Local exact controllability for Berger plate equation

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Abstract

We study the exact controllability of a nonlinear plate equation by the means of a control which acts on an internal region of the plate. The main result asserts that this system is locally exactly controllable if the associated linear Euler-Bernoulli system is exactly controllable. In particular, for rectangular domains we obtain that the Berger system is locally exactly controllable in arbitrarily small time and for every open and nonempty control region.

Keywords: local exact controllability, Berger equation, nonlinear plate equation, spectral criterium.

1 Introduction

During the last decades an important literature has been devoted to the exact controllability of various linear equations modeling the vibrations of elastic plates (see, for instance, Zuazua [20], Lasiecka and Triggiani [10], Jaffard [8]). A case of particular interest is the Euler-Bernoulli model with distributed control, i.e., the initial and boundary value problem

$$\ddot{w}(x,t) + \Delta^2 w(x,t) = u(x,t)\chi_{\mathcal{O}} \quad \text{for } (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

$$w(x,t) = \Delta w(x,t) = 0 \quad \text{for} \quad (x,t) \in \partial\Omega \times (0,\infty), \tag{1.2}$$

$$w(x,0) = \dot{w}(x,0) = 0 \quad \text{for } x \in \Omega. \tag{1.3}$$

In the above equations, $\Omega \subset \mathbb{R}^2$ is an open nonempty set, $\mathcal{O}$ is an open subset of Ω and a dot denotes differentiation with respect to the time t, so that

$$\dot{w} = \frac{\partial w}{\partial t}, \quad \ddot{w} = \frac{\partial^2 w}{\partial t^2}.$$

The state trajectory of the above system is the function $t \mapsto \begin{bmatrix} w \\ \dot{w} \end{bmatrix}$, where w and $\dot{w}$ stand for the transverse displacement and the transverse velocity of the plate, respectively. The input function is $u \in L^2([0,\infty); L^2(\mathcal{O}))$, extended by zero outside $\mathcal{O}$, and $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$.

A general sufficient condition for the exact controllability of (1.1)-(1.3) is that Ω and $\mathcal{O}$ satisfy the geometric optics condition of Bardos, Lebeau and Rauch [2]. This has been originally shown in Lebeau [11], using microlocal analysis. The proof has been successively simplified in Miller [13] and Tucsnak and Weiss [18, Example 11.2.4]. The geometric optics

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condition is not necessary for the exact controllability of (1.1)-(1.3). Indeed, as has been shown in Jaffard [8], if Ω is a rectangle then (1.1)-(1.3) is exactly controllable for every nonempty control region $\mathcal{O}$. More complicated situations in which the geometric optics condition fails but the exact controllability property holds have been recently investigated in Burq and Zworski [4] (see also Tenenbaum and Tucsnak [17] for boundary controllability). Note that the proofs of the exact controllability results in [4] and [8] are based on technics which are quite different of those used for the case in which the geometric optics condition holds.

The aim of this work is to study the local exact controllability of a system modeling the nonlinear vibrations of an elastic plate. This model, which has been proposed by Berger (see Berger [3]), is equivalent in one space dimension to the wider known Von Karman equations (see Perla Menzala and Zuazua [12]). In the two-dimensional case the system we consider can be seen as an asymptotic limit of the Von Karman equations (see Perla Menzala, Pazoto and Zuazua [15], Nayfeh and Mook [14]).

Berger's model for an elastic plate filling the domain Ω and hinged on the boundary $\partial\Omega$ consists in the following initial and boundary value problem:

$$\ddot{w}(x,t) + \Delta^2 w(x,t) - \left(a + b\int_\Omega |\nabla w|^2 \,\mathrm{d}x\right)\Delta w(x,t) = u\chi_{\mathcal{O}} \quad \text{for } (x,t) \in \Omega\times(0,\infty), \tag{1.4}$$

$$w(x,t) = \Delta w(x,t) = 0 \quad \text{for } (x,t)\in\partial\Omega\times(0,\infty), \tag{1.5}$$

$$w(x,0) = 0, \quad \dot{w}(x,0) = 0 \quad \text{for } x\in\Omega. \tag{1.6}$$

In the above system we continue to use the notation described after (1.1)-(1.3). Moreover, the constant a is supposed to be larger then $-\lambda_1$, where $\lambda_1 > 0$ denotes the first eigenvalue of the Dirichlet Laplacian in Ω and it corresponds to the in-plane stretching ($a < 0$) or compression ($a > 0$) of the plate. The constant b is supposed to be positive.

The first main result of the paper is:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with C^2 boundary and let $\mathcal{O}\subset\Omega$ be an open and nonempty subset of Ω such that (1.1)-(1.3) is exactly controllable (in some time $\tau_0 > 0$). Then the nonlinear system (1.4)-(1.6) is locally exactly controllable (in some time $\tau > 0$), i.e., there exist $\tau > 0$, $M > 0$ such that for every $\left[\begin{smallmatrix} w_0 \\ w_1\end{smallmatrix}\right] \in (H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$, with $\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)} \le M^2$, there exists $u \in L^2([0,\tau]; L^2(\mathcal{O}))$ such that the solution w of (1.4)-(1.6) satisfies*

$$w(\cdot,\tau) = w_0, \quad \dot{w}(\cdot,\tau) = w_1.$$

The main interest of Theorem 1.1 is that, being a perturbation result, it relies only on the exact controllability of (1.1)-(1.3), which is a well studied problem. Note that (1.1)-(1.3) is not the linearization around 0 of (1.4)-(1.6). Therefore our perturbation argument is divided in two steps: we first tackle the case $b = 0$, by using frequency domain techniques and then we go back to the original nonlinear problem by a fixed point argument. The main shortcoming of Theorem 1.1 is that it does not provide, as expected, the local exact controllability in arbitrarily small time. Our second main result is Theorem 1.2 below, which fills this gap, at least in the case of rectangular domains.

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^2$ be a rectangle and let $\mathcal{O}$ be an open and nonempty subset of Ω. Then (1.4)-(1.6) is locally exactly controllable in any time $\tau > 0$. In other words, for every $\tau > 0$ there exists a constant $M > 0$ such that for every $\left[\begin{smallmatrix} w_0 \\ w_1\end{smallmatrix}\right] \in (H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$, with*

$$\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)} \le M^2,$$

there exists $u\in L^2([0,\tau]; L^2(\mathcal{O}))$ such that the solution w of (1.4)-(1.6) satisfies

$$w(\cdot,\tau) = w_0, \quad \dot{w}(\cdot,\tau) = w_1.$$

The remaining part of this paper is organized as follows. In Section 2 we give some notation and some background on exact controllability, exact observability and pseudo-periodic functions. In Section 3 we show that if an abstract plate equation is exactly controllable then the same result holds if we perturb the equation by a particular lower order term. Section 4 is devoted to the fixed point argument and to one example in one space dimension. Finally, the main results are proved in Section 5.

2 Notation and preliminaries

In this section we will recall some known results on the observability and controllability of infinite dimensional systems and some results on pseudo-periodic functions. We do not give proofs and we refer to the existing literature.

Let X be a Hilbert space, let $A : \mathcal{D}(A) \to X$ be a densely defined operator with resolvent set $\rho(A) \neq \emptyset$, let $\beta \in \rho(A)$ and let X_1 be $\mathcal{D}(A)$ with the graph norm. Then $A \in \mathcal{L}(X_1, X)$, $(\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$ and this operator is unitary.

In the remaining part of this section X and U are complex Hilbert spaces which are identified with their duals, $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ is a strongly continuous semigroup on X, with generator $A : \mathcal{D}(A) \to X$ and X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|_X$, where $\beta \in \rho(A)$ is fixed. The restriction of $\mathbb{T}_t$ to X_1 is the image of $\mathbb{T}_t \in \mathcal{L}(X)$ through the unitary operator $(\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$. Therefore, these operators form a strongly continuous semigroup on X_1, whose generator is the restriction of A to $\mathcal{D}(A^2)$.

Let us consider the following infinite dimensional system

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = 0, \tag{2.1}$$

where $B \in \mathcal{L}(U, X)$. It is known (see, for instance, [18, Section 4.2]) that, if $\tau > 0$ and $u \in L^2([0, \tau]; U)$, then the solution of (2.1) is $z \in C([0, \tau]; X)$

$$z(t) = \Phi_t u \qquad (t \in (0, \tau)), \tag{2.2}$$

where $\Phi_\tau \in \mathcal{L}(L^2(0, \tau; U), X)$ is defined by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} Bu(\sigma)\, \mathrm{d}\sigma. \tag{2.3}$$

Definition 2.1. Let $\tau > 0$. The pair (A, B) is exactly controllable in time $\tau > 0$ if Ran $\Phi_\tau = X$. The pair (A, B) is exactly controllable if it is exactly controllable in some time $\tau > 0$.

The dual concept of exact controllability is the exact observability. The duality between these two concepts is formalized in the result below, see Dolecki and Russell [7].

Proposition 2.2. *With the above assumptions on A and B, the pair (A, B) is exactly controllable if and only if the pair (A^*, B^*) is exactly observable, i.e., if there exist τ, $C_\tau > 0$ such that*

$$C_\tau^2 \int_0^\tau \|B^* \mathbb{T}_t^* \phi\|_U^2 \geq \|\phi\|_X^2 \qquad (\phi \in \mathcal{D}(A^*)).$$

Proposition 2.3. *Suppose that (A, B) is exactly controllable in time τ. Then there exists an operator $F_\tau \in \mathcal{L}(X, L^2([0, \tau]; U))$ such that*

(1) $\Phi_\tau F_\tau = \mathbb{I}_X$.

(2) If $u \in L^2([0,\tau];U)$ is a control driving the solution z of (2.1) *from* 0 *to* z_0 *in time* τ*, then*

$$\|u\|_{L^2([0,\tau];U)} \geqslant \|F_\tau z_0\|_{L^2([0,\tau];U)}.$$

A simple consequence of the above proposition shows that we can steer the solution of

$$\dot{z}(t) = Az(t) + Bu(t) + F(t), \quad z(0) = 0, \tag{2.4}$$

to an arbitrary state in X by means of a control $u \in L^2([0,\tau];U)$, where $F \in L^2([0,\tau];X)$. More precisely, using (2.2) and Proposition 2.3 we easily obtain the following result.

Corollary 2.4. *Let $\tau > 0$ and assume that the pair (A,B) is exactly controllable in time τ. Let $z_0 \in X$ and*

$$u = F_\tau z_0 - F_\tau \int_0^\tau \mathbb{T}_{\tau-s} F(s)\ ds, \tag{2.5}$$

where F_τ is the operator in Proposition 2.3*. Then the solution z of* (2.4) *satisfies $z(\tau) = z_0$.*

Definition 2.5. Let $V \subset X$ be a closed invariant subspace for $\mathbb{T}$. The part of A in V, denoted by A_V, is the restriction of A to $\mathcal{D}(A_V) = \mathcal{D}(A) \cap V$, regarded as a (possibly unbounded) operator on V.

Clearly, A_V is the generator of the restriction of $\mathbb{T}$ to V. The following proposition follows directly from Proposition 6.4.4 from [18] and Proposition 2.2 (see also Tucsnak and Weiss [19]).

Proposition 2.6. *Assume that there exists an orthonormal basis $(\Phi_n)_{n\in\mathbb{N}}$ formed by eigenvectors of A and the corresponding eigenvalues λ_n satisfy $\lim|\lambda_n| = \infty$. For some bounded set $J \subset \mathbb{C}$ denote*

$$V = span\ \{\Phi_n \mid \lambda_n \in J\}^\perp,$$

let A_V^* be the part of A^* in V and let B_V^* be the restriction of B^* to $\mathcal{D}(A_V^*)$. Assume that (A_V^*, B_V^*) is exactly controllable in time $\tau_0 > 0$ and that $B^*\Phi_n \neq 0$ for every eigenvector Φ_n of A. Then (A,B) is exactly controllable in any time $\tau > \tau_0$.

In this work we use the following spectral characterization of exact controllability of the pair (A,B) in the case where A is skew-adjoint. The proof of this theorem is a straightforward combination of Theorem 1.3 from Ramdani, Takahashi, Tenenbaum and Tucsnak [16], Proposition 2.2 and Proposition 2.6.

Theorem 2.7. *Assume that A is skew-adjoint with compact resolvents and that $B \in \mathcal{L}(U,X)$. Moreover, assume that $(\Phi_n)_{n\in\mathbb{Z}^*}$ is an orthonormal sequence of eigenvectors of A associated to the eigenvalues $(i\mu_n)_{n\in\mathbb{Z}^*}$, where $(\mu_n)_{n\in\mathbb{Z}^*}$ is a sequence of real numbers.*

For $\omega \in \mathbb{R}$ and $\varepsilon > 0$, set

$$J_\varepsilon(\omega) = \{m \in \mathbb{Z}^* \text{ such that } |\mu_m - \omega| < \varepsilon\}. \tag{2.6}$$

Then the pair (A,B) is exactly controllable if and only if there exist $\varepsilon > 0$ and $\delta > 0$ such that for all $\omega \in \mathbb{R}$ and for all $z = \sum_{m\in J_\varepsilon(\omega)} c_m\Phi_m$:

$$\|B^*z\|_Y \geq \delta\|z\|_X. \tag{2.7}$$

We call an element $z = \sum_{m \in J_\varepsilon(\omega)} c_m \Phi_m$ a *wave packet* of A of parameters ω and ε. Notice that $z \in \mathcal{D}(A^\infty) = \bigcap_{n \geq 1} \mathcal{D}(A^n)$.

We next introduce some new notation wich will be useful for second order systems. Let H be a Hilbert space which will be identified with its dual and let $A_0 : \mathcal{D}(A_0) \to H$ be a strictly positive operator. Whenever no confusion is possible, the inner product and the induced norm in H will be simply denoted $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. When saying that A_0 is *strictly positive* we mean that A_0 is self-adjoint and that there exists a constant $\gamma > 0$ such that

$$\langle A_0 \varphi, \varphi \rangle \geqslant \gamma \|\varphi\|^2 \qquad (\varphi \in \mathcal{D}(A_0)).$$

Recall that such an operator A_0 has an orthonormal basis of eigenvectors $(\varphi_n)_{n \in \mathbb{N}^*}$ corresponding to the positive eigenvalues $(\lambda_n)_{n \in \mathbb{N}^*}$. We denote H_1 the Hilbert space $\mathcal{D}(A_0)$ with the inner product $\langle \varphi, \psi \rangle_1 = \langle A_0 \varphi, A_0 \psi \rangle$ and the induced norm

$$\|\varphi\|_1 = \|A_0 \varphi\| \qquad (\varphi \in H_1).$$

The Hilbert space H_2 is $\mathcal{D}(A_0^2)$ with the inner product $\langle \varphi, \psi \rangle = \left\langle A_0^2 \varphi, A_0^2 \psi \right\rangle$ and the induced norm

$$\|\varphi\|_2 = \|A_0^2 \varphi\| \qquad (\varphi \in H_2).$$

Consider the second order evolution equation

$$\ddot{w}(t) + A_0^2 w(t) = B_0 u(t), \quad w(0) = 0, \quad \dot{w}(0) = 0, \tag{2.8}$$

where $B_0 \in \mathcal{L}(U, H)$. In order to write this equation as a first order system we introduce the Hilbert space $X = H_1 \times H$ and the family of operators $(\widetilde{A_a})_{a > -\lambda_1}$, $\widetilde{A_a} : \mathcal{D}(\widetilde{A_a}) \to X$ defined by

$$\mathcal{D}(\widetilde{A_a}) = H_2 \times H_1, \qquad \widetilde{A_a} = \begin{bmatrix} 0 & I \\ -A_0^2 - aA_0 & 0 \end{bmatrix}, \tag{2.9}$$

where λ_1 is the first eigenvalue of the operator A_0. Since A_0 is strictly positive, is easy to prove that $(A_0^2 + aA_0)$ is a strictly positive operator with compact resolvents and so, $\widetilde{A_a}$, defined by (2.9), is a skew-adjoint operator. Applying Stone's theorem, we have that $\widetilde{A_a}$ generates an unitary group $\mathbb{T}$ on $X = H_1 \times H$. Finally, we introduce the control operator $B \in \mathcal{L}(U, X)$ defined by

$$B\mathrm{v} = \begin{bmatrix} 0 \\ B_0 \mathrm{v} \end{bmatrix} \qquad (\mathrm{v} \in U). \tag{2.10}$$

Then (2.8) can be written as

$$\dot{z}(t) = \widetilde{A_0} z(t) + Bu(t), \quad z(0) = 0,$$

where we have denoted $z(t) = \left[\begin{smallmatrix} w(t) \\ \dot{w}(t) \end{smallmatrix}\right]$.

In the remaining part of this section we recall some definitions and results on pseudo-periodic functions, borrowed from Kahane [9], which will be used in the proof of Theorem 1.2.

Let $\mathcal{I} \subset \mathbb{Z}$ an infinite set of integers. We say that $\Lambda = (\lambda_m)_{m \in \mathcal{I}} \subset \mathbb{R}^n$ is a *regular sequence* if there exists $\gamma > 0$ such that

$$\inf_{\substack{m,l \in \mathcal{I} \\ m \neq l}} |\lambda_m - \lambda_l| = \gamma. \tag{2.11}$$

Definition 2.8. An open subset $D \subset \mathbb{R}^n$ is called a domain associated to the regular sequence $\Lambda = (\lambda_m)_{m\in\mathcal{I}}$ if there exist constants $\delta_1(D), \delta_2(D) > 0$ such that, for every sequence of complex numbers $(a_m)_{m\in\mathcal{I}}$ with a finite number of non-vanishing terms, we have

$$\delta_2(D) \sum_{m\in\mathcal{I}} |a_m|^2 \leq \int_D \left| \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m\cdot x} \right|^2 \mathrm{d}x \leq \delta_1(D) \sum_{m\in\mathcal{I}} |a_m|^2. \tag{2.12}$$

Definition 2.9. Let $\Lambda = (\lambda_m)_{m\in\mathcal{I}}$ and $\tilde{\Lambda} = (\tilde{\lambda}_m)_{m\in\mathcal{I}}$ be two regular sequences in $\mathbb{R}^n$. We say that the sequences Λ and $\tilde{\Lambda}$ are asymptotically close if for every $\alpha > 0$ there exists an open ball $B \subset \mathbb{R}^n$ large enough such that

$$|\lambda_m - \tilde{\lambda}_m| < \alpha \qquad (m \in \mathcal{I} \text{ such that } \lambda_m, \tilde{\lambda}_m \in \mathbb{R}^n \setminus B).$$

We use below the following theorem from [9, Theorem III.2.2].

Theorem 2.10. *Let Λ and $\tilde{\Lambda}$ be two regular sequences asymptotically close. Then an open set $D \subset \mathbb{R}^n$ is an associated domain to Λ if and only if is an associated domain to $\tilde{\Lambda}$.*

3 From $\ddot{w} + A_0^2 w = B_0 u$ to $\ddot{w} + A_0^2 w + aA_0 w = B_0 u$

In this section we continue to use the notation introduced in the previous one. In particular, H and U are Hilbert spaces, A_0 is a strictly positive operator (possibly unbounded) on H, $B_0 \in \mathcal{L}(U, H)$ and $\widetilde{A_a}$, B are defined as in (2.9), (2.10). We consider the following differential equation

$$\ddot{w}(t) + A_0^2 w(t) + aA_0 w(t) = B_0 u(t) \tag{3.1}$$

$$w(0) = 0, \quad \dot{w}(0) = 0, \tag{3.2}$$

with $a > -\lambda_1$. With notation from the Section 2, the system (3.1)-(3.2) can be written in the form

$$\dot{z}(t) = \widetilde{A_a} z(t) + Bu(t), \quad z(0) = 0, \tag{3.3}$$

where $z(t) = \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}$.

Our aim is to show that if (3.3) is exactly controllable for $a = 0$ then it is exactly controllable for every $a > -\lambda_1$, where λ_1 is the first eigenvalue of A_0. The main result of this section is the following proposition, which gives no information on the controllability time. Note that $\left[\begin{smallmatrix} 0 \\ -aA_0 \end{smallmatrix}\right]$ is not a compact operator in the state space $X = H_1 \times H$, so that the compactness-uniqueness method introduced in [2] (see also [4]) cannot be applied. Our method is based on a spectral test of Hautus type introduced in [16].

Proposition 3.1. *Assume that the pair $(\widetilde{A_0}, B)$ is exactly controllable. Then for every $a > -\lambda_1$ the pair $(\widetilde{A_a}, B)$ is exactly controllable.*

Proof. Recall from Section 2 that $\widetilde{A_a}$ is skew-adjoint with compact resolvents for every $a > -\lambda_1$. The conclusion is obtained bellow by first showing that in "high frequency" a wave packet (as defined in Section 2) is exactly observable.

Denote $\varphi_{-n} = \varphi_n$ for all $n \in \mathbb{N}^*$. For every $a > -\lambda_1$ we denote by $(\lambda_n(a))_{n\in\mathbb{N}^*}$ the eigenvalues of the operator $(A_0^2 + aA_0)^{\frac{1}{2}}$ associated to the eigenvectors $(\varphi_n)_{n\in\mathbb{N}^*}$. It is easy to verify that

$$\lambda_n(a) = \lambda_n \sqrt{1 + a\lambda_n^{-1}} = \lambda_n + \frac{a}{2} + \frac{a^2}{8} o(\lambda_n^{-1}), \tag{3.4}$$

where, as in Section 2, $\lambda_n = \lambda_n(0)$ are the eigenvalues of the operator A_0. Then the family $(\Phi_n(a))_{n\in\mathbb{Z}^*}$ given by

$$\Phi_n(a) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_n(a)}\varphi_n \\ \varphi_n \end{bmatrix} \qquad (n \in \mathbb{Z}^*) \tag{3.5}$$

is an orthonormal basis of eigenvectors of the operator $\widetilde{A_a}$ associated to the eigenvalues $(i\mu_n(a))_{n\in\mathbb{Z}^*}$, where

$$\mu_n(a) = \begin{cases} -\lambda_n(a), & \text{if } n \in \mathbb{N}^* \\ \lambda_n(a), & \text{if } -n \in \mathbb{N}^*, \end{cases} \tag{3.6}$$

for every $a > -\lambda_1$.

For $\varepsilon > 0$, $\omega \in \mathbb{R}$ and $a > -\lambda_1$ we define

$$J_\varepsilon(\omega, a) = \{m \in \mathbb{Z}^* \text{ such that } |\mu_m(a) - \omega| < \varepsilon\}. \tag{3.7}$$

Since the pair $(\widetilde{A_0}, B)$ is exactly controllable we know from Theorem 2.7 that there exist $\varepsilon, \delta > 0$ such that for all $\omega \in \mathbb{R}$ we have

$$\|B^*\varphi\|_U \geq \delta\|\varphi\|_X, \tag{3.8}$$

for every wave packet $\varphi = \sum\limits_{m\in J_\varepsilon(\omega,0)} c_m\Phi_m(0)$.

The idea is to prove that the inequality (3.8) implies a similar inequality for every wave package $\psi = \sum_{m\in J_\varepsilon(\omega,a)} c_m\Phi_m(a)$. For the remaining part of this proof we consider $a > -\lambda_1$ fixed. Since $|\mu_k(a)| \to \infty$ when $k \to \infty$, there exists an $\omega_a > 0$ such that for every $|\omega| \geq \omega_a$ if $m \in J_\varepsilon(\omega, a)$ then $\frac{a}{8}|\mu_m(a)^{-1}| \leq \frac{1}{2}$. Let ω be such that $|\omega| \geq \omega_a$. Then $m \in J_\varepsilon(\omega, 0)$ is equivalent to

$$|\lambda_m(a) - \frac{a}{2} - \frac{a^2}{8}o(\lambda_m^{-1}) - \omega| < \varepsilon \Leftrightarrow m \in J_\varepsilon(\omega + a, a),$$

so $J_\varepsilon(\omega, 0) = J_\varepsilon(\omega + a, a)$. Since

$$B^*\psi = \sum_{m\in J_\varepsilon(\omega,a)} c_m B^*\Phi_m(a) = \sum_{m\in J_\varepsilon(\omega,a)} c_m \frac{1}{\sqrt{2}} B_0^*\varphi_m = \sum_{m\in J_\varepsilon(\omega-a,0)} c_m B^*\Phi_m(0),$$

from (3.8) we have for every ω with $|\omega| \geq \omega_a$ that

$$\|B^*\psi\|_U \geq \delta\|\psi\|_X.$$

Denote $V = \text{span}\{\Phi_m(a) \mid |\mu_m(a)| < \omega_a\}^\perp$. From the above inequality and Theorem 2.7 we obtain that $(\widetilde{A_a}|_V, B_V)$, with $\widetilde{A_a}|_V$ as in Definition 2.5, is exactly controllable.

From the exact controllability of the pair $(\widetilde{A_0}, B)$ it is clear that

$$B^*\Phi_n(a) = \frac{1}{\sqrt{2}} B_0^*\varphi_n = B^*\Phi(0) \neq 0, \qquad (n \in \mathbb{Z}^*, a > -\lambda_1)$$

and, using Proposition 2.6, the pair $(\widetilde{A_a}, B)$ is exactly controllable. □

The particular case where A_0 is the Dirichlet Laplacian is discussed in the following example.

Example 3.2. We consider the problem of exact controllability of the following linear plate equation

$$\ddot{w}(x,t) + \Delta^2 w(x,t) - a\Delta w(x,t) = u(x,t)\chi_{\mathcal{O}}, \quad \text{for } (x,t) \in \Omega \times (0,\infty) \tag{3.9}$$

$$w(x,t) = \Delta w(x,t) = 0, \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty) \tag{3.10}$$

$$w(x,0) = 0, \quad \dot{w}(x,0) = 0, \quad \text{for } x \in \Omega. \tag{3.11}$$

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with its boundary $\partial\Omega$ of class C^2. Assume that Ω and its open subset $\mathcal{O}$ are such that the Bardos, Lebeau and Rauch geometric control condition is verified, i.e. that there exists $\tau > 0$ such that any light ray traveling in Ω at unit speed and reflected according to geometric optics laws when it hits $\partial\Omega$, will intersect $\mathcal{O}$ in a time smaller than τ (see [2]). Then the problem (3.9)-(3.11) is exactly controllable, i.e., there exists a time $\tau > 0$, such that for every $\left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right] \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ exists a control $u \in L^2([0,\tau]; L^2(\mathcal{O}))$ such that the solution w of (3.9)-(3.11) satisfies

$$w(x,\tau) = w_0(x), \quad \dot{w}(x,\tau) = w_1(x) \qquad (x \in \Omega).$$

Indeed, denote $H = L^2(\Omega)$, $H_1 = H^2(\Omega) \cap H_0^1(\Omega)$,

$$H_2 = \{\varphi \in H^4(\Omega) \mid \varphi = \Delta\varphi = 0 \text{ on } \partial\Omega\}$$

and $U = L^2(\mathcal{O})$. Let $A_0 : H_1 \to H$ be defined by

$$A_0\varphi = -\Delta\varphi \qquad (\varphi \in H_1)$$

and $B_0 u = u\chi_{\mathcal{O}}$. Then, like in Section 2, we introduce the operators $\widetilde{A_a}$ and B, given by (2.9) and (2.10). Then (3.9)-(3.11) can be written as

$$\dot{z}(t) = \widetilde{A}_a z(t) + Bu(t), \quad z(0) = 0,$$

where $z = \left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]$. Since we supposed that Ω and $\mathcal{O}$ satisfy the geometric condition of Bardos, Lebeau and Rauch, $(\widetilde{A_0}, B)$ is exactly controllable in arbitrarily small time (see [11]). Then, from Proposition 3.1, we conclude that $(\widetilde{A_a}, B)$ is exactly controllable for every $a > -\lambda_1$, where λ_1 is the first eigenvalue of the Dirichlet Laplacian in Ω.

4 From $\ddot{w} + A_0^2 w + aA_0 w = B_0 u$ to $\ddot{w} + A_0^2 w + (a + b\|A_0^{\frac{1}{2}} w\|^2)A_0 w = B_0 u$

In this section we consider the following perturbation of the linear differential equation studied in Section 3:

$$\ddot{w}(t) + A_0^2 w(t) + (a + b\|A_0^{\frac{1}{2}} w(t)\|^2)A_0 w(t) = B_0 u(t) \tag{4.1}$$

$$w(0) = \dot{w}(0) = 0, \tag{4.2}$$

where $a > -\lambda_1$, $b > 0$ and $B_0 \in \mathcal{L}(U, H)$.

The principal result of this subsection is the following.

Theorem 4.1. *Assume that* (3.1)-(3.2) *is exactly controllable in time* $\tau > 0$. *Then* (4.1)-(4.2) *is locally exactly controllable in time* τ, *i.e., there exists a constant* $M > 0$ *such that for every* $\left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right] \in H_1 \times H$, *with* $\|w_0\|_{H_1}^2 + \|w_1\|_H^2 \leq M^2$ *exists a control* $u \in L^2([0,\tau]; U)$, *such that the solution* w *of* (4.1)-(4.2) *satisfies*

$$w(\tau) = w_0, \quad \dot{w}(\tau) = w_1.$$

Proof. Recall that since $(A_0^2 + aA_0)$ is a strictly positive operator with compact resolvents, the operator $\widetilde{A_a}$ is skew-adjoint and, applying theorem of Stone, generates a unitary group $\mathbb{T}$ in $X = H_1 \times H$. We denote $G : H_1 \times H \to H_1 \times H$

$$G \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b\|A_0^{\frac{1}{2}} w_1\|^2 A_0 w_1 \end{bmatrix}.$$

Then (4.1)-(4.2) can be written as

$$\dot{z}(t) = \widetilde{A_a} z(t) + Gz(t) + Bu(t), \quad z(0) = 0, \tag{4.3}$$

where $z = \left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]$ and $Bu(t) = \left[\begin{smallmatrix} 0 \\ B_0 u(t) \end{smallmatrix}\right]$. Let us consider the following linear equation

$$\dot{z}(t) = \widetilde{A_a} z(t) + F(t) + Bu(t), \quad z(0) = 0, \tag{4.4}$$

where $F = \left[\begin{smallmatrix} 0 \\ f(t) \end{smallmatrix}\right]$ and $f \in L^2([0,T]; H)$. Let $z_0 \in X$. Since the pair $(\widetilde{A_a}, B)$ is exactly controllable in time τ we can consider a control operator $F_\tau \in \mathcal{L}(X, L^2([0,\tau]; U))$ as in Proposition 2.3. Using Corollary 2.4 we see that the input function u given by (2.5) is such that $z(\tau) = z_0$.

Consider the mapping $\mathcal{F} : L^2([0,\tau]; H) \to L^2([0,\tau]; H)$ defined by

$$\mathcal{F}(f) = b\|A_0^{\frac{1}{2}} w\|_{L^2(0,\tau;H)} A_0 w,$$

where $\left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]$ is the solution of (4.4) with u given by (2.5).

To obtain the conclusion of the theorem it suffices to show that $\mathcal{F}$ has a fixed point. Let $M > 0$ to be fixed later and $f \in L^2([0,\tau]; H)$ with $\|f\|_{L^2([0,\tau];H)} \leq M$. We first show that if M is small enough the ball $B(0, M)$ of center 0 and radius M is invariant for $\mathcal{F}$ in $L^2([0,\tau]; H)$. Since the operator F_τ given by Proposition 2.3 is bounded, from (2.5) we obtain easily that there exists a constant $C_\tau > 0$ such that

$$\|u\|_{L^2([0,\tau];U)} \leq C_\tau(\|z_0\|_X + \|f\|_{L^2([0,\tau];H)}). \tag{4.5}$$

Using the formula

$$z(\tau) = \int_0^\tau \mathbb{T}_{\tau-s} F(s)\, \mathrm{d}s + \Phi_\tau u$$

combined to the inequality (4.5) and renoting the constant, we obtain

$$\|z\|_{C([0,\tau];X)} = \|w\|_{C([0,\tau];H_1)} + \|\dot{w}\|_{C([0,\tau];H)} \leq C_\tau(\|z_0\|_X + \|f\|_{L^2([0,\tau];H)}).$$

From the last estimate we can conclude that

$$\begin{aligned} \|\mathcal{F}(f)\|_{L^2([0,\tau];H)} = b\|A_0^{\frac{1}{2}} w\|^2_{L^2([0,\tau];H)} \|A_0 w\|_{L^2([0,T];H)} &\leq bC_\tau \|z\|^3_{C([0,\tau];X)} \\ &\leq bC_\tau \left(\|z_0\|_X + \|f\|_{L^2([0,\tau];H)}\right)^3. \end{aligned}$$

Then, assuming that $0 < M < \frac{1}{2bC_\tau\sqrt{2}}$ and $\|z_0\|_X < M$, we obtain from the previous estimate that $\mathcal{F}(B(0, M)) \subset B(0, M)$.

We next show that $\mathcal{F}$ is a contraction of $B(0, M)$. Let $f_1, f_2 \in L^2([0,\tau]; H)$ two functions such that $\|f_1\|, \|f_2\| \leq M$. Let $u_1, u_2 \in L^2([0,\tau]; U)$ be the controls given by (2.5) for $f = f_1$, respectively $f = f_2$, and denote z_1 and z_2 the solutions of

$$\dot{z}_1(t) = \widetilde{A_a} z_1(t) + \begin{bmatrix} 0 \\ f_1(t) \end{bmatrix} + Bu_1(t), \quad z_1(0) = 0, \tag{4.6}$$

$$\dot{z}_2(t) = \widetilde{A_a} z_2(t) + \begin{bmatrix} 0 \\ f_2(t) \end{bmatrix} + Bu_2(t), \quad z_2(0) = 0. \tag{4.7}$$

Then we have

$$\|\mathcal{F}(f_1) - \mathcal{F}(f_2)\|_{L^2([0,\tau];H)} = \left\| \|A_0^{\frac{1}{2}} w_1\|^2_{L^2([0,\tau];H)} A_0 w_1 - \|A_0^{\frac{1}{2}} w_2\|^2_{L^2([0,\tau];H)} A_0 w_2 \right\|_{L^2([0,\tau];H)}$$

$$\leq \|A_0^{\frac{1}{2}} w_1\|^2_{L^2([0,\tau];H)} \|A_0(w_1 - w_2)\|_{L^2([0,\tau];H)}$$

$$+ \left\| A_0^{\frac{1}{2}}(w_1 - w_2) \right\|_{L^2([0,\tau];H)} \left(\|A_0^{\frac{1}{2}} w_1\|_{L^2([0,\tau];H)} + \|A_0^{\frac{1}{2}} w_2\|_{L^2([0,\tau];H)} \right) \|A_0 w_2\|_{L^2([0,\tau];H)},$$

where $z_1 = \left[\begin{smallmatrix} w_1 \\ \dot{w}_1 \end{smallmatrix} \right]$ and $z_2 = \left[\begin{smallmatrix} w_2 \\ \dot{w}_2 \end{smallmatrix} \right]$. If we denote $\tilde{z} = z_1 - z_2$ then we have

$$\dot{\tilde{z}}(t) = \widetilde{A_a} \tilde{z}(t) + \begin{bmatrix} 0 \\ (f_1 - f_2)(t) \end{bmatrix} + B(u_1(t) - u_2(t)), \quad \tilde{z}(0) = \tilde{z}(\tau) = 0.$$

It is easily to prove that there exists a constant $C > 0$ such that

$$\|w_1 - w_2\|_{C([0,\tau];\mathcal{D}(A_0)} \leq C(\|f_1 - f_2\|_{L^2([0,\tau],H)} + \|u_1 - u_2\|_{L^2([0,\tau];U)}).$$

Moreover, using the form of u_1 and u_2, we have

$$u_1 - u_2 = -F_\tau \int_0^\tau \mathbb{T}_{\tau - s} \left[\begin{smallmatrix} 0 \\ (f_1 - f_2)(t) \end{smallmatrix} \right] \, \mathrm{d}s.$$

Therefore, there exist two constants $C_1, C_2 > 0$ such that

$$\|A_0(w_1 - w_2)\|_{L^2(0,\tau;H)} \leq C_1 \|f_1 - f_2\|_{L^2(0,\tau;H)},$$

$$\|A_0^{\frac{1}{2}}(w_1 - w_2)\|_{L^2(0,\tau;H)} \leq C_2 \|f_1 - f_2\|_{L^2(0,\tau;H)}.$$

Using the above estimates there exists a constant $\alpha > 0$, depending on the time τ, such that

$$\|\mathcal{F}(f_1) - \mathcal{F}(f_2)\|_{L^2([0,\tau];H)} \leq \alpha \left[(\|z_0\| + \|f_1\|_{L^2([0,\tau];H)})^2 \right.$$
$$\left. + (\|z_0\| + \|f_1\|_{L^2([0,\tau];H)})(\|z_0\| + \|f_2\|_{L^2([0,\tau];H)}) + (\|z_0\| + \|f_2\|_{L^2([0,\tau];H)})^2 \right] \|f_1 - f_2\|_{L^2([0,\tau];H)},$$

for all $f_1, f_2 \in L^2([0,\tau];H)$ with $\|f_1\|_{L^2([0,\tau];H)}, \|f_2\|_{L^2([0,\tau];H)} \leq M$.

Then, choosing M small enough, $\|z_0\|_X < M$ and denoting $C = 12\alpha M^2$, we can write the previous estimate as

$$\|\mathcal{F}(f_1) - \mathcal{F}(f_2)\|_{L^2([0,\tau];H)} \leq C \|f_1 - f_2\|_{L^2([0,\tau];H)},$$

so $\mathcal{F}$ is a contraction in $B(0, M)$.

Using Banach fixed point theorem, we obtain that $\mathcal{F}$ has a fixed point, and the proof of the theorem is completed. □

Example 4.2. In one space dimension, i.e. $\Omega = (0, \pi)$, the initial system (1.4)-(1.6) becomes

$$\ddot{w}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) - \left(a + b \int_0^\pi \left| \frac{\partial w}{\partial x}(x,t) \right|^2 \, \mathrm{d}x \right) \frac{\partial^2 w}{\partial x^2}(x,t) = u\chi_{\mathcal{O}}, \quad x \in (0,\pi), t > 0 \tag{4.8}$$

$$w(0,t) = w(\pi,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\pi,t), \quad t \in (0,\infty) \tag{4.9}$$

$$w(x,0)=0, \quad \dot{w}(x,0)=0, \quad x\in(0,\pi), \tag{4.10}$$

where $a>-1$, $b>0$ and $\mathcal{O}\in\Omega$ is an open interval. The above equations are a model for the free vibrations of a hinged extensible beam compressed (if $a>0$) or stretched (if $a<0$) by an axial force, which has been extensively studied in the literature (see Dickey [5], [6], Ball [1]).

We claim that this problem is locally exactly controllable in arbitrarily small time. Indeed, denote $H=L^2(0,\pi)$, $H_1=H^2(0,\pi)\cap H_0^1(0,\pi)$ and

$$H_2=\left\{\varphi\in H^4(0,\pi)\cap H_0^1(0,\pi) \text{ such that } \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(0)=\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(\pi)=0\right\}.$$

Let $A_0: H_1\to H$ be the operator defined by $A_0\varphi=-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}$ for all $\varphi\in H_1$. Using the operator $\widetilde{A_a}$, defined by (2.9), we can write (4.8)-(4.10) on the form of system (4.1)-(4.2). Denote $\varphi_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$ the eigenvectors of $(A_0^2+aA_0)^{\frac{1}{2}}$ associated to the eigenvalues $\lambda_n(a)=\sqrt{n^4+an^2}$ and $\varphi_{-n}=\varphi_n$ for all $n\in\mathbb{N}^*$. It follows that $(\Phi_n)_{n\in\mathbb{Z}^*}$ given by

$$\Phi_n(x)=\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{i\mu_n(a)}\sqrt{\frac{2}{\pi}}\sin(nx)\\ \sqrt{\frac{2}{\pi}}\sin(nx)\end{bmatrix}$$

is an orthonormal basis formed by the eigenvectors of $\widetilde{A_a}$ associated to the eigenvalues $(i\mu_n(a))_{n\in\mathbb{Z}^*}$, where

$$\mu_n(a)=\begin{cases}-\lambda_n(a), & \text{if } n\in\mathbb{N}^*\\ \lambda_n(a), & \text{if } -n\in\mathbb{N}^*,\end{cases}$$

Is easy to verify that for any $a>-1$ we have

$$\lim_{n\to\infty}|\mu_{n+1}(a)-\mu_n(a)|=\infty.$$

Using Proposition 8.1.3 from [18], we obtain that the linear part of the (4.8) is exactly controllable in a time arbitrarily small. So, applying Theorem 4.1 we obtain the local exact controllability of equation (4.8)-(4.10) in a time arbitrarily small.

5 Proof of main results

In this section $\Omega\subset\mathbb{R}^2$ is a bounded open set with C^2 boundary or Ω is a rectangle and the operator A_0 is as in Example 3.2, i.e., $A_0\in\mathcal{L}(H_1,H)$, where

$$H_1=H^2(\Omega)\cap H_0^1(\Omega), \quad A_0\varphi=-\Delta\varphi \qquad (\varphi\in H_1).$$

Then, $H_2=\mathcal{D}(A_0^2)=\{\varphi\in H^4(\Omega)\mid\varphi=\Delta\varphi=0 \text{ on } \partial\Omega\}$ and

$$A_0^2\varphi=\Delta^2\varphi \qquad (\varphi\in H_2).$$

If $\mathcal{O}\subset\Omega$ is an open and nonempty set we denote $U=L^2(\mathcal{O})$ and we consider $B_0\in\mathcal{L}(U,H)$ defined by $B_0u=u\chi_{\mathcal{O}}$.

Proof of Theorem 1.1. With the above notation, the problem (1.4)-(1.6) can be written as

$$\ddot{w}(t)+A_0^2w(t)+(a+b\|A_0^{\frac{1}{2}}w(t)\|^2)A_0w(t)=B_0u(t) \tag{5.1}$$

$$w(0)=0, \quad \dot{w}(0)=0. \tag{5.2}$$

By the hypothesis, for $a = b = 0$, the system (5.1)-(5.2) is exactly controllable in some time $\tau_0 > 0$. Applying Proposition 3.1 we have that (5.1)-(5.2), with A_0 and B_0 chosen like above, is exactly controllable in a time $\tau > 0$ for every $a > -\lambda_1$ and $b = 0$, where λ_1 is the first eigenvalue of Dirichlet Laplacian in Ω. Then from Theorem 4.1 we obtain that (5.1)-(5.2) is locally exactly controllable in time τ, which means that (1.4)-(1.6) is locally exactly controllable. □

As already mentioned the above result gives no information on the controllability time of (1.4)-(1.6). This shortcoming can be remedied in the case of a rectangular domain Ω by using the explicit knowledge of the eigenvectors and of the eigenvalues of A_0. We prove first an exact observability result for plate equation in a rectangular domain. Consider the initial and boundary value problem:

$$\ddot{w}(x,t) + \Delta^2 w(x,t) - a\Delta w(x,t) = 0, \quad \text{for } (x,t) \in \Omega \times (0,\infty) \tag{5.3}$$

$$w(x,t) = \Delta w(x,t) = 0, \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty), \tag{5.4}$$

$$w(x,0) = w_0, \quad \dot{w}(x,0) = w_1, \quad \text{for } x \in \Omega, \tag{5.5}$$

where $a > -\lambda_1$. We consider the following output:

$$y(t) = \dot{w}(\cdot,t)|_{\mathcal{O}}. \tag{5.6}$$

Proposition 5.1. *Let $\Omega = (0,l_1) \times (0,l_2)$ be a rectangle in $\mathbb{R}^2$ and $\mathcal{O}$ an open and nonempty subset of Ω. Then (5.3)-(5.6) is exactly observable in the state space $H_1 \times H$ in any time $\tau > 0$.*

We use notation and results on pseudo-periodic functions which have been recalled in Section 2. The following proposition plays a central role in the proof of Proposition 5.1.

Proposition 5.2. *Let $r, s > 0$, $a > -(r+s)$ and let $\Lambda = (\mu_{mn})_{m,n\in\mathbb{Z}^*}$ be a sequence defined by*

$$\mu_{mn} = \begin{bmatrix} m\sqrt{r} \\ n\sqrt{s} \\ \sqrt{(rm^2+sn^2)^2 + a(rm^2+sn^2)} \end{bmatrix}. \tag{5.7}$$

Then any open and nonempty set in $\mathbb{R}^3$ is a domain associated to Λ in the sense of Definition 2.8.

Proof. Consider the sequence $(\alpha_{mn})_{m,n\in\mathbb{Z}^*}$ in $\mathbb{R}^3$ defined by

$$\alpha_{mn} = \begin{bmatrix} m\sqrt{r} \\ n\sqrt{s} \\ rm^2+sn^2 \end{bmatrix} \qquad (m,n \in \mathbb{Z}^*). \tag{5.8}$$

According to Jaffard [8], every open and nonempty set $D \subset \mathbb{R}^3$ is an associated domain to (α_{mn}). This clearly implies that every open and nonempty set in $\mathbb{R}^3$ is an associated domain to the sequence $\tilde{\Lambda} = (\tilde{\mu}_{m,n})_{m,n\in\mathbb{Z}^*}$ defined by

$$\tilde{\mu}_{mn} = \begin{bmatrix} m\sqrt{r} \\ n\sqrt{s} \\ rm^2+sn^2+\frac{a}{2} \end{bmatrix}.$$

It is easy to check that

$$\sqrt{(rm^2+sn^2)^2 + a(rm^2+sn^2)} = rm^2+sn^2+\frac{a}{2}+\varepsilon_{mn},$$

with $\lim_{m^2+n^2\to\infty} \varepsilon_{mn} = 0$. From that it follows that the sequences Λ and $\tilde{\Lambda}$ are asymptotically close in the sense of Definition 2.9. Therefore, applying Theorem 2.10, every open nonempty set in $\mathbb{R}^3$ is an associated domain to Λ. □

To prove Proposition 5.1 we also need the following lemma:

Lemma 5.3. *With notation from the beginning of this section, let $C_0 \in \mathcal{L}(H, U)$ be the operator defined by $C_0 w = w|_{\mathcal{O}}$. The pair $(i(A_0^2 + aA_0)^{\frac{1}{2}}, C_0)$ is exactly observable in H in every time $\tau > 0$, i.e., there exists a constant $k_\tau > 0$ such that*

$$\int_0^\tau \|C_0 \mathbb{S}_t w_0\|_U^2 \, dt \geq k_\tau^2 \|w_0\|^2 \qquad (w_0 \in H),$$

where $(\mathbb{S}_t)$ is the semigroup generated by $i(A_0^2 + aA_0)^{\frac{1}{2}}$.

Proof. Denote $(\lambda_{mn})_{m,n\in\mathbb{N}^*}$ the eigenvalues of Dirichlet Laplacian in Ω, given by

$$\lambda_{mn} = rm^2 + sn^2 \qquad (m, n \in \mathbb{N}^*),$$

where $r = \left(\frac{\pi}{l_1}\right)^2$, $s = \left(\frac{\pi}{l_2}\right)^2$ and denote by (φ_{mn}) a corresponding orthonormal basis formed by eigenvectors of A_0

$$\varphi_{mn}(x, y) = \frac{2}{\sqrt{l_1 l_2}} \sin(\sqrt{r}mx) \sin(\sqrt{s}ny) \qquad (m, n \in \mathbb{N}^*,\ (x, y) \in \Omega).$$

It is easy to check that for every $m, n \in \mathbb{N}^*$ φ_{mn} is an eigenvector of $(A_0^2 + aA_0)^{\frac{1}{2}}$ with corresponding eigenvalue

$$\lambda_{mn}(a) = \sqrt{(rm^2 + sn^2)^2 + a(rm^2 + sn^2)}.$$

The above facts imply that for every $a > -\lambda_{11}(0)$ the semigroup $\mathbb{S}$ generated by $i(A_0^2 + aA_0)^{\frac{1}{2}}$ verifies

$$\mathbb{S}_t z = \sum_{m,n} z_{mn} e^{i\lambda_{mn}(a)t} \varphi_{mn} \qquad (z \in H_1),$$

where we have denoted

$$z_{mn} = \langle z, \varphi_{mn} \rangle \qquad (m, n \in \mathbb{N}^*).$$

Therefore, we have

$$\begin{aligned}
\int_0^\tau \|C_0 \mathbb{S}_t z\|^2 \, \mathrm{d}t &= \int_0^\tau \int_{\mathcal{O}} \left| \sum_{m,n\in\mathbb{N}^*} z_{mn} e^{i\lambda_{mn}(a)t} \varphi_{mn}(x, y) \right|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= \frac{4}{l_1 l_2} \int_0^\tau \int_{\mathcal{O}} \left| \sum_{m,n\in\mathbb{N}^*} z_{mn} e^{i\lambda_{mn}(a)t} \sin(\sqrt{r}mx) \sin(\sqrt{s}ny) \right|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t \qquad (5.9)
\end{aligned}$$

Let us extend the sequence (z_{mn}) by setting

$$z_{-m,n} = -z_{mn},\ z_{m,-n} = -z_{mn},\ z_{-m,-n} = z_{mn}, \qquad (m, n \in \mathbb{N}^*).$$

Then (5.9) can be written as

$$\int_0^\tau \|C_0 \mathbb{S}_t z\|^2 \, \mathrm{d}t = \frac{1}{l_1 l_2} \int_0^\tau \int_{\mathcal{O}} \left| \sum_{m,n\in\mathbb{Z}^*} z_{mn} e^{i\mu_{mn} \cdot \left[\begin{smallmatrix} x \\ y \\ t \end{smallmatrix}\right]} \right|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t,$$

where (μ_{mn}) is defined by (5.7). By Proposition 5.2, $(\mu_{mn})_{mn}$ is a sequence associated to the domain $\mathcal{O} \times (0, \tau)$ for any $\tau > 0$ and any open and nonempty $\mathcal{O} \in \Omega$. Using the definition of an associated sequence to a domain it follows that there exists a constant $c > 0$ such that

$$\int_0^\tau \|C_0 \mathbb{S}_t z\|^2 \, \mathrm{d}t \geq c^2 \sum_{m,n \in \mathbb{Z}^*} |z_{mn}|^2.$$

We have thus shown that the pair $(i(A_0^2 + aA_0)^{\frac{1}{2}}, C_0)$ is exactly observable in any time $\tau > 0$. □

Proof of Proposition 5.1. Let $C_0 \in \mathcal{L}(U, H)$ be the operator introduced by Lemma 5.3. From this lemma we have that the pair $(i(A_0^2 + aA_0)^{\frac{1}{2}}, C_0)$ is exactly observable in any time $\tau > 0$. Applying Proposition 6.8.2 from [18] (with A_0 replaced by $(A_0^2 + aA_0)^{\frac{1}{2}}$), we obtain that the pair $(\widetilde{A_a}, C)$, with $C = \left[\begin{smallmatrix} 0 \\ C_0 \end{smallmatrix}\right]$, is exactly observable in any time $\tau > 0$. □

A direct consequence of Proposition 5.1 and Proposition 2.2 is the following corollary.

Corollary 5.4. *Let $\Omega = (0, l_1) \times (0, l_2)$ be a rectangle and $\mathcal{O}$ an open and nonempty subset of Ω. Then for every $a > -\lambda_1$ and $b = 0$ the system* (1.4)-(1.6) *is exactly controllable in any time $\tau > 0$.*

Proof of Theorem 1.2. With notation from the beginning of this section the system (1.4)-(1.6) can be written, like in the proof of Theorem 1.1, as an abstract system (5.1)-(5.2). From Corollary 5.4 we obtain that for $b = 0$ and for every $a > -\lambda_1$ the system (5.1)-(5.2) is exactly controllable in a time $\tau > 0$ arbitrarily small. Then, applying Theorem 4.1, we can conclude that (5.1)-(**??**) is locally exactly controllable in time $\tau > 0$, so we proved that, if Ω is a rectangle, Berger equation (1.4)-(1.6) is locally exactly controllable in a time arbitrarily small. □

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