Fast and strongly localized observation for a perturbed plate equation

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Abstract. The aim of this work is to study the exact observability of a perturbed plate equation. A fast and strongly localized observation result was proven using a perturbation argument of an Euler-Bernoulli plate equation and a unique continuation result for bi-Laplacian.

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1. Introduction

Various observability and controllability properties for the system of partial differential equations modeling the vibrations of an Euler-Bernoulli plate have been investigated in the literature. In most of the existing references it assumed that the observation region satisfies the geometric optics condition of Bardos, Lebeau and Rauch [1], which is known to be necessary and sufficient for the exact observability of the wave equation (see, for instance, Lasiecka and Triggiani [9], Lebeau [10], Burq and Zworski [2] and references therein). In the case of *internal control*, the first result asserting that exact observability for the Schrödinger equation holds for an *arbitrarily small control region* has been given by Jaffard [7], who shows, in particular, that for systems governed by the Schrödinger equation in a rectangle we have exact internal observability with an arbitrary observation region and in arbitrarily small time. Jaffard's method has been adapted by Komornik [8] to an n-dimensional context. The similar results for boundary observation have been given in Ramdani, Takahashi, Tenenbaum and Tucsnak [13] and Tenenbaum and Tucsnak [14]. The aim of this work is to extend some of these results, namely those in [7], for the case of an Euler-Bernoulli plate perturbed by a zero order term. Note that the above mentioned papers tackling arbitrarily small observation regions use the explicit knowledge of the eigenvalues and of the eigenvectors of the Laplace operator in rectangular domains. Such an information is not available for the plate equations perturbed by lower order terms. On the other hand, as far as we know, the method based on Carleman estimates, which is generally used to tackle lower order terms, does not yield exact observability with arbitrarily small observation region. This is why we consider a different method, in which our problem is tackled as a perturbation of the case considered in [7] and [8], using recent results from Hadd [4] and Tucsnak and Weiss [15].

Let us now give the precise statement of the problem and of the main results. In the remaining part of this work $n \in \mathbb{N}$ and Ω is a rectangular domain in $\mathbb{R}^n$, say

$$\Omega = [0, a_1] \times [0, a_2] \times \dots [0, a_n],$$

with $a_1, a_2, \dots, a_n > 0$.

We consider the initial and boundary value problem

$$\frac{\partial^2 \eta}{\partial t^2} + \Delta^2 \eta + a\eta = 0, \quad \text{in } \Omega \times (0, \infty) \tag{1.1}$$

$$\eta = \Delta\eta = 0, \quad \text{on } \Gamma \times (0, \infty) \tag{1.2}$$

$$\eta(0) = f, \quad \dot{\eta}(0) = g \quad \text{in } \Omega, \tag{1.3}$$

where $a \in L^\infty(\Omega)$, $f \in H^2(\Omega) \cap H_0^1(\Omega)$ and $g \in L^2(\Omega)$. For $n = 2$ the above equations model the vibrations of an Euler-Bernoulli plate with a hinged boundary. The output of this system is

$$y(t) = \dot{\eta}(\cdot, t)|_{\mathcal{O}}, \tag{1.4}$$

where $\mathcal{O}$ is an open subset of Ω. Here, and in the remaining part of this paper we denote

$$\dot{\eta} = \frac{\partial \eta}{\partial t}.$$

Our main result is:

Theorem 1.1. *For any subset $\mathcal{O} \subset \Omega$ the system* (1.1)-(1.4) *is exactly observable in time any time $\tau > 0$, i.e., there exists a constant $k_\tau > 0$ such that*

$$\int_0^\tau \|\dot{\eta}(t)\|^2_{L^2(\mathcal{O})} \; dt \geq k_\tau^2 \left(\|f\|^2_{H^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \right) \forall f \in H^2(\Omega) \cap H_0^1(\Omega),\, g \in L^2(\Omega).$$

The above theorem has two consequences concerning exact controllability and uniform stabilizability for the plate equations. The first one follows from a standard duality argument, see for instance, Lions [11].

Corollary 1.2. *For any open subset $\mathcal{O} \subset \Omega$ the following problem*

$$\frac{\partial^2 \eta}{\partial t^2} + \Delta^2 \eta + a\eta + u\chi_{\mathcal{O}} = 0, \quad \textit{in } \Omega \times (0, \infty) \tag{1.5}$$

$$\eta = \Delta\eta = 0, \quad \textit{on } \Gamma \times (0, \infty) \tag{1.6}$$

$$\eta(0) = f, \quad \dot{\eta}(0) = g \quad \textit{in } \Omega, \tag{1.7}$$

is exactly controllable in any time $\tau > 0$, i.e., for any $\left[\begin{smallmatrix} f_1 \\ g_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} f_2 \\ g_2 \end{smallmatrix}\right] \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ there exists a control $u \in L^2(\mathcal{O})$ such that

$$\begin{bmatrix} \eta(0) \\ \dot{\eta}(0) \end{bmatrix} = \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \eta(\tau) \\ \dot{\eta}(\tau) \end{bmatrix} = \begin{bmatrix} f_2 \\ g_2 \end{bmatrix},$$

where by $\chi_{\mathcal{O}}(x)$ *we denote the function that is* 1 *for* $x \in \mathcal{O}$ *and* 0 *otherwise.*

Moreover, from Theorem 1.1 and the general result in Haraux [5] it follows that the system (1.5)-(1.7) can be exponentially stabilized by using a simple feedback. More precisely, the following result holds.

Corollary 1.3. *Let* $\mathcal{O}$ *be an open subset of* Ω *and let* $a, b \in L^\infty(\Omega, [0,\infty))$ *with* $b(x) \geq b_0 > 0$ *for almost every* $x \in \mathcal{O}$. *Then the system determined by initial and boundary value problem* (1.5)-(1.7) *with* $u(x,t) = -b(x)\dot{\eta}(x,t)$, *is exponentially stable, i.e. there exist* $M, \omega > 0$ *such that*

$$\|\dot{\eta}(t)\|_{L^2(\Omega)} + \|\eta(t)\|_{H^2(\Omega)} \leq Me^{-\omega t}\left(\|f\|_{H^2(\Omega)} + \|g\|_{L^2(\Omega)}\right) \qquad (t \geq 0).$$

The plan of this work is as follows. In Section 2 we fix some notation and we recall some basic results. Section 3 contains the proofs of the main results. In section 4 we prove a Carleman estimate for the bilaplacian, which has been used for the proof of the main result.

2. Notation and preliminaries

In the remaining part of the paper we denote $H = L^2(\Omega)$ and

$$H_1 = \left\{\varphi \in H^4(\Omega) | \varphi = \Delta\varphi = 0 \text{ on } \Gamma\right\}.$$

Let $A_0 : H_1 \to H$ be the operator defined by $A_0\varphi = \Delta^2\varphi$, $\forall \varphi \in H_1$. Let $H_{\frac{1}{2}} = H^2(\Omega) \cap H_0^1(\Omega)$, $X = H_{\frac{1}{2}} \times H$, $X_1 = H_1 \times H_{\frac{1}{2}}$ and

$$A : X_1 \to X, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}.$$

It is well-known that A is skew-adjoint so that, according to Stone's theorem, it generates a strongly continuous group of isometries $\mathbb{T}$ on X. By $\|\cdot\|$ without any index we design the standard norm in $L^2(\Omega)$. We denote $Y = L^2(\mathcal{O})$, with $\mathcal{O} \subset \Omega$ an open set. The operator $C \in \mathcal{L}(X_1, Y)$ corresponding to the observation (1.4) is

$$C\begin{bmatrix} f \\ g \end{bmatrix} = g|_{\mathcal{O}} \qquad \left(\begin{bmatrix} f \\ g \end{bmatrix} \in X\right). \tag{2.1}$$

Let $P_0 \in \mathcal{L}(H)$ be the linear operator defined by $P_0 f = -af$ for all $f \in H$, and $P \in \mathcal{L}(X)$ given by

$$P = \begin{bmatrix} 0 & 0 \\ P_0 & 0 \end{bmatrix}, \quad P\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ P_0 f \end{bmatrix}.$$

We define $A_P : \mathcal{D}(A_P) \to X$ by

$$\mathcal{D}(A_P) = \mathcal{D}(A), \quad A_P = A + P. \tag{2.2}$$

We note that

$$\|P\|_{\mathcal{L}(X)} = \sup\left\{\left\|P\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]\right\|_X\right\} = \sup_{\|f\|\leq 1} \|af\| \leq \|a\|_{L^\infty}.$$

We know from Pazy [12] (Theorem 1.1 p.76) that A_P is the generator of a strongly continuous semigroup $\mathbb{T}^P$ satisfying

$$\|\mathbb{T}_t^P\| \leq Me^{\alpha t}, \quad t \geq 0, \tag{2.3}$$

where $\alpha = \omega + M\|P\|$, and ω and M are such that $\|\mathbb{T}_t\| \leq Me^{\omega t}$ for all $t \geq 0$.

In this context the problem (1.1)-(1.3) can be written as a first order equation

$$\dot{z}(t) = A_P z(t), \quad t \geq 0 \tag{2.4}$$

$$z(0) = z_0, \tag{2.5}$$

where $z(t) = \left[\begin{smallmatrix} \eta \\ \dot{\eta}(t) \end{smallmatrix}\right]$ and $z_0 = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$.

The proof of Theorem 1.1 is based on two abstract results, which are stated below. The first one concerns the robustness of the exact observability with respect to bounded small norm perturbations of the generator and it can be proved by a simple duality argument from Theorem 3.3 in [4].

Proposition 2.1. *Suppose that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Assume that (A, C) is exactly observable in time $\tau > 0$, i.e., there exists $k_\tau > 0$ such that*

$$\left(\int_0^\tau \|C\mathbb{T}_t z_0\|^2 \, dt\right)^{\frac{1}{2}} \geq k_\tau \|z_0\| \quad \forall z_0 \in \mathcal{D}(A).$$

Let $P \in \mathcal{L}(X)$ and let $\mathbb{T}^P$ be the strongly continuous semigroup generated by $A+P$. If there exists a constant $\mathcal{K} > 0$ such that

$$\|P\| \leq \mathcal{K}, \tag{2.6}$$

then $(A + P, C)$ is exactly observable in time τ, i.e., there exists $k_\tau^P > 0$ such that

$$\left(\int_0^\tau \|C\mathbb{T}_t^P z_0\|^2 \, dt\right)^{\frac{1}{2}} \geq k_\tau^P \|z_0\| \quad \forall z_0 \in \mathcal{D}(A).$$

The second result says, roughly speaking, that for systems with diagonalisable generators that in order to prove the exact observability it is is sufficient to check the exact observability of the high frequency part and the observability of eigenvectors. More precisely, we have the following result, borrowed from [15].

Proposition 2.2. *Assume that there exists an orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ formed of eigenvectors of A and the corresponding eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ satisfy $\lim \lambda_k = \infty$. Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$. For some bounded set $J \subset C$ denote*

$$V = span \ \{\phi_k \mid \lambda_k \in J\}^\perp$$

and let A_V be the part of A in V . Let C_V be the restriction of C to $\mathcal{D}(A_V)$. Assume that (A_V, C_V) is exactly observable in time $\tau_0 > 0$ and that $C\phi \neq 0$ for every eigenvector ϕ of A. Then (A, C) is exactly observable in any time $\tau > \tau_0$.

3. Main results

The proof of Theorem 1.1 follows the same idea like in [15] (Theorem 6.3.2), where a similar result is proved for the waves equation. Also, we will use an appropriate decomposition of X as a direct sum of invariant subspaces. To obtain this decomposition, we need the following characterization of the eigenvalues and eigenvectors of A_P.

Proposition 3.1. *With the above notation, $\phi = \left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right] \in \mathcal{D}(A_P)$ is an eigenvector of A_P, associated to the eigenvalue $i\mu$, if and only if φ is an eigenvector of $A_0 - P_0$, associated to the eigenvalue μ^2, and $\psi = i\mu\varphi$.*

Proof. Suppose that $\mu \in \mathbb{C}$ and $\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right] \in X \setminus \{\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]\}$. According to the definition of A_P this is equivalent to

$$\begin{cases} \psi = i\mu\varphi \\ (-A_0 + P_0)\varphi = i\mu\psi. \end{cases}$$

The above conditions hold iff

$$(-A_0 + P_0)\varphi = -\mu^2\varphi \text{ and } \psi = i\mu\varphi.$$

□

Clearly, $A_0 - P_0$ is self-adjoint and it has compact resolvent. Then $A_0 - P_0$ is diagonalisable with an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}^*}$ of eigenvectors and the corresponding family of real eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ satisfies $\lim_{k\to\infty}|\lambda_k| = \infty$. Since $A_0 - P_0 + \|P_0\|I \geq 0$, it follows that all the eigenvalues λ of $A_0 - P_0$ satisfy $\lambda > -\|P_0\|$. Hence, $\lim_{k\to\infty}\lambda_k = \infty$. Without loss of generality we may assume that the sequence $(\lambda_k)_{k\in\mathbb{N}^*}$ is non-decreasing. We extend the sequence (φ_k) to a sequence indexed by $\mathbb{Z}^*$ by setting $\varphi_k = -\varphi_{-k}$ for every $k \in \mathbb{Z}_-$. We introduce the real sequence $(\mu_k)_{k\in\mathbb{Z}^*}$ by

$$\mu_k = \sqrt{|\lambda_k|} \;\text{ if }\; k > 0 \;\text{ and }\; \mu_k = -\mu_{-k} \;\text{ if }\; k < 0.$$

We denote

$$W_0 = \text{span}\left\{ \begin{bmatrix} \frac{1}{i\text{sign}(k)}\varphi_k \\ \varphi_k \end{bmatrix} \middle| \; k \in \mathbb{Z}^*,\ \mu_k = 0 \right\}.$$

If $\text{Ker}(A_0 - P_0) = \{0\}$ then of course W_0 is the zero subspace of X. Let $N \in \mathbb{N}^*$ be such that $\lambda_N > 0$. We denote

$$W_N = \text{span}\left\{ \begin{bmatrix} \frac{1}{i\mu_k}\varphi_k \\ \varphi_k \end{bmatrix} \middle| \; k \in \mathbb{Z}^*,\ |k| < N,\ \mu_k \neq 0 \right\},$$

and define $Y_N = W_0 + W_N$. We also introduce the space

$$V_N = \text{clos span}\left\{ \begin{bmatrix} \frac{1}{i\mu_k}\varphi_k \\ \varphi_k \end{bmatrix} \middle| \; |k| \geq N \right\}. \tag{3.1}$$

Lemma 3.2. *We have $X = Y_N \oplus V_N$ and Y_N, V_N are invariant under $\mathbb{T}^P$.*

By $X = Y_N \oplus V_N$ we mean that $X = Y_N + V_N$ and $Y_N \cap V_N = \{0\}$.

Proof. Let $A_1 : \mathcal{D}(A_0) \to H$ be defined by

$$A_1 f = \sum_{\lambda_k = 0} \langle f, \varphi_k \rangle \varphi_k + \sum_{\lambda_k \neq 0} |\lambda_k| \langle f, \varphi_k \rangle \varphi_k, \quad \forall\, f \in \mathcal{D}(A_0).$$

Since the family $(\varphi_k)_{k \in \mathbb{N}^*}$ is an orthonormal basis in H and each φ_k is an eigenvector of A_1, it follows that A_1 is diagonalisable. Moreover, since the eigenvalues of A_1 are strictly positive, it follows that $A_1 > 0$. Is easy to see that the inner product on X defined by

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_1 = \langle A_1^{\frac{1}{2}} f_1, A_1^{\frac{1}{2}} f_2 \rangle + \langle g_1, g_2 \rangle, \quad \forall \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \in X,$$

is equivalent to the original one (meaning that it induces a norm equivalent to the original norm). Let $\mathcal{A}_1$ be the operator on X defined by

$$\mathcal{D}(\mathcal{A}_1) = H_1 \times H_{\frac{1}{2}}, \quad \mathcal{A}_1 = \begin{bmatrix} 0 & I \\ -A_1 & 0 \end{bmatrix}.$$

We can verify that $\mathcal{A}_1$ is skew-adjoint on X (if endowed with the inner product $\langle \cdot, \cdot \rangle_1$). Consequently we obtain that $Y_N = V_N^{\perp}$ (with respect to this inner product $\langle \cdot, \cdot \rangle_1$). It follows that $X = Y_N \oplus V_N$.

We still have to show that V_N and Y_N are invariant subspaces under $\mathbb{T}^P$. Since V_N is the closed span of a set of eigenvectors of A_P, its invariance under the action of $\mathbb{T}^P$ is clear. If $\mu_k = 0$, then

$$A_P \begin{bmatrix} \frac{1}{i\mathrm{sign}(k)} \varphi_k \\ \varphi_k \end{bmatrix} = \begin{bmatrix} \varphi_k \\ 0 \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} \frac{1}{i\mathrm{sign}(k)} \varphi_k \\ \varphi_k \end{bmatrix} + \begin{bmatrix} \frac{1}{i\mathrm{sign}(-k)} \varphi_{-k} \\ \varphi_{-k} \end{bmatrix} \right) \in W_0,$$

so that W_0 is invariant under $\mathbb{T}^P$. If $|k| < N$ and $\lambda_k < 0$ then

$$(A_0 - P_0)\varphi_k = -\mu_k^2 \varphi_k,$$

so that

$$A_P \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} = \begin{bmatrix} \varphi_k \\ \frac{\mu_k}{i} \varphi_k \end{bmatrix} = i\mu_k \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ -\varphi_k \end{bmatrix} = i\mu_k \begin{bmatrix} \frac{1}{i\mu_{-k}} \varphi_{-k} \\ \varphi_{-k} \end{bmatrix} \in W_N.$$

If $|k| < N$ and $\lambda_k > 0$, then

$$A_P \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} = i\mu_k \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} \in W_N.$$

Thus W_N, and hence also $Y_N = W_0 + W_N$, are invariant for $\mathbb{T}$. □

Lemma 3.3. *With the previous notation and* (3.1)*, let* $N \in \mathbb{N}^*$ *be such that* $\lambda_N > \|a\|_{L^\infty}$*. Let us denote by* $P_{V_N} \in \mathcal{L}(V_N, X)$ *the restriction of* P *to* V_N*. Then*

$$\|P_{V_N}\| \leq \frac{\|a\|_{L^\infty}}{\sqrt{\lambda_N - \|a\|_{L^\infty}}}. \tag{3.2}$$

Proof. Take a finite linear combination of the vectors φ_k with $k \geq N$:

$$f = \sum_{k=N}^{M} \alpha_k \varphi_k, \tag{3.3}$$

so that $\|f\|^2 = \sum_{k=N}^{M} |\alpha_k|^2$. Then

$$\begin{aligned}
\|\Delta f\|^2 + \langle af,\, f\rangle &= \int_\Omega \Delta f \Delta \overline{f}\ \mathrm{d}x + \int_\Omega a(x) f\overline{f}\ \mathrm{d}x = \\
&= \int_\Omega \Delta^2 f\overline{f} + af\overline{f}\ \mathrm{d}x = \int_\Omega (A_0 - P_0) f\overline{f}\ \mathrm{d}x = \\
&= \sum_{k,l=N}^{M} \alpha_k \overline{\alpha_l}\, \langle (A_0 - P_0)\varphi_k,\, \varphi_l\rangle = \sum_{k=N}^{M} |\alpha_k|^2 \lambda_k \geq \lambda_N \|f\|^2.
\end{aligned}$$

From here we see that

$$\|\Delta f\|^2 \geq (\lambda_N - \|a\|_{L^\infty})\, \|f\|^2.$$

Now take z to be a finite linear combination of the eigenvectors of A_P in V_N:

$$z \in \ \mathrm{span}\left\{ \begin{bmatrix} \frac{1}{i\mu_k}\varphi_k \\ \varphi_k \end{bmatrix} \middle|\ |k| \geq N \right\},$$

so that in particular $z \in V_N$ and $z = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$, with f as in (3.3). Therefore

$$\begin{aligned}
\|P_{V_N} z\|_X &= \|Pz\|_X = \|af\| \leq \|a\|_{L^\infty} \|f\| \\
&\leq \frac{\|a\|_{L^\infty}}{\sqrt{\lambda_N - \|a\|_\infty}} \|\Delta f\| \leq \frac{\|a\|_{L^\infty}}{\sqrt{\lambda_N - \|a\|_\infty}} \|z\|_X.
\end{aligned}$$

Since all the vectors like our z are dense in V_N, it follows that the above estimate holds for all $z \in V_N$, and this implies the estimate in the lemma. □

Lemma 3.4. *Let $a \in L^\infty(\Omega)$ and let u be a function such that*

$$\Delta^2 u + au = \mu^2 u \qquad \text{in } \Omega \tag{3.4}$$

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega \tag{3.5}$$

and

$$u = 0 \quad \text{in } \mathcal{O}. \tag{3.6}$$

Then $u = 0$ in all Ω.

Proof. The proof of this lemma is an direct consequence of the Theorem 4.3 given in the next section. Let denote $g = (\mu^2 - a)u \in L^2(\Omega)$. Now we apply Theorem 4.3 for (3.4)-(3.5) and using (3.6) we obtain

$$\begin{aligned}
s\lambda^2 \int_\Omega |\nabla(\Delta u)|^2 e^{2s\varphi} \mathrm{d}x + s^4\lambda^6 \int_\Omega |\nabla u|^2 e^{2s\varphi} \mathrm{d}x + s^6\lambda^8 \int_\Omega |u|^2 \varphi^2 e^{2s\varphi} \mathrm{d}x \\
\leq C \int_\Omega \frac{|g|^2}{\varphi} e^{2s\varphi} \mathrm{d}x.
\end{aligned}$$

After some small calculations we can prove the estimate

$$\int_\Omega \frac{|g|^2}{\varphi} e^{2s\varphi} \mathrm{d}x \le C_1(\lambda) \int_\Omega (\mu^2 + \|a\|_{L^\infty}^2)|u|^2 \varphi^2 e^{2s\varphi} \mathrm{d}x + C_2(\lambda),$$

where C_1, C_2 depend only of λ. Coupling the last two equations and taking $s \to \infty$ we obtain that $u = 0$ in Ω. □

Proof of Theorem 1.1. Let $N \in \mathbb{N}^*$ be such that $\lambda_N > 0$ and let A_N and C_N be the parts of A_P, respectively of C, in V_N, where V_N has been defined in (3.1). (Thus, $A_N = (A+P)|_{V_N}$ and $C_N = C|_{V_N}$.) We claim that for $N \in \mathbb{N}^*$ large enough the pair (A_N, C_N) (with state space V_N) is exactly observable in time τ_0.

For a given constant $\mathcal{K} > 0$, from the estimation (3.2), there exists a $N \in \mathbb{N}^*$, big enough, such that

$$\|P_{V_N}\| \le \mathcal{K}.$$

Because (A, C) is exactly observable in any time $\tau > 0$, using Proposition 2.1 we obtain that (A_N, C_N) is exactly observable in time τ in V_N.

On the other hand, if $\phi = \left[\begin{smallmatrix}\varphi \\ \psi\end{smallmatrix}\right] \in \mathcal{D}(A_P)$ is an eigenvector of A_P, associated to the eigenvalue $i\mu$, such that $C\phi = 0$ then, according to Proposition 3.1, $\varphi \in H_1$ is an eigenvector of $A_0 - P_0$, associated to the eigenvalue μ^2, i.e., $\varphi \in H_1$ satisfies

$$\Delta^2 \varphi + a\varphi = \mu^2 \varphi. \tag{3.7}$$

Moreover, the condition $C\phi = 0$ is equivalent to

$$\varphi = 0 \quad \text{in } \mathcal{O}.$$

As shown in Lemma 3.4, the only function $\varphi \in H_1$ satisfying above conditions is $\varphi = 0$. Now, from Proposition 2.2 we can conclude that (A, C) is exactly observable in any time $\tau > 0$. □

4. A global Carleman estimate for bi-Laplacian

In this section we will prove a global Carleman estimate for bi-Laplacian, applying two times a particular case of the global Carleman estimate proved in [6].

Let Ω be an nonempty open set of class C^2. Let $y \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the problem

$$\Delta y = f, \quad \text{in } \Omega \tag{4.1}$$

$$y = 0, \quad \text{on } \partial\Omega, \tag{4.2}$$

where $f \in L^2(\Omega)$. We use the following classic lemma stated in [6], and proved in [3].

Lemma 4.1. *Let $\mathcal{O}$ be an nonempty open set $\mathcal{O} \subset \Omega$. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\psi = 0, \qquad on\ \partial\Omega \tag{4.3}$$

$$\psi(x) > 0, \qquad \forall\, x \in \Omega \tag{4.4}$$

$$|\nabla\psi(x)| > 0, \quad \forall\, x \in \overline{\Omega \setminus \mathcal{O}}. \tag{4.5}$$

We consider a weight function

$$\varphi(x) = e^{\lambda\psi(x)}, \tag{4.6}$$

where $\lambda \in \mathbb{R}$, $\lambda \geq 1$ will be chosen later. The following theorem is a particular case of the Carleman estimate proved by Imanuvilov-Puel in [6] for the general elliptic operators.

Theorem 4.2. *Assume that the hypotheses* (4.3)-(4.6) *are verified and let $y \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of* (4.1)-(4.2)*. Then there exists a constant $C > 0$ independent of s and λ, and parameters $\widehat{\lambda} > 1$ and $\widehat{s} > 1$ such that for all $\lambda \geq \widehat{\lambda}$ and for all $s > \widehat{s}$ we have*

$$\int_\Omega |\nabla y|^2 e^{2s\varphi} \mathrm{d}x + s^2\lambda^2 \int_\Omega |y|^2\varphi^2 e^{2s\varphi}\mathrm{d}x \leq \\ C\left(\frac{1}{s\lambda^2}\int_\Omega \frac{|f|^2}{\varphi}e^{2s\varphi}\mathrm{d}x + \int_{\mathcal{O}} \left(|\nabla y|^2 + s^2\lambda^2\varphi^2|y|^2\right)e^{2s\varphi}\mathrm{d}x\right). \tag{4.7}$$

Let $u \in H_1$ be the solution of the problem

$$\Delta^2 u = g, \qquad \text{in } \Omega \tag{4.8}$$

$$u = \Delta u = 0, \quad \text{on } \partial\Omega, \tag{4.9}$$

where $g \in L^2(\Omega)$.

Theorem 4.3. *Let $\psi \in C^2(\overline{\Omega})$ be a function such that* (4.3)-(4.5) *are verified, let φ given by* (4.6)*, and let $u \in H_1$ be the solution of* (4.8)-(4.9)*. Then there exist $\widehat{s} > 1$, $\lambda > 1$ and a constant $C > 0$ independent of $s \geq \widehat{s}$ such that*

$$s\lambda^2 \int_\Omega \left(|\nabla(\Delta u)|^2 e^{2s\varphi} + s^3\lambda^4|\nabla u|^2 e^{2s\varphi} + s^5\lambda^6|u|^2\varphi^2 e^{2s\varphi}\right) \leq C\left(\int_\Omega \frac{|g|^2}{\varphi}e^{2s\varphi}\mathrm{d}x\right. \\ \left. + s\lambda^2\int_{\mathcal{O}} (|\nabla(\Delta u)|^2 + s^2\lambda^2\varphi^2|\Delta u|^2 + s^3\lambda^4|\nabla u|^2 + s^5\lambda^6\varphi^2|u|^2)e^{2s\varphi}\mathrm{d}x\right). \tag{4.10}$$

Proof. We denote $y = \Delta u$. Then (4.8) and the last part of (4.9) can be written as

$$\Delta y = g, \qquad \text{in } \Omega \tag{4.11}$$

$$y = 0, \quad \text{on } \partial\Omega \tag{4.12}$$

Applying the Theorem 4.2 there exist $s_1 > 1$, $\lambda_1 > 1$ and $C_1 > 0$ independent of s and λ such that for all $s \geq s_1$, $\lambda \geq \lambda_1$ the following estimate is satisfied

$$s\lambda^2 \int_\Omega |\nabla y|^2 e^{2s\varphi} \mathrm{d}x + s^3\lambda^4 \int_\Omega |y|^2 \varphi^2 e^{2s\varphi} \mathrm{d}x \leq C_1 \left(\int_\Omega |g|^2 \varphi^{-1} e^{2s\varphi} \mathrm{d}x + \int_{\mathcal{O}} \left(s\lambda^2 |\nabla y|^2 + s^3\lambda^4 \varphi^2 |y|^2 \right) e^{2s\varphi} \mathrm{d}x \right).$$

Replacing y with Δu in the previous estimate we obtain

$$s\lambda^2 \int_\Omega |\nabla(\Delta u)|^2 e^{2s\varphi} \mathrm{d}x + s^3\lambda^4 \int_\Omega |\Delta u|^2 \varphi^2 e^{2s\varphi} \mathrm{d}x \leq C_1 \left(\int_\Omega |g|^2 \varphi^{-1} e^{2s\varphi} \mathrm{d}x + \int_{\mathcal{O}} \left(s\lambda^2 |\nabla(\Delta u)|^2 + s^3\lambda^4 \varphi^2 |\Delta u|^2 \right) e^{2s\varphi} \mathrm{d}x \right). \quad (4.13)$$

Now consider the problem

$$\Delta u = y, \qquad \text{in } \Omega \quad (4.14)$$

$$u = 0, \qquad \text{on } \partial\Omega, \quad (4.15)$$

and apply the Theorem 4.2. Then there exist $C_2 > 0$, $s_2 > 1$, $\lambda_2 > 1$ such that for $s \geq s_2$ and $\lambda \geq \lambda_2$ we have

$$\begin{aligned} s\lambda^2 \int_\Omega |\nabla u|^2 e^{2s\varphi} \mathrm{d}x + s^3\lambda^4 \int_\Omega |u|^2 \varphi^2 e^{2s\varphi} \mathrm{d}x &\leq \\ C_2 \left(\int_\Omega |\Delta u|^2 \varphi^{-1} e^{2s\varphi} \mathrm{d}x + \int_{\mathcal{O}} \left(s\lambda^2 |\nabla u|^2 + s^3\lambda^4 \varphi^2 |u|^2 \right) e^{2s\varphi} \mathrm{d}x \right) &\leq \\ C_3 \left(\int_\Omega |\Delta u|^2 \varphi^2 e^{2s\varphi} \mathrm{d}x + \int_{\mathcal{O}} \left(s\lambda^2 |\nabla u|^2 + s^3\lambda^4 \varphi^2 |u|^2 \right) e^{2s\varphi} \mathrm{d}x \right). &\quad (4.16) \end{aligned}$$

We denote $\lambda = \max\{\lambda_1, \lambda_2\}$ and $\widehat{s} = \max\{s_1, s_2\}$. For $s \geq \widehat{s}$, combining (4.13) and (4.16) we have

$$\begin{aligned} s\lambda^2 \int_\Omega |\nabla(\Delta u)|^2 e^{2s\varphi} \mathrm{d}x + \frac{s^3\lambda^4}{C_3} \left(s\lambda^2 \int_\Omega |\nabla u|^2 e^{2s\varphi} \mathrm{d}x + s^3\lambda^4 \int_\Omega |u|^2 \varphi^2 e^{2s\varphi} \mathrm{d}x \right) - \\ - s^3\lambda^4 \left(\int_{\mathcal{O}} \left(s\lambda^2 |\nabla u|^2 + s^3\lambda^4 \varphi^2 |u|^2 \right) e^{2s\varphi} \mathrm{d}x \right) \leq \\ C_1 \left(\int_\Omega |g|^2 \varphi^{-1} e^{2s\varphi} \mathrm{d}x + \int_{\mathcal{O}} \left(s\lambda^2 |\nabla(\Delta u)|^2 + s^3\lambda^4 \varphi^2 |\Delta u|^2 \right) e^{2s\varphi} \mathrm{d}x \right). \quad (4.17) \end{aligned}$$

How λ is fixed in 4.17, we affirm that exists a constant $C > 0$ such that (4.10) is verified. So, the proof of theorem is completed. □

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