chi_1(g)q_1, \quad g(p_1) = \chi_1(g)^{-1}p_1, \qquad \text{for all } g \in G$$

where χ_1 is a character $G \to \mathbb{k}^\times$. The element $w_1 = p_1q_1$ is invariant under G. We define in $D_1(\mathbb{k}) = \mathbb{k}(w_1)(p_1,\ \sigma_1)$, see (38), the subalgebra $S_1 = \mathbb{k}(w_1)[p_1,\ \sigma_1]$. We have $\operatorname{Frac} S_1 = D_1(\mathbb{k})$. Any $g \in G$ fixes w_1 and acts on p_1 by $g(p_1) = \chi_1(g)p_1$. We can apply the theorem of 3.3.1. If $S_1^G \subseteq \mathbb{k}(w_1)$, then $D_1(\mathbb{k})^G = S_1^G = \mathbb{k}(w_1)^G = \mathbb{k}(w_1)$; we deduce that in this case $D_1(\mathbb{k})^G \simeq \mathcal{D}_{1-s,s}(\mathbb{k})$ with $s = 1$. If $S_1^G \not\subseteq \mathbb{k}(w_1)$, then S_1^G is an Ore extension $\mathbb{k}(w_1)[u\,;\,\sigma', \delta']$ for some automorphism σ' and some σ'-derivation δ' of $\mathbb{k}(w_1)$, and some polynomial u in the variable p_1 with coefficients in $\mathbb{k}(w_1)$ such that $g(u) = u$ for all $g \in G$ and of minimal degree. Because of the form of the action of G on p_1, we can choose without any restriction $u = p_1^a$ for an integer $a \geq 1$, and so $\sigma' = \sigma_1^a$ and $\delta' = 0$. To sum up, $D_1(\mathbb{k})^G = \operatorname{Frac} S_1^G = \mathbb{k}(w_1)(p_1^a\,;\,\sigma_1^a)$. This skewfield is also generated by $x = p_1^a$ and $y = a^{-1}w_1p_1^{-a}$ which satisfy $xy - yx = 1$. We conclude that $D_1(\mathbb{k})^G \simeq D_1(\mathbb{k}) = \mathcal{D}_{1-s,s}(\mathbb{k})$ for $s = 0$.

2) Now suppose that the theorem is true for any direct summand of $n-1$ representations of dimension one of any group over any field of characteristic zero. Let us consider a direct summand of n representations of dimension one of a group G over $\mathbb{k}$. Then G acts on $A_n(\mathbb{k})$ by automorphisms of the form:

$$g(q_i) = \chi_i(g)q_i, \quad g(p_i) = \chi_i(g)^{-1}p_i, \qquad \text{for all } g \in G \text{ and } 1 \leq i \leq n,$$

where $\chi_1, \chi_2, \dots, \chi_n$ are characters $G \to \mathbb{k}^\times$. Thus, recalling the notation $w_i = p_iq_i$, we have:

$$g(w_i) = w_i, \qquad \text{for any } g \in G \text{ and any } 1 \leq i \leq n.$$

In $D_n(\mathbb{k}) = \mathbb{k}(w_1, w_2, \dots, w_n)(p_1\,;\,\sigma_1)(p_2\,;\,\sigma_2)\cdots(p_{n-1}\,;\,\sigma_{n-1})(p_n\,;\,\sigma_n)$, see (44), let us consider the subfields:

$$\begin{array}{rcl} L & = & \Bbbk(w_n), \\ K & = & \Bbbk(w_1,w_2,\dots,w_n)(p_1\,;\,\sigma_1)(p_2\,;\,\sigma_2)\cdots(p_{n-1}\,;\,\sigma_{n-1}) \\ & = & \Bbbk(w_n)(w_1,w_2,\dots,w_{n-1})(p_1\,;\,\sigma_1)(p_2\,;\,\sigma_2)\cdots(p_{n-1}\,;\,\sigma_{n-1}) \\ & \simeq & D_{n-1}(L), \end{array}$$

and the subalgebra $S_n = K[p_n\,;\,\sigma_n]$ which satisfies $\mathrm{Frac}\, S_n = D_n(\Bbbk)$. Applying the induction hypothesis to the restriction of the action of G by L-automorphisms on $A_{n-1}(L)$, there exists an integer $0 \le s \le n-1$ such that: $D_{n-1}(L)^G \simeq \mathcal{D}_{n-1-s,s}(L) \simeq \mathcal{D}_{n-(s+1),s+1}(\Bbbk)$. Since K is stable under the action of G, we can apply the theorem of 3.3.1 to the ring $S_n = K[p_n\,;\,\sigma_n]$. Two cases are possible.

First case: $S_n^G = K^G$. Then we obtain: $D_n(\Bbbk)^G = \mathrm{Frac}\,(S_n^G) = K^G \simeq D_{n-1}(L)^G \simeq \mathcal{D}_{n-(s+1),s+1}(\Bbbk)$.

Second case: there exists a polynomial $u \in S_n$ with $\deg_{p_n} u \ge 1$ such that $g(u) = u$ for all $g \in G$. Choosing u such that $\deg_{p_n} u$ is minimal, there exist an automorphism σ' and a σ'-derivation δ' of K^G such that $S_n^G = K^G[u\,;\,\sigma',\delta']$ and $D_n(\Bbbk)^G = \mathrm{Frac}\, S_n^G = K^G(u\,;\,\sigma',\delta')$.
Let us develop $u = f_m p_n^m + \cdots + f_1 p_n + f_0$ with $m \ge 1$ and $f_i \in K^G$ for all $0 \le i \le m$. In view of the action of G on p_n, it's clear that the monomial $f_m p_n^m$ is then invariant under G. Using the embedding in skewfield of Laurent series (see 3.2.2), we can develop f_m in:

$$\overline{K} = \Bbbk(w_1,w_2,\dots,w_n)((p_1^{-1}\,;\,\sigma_1^{-1}))((p_2^{-1}\,;\,\sigma_2^{-1}))\cdots((p_{n-1}^{-1}\,;\,\sigma_{n-1}^{-1})).$$

The action of G extends to $\overline{K}$ acting diagonally on the p_i's and fixing w_i's. Therefore we can choose without any restriction a monomial u:

$$u = p_1^{a_1}\dots p_n^{a_n} \text{ with } (a_1,\dots,a_n) \in \mathbb{Z}^n, \text{ et } a_n \ge 1.$$

For any $1 \le j \le n$, we have $uw_j = (w_j + a_j)u$. Let us introduce the elements:

$$w_1' = w_1 - a_n^{-1}a_1 w_n, \quad w_2' = a_n w_2 - a_n^{-1}a_2 w_n, \quad \dots, \quad w_{n-1}' = a_n w_{n-1} - a_n^{-1}a_{n-1}w_n.$$

We obtain: $w_j' u = u w_j'$ for any $1 \le j \le n-1$. Since $\sigma_i(w_j') = w_j' + \delta_{i,j}$ pour $1 \le i,j \le n-1$, the skewfield $F_{n-1} = \Bbbk(w_1',w_2',\dots,w_{n-1}')(p_1\,;\,\sigma_1)(p_2\,;\,\sigma_2)\cdots(p_{n-1}\,;\,\sigma_{n-1})$ is isomorphic to $D_{n-1}(\Bbbk)$. More precisely, F_{n-1} is the skewfield of fractions of the algebra $\Bbbk[q_1',\dots,q_{n-1}'][p_1\,;\,\partial_{q_1'}]\dots[p_{n-1}\,;\,\partial_{q_{n-1}'}]$, where $q_i' = w_i p_i^{-1}$ for any $1 \le i \le n-1$. This algebra is isomorphic to the Weyl algebra $A_{n-1}(\Bbbk)$. Applying the induction hypothesis, there exists $0 \le s \le n-1$ such that $F_{n-1}^G \simeq \mathcal{D}_{n-1-s,s}(\Bbbk)$. It's clear by definition of the w_j''s that $\Bbbk(w_n)(w_1',w_2',\dots,w_{n-1}') = \Bbbk(w_n)(w_1,w_2,\dots,w_{n-1})$; since w_n commutes with all the elements of F_{n-1}, we deduce that $K = F_{n-1}(w_n)$. The algebra $S_n^G = K^G[u\,;\,\sigma',\delta']$ can then be written $S_n^G = F_{n-1}^G(w_n)[u\,;\,\sigma',\delta']$. The generator u commutes with w_j' for any $0 \le j \le n-1$ as we have seen above, commutes with all the p_i's by definition, and satisfies with w_n the relation $uw_n = (w_n + a_n)u$. Therefore the change of variables $u' = a_n^{-1}u$ implies: $S_n^G = F_{n-1}^G(w_n)[u'\,;\,\sigma'']$, with σ'' which is the identity map on F_{n-1}^G and satisfies $\sigma''(w_n) = w_n + 1$. It follows that: $\mathrm{Frac}\, S_n^G \simeq D_1(F_{n-1}^G) \simeq D_1(\mathcal{D}_{n-1-s,s}(\Bbbk)) \simeq \mathcal{D}_{n-s,s}(\Bbbk)$. □

Corollary (Application to finite abelian groups). *We suppose here that $\Bbbk$ is algebraically closed. Then, for any finite dimensional representation of a finite abelian group G, we have $D_n(\Bbbk)^G \simeq D_n(\Bbbk)$.*

Proof. By Schur's lemma and total reducibility, any finite representation of G is a direct summand of one dimensional representations (see 2.4.3). Then the result follows from the previous theorem and remark 2 of 3.4.1. □

This result already appears in [6]. The following corollary proves in particular that for non necessarily finite groups G, all possible values of s can be obtained in the previous theorem.

Corollary (Application to the canonical action of the subgroups of a torus). *Let n be a positive integer and $\mathbb{T}_n$ be the torus $(\Bbbk^\times)^n$ acting canonically on the vector space $\Bbbk^n$. Then:*

(i) *for any subgroup G of $\mathbb{T}_n$, there exists a unique integer $0 \leq s \leq n$ such that $D_n(\Bbbk)^G \simeq \mathcal{D}_{n-s,s}(\Bbbk)$;*
(ii) *for any integer $0 \leq s \leq n$ there exists at least one subgroup G of $\mathbb{T}_n$ such that $D_n(\Bbbk)^G \simeq \mathcal{D}_{n-s,s}(\Bbbk)$;*
(iii) *in particular $s = n$ if $G = \mathbb{T}_n$, and $s = 0$ if G is finite.*

Proof. Point (i) is just the application of the previous theorem. For (ii), let us fix an integer $0 \leq s \leq n$ and consider in $\mathbb{T}_n$ the subgroup:

$$G = \{\mathrm{Diag}\,(\alpha_1, \dots, \alpha_s, 1, \dots, 1)\ ;\ (\alpha_1, \dots, \alpha_s) \in (\Bbbk^\times)^s\} \simeq \mathbb{T}_s,$$

acting by automorphisms on $A_n(\Bbbk)$:

$$\begin{array}{lll} q_i \mapsto \alpha_i q_i, & p_i \mapsto \alpha_i^{-1} p_i, & \text{pour tout } \ 1 \leq i \leq s, \\ q_i \mapsto q_i, & p_i \mapsto p_i, & \text{pour tout } \ s+1 \leq i \leq n. \end{array}$$

In the skewfield $D_n(\Bbbk) = \Bbbk(w_1, w_2, \dots, w_n)(p_1\,;\,\sigma_1)(p_2\,;\,\sigma_2)\cdots(p_n\,;\,\sigma_n)$, we introduce the subfield $K = \Bbbk(w_1, w_2, \dots, w_n)(p_{s+1}\,;\,\sigma_{s+1})(p_{s+2}\,;\,\sigma_{s+2})\cdots(p_n\,;\,\sigma_n)$. Then the subalgebra $S = K[p_1\,;\,\sigma_1]\cdots[p_s\,;\,\sigma_s]$ satisfies $\mathrm{Frac}\, S = D_n(\Bbbk)$. It's clear that K is invariant under the action of G. If $S^G \not\subset K$, we can find in particular in S^G a monomial:

$$u = v p_1^{d_1} p_2^{d_2} \cdots p_s^{d_s}, \quad v \in K,\ v \neq 0,\ d_1, \dots, d_s \in \mathbb{N},\ (d_1, \dots, d_s) \neq (0, \dots, 0),$$

then $\alpha_1^{d_1}\alpha_2^{d_2}\cdots\alpha_s^{d_s} = 1$ for all $(\alpha_1, \alpha_2, \dots, \alpha_s) \in (\Bbbk^\times)^s$, and so a contradiction. We conclude with theorem 3.3.1 that $(\mathrm{Frac}\, S)^G = S^G = K^G$, and so $D_n(\Bbbk)^G = K$. It's clear that $K \simeq \mathcal{D}_{n-s,s}(\Bbbk)$; this achieves the proof of point (ii). Point (iii) follows then from the previous corollary. □

The actions of tori $\mathbb{T}_n$ on the Weyl algebras have been studied in particular in [43].

3.4.3. *Rational invariants for the differential operators on Kleinian surfaces.* Another situation where it's possible to give a positive answer to the question of 3.4.1 is the case of a 2-dimensional representation. Using the main theorem 3.3.1 as a key argument, one can the prove (by technical developments which cannot be detailed her; see [8] for a complete proof) the following general result.

Theorem ([8]).
(i) *For any 2-dimensional representation of a group G, there exist two nonnegative integers m, t with $1 \leq m + t \leq 2$ such that $D_2(\Bbbk)^G \simeq \mathcal{D}_{m,t}(\Bbbk)$.*
(ii) *In particular, for any 2-dimensional representation of a finite group G, we have $D_2(\Bbbk)^G \simeq D_2(\Bbbk)$.*

As an application, let us consider again the canonical action (see 1.3) of a finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$ on $S = \mathbb{C}[x, y] = \mathbb{C}[V]$ for $V = \mathbb{C}^2$. The corresponding invariant algebra S^G is one of the Kleinian surfaces studied in 1.3.2. This action extends to the rational functions field $K = \mathrm{Frac}\, S = \mathbb{C}(x, y)$ and it follows from Castelnuovo or Burnside theorems (see 3.1.2 and 3.1.3) that $K^G \simeq K$. Considering the first Weyl algebra $A_1(\mathbb{C})$ as a noncommutative deformation of $\mathbb{C}[x, y]$, we have studied in 2.4.2 the action of G on $A_1(\mathbb{C})$ and the associated deformation $A_1(\mathbb{C})^G$ of the Kleinian surface S^G. The extension of the action to $\mathrm{Frac}\, A_1(\mathbb{C}) = D_1(\mathbb{C})$ has been considered in 3.3.2, and we have proved that $D_1(\mathbb{C})^G \simeq D_1(\mathbb{C})$. From another point of view, we can apply to the action of G on S the duality extension process described in 2.4.4 in order to obtain an action on $A_2(\mathbb{C})$. As explained in second example 2.4.4, the invariant algebra $A_2(\mathbb{C})^G = (\mathrm{Diff}\, S)^G$ is then isomorphic to $\mathrm{Diff}\,(S^G)$; in other words the invariants of differential operators on S are isomorphic to the differential operators on the Kleinian surface S^G (by theorem 5 of [37]). Of course the action extends to $D_2(\mathbb{C}) = \mathrm{Frac}\, A_2(\mathbb{C})$ and the following corollary follows then from point (ii) of the previous theorem.

Corollary ([8]). *Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. For the action of G on $A_2(\mathbb{C}) = \mathrm{Diff}\, S$ canonically deduced from the natural action of G on $S = \mathbb{C}[x, y]$, we have $D_2(\mathbb{C})^G \simeq D_2(\mathbb{C})$.*

The method used in [8] to prove this result allows to compute explicitly, according to each type of G in the classification of 1.3.2, some generators P_1, P_2, Q_1, Q_2 of $D_2(\mathbb{C})^G$ satisfying canonical relations $[P_1, Q_1] = [P_2, Q_2] = 1$ and $[P_i, P_j] = [Q_i, Q_j] = [P_i, Q_j] = 0$ for $i \neq j$. For instance, starting with $A_2(\mathbb{C}) = \mathbb{C}[q_1, q_2][p_1 ; \partial_{q_1}][p_1 ; \partial_{q_2}]$, a solution for the type A_{n-1} is:

First case: $n = 2p + 1$.

$$Q_1 = q_1^{2p+1} q_2^{-2p-1} \qquad Q_2 = q_1^{p+1} q_2^{-p}$$

$$P_1 = -\tfrac{p}{2p+1}\, q_1^{-2p} q_2^{2p+1} p_1 - \tfrac{p+1}{2p+1}\, q_1^{-2p-1} q_2^{2p+2} p_2 \qquad P_2 = q_1^{-p} q_2^{p} p_1 + q_1^{-p-1} q_2^{p+1} p_2$$

Second case: $n = 2p$.

$$Q_1 = q_1^{p} q_2^{-p} \qquad Q_2 = q_1 q_2$$

$$P_1 = \tfrac{1}{2p}\, q_1^{1-p} q_2^{p} p_1 - \tfrac{1}{2p}\, q_1^{-p} q_2^{p+1} p_2 \qquad P_2 = \tfrac{1}{2}(q_2^{-1} p_1 + q_1^{-1} p_2)$$

4. Actions on power series

4.1. Actions on pseudo-differential operators and related invariants.

4.1.1. *Preliminary results.* We fix R a commutative domain (related to forthcoming applications, we'll sometimes refer to R as the "ring of functions"). For any derivation d of R, the ring of formal operators in one variable t over R is by definition the Ore extension $T = R[t ; d]$ in the sense of 2.3.1. Let us recall that the elements of T are the finite sums $\sum_i a_i t^i$ where the a_i's are in R, with usual addition and noncommutative multiplication defined from the law:

$$ta = at + d(a) \qquad \text{for all } a \in R. \tag{50}$$

For any derivation δ of R, the ring $A = R[[x ; \delta]]$ of formal power series in one variable x over R is by definition the set of infinite sums $\sum_{i \geq 0} a_i x^i$ where the a_i's are in R, with usual addition and noncommutative multiplication defined from the law:

$$xa = ax + \delta(a)x^2 + \delta^2(a)x^3 + \cdots \qquad \text{for all } a \in R. \tag{51}$$

It's clear that x generates a two-sided ideal in A; the localized ring of A with respect of the powers of x is named the ring of formal pseudo-differential operators in one variable x with coefficients in R, and is denoted $B = R(\!(x ; \delta)\!)$. The elements of B are the Laurent series $\sum_{i > -\infty} a_i x^i$ where the a_i's are in R, with usual addition and noncommutative multiplication defined from (51) and

$$x^{-1}a = ax^{-1} - \delta(a) \qquad \text{for all } a \in R. \tag{52}$$

It follows from (50) and (52), and we have already observed in 3.2.2, that $T = R[x^{-1} ; -\delta]$ is a subring of $B = R(\!(x ; \delta)\!)$.

For any nonzero series $f \in B$, there exist an integer $m \in \mathbb{Z}$ and a sequence $(a_i)_{i \geq m}$ of elements of R such that $f = \sum_{i \geq m} a_i x^i$ and $a_m \neq 0$. The integer m is the valuation of f, denoted by $v_x(f)$, and the element a_m is the coefficient of lowest valuation of f, denoted by $\varphi(f)$. By convention, we set $v_x(0) = +\infty$ and $\varphi(0) = 0$. It's easy to check that $v_x : B \to \mathbb{Z}$ is a discrete valuation and that $\varphi : B \to R$ is a multiplicative map. It follows that A and B are domains. We have $A = \{f \in B\,;\, v_x(f) \geq 0\}$ and

(53) for all $f \in B$ with $v_x(f) = m \in \mathbb{Z}$, there exists $h \in A$ with $v_x(f) = 0$ s.t. $f = hx^m$.

For any integer $k \in \mathbb{Z}$, we denote $B_k = \{f \in B\,;\, v_x(f) \geq k\}$ and π_k the morphism $B_k \to R$ defined by $\pi_k(\sum_{i \geq k} a_i x^i) = a_k$. In particular $B_0 = A$.

Remarks

(i) Let $U(A)$ be the group of invertible elements of A. An element $f = \sum_{i\geq 0} a_i x^i$ of A lies in $U(A)$ if and only if $v_x(f) = 0$ and $\varphi(f) = a_0$ lies in the group $U(R)$ of invertible elements of R (although the calculations in A are twisted by δ, the proof is similar to the commutative case). It follows that an element of B is invertible in B if and only if its coefficient of lowest valuation is invertible in R.

(ii) Let $f = \sum_{i\geq 0} a_i x^i$ be an element of A with $v_x(f) = 0$ and $\varphi(f) = a_0 = 1$. Then, for any positive integer p such that $p.1 \in U(R)$, there exist $h \in A$ satisfying $v_x(h) = 0$ and $\varphi(h) = 1$ such that $f = h^p$ (the proof is a simple calculation by identification and is left to the reader).

Proposition. *We assume here that R is a field. Then:*

(i) $B = R(\!(x\,;\,\delta)\!)$ *is a skewfield, and* $B = \operatorname{Frac} A$ *where* $A = R[[x\,;\,\delta]]$*;*
(ii) $R(x^{-1}\,;\,-\delta) = \operatorname{Frac} R[x^{-1}\,;\,-\delta]$ *is a subfield of B;*
(iii) *for any $f \in B$, we have $f \in A$, or $f \neq 0$ and $f^{-1} \in A$.*

Proof. Straightforward by remark (i) and (53). $\square$

The following lemma, which will be fundamental in the following, asserts that any automorphism of B such that $\theta(R) = R$ is continuous for the x-adic topology. The arguments of the proof are somewhat similar to the ones of [4].

Lemma. *Let θ be an automorphism of $R(\!(x\,;\,\delta)\!)$ such that $\theta(R) = R$. Then $v_x(\theta(f)) = v_x(f)$ for all $f \in R(\!(x\,;\,\delta)\!)$.*

Proof. It's clear that $\theta(x) \neq 0$. Denote $s = v_x(\theta(x)) \in \mathbb{Z}$. First we prove that $s \geq 0$. Suppose that $s < 0$. We set $u = 1 + x^{-1} \in B$. Since $v_x(\theta(x)^{-1}) = -s > 0$, we have $\theta(u) = 1 + \theta(x)^{-1} \in A$. We can apply to $\theta(u)$ the remark (ii) above. For an integer $p \geq 2$ such that $p.1$ is invertible in R, there exists $f \in A$ such that $\theta(u) = f^p$. Applying the automorphism θ^{-1}, we obtain $v_x(u) = pv_x(\theta^{-1}(f))$, so a contradiction since $v_x(u) = -1$ by definition. We have proved that $s \geq 0$. In particular the restriction of θ to A is an automorphism of A.

We can write $\theta(x) = a(1+w)x^s$ with nonzero $a \in R$ and $w \in A$ such that $v_x(w) \geq 1$. Applying θ^{-1}, we obtain $x = \theta^{-1}(a)\theta^{-1}(1+w)\theta^{-1}(x)^s$, and then:

$$v_x(\theta^{-1}(a)) + v_x(\theta^{-1}(1+w)) + sv_x(\theta^{-1}(x)) = 1.$$

From the one hand, $\theta(R) = R$ implies $\theta^{-1}(R) = R$, thus $\theta^{-1}(a)$ is a nonzero element of R, and so $v_x(\theta^{-1}(a)) = 0$. From the other hand, it follows from remark (i) above that $1 + w \in U(A)$; since $U(A)$ is stable by θ^{-1} (which is an automorphism of A by the first step of the proof), we deduce that $v_x(\theta^{-1}(1+w)) = 0$. We deduce that $sv_x(\theta^{-1}(x)) = 1$. As $s \geq 0$, we conclude that $s = 1$ and the result follows. $\square$

4.1.2. *Extension of an action from functions to pseudo-differential operators.* We fix R a commutative domain of characteristic zero and δ a nonzero derivation of R. We denote by $U(R)$ the group of invertible elements in R. We consider a group Γ acting by automorphisms on R.

Definitions. We say that the action of Γ on R is δ-compatible if δ is an eigenvector for the action of Γ by conjugation on Der$R$, i.e. equivalently when the following condition is satisfied:

$$\text{for all } \theta \in \Gamma, \text{ there exists } p_\theta \in U(R), \text{ such that } \theta \circ \delta = p_\theta \, \delta \circ \theta. \tag{54}$$

It's clear that $\theta \mapsto p_\theta$ defines then an application $p : \Gamma \to U(R)$ which is multiplicative 1-cocycle for the canonical action of Γ on $U(R)$, that is which satisfies:

$$p_{\theta\theta'} = p_\theta \theta(p_{\theta'}) \qquad \text{for all } \theta, \theta' \in \Gamma. \tag{55}$$

It follows that, if we set

$$\langle\, f \,|_k\, \theta \,\rangle := p_\theta^{-k} \theta(f) \qquad \text{for all } k \in \mathbb{Z},\ \theta \in \Gamma,\ f \in R, \tag{56}$$

then the map $(\theta, f) \mapsto \langle\, f \,|_k\, \theta \,\rangle$ defines a left action $\Gamma \times R \to R$. This action is named the left action of weight k of Γ on R. The weight 0 action is just the canonical action. For the weight one action, a 1-cocycle for the weight one action is a map $r : \Gamma \to R$ which satisfies:

$$r_{\theta\theta'} = r_\theta + p_\theta^{-1} \theta(r_{\theta'}) = r_\theta + \langle\, r_{\theta'} \,|_1\, \theta \,\rangle \qquad \text{for all } \theta, \theta' \in \Gamma. \tag{57}$$

We denote by $Z^1(\Gamma, R)$ the left R^Γ-module of such 1-cocycles. For all $k \in \mathbb{Z}$, we define the additive subgroup of R of weight k invariants:

$$I_k := \{\ f \in R;\ \langle\, f \,|_k\, \theta \,\rangle = f \ \text{ for all } \theta \in \Gamma\}. \tag{58}$$

In particular, $I_0 = R^\Gamma$ and $I_k I_\ell \subseteq I_{k+\ell}$.

Theorem ([28]). *With the previous data and notations, the action of Γ on R extends into an action by automorphisms on $B = R((x\,;\,\delta))$ if and only if this action is δ-compatible, and we have then:*

$$\theta(x^{-1}) = p_\theta x^{-1} + p_\theta r_\theta \qquad \textit{for all } \theta \in \Gamma, \tag{59}$$

where $p : \Gamma \to U(R)$ is the multiplicative 1-cocycle uniquely determined by condition (54) *of δ-compatibility and $r : \Gamma \to R$ is a 1-cocycle for the weight one action arbitrarily chosen in $Z^1(\Gamma, R)$.*

Proof. Let θ be an automorphism of B such that the restriction of θ to R is an element of Γ. In particular, we have $\theta(R) = R$. We can apply the lemma of 4.1.1 to write $\theta(x^{-1}) = c_{-1}x^{-1} + c_0 + c_1 x + \cdots$, with $c_i \in R$ for any $i \geq -1$ and $c_{-1} \neq 0$. Moreover $x^{-1} \in U(B)$ implies $\theta(x^{-1}) \in U(B)$ and then $c_{-1} \in U(R)$ by remark (i) of 4.1.1. Applying θ to (52), we obtain:

$$\theta(x^{-1})\theta(a) - \theta(a)\theta(x^{-1}) = -\theta(\delta(a)) \quad \text{ for any } a \in R.$$

Since $\theta(a) \in R$, we can develop this identity:

$$[c_{-1}x^{-1}\theta(a) - \theta(a)c_{-1}x^{-1}] + [c_0\theta(a) - \theta(a)c_0] + \textstyle\sum_{j\geq 1}[c_j x^j \theta(a) - \theta(a)c_j x^j] = -\theta(\delta(a)).$$

The first term is: $c_{-1}[x^{-1}\theta(a) - \theta(a)x^{-1}] = -c_{-1}\delta(\theta(a)) \in R$. The second is zero by commutativity of R. The third is of valuation ≥ 1. So we deduce that:

$$-c_{-1}\delta(\theta(a)) = -\theta(\delta(a)) \qquad \text{and} \qquad \textstyle\sum_{j\geq 1}[c_j x^j \theta(a) - \theta(a)c_j x^j] = 0.$$

Denote $p_\theta := c_{-1}$; we have $p_\theta \in U(R)$ and the first assertion above implies that $p_\theta\delta(\theta(a)) = \theta(\delta(a))$ for all $a \in R$. Now we claim that the second assertion implies that $c_j = 0$ for all $j \geq 1$. To see that, suppose that there exists a minimal index $m \geq 1$ such that $c_m \neq 0$; then $\sum_{j\geq m}[c_j x^j \theta(a) - \theta(a)c_j x^j] = 0$ implies by identification of the coefficients of lowest valuation that $c_m m\delta(\theta(a))x^{m+1} + \cdots = 0$. Therefore

$c_m m\delta(\theta(a)) = 0$. If we choose $a \in R$ such that $\delta(a) \neq 0$, then $\theta(\delta(a)) \neq 0$; hence $\delta(\theta(a)) \neq 0$ [by the condition $p_\theta\delta(\theta(a)) = \theta(\delta(a))$ that we have proved previously], and we obtain a contradiction since R is a domain of characteristic zero and $c_m \neq 0$. We conclude that $c_j = 0$ for all $j \geq 1$.
We have finally checked that $\theta(x^{-1}) = c_{-1}x^{-1} + c_0$. We have already observed that $p_\theta = c_{-1}$ satisfies (54). Now we set $r_\theta = (c_{-1})^{-1}c_0$. We have $\theta(x^{-1}) = p_\theta x^{-1} + p_\theta r_\theta$. Relations (55) and (57) follow then from a straightforward calculation of $\theta(\theta'(x^{-1}))$.

Conversely, let us assume that the action of Γ on R is δ-compatible. Denote by p the map $\Gamma \to U(R)$ uniquely determined by (54), which satisfies necessarily (55). Let us choose a 1-cocycle $r : \Gamma \to R$ arbitrarily in $Z^1(\Gamma, R)$. We consider any $\theta \in \Gamma$; denoting $q_\theta = p_\theta r_\theta$, we calculate for all $a \in R$:

$$(p_\theta x^{-1} + q_\theta)\theta(a) - \theta(a)(p_\theta x^{-1} + q_\theta) = p_\theta(x^{-1}\theta(a) - \theta(a)x^{-1}) = -p_\theta\delta(\theta(a)) = -\theta(\delta(a)).$$

Hence we can define an automorphism θ_r of $T = R[t\,;-\delta] = R[x^{-1}\,;-\delta]$ such that the restriction of θ_r to R is θ and $\theta_r(t) = p_\theta t + p_\theta r_\theta$; (observe that $p_\theta \in U(R)$ implies the bijectivity of θ_r). Since $p_\theta \in U(R)$, the element $\theta_r(x^{-1}) = p_\theta x^{-1} + q_\theta$ is invertible in B by remark (i) of 4.1.1. Then we define: $\theta_r(x) = \theta_r(x^{-1})^{-1} = x(p_\theta + q_\theta x)^{-1}$ with $p_\theta + q_\theta x$ which is invertible in $A = R[[x\,;\delta]]$. So we have built for any $\theta \in \Gamma$ an automorphism θ_r of B which extends θ. It follows immediately from the assumptions (55) on p and (57) on r that $(\theta\theta')_r = \theta_r\theta'_r$ for all $\theta, \theta' \in \Gamma$. □

Remark. Computing $(p_\theta + q_\theta x)^{-1} = \big(\sum_{j\geq 0}(-1)^j(p_\theta^{-1}q_\theta x)^j\big)p_\theta^{-1} \in A$, we deduce that, under the hypothesis of the theorem, we have:

$$\theta(x) = x\Big(\sum_{j\geq 0}(-1)^j(r_\theta x)^j\Big)p_\theta^{-1} = p_\theta^{-1}x + \cdots \quad \text{for all } \theta \in \Gamma \tag{60}$$

In particular, the restriction to B_k of the action of Γ on B defines an action on B_k for any $k \in \mathbb{Z}$.

Corollary. *Under the assumptions of the theorem, the action of Γ on R extends into an action by automorphisms on $B = R((x\,;\delta))$ if and only if it extends into an action by automorphisms on $T = R[x^{-1}\,;-\delta]$.*

Examples. We suppose that the action of Γ on R is δ-compatible; thus the map $p : \Gamma \to U(R)$ defined by (54) is uniquely determined and satisfies (55), and we consider here various examples for the choice of $r \in Z^1(\Gamma, R)$.

1. If we take $r = 0$, the action of Γ on B is defined by $\theta(x^{-1}) = p_\theta x^{-1}$, and then $\theta(x) = xp_\theta^{-1} = \sum_{j\geq 0}\delta^j(p_\theta^{-1})x^{j+1}$ for any $\theta \in \Gamma$.
2. If r is a coboundary (i.e. there exists $f \in R$ such that: $r_\theta = \langle f \,|_1\, \theta\rangle - f = p_\theta^{-1}\theta(f) - f$ for any $\theta \in \Gamma$), then the element $y = (x^{-1} - f)^{-1}$ satisfies $B = R((x\,;\delta)) = R((y\,;\delta))$ and $\theta(y^{-1}) = p_\theta y^{-1}$ for any $\theta \in \Gamma$. Thus we find the situation of example 1.
3. We can take for r the map $\Gamma \to R$ defined by: $r_\theta = -p_\theta^{-1}\delta(p_\theta)$ for any $\theta \in \Gamma$, which is an element of $Z^1(\Gamma, R)$ by (54) and (55). The corresponding action of Γ on B is given by: $\theta(x^{-1}) = p_\theta x^{-1} - \delta(p_\theta) = x^{-1}p_\theta$ for any $\theta \in \Gamma$.
4. For any $r \in Z^1(\Gamma, G)$ and any $f \in R$, the map $\theta \mapsto r_\theta + p_\theta^{-1}\theta(f) - f$ is an element of $Z^1(\Gamma, R)$. The corresponding action of Γ on B is defined by $\theta(x^{-1}) = p_\theta x^{-1} + p_\theta r_\theta + \theta(f) - p_\theta f$. As in example 2, $y = (x^{-1} - f)^{-1}$ satisfies $B = R((x\,;\delta)) = R((y\,;\delta))$ and allows to express the action by $\theta(y^{-1}) = p_\theta y^{-1} + p_\theta r_\theta$ for any $\theta \in \Gamma$.
5. Since $Z^1(\Gamma, R)$ is a left R^Γ-module, the map κr is an element of $Z^1(\Gamma, R)$ for any $r \in Z^1(\Gamma, G)$ and any $\kappa \in R^\Gamma$. The corresponding action of Γ on B is given by: $\theta(x^{-1}) = p_\theta x^{-1} + \kappa\, p_\theta r_\theta$ for any $\theta \in \Gamma$. If we suppose moreover that $\kappa \in U(R)$, then $y = (\kappa^{-1}x^{-1})^{-1}$ satisfies $B = R((x\,;\delta)) = R((y\,;\kappa^{-1}\delta))$, and we find $\theta(y^{-1}) = p_\theta y^{-1} + p_\theta r_\theta$ for any $\theta \in \Gamma$.

4.1.3. *Invariant pseudo-differential operators.* We fix R a commutative domain, δ a nonzero derivation of R, and Γ a group acting by automorphisms on R. We suppose that the action of Γ is δ-compatible and so extends to $B = R(\!(x\,;\,\delta)\!)$ by (59) where r is an arbitrarily chosen element of $Z^1(\Gamma, R)$. We denote by $B^{\Gamma,r}$ (respectively $A^{\Gamma,r}$) the subring of invariant elements of B (respectively A) under this action.

Remarks. For any $k \in \mathbb{Z}$, we denote $B_k^{\Gamma,r} = B_k \cap B^{\Gamma,r}$. The following observations precise some relations between invariant pseudo-differential operators of valuation k (i.e. elements of $B_k^{\Gamma,r}$) and weight k invariant functions (i.e. elements of I_k, see (58)).

(i) If $B^{\Gamma,r} \neq R^\Gamma$, then there exists some nonzero integer k such that $I_k \neq \{0\}$.

Proof. Suppose that there exists $y \in B^{\Gamma,r}$ such that $y \notin R^\Gamma$. Set $k = v_x(y)$, thus $y \in B_k^{\Gamma,r}$. If $k \neq 0$, then $\pi_k(y)$ is a non zero element of I_k by remark (i). If $k = 0$, then $\pi_0(y) \in I_0 = R^\Gamma$, thus $y' = y - \pi_0(y)$ is a nonzero element of $B_{k'}^{\Gamma,r}$ for some integer $k' > 0$ and we apply the first case.

(ii) For any $k \in \mathbb{Z}$ and $y \in B$, we have: ($y \in B_k^{\Gamma,r} \Rightarrow \pi_k(y) \in I_k$); this is a straightforward consequence of (56), (58), (59) and (60). If we assume that

$$0 \longrightarrow B_{k+1}^{\Gamma,r} \xrightarrow[\text{inj}]{\text{can}} B_k^{\Gamma,r} \xrightarrow{\pi_k} I_k \longrightarrow 0$$

is a split exact sequence, then $B^{\Gamma,r} \neq R^\Gamma$ if and only if there exists some nonzero integer k such that $I_k \neq \{0\}$.

Proof. Suppose that there exists a nonzero integer k and a nonzero element f in I_k. By assumption, we can consider $\psi_k : I_k \to B_k^{\Gamma,r}$ such that $\pi_k \circ \psi_k = \mathrm{id}_{I_k}$. Then $\psi_k(f) = fx^k + \cdots$ lies in $B_k^{\Gamma,r}$ with valuation $k \neq 0$. Therefore $\psi_k(f) \notin R^\Gamma$.

The following theorem gives an explicit description of the ring $B_k^{\Gamma,r}$ when the functions ring R is a field. It can be viewed as an analogue for noncommutative power series of the theorem previously proved in 3.3.1 for noncommutative rational functions.

Theorem ([28]). *Let R be a commutative field of characteristic zero. Let δ be a nonzero derivation of R, $A = R[[x\,;\,\delta]]$ and $B = R(\!(x\,;\,\delta)\!) = \operatorname{Frac} A$. For any δ-compatible action of a group Γ on R and for any $r \in Z^1(\Gamma, R)$, we have:*

(i) *if $A^{\Gamma,r} \subseteq R$, then $A^{\Gamma,r} = B^{\Gamma,r} = R^\Gamma$;*

(ii) *if $A^{\Gamma,r} \not\subseteq R$ and $R^\Gamma \subset \ker\delta$, then there exist elements of positive valuation in $A^{\Gamma,r}$ and, for any $u \in A^{\Gamma,r}$ of valuation $e = \min\{v_x(y)\,;\, y \in A^{\Gamma,r}, v_x(y) \geq 1\}$, we have $A^{\Gamma,r} = R^\Gamma[[u]]$ and $B^{\Gamma,r} = \operatorname{Frac}(A^{\Gamma,r}) = R^\Gamma(\!(u)\!)$;*

(iii) *if $A^{\Gamma,r} \not\subseteq R$ and $R^\Gamma \not\subset \ker\delta$, then there exists an element u of valuation 1 in $A^{\Gamma,r}$ and a nonzero derivation δ' of R^Γ such that $A^{\Gamma,r} = R^\Gamma[[u\,;\,\delta']]$ and $B^{\Gamma,r} = \operatorname{Frac}(A^{\Gamma,r}) = R^\Gamma(\!(u\,;\,\delta')\!)$.*

The proof of this theorem is somewhat long and technical and cannot take place here. It uses in an essential way the notion of higher derivation and related results (see [27] for a survey).

Some comments.

1. In point (iii) of the theorem, $\delta' = c_1^{-1}\delta$ where $u = c_1x + c_2x^2 + \cdots$ with $c_i \in R$, $c_1 \neq 0$.

2. The equality $\mathrm{Frac}\,(A^{\Gamma,r}) = (\mathrm{Frac}\,A)^{\Gamma,r}$, which can be nontrivial in some cases (see the proof of 3.3.1 and remark 1 in 3.2.1) follows here immediately from point (iii) of the proposition in 4.1.1.

3. Under the assumptions of the theorem, if r and r' are two 1-cocycles in $Z^1(\Gamma, R)$ such that $B^{\Gamma,r} \not\subseteq R$ and $B^{\Gamma,r'} \not\subseteq R$, then $B^{\Gamma,r} \simeq B^{\Gamma,r'}$.

4. Under the assumptions of the theorem, if the exact sequence of remark (ii) is split for r and r' two 1-cocycles in $Z^1(\Gamma, R)$, then $B^{\Gamma,r} \simeq B^{\Gamma,r'}$.

5. If we don't assume that R is a field, we don't have a general theorem, but some particular results can be useful for further arithmetical applications. In particular it is proved in [28] that: *if there exists in $B^{\Gamma,r}$ an element $w = bx^{-1} + c$ with $b \in U(R)$ and $c \in R$, then the derivation $D = b\delta$ restricts into a derivation of R^Γ, and we have then $A^{\Gamma,r} = R^\Gamma[[u\,;\,D]]$ and $B^{\Gamma,r} = R^\Gamma(\!(u\,;\,D)\!)$ for $u = w^{-1}$.*

4.1.4. *Application to the first Weyl local skewfield.* We take here $R = \mathbb{C}(z)$ and $\delta = \partial_z$. We consider the ring $A = R[[x\,;\,\delta]]$ and its skewfield of fractions $F = R(\!(x\,;\,\delta)\!)$. The skewfield $Q = R(t\,;\,d)$ where $t = x^{-1}$ and $d = -\delta$ is a subfield of F (see point (ii) of proposition 4.1.1) which is clearly isomorphic to the Weyl skewfield $D_1(\mathbb{C})$ (see 3.2.3). We have:

$$xz - zx = x^2, \quad \text{or equivalently} \quad zt - tz = 1.$$

We name F the first local skewfield. It's well known that any $\mathbb{C}$-automorphism θ of R is of the form $z \mapsto \frac{az+b}{cz+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C})$. For any $f(z) \in R$, we compute:

$$\partial_z(\theta(f)) = \partial_z(f(\tfrac{az+b}{cz+d})) = \tfrac{ad-bc}{(cz+d)^2} f'(\tfrac{az+b}{cz+d}) = \tfrac{ad-bc}{(cz+d)^2}\,\theta(\partial_z(f)).$$

By (54), it follows that the action of any $\theta \in \mathrm{Aut}\,R$ is δ-compatible, with $p_\theta = \frac{(cz+d)^2}{ad-bc}$. We conclude with the theorem of 4.1.2 that any automorphism θ of F which restricts into an automorphism of R is of the form:

$$\theta \;:\; z \mapsto \tfrac{az+b}{cz+d}, \quad x^{-1} \mapsto \tfrac{(cz+d)^2}{ad-bc} x^{-1} + q_\theta(z).$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C})$ and $q_\theta(z) \in \mathbb{C}(z)$. Then, using remark 2 of 3.2.1, we can prove that point (iii) of the theorem of 4.1.3 applies and it's easy to deduce with Lüroth's theorem that:

Proposition. *For any finite subgroup Γ of $\mathbb{C}$-automorphisms of $F = \mathbb{C}(z)(\!(x\,;\,\partial_z)\!)$ stabilizing $\mathbb{C}(z)$, we have $F^\Gamma \simeq F$.*

4.2. **Applications to modular forms.** In order to give an overview about some applications of the previous results in number theory, we fix the following data and notations.

4.2.1. *Data and notations.* In the following, Γ is a subgroup of $\mathrm{SL}(2,\mathbb{C})$, and R is a commutative $\mathbb{C}$-algebra R of functions in one variable z such that:

(i) Γ acts (on the right) by homographic automorphisms on R

$$(f\,|_0\,\gamma) = f(\tfrac{az+b}{cz+d}) \quad \text{for all } f \in R \text{ and } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

(ii) the function $z \mapsto cz + d$ is invertible in R for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

(iii) R is stable by the derivation ∂_z.

The case where $R = \mathbb{C}(z)$ corresponds to the formal situation studied in 4.1.4. In many arithmetical situations, R is some particular subalgebra of $\mathcal{F}_{\rm der}(\Delta, \mathbb{C})$ with $\Delta \subseteq \mathbb{C}$ stable by the homographic action of a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$. We denote:

$$(f\,|_k\,\gamma)(z) = (cz+d)^{-k} f(\tfrac{az+b}{cz+d}) \qquad \text{for all } f \in R,\ \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma,\ k \in \mathbb{Z}. \tag{61}$$

Let us observe that $(\,(f\,|_k\,\gamma')\,|_k\,\gamma\,) = (f\,|_k\,\gamma'\gamma)$ for all $\gamma, \gamma' \in \Gamma$ and $f \in R$. For any $k \in \mathbb{Z}$, we define the $\mathbb{C}$-vector space of weight k modular forms:

$$M_k(\Gamma, R) = \{\ f \in R\,;\ (f\,|_k\,\gamma) = f \text{ for all } \gamma \in \Gamma\ \} \tag{62}$$

Remarks.

1. $M_0(\Gamma, R) = R^\Gamma$.
2. If $\Gamma \ni (\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix})$, then $M_k(\Gamma, R) = (0)$ for any odd k.
3. If Γ contains at least one element $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ such that $(c, d) \notin \{0\} \times \mathbb{U}_\infty$, we have $M_k(\Gamma, R) \cap M_\ell(\Gamma, R) = (0)$ pour $k \neq \ell$.
4. For all $f \in M_k(\Gamma, R)$ and $g \in M_\ell(\Gamma, R)$, we have $fg \in M_{k+\ell}(\Gamma, R)$.
5. For any $f \in M_k(\Gamma, R)$, the function $f' = \partial_z(f)$ satisfies $(f'\,|_{k+2}\,\gamma)(z) = f'(z) + k\frac{c}{cz+d} f(z)$. Thus f' is not necessarily a modular form (unless for $k = 0$).

Comment: Rankin-Cohen brackets (see [19]). It follows from remark 5 above that, for $f \in M_k(\Gamma, R)$ and $g \in M_\ell(\Gamma, R)$, and r, s nonnegative integers, the product $f^{(r)} g^{(s)}$ is not necessarily an element of $M_{k+\ell+2r+2s}(\Gamma, R)$. For any integer $n \geq 0$, we denote by $[\ ,\ ]_n$ the n-th Rankin-Cohen bracket, defined as the linear combination:

$$\begin{aligned}
&[f, g]_0 = fg, \\
&[f, g]_1 = kfg' - \ell f'g, \\
&[f, g]_2 = k(k+1)fg'' - (k+1)(\ell+1)f'g' + \ell(\ell+1)f''g, \\
&\ldots \\
&[f, g]_n = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r} \binom{\ell+n-1}{r} f^{(r)} g^{(n-r)},
\end{aligned}$$

and satisfies the characteristic property:

$$\text{for } f \in M_k(\Gamma, R) \text{ and } g \in M_\ell(\Gamma, R), \text{ we have } [f, g]_n \in M_{k+\ell+2n}(\Gamma, R).$$

(In fact it is possible to prove that any linear combination of $f^{(r)} g^{(s)}$ satisfying this property is a scalar multiple of the n-th Rankin-Cohen bracket). It follows from the definition that $[g, f]_n = (-1)^n [f, g]_n$, and that $[\ ,\ ]_1$ satisfies Jacobi identity.

4.2.2. *Action on the pseudo-differential operators.* For $\delta = -\partial_z$, we compute: $\delta(f\,|_0\,\gamma)(z) = -\partial_z(f(\frac{az+b}{cz+d})) = -f'(\frac{az+b}{cz+d}) \times \frac{1}{(cz+d)^2}$, and thus: $(\,\delta(f)\,|_0\,\gamma\,)(z) = (cz+d)^2 \delta(\,f\,|_0\,\gamma\,)(z)$. Then the homographic action of Γ on R is δ-compatible. The associated multiplicative 1-cocycle $p : \Gamma \to U(R)$ defined by (54) is:

$$p_\gamma = (cz+d)^2 \text{ for any } \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma \tag{63}$$

For any $k \in \mathbb{Z}$, the weight k action in the sense of (57) corresponds to the weight $2k$ action in the sense (61) of modular forms:

$$\langle f\,|_{k}\,\gamma \rangle(z) = (cz+d)^{-2k} f(\frac{az+b}{cz+d}) = (f\,|_{2k}\,\gamma)(z) \quad \text{for all } \ \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma,\ f \in R. \tag{64}$$

and then $I_k = M_{2k}(\Gamma, R)$.

We know by example 3 of 4.1.2 that $r'_\gamma = -p_\gamma^{-1}\delta(p_\gamma) = (cz+d)^{-2}\partial_z((cz+d)^2) = 2c(cz+d)^{-1}$ defines an additive 1-cocycle $r' : \Gamma \to R$. Then by example 5 of 4.1.2, we can consider for any $\kappa \in \mathbb{C}$ the additive 1-cocycle $r = \frac{\kappa}{2}r'$:

$$r_\gamma = \kappa\, c(cz+d)^{-1} \quad \text{for all} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma. \tag{65}$$

Applying the theorem of 4.1.2, the action of Γ on R extends for any $\kappa \in \mathbb{C}$ into an action by automorphisms on $B = R(\!(x\,;\,-\partial_z)\!)$ by

$$\gamma(x^{-1}) = (cz+d)^2 x^{-1} + \kappa\, c(cz+d) \quad \text{for all} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma. \tag{66}$$

We denote by $B^{\Gamma,\kappa}$ the subalgebra of invariant elements of B under this action.

4.2.3. *Invariant pseudo-differential operators.* We fix $\kappa \in \mathbb{C}$. For any $f \in R$ and any integer $k \geq 1$, we define:

$$\psi_k(f) = fx^k + \sum_{n\geq 1} (-1)^n \tfrac{(n+k-1)!}{n!\,(n+2k-1)!} \times k!\,(-\kappa+k+1)(-\kappa+k+2)\cdots(-\kappa+k+n)\, f^{(n)}x^{k+n} \in B,$$

$$\psi_0(f) = f \;\in\; R,$$

$$\psi_{-k}(f) = fx^{-k} + \sum_{n=1}^{k} \tfrac{(2k-n)!}{n!\,(k-n)!} \times \tfrac{(\kappa+k-n)(\kappa+k-n+1)\cdots(\kappa+k-1)}{(k-1)!}\, f^{(n)}x^{-k+n} \in B,$$

with the notation $f^{(n)} = \partial_z^n(f)$. The following two results by P. Cohen, Y. Manin and Don Zagier allow to define a vector space isomorphism between the invariant pseudo-differential operators and the product of even weight modular forms.

Lemma ([19]). *For all $f \in R$, $k \in \mathbb{Z}$, $\gamma \in \Gamma$, we have: $\psi_k(\,(f\,|_{2k}\,\gamma)\,) = \gamma(\,\psi_k(f)\,)$, thus:*

$$(\; f \in M_{2k}(R;\Gamma)\;) \;\Leftrightarrow\; (\; \psi_k(f) \in B_k^{\Gamma,\kappa}\;),$$

and then:

$$0 \longrightarrow B_{k+1}^{\Gamma,r} \xrightarrow[\text{inj}]{\text{can}} B_k^{\Gamma,r} \underset{\psi_k}{\overset{\pi_k}{\rightleftarrows}} M_{2k}(\Gamma,R) \longrightarrow 0$$

is a split exact sequence.

Theorem ([19]).

(i) *For any $j \in \mathbb{Z}$, the map*

$$\Psi_{2j} : \mathcal{M}_{2j} := \prod_{k\geq j} M_{2k}(\Gamma,R) \longrightarrow B_j^{\Gamma,\kappa}\;;\; (f_{2k})_{k\geq j} \longmapsto \sum_{k\geq j} \psi_k(f_{2k})$$

is a vector space isomorphism.

(ii) *The map $\Psi_{2*} : \mathcal{M}_{2*} := \bigcup_{j\in\mathbb{Z}} \mathcal{M}_{2j} \longrightarrow \bigcup_{j\in\mathbb{Z}} B_j^{\Gamma,\kappa} = B^{\Gamma,\kappa} = R(\!(x\,;\,-\partial_z)\!)^{\Gamma,\kappa}$ canonically induced by the Ψ_{2j}'s is vector space isomorphism.*

It's not possible to give here the proofs of these results and we can only refer the reader to the original article [19]. In order to illustrate the construction, let us give some explicit calculations for Ψ_0 in the particular case where $\kappa = 0$.

Example.

$$\Psi_0 : \mathcal{M}_0 = \prod_{k\geq 0} M_{2k}(\Gamma,R) \longrightarrow A^{\Gamma,0} = R[[x\,;\,-\partial_z]]^{\Gamma,0} = B_0^{\Gamma,0}\;;\; (f_{2k})_{k\geq 0} \longmapsto \sum_{k\geq 0} \psi_k(f_{2k})$$

For any $(f_0, f_2, f_4, \ldots) \in \mathcal{M}_0$, we have:

$$\psi_0(f_0) = f_0$$
$$\psi_1(f_2) = f_2x - f_2'x^2 + f_2''x^3 - f_2'''x^4 + \cdots = xf_2$$

$$\begin{aligned}
\psi_2(f_4) &= \tfrac{1}{3} f_4 x^2 - \tfrac{1}{2} f_4' x^3 + \tfrac{3}{5} f_4'' x^4 + \cdots \\
\psi_3(f_6) &= \tfrac{1}{10} f_6 x^3 - \tfrac{1}{5} f_6' x^4 + \cdots \\
\psi_4(f_8) &= \tfrac{1}{35} f_8 x^4 + \cdots \\
&\ldots.
\end{aligned}$$

thus:

$$\Psi_0 : \mathcal{M}_0 \longrightarrow A^{\Gamma,0} \quad ; \quad (f_{2k})_{k\geq 0} \longmapsto \sum_{n\geq 0} h_n x^n$$

$$\Psi_0^{-1} : A^{\Gamma,0} \longrightarrow \mathcal{M}_0 \quad ; \quad \sum_{n\geq 0} h_n x^n \longmapsto (f_{2k})_{k\geq 0},$$

with:

$$\begin{array}{ll}
h_0 = f_0 & f_0 = h_0 \\
h_1 = f_2 & f_2 = h_1 \\
h_2 = \frac{1}{3} f_4 - f_2' & f_4 = 3h_2 + 3h_1' \\
h_3 = \frac{1}{10} f_6 - \frac{1}{2} f_4' + f_2'' & f_6 = 10h_3 + 15h_2' + 5h_1'' \\
h_4 = \frac{1}{35} f_8 - \frac{1}{5} f_6' + \frac{3}{5} f_4'' - f_2''' & f_8 = 35h_4 + 70h_3' + 42h_2'' + h_1''' \\
\ldots. & \ldots \\
h_n = \sum_{r=0}^{n-1} (-1)^r \frac{n!\,(n-1)!}{r!\,(2n-r-1)!}\, f_{2(n-r)}^{(r)} & f_{2k} = \sum_{r=0}^{k-1} \frac{(2k-1)\,(2k-2-r)!}{r!\,(k-r)!\,(k-r-1)!}\, h_{k-r}^{(r)}
\end{array}$$

4.2.4. *Non commutative structure on $\mathcal{M}_{2*}$ and Rankin-Cohen brackets.* By transfer of structures, the vector space isomorphisms

$$\Psi_{2*} : \mathcal{M}_{2*} \to B^{\Gamma,\kappa} \quad \text{et} \quad \Psi_{2*}^{-1} : B^{\Gamma,\kappa} \to \mathcal{M}_{2*}$$

resulting of point (ii) of the theorem of 4.2.3 allow to equip M_{2*} with a structure of non commutative $\mathbb{C}$-algebra. We denote it by $\mathcal{M}_{2*}^{\kappa}$ which depends in principle on the parameter κ fixed in the definition of the extension of the action form R to B.

$$\mathcal{M}_{2*}^{\kappa} \simeq B^{\Gamma,\kappa} \text{ for any } \kappa \in \mathbb{C}.$$

The description given in 4.1.3 of the rings $B^{\Gamma,\kappa}$ allows to deduce some algebraic properties (center, centralizers,...) of the algebras $\mathcal{M}_{2*}^{\kappa}$. In particular, supposing that R is a field of characteristic zero, the corollary of the theorem on 4.1.3 given in the comment 4 applies by the lemma of 4.2.3, and we prove so that:

Theorem. *If R is a commutative field of characteristic zero, then $\mathcal{M}_{2*}^{\kappa} \simeq \mathcal{M}_{2*}^{\kappa'}$ for all $\kappa, \kappa' \in \mathbb{C}$.*

Application to the noncommutative product of two modular forms. Let us fix $f \in M_{2k}(\Gamma, R)$ and $g \in M_{2\ell}(\Gamma, R)$. With the identifications:

$$f \equiv (f, 0, 0, \ldots) \in \mathcal{M}_{2k} \ \text{ and } \ g \equiv (g, 0, 0, \ldots) \in \mathcal{M}_{2\ell},$$

the noncommutative product of f by g in $\mathcal{M}_{2*}^{\kappa}$, for an arbitrary choice of $\kappa \in \mathbb{C}$, is given by:

$$\mu^{\kappa}(f, g) = \Psi_{2*}^{-1}\big(\Psi_{2*}(f).\Psi_{2*}(g)\big) = \Psi_{2(k+\ell)}^{-1}\big(\psi_k(f).\psi_\ell(g)\big) \in \mathcal{M}_{2(k+\ell)}.$$

The authors of [19] prove then that:

$$\mu^{\kappa}(f, g) = \sum_{n\geq 0} t_n^{\kappa}(k, \ell)\, [f, g]_n,$$

where $[\ ,\ ]_n : M_{2k}(\Gamma, R) \times M_{2\ell}(\Gamma, R) \to M_{2(k+\ell+n)}(\Gamma, R)$ is the n-th Rankin-Cohen bracket (see comment in 4.2.1), and $t_n^{\kappa}(k, \ell) \in \mathbb{Q}$ is defined by:

$$t_n^\kappa(k,\ell) = \frac{1}{\binom{-2\ell}{n}} \sum_{r+s=n} \frac{\binom{-k}{r}\binom{-k-1+\kappa}{r}}{\binom{-2k}{r}} \frac{\binom{n+k+\ell-\kappa}{s}\binom{n+k+\ell-1}{s}}{\binom{2n+2k+2\ell-2}{s}}$$

These coefficients satisfy $t_n^\kappa(k,\ell) = t_n^{2-\kappa}(k,\ell)$. In particular for $\kappa = \frac{1}{2}$ or $\kappa = \frac{3}{2}$, the product $\mu^{\frac{1}{2}}(f,g)$ is the well known associative Eholzer product $f \star g = \mu^{\frac{1}{2}}(f,g) = \sum_{n\geq 0} [f,g]_n$.

References

[1] J. Alev *Un automorphisme non modéré de $U(g_3)$*, Commun. Algebra **14** (1986), 1365-1378.

[2] J. Alev and M. Chamarie, *Automorphismes et dérivations de quelques algèbres quantiques*, Commun. Algebra, **20** (1992), 1787-1802.

[3] J. Alev and F. Dumas, *Rigidité des plongements des quotients primitifs minimaux de $U_q(\mathrm{sl}(2))$ dans l'algèbre quantique de Weyl-Hayashi*, Nagoya Math. J. **143** (1996), 119–146.

[4] ______, *Automorphismes de certains complétés du corps de Weyl quantique*, Collec. Math. **46** (1995), 1–9.

[5] ______, *Invariants du corps de Weyl sous l'action de groupes finis*, Commun. Algebra **25** (1997), 1655–1672.

[6] ______, *Sur les invariants des algèbres de Weyl et de leurs corps de fractions*, Lectures Notes Pure and Applied Math. **197** (1998), 1–10.

[7] ______, *Corps de Weyl mixtes*, Bol. Acad. Nac. Ciencias, Cordoba, **65** (2000), 29–43.

[8] ______, *Opérateurs différentiels invariants et problème de Noether*, Studies in Lie Theory: A. Joseph Festschrift (eds. J. Bernstein, V. Hinich and A. Melnikov), Birkäuser, 2006 (to appear).

[9] J. Alev, T. J. Hodges and J. -D. Velez, *Fixed Rings of the Weyl Algebra $A_1(\mathbb{C})$* J. Algebra **130** (1990), 83–96

[10] J. Alev and Th. Lambre, *Comparaison de l'homologie de Hochschild et de l'homologie de Poisson pour une déformation des surfaces de Klein*, in "Algebra and operator theory" (Tashkent, 1997), 25–38, Kluwer Acad. Publ., Dordrecht, 1998.

[11] ______, *Homologie des invariants d'une algèbre de Weyl*, K-Theory, **18** (1999), 401–411.

[12] J. Alev, M. Farinati, Th. Lambre, A. Solotar, *Homologie des invariants d'une algèbre de Weyl sous l'action d'un groupe fini*, J. Algebra, **232**, (2000), 564-577.

[13] J. Alev, A. Oooms, M. Vand Den Bergh, *A class of counter examples to the Gelfand-Kirillov conjecture*, Trans. Amer. Math. Soc., **348** (1996), 1709–1716.

[14] I. Assem, *Algèbres et modules*, Les Presses de l'Université d'Ottawa, Masson, Paris, Mila, Barcelone, 1997.

[15] G. M. Bergman and I. M. Isaacs, *Rings with fixed point free group actions*, Proc. London Math. Soc., **27** (1973), 69–87.

[16] K. A. Brown, K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Course in Math. CRM Barcelona, vol 2, Birkhaüser Verlag, Basel, 2002.

[17] A. Charnow, *On the fixed field of a linear abelian group*, J. London Math. Soc. **1** (1969) 348–380.

[18] L. Chiang, H. Chu and M. Kang, *Generation of invariants*, J. Algebra **221** (1999) 232–241.

[19] P. Cohen, Y. Manin, Don Zagier, *Automorphic pseudodifferential operators.* Algebraic aspects of integrable systems, 17–47, Progr. Nonlinear Differential Equations Appl., 26, Birkhäuser Boston, Boston, MA, 1997.

[20] P. M. Cohn, *Free rings and their relations (second edition),* Academic Press, London, 1985.

[21] ______, *Skew Fields, theory of general division rings*, Cambridge University Press, Cambridge, 1995.

[22] S.C. Coutinho, *A primer of algebraic D-modules*, Cambridge University Press, Cambridge, 1995.

[23] F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics, Vol. 228, Springer-Verlag, 2005.

[24] J. Dixmier, *Sur les algèbres de Weyl*, Bull. Soc. Math. France, **96** (1968), 209–242.

[25] ______, *Algèbres enveloppantes*, Gauthier-Villars, 1974.

[26] I. V. Dolgachev, *Rationality of fields of invariants*, Proc. Symposia Pure Math. 4**6** (1987), 3–16.

[27] F. Dumas, *Skew power series rings with general commutation fomula*, Theoret. Computer. Sci, **98** (1992), 99–114.

[28] F. Dumas and F. Martin, *Invariants d'opérateurs pseudo-différentiels et formes modulaires*, preprint (2006)

[29] F. Dumas and L. Rigal, *Prime spectrum and automorphisms for 2×2 Jordanian matrices*, Commun. Algebra **3**0 (2002), 2805-2828

[30] K. R. Goodearl and R. B. Warfield, *An introduction to noncommutative Noetherian rings*, Cambridge University Press, London 1985.

[31] I. M. Gelfand and K. K. Kirillov, *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie*, Inst. Hautes Etudes Sci. Publ. Math. **31** (1966), 509–523.
[32] N. Jacobson, *Basic Algebra II*, (second edition) W. H. Freeman and company, New York 1989.
[33] P. I. Katsylo, *Rationality of fields of invariants of reducible representations of the group* SL_2, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **39** (1984), 77–79.
[34] M. Kervaire et T. Vust, *Fractions rationnelles invariantes par un groupe fini*, in "Algebraische Transformationsgruppen und Invariantentheorie", D.M.V. Sem. **13** Birkhäuser, Basel, (1989), 157–179.
[35] H. Kraft, C. Procesi, *Classical invariant theory: a primer*, (1996).
[36] S. Lang, *Algebra*, Addison-Wesley, 1978.
[37] Th. Levasseur, *Anneaux d'opérateurs différentiels* pp. 157–173, Lecture Notes in Math. **867**, Springer, Berlin-New York, 1981.
[38] T. Maeda, *Noether's problem for* A_5, J. Algebra **125** (1989), 418–430
[39] J. C. Mc Connell and J. C. Robson, *Noncommutative Noetherian rings*, Wiley, Chichester, 1987.
[40] T. Miyata, *Invariants of certain groups I*, Nagoya Math. J. **41** (1971), 68–73.
[41] S. Montgomery, *Fixed Rings of Finite Automorphism Groups of Associative Rings*, Lecture Notes in Math. **818** Springer-Verlag, Berlin, 1980
[42] S. Montgomery and L. W. Small, *Fixed rings of noetherian rings*, Bull. London Math. Soc. **1**3 (1981), 33–38.
[43] I. M. Musson, *Actions of tori on Weyl algebras*, Commun. Algebra **1**6 (1988), 139–148.
[44] L. Richard, *Equivalence rationnelle d'algèbres polynomiales classiques et quantiques*, J. Algebra **287** (2005), 52–87.
[45] O. Riemenschneider, *Die Invarianten der endlichen Untergruppen von* $GL(2,\mathbb{C})$, Math. Z. **153** (1977), 37–50.
[46] D. Saltman, *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984) 71–84.
[47] ———, *Groups acting on fields: Noether's problem*, in "Group actions on rings" (Brunswick, Maine, 1984), 267–277, Contemp. Math., **43**, Amer. Math. Soc., Providence, RI, 1985.
[48] B. Schmid, *Finite groups and invariant theory*, in "Topics in Invariant Theory", Lecture Notes in Math. **1478** Springer-Verlag, Berlin, 1991
[49] J.-P. Serre, *Amalgames,* SL_2, Astérisque **46**, Société Mathématique de France, 1977.
[50] L. Smith *Polynomial invariants of finite groups*, Research Notes in Mathematics volume 6, A.K. Peters, Wellesley Massachusetts, 1995.
[51] T. A. Springer, *Invariant Theory*, Lecture Notes in Math. **585** Springer-Verlag, Berlin, 1977.
[52] A. van den Essen *Polynomial automorphisms and the jacobian conjeture*, Progress in Mathematics volume 190, Birkhäuser Verlag, Basel, Boston, Berlin, 2000.
[53] E. B. Vinberg, *Rationality of the field of invariants of a triangular group*, Vestnik Mosk. Univ. Mat. **37** (1982), 23–24.

Université Blaise Pascal, Laboratoire de Mathématiques (UMR 6620 du CNRS), F-63177, Aubière (France)
E-mail address: `Francois.Dumas@math.univ-bpclermont.fr`