The multi-slopes MUSCL method

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ABSTRACT. *A new class of finite volume MUSCL-type method for unstructured mesh using the cell centered approach introduced by Buffard and Clain [3] is extended to the 3D case. Numerical tests are performed in order to compare the accuracy, numerical viscosity and efficiency of the new multi-slopes methods with the first order method and the standard gradient MUSCL method.*

KEYWORDS: *MUSCL method, multi-slope, unstructured meshes*

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1. Introduction

MUSCL methods have been first introduced by Van-Leer [8] in order to provide a more acurate approximation of the conservation equation solutions. The main idea is to maintain stability and monotonicity while the scheme order is greater. The main advantage of the MUSCL technique is its ability to enhance the solution precision without alteration of the numerical flux. This point is of major interest from a numerical point of view since a specific MUSCL subroutine is only added without modification of the solver. After Van-Leer works, extensions for multidimensional geometries has been proposed using a regular mesh [6] and a version for unstructured meshes has been introduced by [1]. A general presentation of the classical MUSCL can be found in [5] and in [7].

Roughly speaking, the MUSCL technique consists in providing new approximation of the unknow on both side of each interface between two elements. To this end, a predicted vectorial slope representing the gradient of the approximated solution at the centroid point of the control volume is computed and a correction factor is introduced to maintain the monotonicity of the reconstruction. The unknowns are evaluated at a point on each interface.

We propose a new MUSCL technique where a scalar slope is introduced for each direction leading to a one-dimensional MUSCL method where each slope is computed independently. The revealant points of the presented method are the simplicity, since we only use one-dimensional MUSCL reconstructions, and the stability.

2. The classical MUSCL methods

Let us consider a bounded domain $\Omega \subset \mathbb{R}^3$ et $\boldsymbol{\nu}$ denotes the outward normal vector on the boundary $\partial\Omega$. Let $\boldsymbol{F}(s)$ a $\mathbb{R}^3$-valued regular function, for any scalar function $u_b(\boldsymbol{x},t)$ with $\boldsymbol{x} \in \partial\Omega$ and $t \in [0,T]$, we define

$$\Gamma^-(u_b) = \{(\boldsymbol{x},t);\ \boldsymbol{x} \in \partial\Omega, t \in [0,T] \text{ such that } \boldsymbol{F}(u_b(\boldsymbol{x},t)).\boldsymbol{\nu}(\boldsymbol{x}) < 0\}.$$

We consider the scalar conservation equation :

$$\partial_t u(\boldsymbol{x},t) + \boldsymbol{\nabla}.\boldsymbol{F}\big(u(\boldsymbol{x},t)\big) = 0 \qquad \boldsymbol{x} \in \Omega, \qquad t \in]0,T[, \tag{1}$$

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega, \tag{2}$$

$$u(\boldsymbol{x},t) = u_b(\boldsymbol{x},t) \qquad \boldsymbol{x},t \in \Gamma^-(u_b), \tag{3}$$

with u^0 a prescribed initial function and u_b the Dirichlet condition. Assuming Ω polygonal, we denote by $\mathscr{T}_h$ a discretization of Ω with tetrahedron K_i of centroid $\boldsymbol{B}_i$. For each volume K_i, $\mathscr{V}(i)$ is the set of index of all the neighbour elements K_j with a common side S_{ij} (see Figure 1 for a 2D configuration for the sake of simplicity). In the sequel for any $K_i \in \mathscr{T}_h$ and $j \in \mathscr{V}(i)$ we denote by $|K_i|$, $|S_{ij}|$ and $|\boldsymbol{B}_i\boldsymbol{B}_j|$ the volume, the area and the length. Moreover, $\boldsymbol{n}_{ij}$ stands for the exterior normal of K_i on interface S_{ij}.

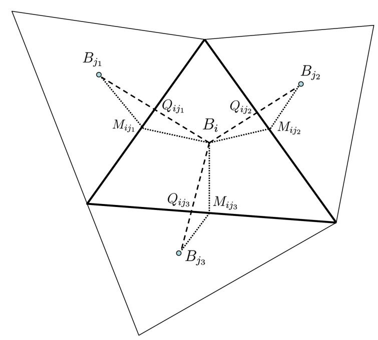

Figure 1. *Geometry, 2D case.*

Let $(t^n)_{n\in[0,N]}$ be a partition of $[0,T]$ and $\Delta t^n = t^{n+1} - t^n$ the time step, U_i^n represents an approximation of the mean value of $u(\boldsymbol{x},t)$ on K_i at time t^n and a generic explicite finite volume scheme writes

$$U_i^{n+1} = U_i^n - \Delta t^n \sum_{j\in\mathscr{V}(i)} \frac{|S_{ij}|}{|K_i|} \boldsymbol{F}_{ij}(U_i^n, U_j^n).\boldsymbol{n}_{ij}, \quad (4)$$

where we assume that the numerical flux $G_{ij}(U,V) = \boldsymbol{F}_{ij}(U,V).\boldsymbol{n}_{ij}$ is monotone in order to provide L^∞ stability under the CFL condition [4].

In order to get a better approximation, the MUSCL technique consists in providing new values U_{ij}^n and U_{ji}^n on both side of the interface S_{ij} and the second order scheme then writes

$$U_i^{n+1} = U_i^n - \Delta t^n \sum_{j\in\mathscr{V}(i)} \frac{|S_{ij}|}{|K_i|} \boldsymbol{F}_{ij}(U_{ij}^n, U_{ji}^n).\boldsymbol{n}_{ij}, \quad (5)$$

such that we satisfy a maximum principle, for instance

$$\min(U_i^n, U_j^n) \leq U_{ij}^n, U_{ji}^n \leq \max(U_i^n, U_j^n). \quad (6)$$

The valuable point is that the numerical flux has not to be modified but only its arguments leading to a numerical method easy to implement: the flux and the reconstruction are indeed independent.

2.1. *Classical mono-slope methods*

Classical MUSCL technique is based on the following local reconstruction

$$\widetilde{u}_i(\boldsymbol{x}) = U_i + \boldsymbol{a}_i.\boldsymbol{B}_i\boldsymbol{x},$$

where the slope $\boldsymbol{a}_i$ is an approximation of $\boldsymbol{\nabla} U$ at the element centroid $\boldsymbol{B}_i$. We drop the time index n for the sake of simplicity. Note that such a reconstruction is conservative since $\dfrac{1}{|K_i|}\displaystyle\int_{K_i} \widetilde{u}_i(\boldsymbol{x})\ dx = U_i$. For a given point $\boldsymbol{X}_{ij}$ on side S_{ij} we define $U_{ij} = U_i + \boldsymbol{a}_i.\boldsymbol{B}_i\boldsymbol{X}_{ij}$. Unfortunately, instabilities appear since relation (6) is not *a priori* satisfied and a correction is provided using a limiter coefficient ϕ_i. Thus the reconstruction is given by

$$U_{ij} = U_i + \phi_i \boldsymbol{a}_i.\boldsymbol{B}_i\boldsymbol{X}_{ij}. \tag{7}$$

We name this class of MUSCL method *mono-slope method* since the values U_{ij}, $j \in \mathscr{V}(i)$ are obtained using the same slope and the same limitation for all the sides of K_i.

2.2. *Multi-slope methods*

We now consider a new MUSCL method where the scalar slope p_{ij} in direction $\boldsymbol{B}_i\boldsymbol{X}_{ij}$ is evaluated for each point X_{ij}:

$$U_{ij} = U_i + p_{ij}|\boldsymbol{B}_i\boldsymbol{X}_{ij}|. \tag{8}$$

We name this class of MUSCL method *multi-slope method* as we compute a specific slope for each direction.

REMARK. — A mono-slope method can be formulated using the multi-slope framework. Indeed, for a given slope $\boldsymbol{a}_i$ and a point $\boldsymbol{X}_{ij} \in S_{ij}$, we set

$$p_{ij} = \boldsymbol{a}_i.\frac{\boldsymbol{B}_i\boldsymbol{X}_{ij}}{|\boldsymbol{B}_i\boldsymbol{X}_{ij}|}$$

and relation (7) becomes (8) provided that $\phi_i = 1$. Consequently, the MUSCL multi-slope method can be considered as an extention of the classical MUSCL method. □

Two particular choices are of importance (see Figure 1): in one hand the intersection point $\boldsymbol{Q}_{ij}$ between S_{ij} and the segment $[\boldsymbol{B}_i\boldsymbol{B}_j]$ for geometrical reason (interpolation) and, in the other hand, the median point $\boldsymbol{M}_{ij}$ of S_{ij} to provide the best numerical integration of the flux.

3. Multi-slope method at point *Q*

We use point $\boldsymbol{Q}_{ij}$ to evaluate U_{ij} with formula (8) and outline the construction of slope p_{ij}. We shall only consider inner tetrahedra K_i such that $\#\mathscr{V}(i) = 4$. Let us denote $\boldsymbol{t}_{ij} = \boldsymbol{B}_i\boldsymbol{B}_j/|\boldsymbol{B}_i\boldsymbol{B}_j|$ the nomalized neighbour directions. Since we assume $|K_i| > 0$ we have the fundamental decomposition

$$\boldsymbol{t}_{ij} = \sum_{\substack{k\in\mathscr{V}(i)\\ k\neq j}} \beta_{ijk}\ \boldsymbol{t}_{ik}. \tag{9}$$

where β_{ijk} are real constants. Mesh $\mathscr{T}_h$ is required to respect the following properties

– $(\mathscr{P}_1)$: $\boldsymbol{Q}_{ij}$ *is a point belonging to the side* S_{ij},

– $(\mathscr{P}_2)$: $\boldsymbol{B}_i$ *is strictly inside the tetrahedra delimited by points* $\boldsymbol{B}_j,\ j \in \mathscr{V}(i)$.

Under assumption $(\mathscr{P}_2)$ we get that $\beta_{ijk} < 0$ since the barycentric coordinates of $\boldsymbol{B}_i$ with respect to $\boldsymbol{B}_j$, $j \in \mathscr{V}(i)$ are positive. We now define the forward slopes in direction $\boldsymbol{t}_{ij}$, $j \in \mathscr{V}(i)$ setting

$$p_{ij}^{+} = \frac{U_j - U_i}{|\boldsymbol{B}_j \boldsymbol{B}_i|}. \tag{10}$$

Using the fondamental decomposition (9), we build the backward slopes

$$p_{ij}^{-} = \sum_{\substack{k \in \mathscr{V}(i) \\ k \neq j}} \beta_{ijk}\ p_{ik}^{+}. \tag{11}$$

At last, we obtain the slope $p_{ij} = \text{minmod}(p_{ij}^{+}, p_{ij}^{-})$ and the reconstructed values are $U_{ij} = U_i + p_{ij} |\boldsymbol{B}_i \boldsymbol{Q}_{ij}|$.
REMARK. — Other one dimensional limiter are available, for exemple the Sweeby or the Van-Leer limiter. For a general discution on the limiter choice see [4]. □

3.1. *Properties of the reconstruction*

Property 3.1 *We list here some basic properties of the multi-slope* $\boldsymbol{Q}$ *reconstruction.*

1) We have a second order method in the sense that for any linear function $U(\boldsymbol{x})$*:* $U_{ij} = U_{ji} = U(\boldsymbol{Q}_{ij})$.

2) The scheme is first order at the extrema.

To prove the first assertion, let us consider a linear function $U(\boldsymbol{x}) = U_0 + \boldsymbol{L}.\boldsymbol{x}$, then we have $p_{ij}^{+} = \boldsymbol{L}.\boldsymbol{t}_{ij}$. Moreover, we write thanks to definition (11)

$$\begin{aligned} p_{ij}^{-} &= \sum_{\substack{k \in \mathscr{V}(i) \\ k \neq j}} \beta_{ijk}\ p_{ik}^{+} = \sum_{\substack{k \in \mathscr{V}(i) \\ k \neq j}} \beta_{ijk}\ \boldsymbol{L}.\boldsymbol{t}_{ik} \\ &= \boldsymbol{L}. \sum_{\substack{k \in \mathscr{V}(i) \\ k \neq j}} \beta_{ijk}\ \boldsymbol{t}_{ik} = \boldsymbol{L}.\boldsymbol{t}_{ij} = p_{ij}^{+}. \end{aligned}$$

We deduce p_{ij} =minmod(p_{ij}^{+}, p_{ij}^{-})=minmod$(p_{ij}^{+}, p_{ij}^{+}) = p_{ij}^{+}$ and thus $U_{ij} = U(\boldsymbol{Q}_{ij})$.

To prove the second assertion, let assume that U_i is a local extremum. Then all the slopes p_{ij}^{+} have the same sign and since the β_{ijk} coefficients are non positive, the slopes p_{ij}^{-} have the opposite sign. Consequently $p_{ij}^{+} p_{ij}^{-} \leq 0$ and the *minmod* limiter yields to $p_{ij} = 0$, hence $U_{ij} = U_i$. □
REMARK. — A L^{∞} stability result is proved in [4] based on a sharp analysis of coefficients β_{ijk}.

4. Multi-slope method at point M

We now intend to compute the values U_{ij} at point $\boldsymbol{M}_{ij}$ to provide a better numerical integration of the flux. Indeed, even if we use a second order reconstruction at point $\boldsymbol{Q}_{ij}$, the numerical integration is only a first order one and the scheme precision is reduced. The main difficulty is that informations are given at points $\boldsymbol{B}_i$ and $\boldsymbol{B}_j$ (hence $\boldsymbol{Q}_{ij}$ by interpolation) and we need here a representative value at point $\boldsymbol{M}_{ij}$. To this end, we introduce the normalized directions

$$\boldsymbol{s}_{ij} = \frac{\boldsymbol{B}_i\boldsymbol{M}_{ij}}{|\boldsymbol{B}_i\boldsymbol{M}_{ij}|}, \tag{12}$$

and the decomposition $\boldsymbol{s}_{ij} = \alpha_{ij}\ \boldsymbol{t}_{ij} + \sqrt{1-\alpha_{ij}^2}\ \boldsymbol{t}_{ij}^{\perp}$, with $\alpha_{ij} = \boldsymbol{s}_{ij}.\boldsymbol{t}_{ij}$, where the orthogonal vector is given by the unique representation

$$\boldsymbol{t}_{ij}^{\perp} = \sum_{\substack{k\in\mathscr{V}(i)\\ k\neq j}} \pi_{ijk}\ \boldsymbol{t}_{ik}. \tag{13}$$

Coupling the above relations, we draw the combinaition

$$\boldsymbol{s}_{ij} = \alpha_{ij}\ \boldsymbol{t}_{ij} + \sqrt{1-\alpha_{ij}^2} \sum_{\substack{k\in\mathscr{V}(i)\\ k\neq j}} \pi_{ijk}\ \boldsymbol{t}_{ik}. \tag{14}$$

Thanks to relation (14), we define the forward slope in $\boldsymbol{s}_{ij}$ direction by

$$q_{ij}^{+} = \alpha_{ij}\ p_{ij}^{+} + \sqrt{1-\alpha_{ij}^2} \sum_{\substack{k\in\mathscr{V}(i)\\ k\neq j}} \pi_{ijk}\ p_{ik}^{+}. \tag{15}$$

To compute the backward slope, we use the fundamental decomposition

$$\boldsymbol{s}_{ij} = \sum_{\substack{k\in\mathscr{V}(i)\\ k\neq j}} \beta'_{ijk}\ \boldsymbol{s}_{ik}, \tag{16}$$

where coefficients β'_{ijk} are unique and negative and we set

$$q_{ij}^{-} = \sum_{\substack{k\in\mathscr{V}(i)\\ k\neq j}} \beta'_{ijk}\ q_{ik}^{+}. \tag{17}$$

We obtain the slope in direction $\boldsymbol{s}_{ij}$ with $q_{ij} = \text{minmod}(q_{ij}^{+}, q_{ij}^{-})$ and the reconstructed values are $U_{ij} = U_i + q_{ij}|\boldsymbol{B}_i\boldsymbol{M}_{ij}|$.

REMARK. — It is important to note that both in the $\boldsymbol{M}$ method and the $\boldsymbol{Q}$ method, all introduced coefficients depend only on the mesh caracteristics and should be computed in a preprocessing step. Only the slope evaluation and the limitation process have to be performed to compute the approximation. It results in a very simplified method and the computational cost is then strongly reduced with regard to the classical MUSCL technique. □

Property 4.1 *For any linear function $U(\boldsymbol{x})$, using the multi-slope $\boldsymbol{M}$ reconstruction, we have $U_{ij} = U_{ji} = U(\boldsymbol{M}_{ij})$.*

To prove the assertion, let us consider a linear function $U(\boldsymbol{x}) = U_0 + \boldsymbol{L}.\boldsymbol{x}$. Then we have $p^+_{ij} = \boldsymbol{L}.\boldsymbol{t}_{ij}$. Thanks to the relation (14), we get

$$\begin{aligned} q^+_{ij} &= \alpha_{ij}\ p^+_{ij} + \sqrt{1-\alpha^2_{ij}} \sum_{\substack{k\in\mathscr{V}(i)\\ k\neq j}} \pi_{ijk}\ p^+_{ik} \\ &= \alpha_{ij}\ \boldsymbol{L}.\boldsymbol{t}_{ij} + \sqrt{1-\alpha^2_{ij}} \sum_{\substack{k\in\mathscr{V}(i)\\ k\neq j}} \pi_{ijk}\ \boldsymbol{L}.\boldsymbol{t}_{ik} \\ &= \boldsymbol{L}.\left(\alpha_{ij}\ \boldsymbol{t}_{ij} + \sqrt{1-\alpha^2_{ij}} \sum_{\substack{k\in\mathscr{V}(i)\\ k\neq j}} \pi_{ijk}\ \boldsymbol{t}_{ik}\right) = \boldsymbol{L}.\boldsymbol{s}_{ij}. \end{aligned}$$

Moreover, as in proposition (3.1), the fundamental decomposition (16) yields $q_{ij} = q^-_{ij} = q^+_{ij}$ and thus $U_{ij} = U(\boldsymbol{M}_{ij})$.$\square$

5. Numerical tests

We present some numerical tests in the case of a linear convection problem to show the performance of the MUSCL multi-slope method. The set Ω is the open cube $]0,1[^3$ in which we consider the convection equation of variable u

$$\partial_t u + \boldsymbol{\nabla}.\big(\boldsymbol{\lambda} u\big) = 0 \tag{18}$$

where the convection velocity is $\boldsymbol{\lambda} = (1,1,1)$. Let u^0 be the initial condition, the exact solution is then $u^e(\boldsymbol{x},t) = u^0(\boldsymbol{x} - \boldsymbol{\lambda} t)$.

Let $\boldsymbol{x}_0 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, we consider two different initial conditions with a compact support in the ball $B = \{\boldsymbol{x} \in \Omega,\ |\boldsymbol{x} - \boldsymbol{x}_0| \leq \frac{1}{4}\}$ given by :

$$f_1(\boldsymbol{x}) = 0.5(1 + \cos(4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)), \quad f_2(\boldsymbol{x}) = 1.$$

Note that f_1 is a C^1 function whereas f_2 is a discontinuous function on Ω. We compute the approximation until a final time T in the finite volume framework using MUSCL reconstructions with the minmod limiter and the upwind flux. On the other hand, we use the limiter proposed in [1] to compute the limiter for the mono-slope method. On the boundary, we impose a homogeneous Dirichlet condition since for $T < 1/2$ the support of the exact solution is contained in domain Ω. Consequently, we only employ a first order scheme for the boundary elements. We consider several 3D arbitrary meshes with repectively 26201, 98254, 247802, 784553 and 1803439 elements. Let $h = \min\limits_{\substack{K_i\in\mathscr{T}_h\\ j\in\mathscr{V}(i)}} \dfrac{|K_i|}{|S_{ij}|}$ represents the characteristic length of the mesh. We assume that the method order behaves asymptotically as follows $\|u(.,T) - u_e(.,T)\|_{L^\gamma(\Omega)} = Ch^\alpha$ where $\gamma = 1, \infty$ and α is the coefficient order. When two methods have quite the same order α, it is worth comparing their constant C to determine the accuracy.

	First order		Gradient		multi-slope $\boldsymbol{Q}_{ij}$		multi-slope $\boldsymbol{M}_{ij}$	
	C	α	C	α	C	α	C	α
f_1	5.1	0.7	3.8	0.8	4.3	0.95	3.0	1.2

Table 1. *Order and constant, L^∞ norm.*

For the regular solution, we compute the L^∞ norm of the methods. As expected, the multi-slope $\boldsymbol{M}_{ij}$ method furnishes the highest order followed by the multi-slope $\boldsymbol{Q}_{ij}$ method. The L_1 norm of the error confirms that the multi-slope method provides

	First order		Gradient		multi-slope $\boldsymbol{Q}_{ij}$		multi-slope $\boldsymbol{M}_{ij}$	
	$100 \times C$	α	$100 \times C$	α	$100 \times C$	α	$100 \times C$	α
f_1	1.7	0.7	1.3	0.8	1.8	1.0	4.2	1.3
f_2	7.5	0.4	7.4	0.4	5.3	0.5	5.9	0.6

Table 2. *Order and constant, L^1 norm.*

an higher order than the first order and the standard gradient methods. The $\boldsymbol{M}$ method is more accurate than the $\boldsymbol{Q}$ one which highlights the crucial importance of using a second order numerical method for the flux integration. For the discontinuous function test, the constants C are also of interest since the order is low. The $\boldsymbol{M}$ method and $\boldsymbol{Q}$ method provide the smallest constant. The MUSCL multi-slope methods have also been tested for the Euler system (see [4]) and we get similar results : the multi-slope $\boldsymbol{M}_{ij}$ method has the highest order while the diffusion is strongly reduced.

6. References

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