Under-sampled in time observers for the wave equation

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Wave equation

Wave equation :

$$\begin{cases} \ddot{w}(t,x) - \Delta w(t,x) = 0, & t > 0, x \in \Omega \\ w(t,x) = 0 & t > 0, x \in \partial\Omega \\ w(0,x) = w_0(x), \qquad \dot{w}(0,x) = w_1(x), & x \in \Omega. \end{cases}$$

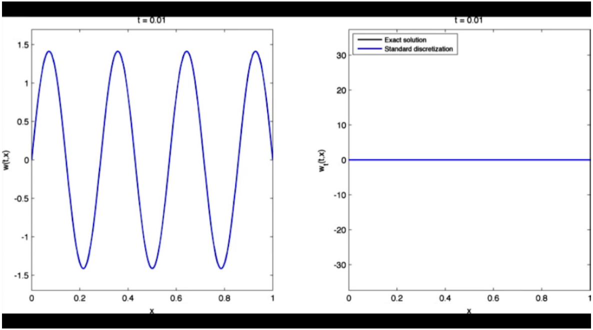
Energy of the solution :

$$E(t) = \frac{1}{2}\left(\|w(t,\cdot)\|^2_{H^1_0(\Omega)} + \|\dot{w}(t,\cdot)\|^2_{L^2(\Omega)}\right).$$

Discretization :

- P_1 finite elements in space.
- midpoint finite differences scheme in time.

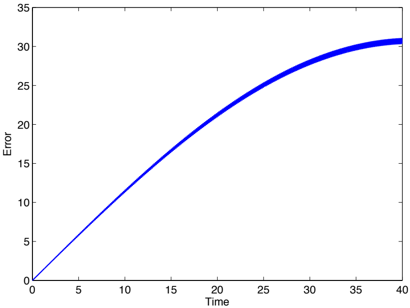
Wave equation
Numerical simulation



One-dimensional wave equation discretized using :

- P_1 finite elements in space.
- midpoint finite differences scheme in time.

Wave equation
Error increases with time



- h is the discretization step in space
- Δt is the discretization step in time

Relative error at time $k\Delta t$:

$$\|E_h^k\| \leq C(w_0, w_1) k \kappa(h, \Delta t), \qquad k \in \mathbb{N}.$$

Go Back

Outline

Outline

Aim of the talk

Supposing that measurements of the system's state are available, we propose a method to fully discretize second order conservative systems

$$\begin{cases} \ddot{w}(t) + A_0 w(t) = 0, & t > 0 \\ w(0) = w_0, & \dot{w}(0) = w_1, \end{cases}$$

such that the discretization error E_h^k at the time $k\Delta t$ verifies

$$\|E_h^k\| \leq C(w_0, w_1)\kappa(h, \Delta t).$$

Known data :

$$z(t) = B_0 w(t), \qquad \text{where}$$

- $A_0 : \mathcal{D}(A_0) \to H$ self-adjoint, positive, with compact resolvents.
- $B_0 \in \mathcal{L}(\mathcal{D}(A_0^{\frac{1}{2}}), Z)$.

A first order system

We write this system as a first order system

$$\begin{cases} \dot{x}(t) = Ax(t), & t > 0 \\ x(0) = x_0, \end{cases}$$

where $A : \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}) \to X = \mathcal{D}(A_0^{\frac{1}{2}}) \times H$ and

$$x(t) = \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}.$$

The known data are now given by

$$z(t) = Bx(t), \qquad B = \begin{pmatrix} B_0 & 0 \end{pmatrix}.$$

Some more notation

- $(V_h)_h$ a family of finite dimensional subspaces of $\mathcal{D}(A_0^{\frac{1}{2}})$.
- $\pi_h : H \to V_h$, $\widetilde{\pi}_h : \mathcal{D}(A_0^{\frac{1}{2}}) \to V_h$ orthogonal projectors with respect to inner product in H and $\mathcal{D}(A_0^{\frac{1}{2}})$, respectively.
- $A_{0h} \in \mathcal{L}(V_h)$:

$$\langle A_{0h}\phi_h, \psi_h \rangle = \langle A_0^{\frac{1}{2}}\phi_h, A_0^{\frac{1}{2}}\psi_h \rangle, \qquad \forall \phi_h, \psi_h \in V_h.$$

- $A_h \in \mathcal{L}(V_h \times V_h)$, $\qquad A_h = \begin{pmatrix} 0_h & I_h \\ -A_{0h} & 0_h \end{pmatrix}$.
- $\Pi_h : \mathcal{D}(A_0^{\frac{1}{2}}) \times H \to V_h \times V_h$, $\qquad \Pi_h = \begin{pmatrix} \widetilde{\pi}_h & 0 \\ 0 & \pi_h \end{pmatrix}$.

Fully discretized system

First order continuous system :

$$\begin{cases} \dot{x}(t) = Ax(t), & t > 0 \\ x(0) = x_0. \end{cases}$$

Full discretization :

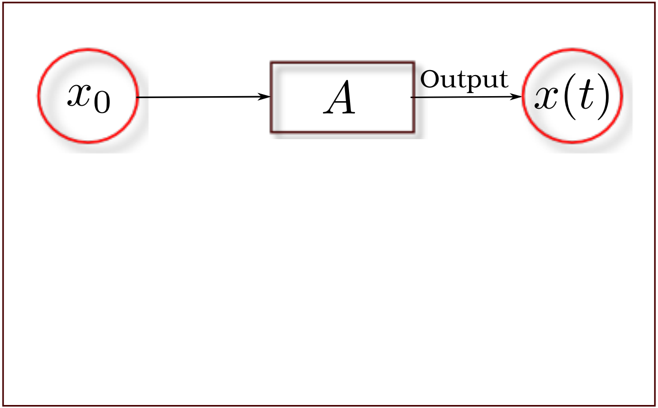
- Galerkin method in space.
- finite differences midpoint scheme in time.

$$\boxed{\begin{cases} \dfrac{x_h^{k+1} - x_h^k}{\Delta t} = A_h \dfrac{x_h^k + x_h^{k+1}}{2}, & k \in \mathbb{N}^* \\ x_h^0 = \Pi_h x_0. \end{cases}}$$

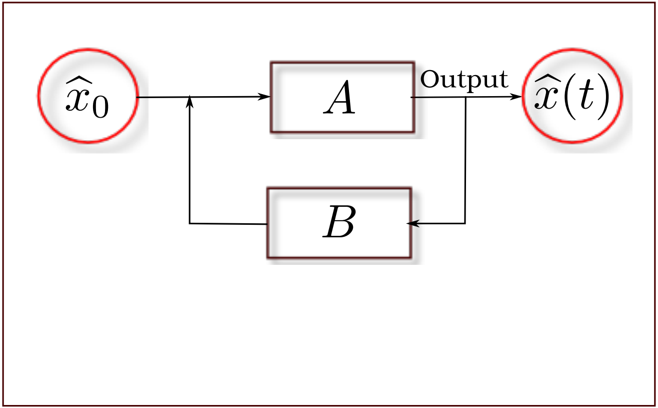
$$E_h^k = x_h^k - \Pi_h x(k\Delta t).$$

Error estimate

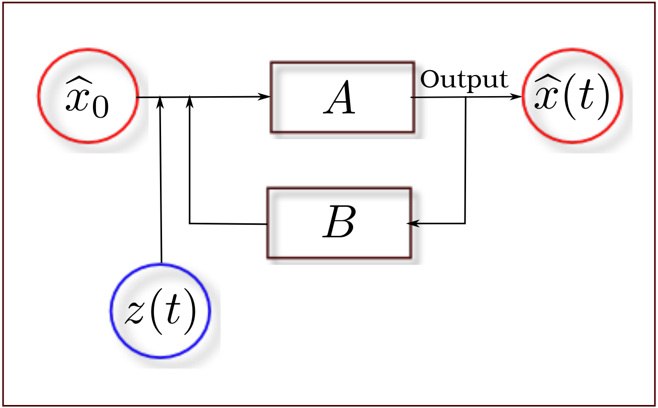
Luenberger observer



Luenberger observer



Luenberger observer



Luenberger observer

D. G. Luenberger, *An introduction to observers*. IEEE Trans. Autom. Control 16 (1971) 596-602.

We consider the following observer

$$\begin{cases} \dot{\widehat{x}}(t) = A\widehat{x}(t) + \gamma B^*(z(t) - B\widehat{x}(t)), & t > 0 \\ \widehat{x}(0) = \widehat{x}_0. \end{cases}$$

We denote $e(t) = x(t) - \widehat{x}(t)$ the error between the observer and the observed system :

$$\begin{cases} \dot{e}(t) = (A - \gamma B^* B)e(t), & t > 0 \\ e(0) = x_0 - \widehat{x}_0 \end{cases}$$

If (A, B) is exactly observable it is well known that there exist $M, \mu > 0$ such that

$$\|e(t)\|_X^2 \leq M e^{-\mu t} \|e(0)\|_X^2, \qquad \forall t > 0.$$

Discrete Luenberger observer

Continuous observer :

$$\begin{cases} \dot{\widehat{x}}(t) = A\widehat{x}(t) + \gamma B^*(z(t) - B\widehat{x}(t)), & t > 0 \\ \widehat{x}(0) = \widehat{x}_0. \end{cases}$$

Fully discretized observer :

- Galerkin's method in space.
- Newmark midpoint scheme in time.

$$\boxed{\begin{array}{l} \dfrac{\widehat{x}_h^{k+1} - \widehat{x}_h^k}{\Delta t} = A_h \dfrac{\widehat{x}_h^{k+1} + \widehat{x}_h^k}{2} + \gamma B_h^* \left(\dfrac{z_h^k + z_h^{k+1}}{2} - B_h \dfrac{\widehat{x}_h^{k+1} + \widehat{x}_h^k}{2} \right) \\ \widehat{x}_h^0 = \Pi_h \widehat{x}_0. \end{array}}$$

⚠ We assume that data $z(t)$ are available continuously in time.

Error for the Luenberger observer

The error is given by

$$E_h^k = \widehat{x}_h^k - \Pi_h x(k\Delta t).$$

It is easy to see that E_h^k satisfies

$$\boxed{\begin{array}{l} \dfrac{E_h^{k+1} - E_h^k}{\Delta t} = (A_h - B_h^* B_h)\dfrac{E_h^k + E_h^{k+1}}{2} + G_h^k \\ E_h^0 = 0, \end{array}}$$

where

$$G_h^k = (A_h \Pi_h - \Pi_h A)\frac{x(k\Delta t) + x((k+1)\Delta t)}{2} + \Delta t H_h^k.$$

Error estimate for the Luenberger observer

$$\boxed{\begin{array}{l} \dfrac{E_h^{k+1} - E_h^k}{\Delta t} = (A_h - B_h^* B_h)\dfrac{E_h^k + E_h^{k+1}}{2} + G_h^k \\ E_h^0 = 0, \end{array}}$$

Homogeneous system :

$$\boxed{\dfrac{F_h^{k+1} - F_h^k}{\Delta t} = (A_h - B_h^* B_h)\dfrac{F_h^k + F_h^{k+1}}{2}.}$$

Assuming that (A, B) is exactly observable, there exists $M_{h,\Delta t},\ \mu_{h,\Delta t} > 0$ such that

$$\|F_h^k\|_X \leq M_{h,\Delta t} e^{-\mu_{h,\Delta t} k \Delta t} \|F_h^0\|_X.$$

Therefore,

$$\|E_h^k\|_X \leq \frac{M_{h,\Delta t}\Delta t}{1 - e^{-\mu_{h,\Delta t} k \Delta t}} \max_{0 \leq i \leq k} \|G_h^k\|_X.$$

Outline

1. Introduction
   - An abstract framework
   - Luenberger observers
2. Measurements continuously available in time
   - A viscous observer
   - Uniform observability of a space semi-discrete system
   - Numerical simulations
3. Under-sampled in time measurements
   - An on/off switch observer
   - An observer using interpolated data
   - Numerical simulations

A viscous Luenberger observer

- A discrete viscous observer

$$\boxed{\begin{array}{l} \dfrac{\widetilde{\widehat{x}}_h^{k+1} - \widehat{x}_h^k}{\Delta t} = A_h \dfrac{\widehat{x}_h^k + \widetilde{\widehat{x}}_h^{k+1}}{2} + \gamma B_h^* \left(\dfrac{z_h^k + z_h^{k+1}}{2} - B_h \dfrac{\widehat{x}_h^k + \widetilde{\widehat{x}}_h^{k+1}}{2} \right) \\ \dfrac{\widehat{x}_h^{k+1} - \widetilde{\widehat{x}}_h^{k+1}}{\Delta t} = \varepsilon \mathcal{V}_\varepsilon \widehat{x}_h^{k+1} \\ \widehat{x}_h^0 = \Pi_h x_0. \end{array}}$$

See also :

S. Ervedoza, E. Zuazua, *Uniformly exponentially stable approximations for a class of damped systems*. J. Math. Pures Appl. (9), 91(1) :20–48, 2009.

Main result

Theorem (D. Chapelle, N.C., P. Moireau (2012))

Suppose that (A, B) is exactly observable. With some technical assumptions on

- *the projector $\Pi_h : \mathcal{D}(A_0^{\frac{1}{2}}) \times H \to V_h \times V_h$*
- *the viscosity operator $\mathcal{V}_\varepsilon \in \mathcal{L}(V_h \times V_h)$*

and choosing $\varepsilon = \max\{\Delta t, h^\theta\}$, there exists a positive constant $C(x_0)$ depending on $x_0 \in \mathcal{D}(A)$ such that the following estimate holds

$$\|E_h^k\|_X \leq C(x_0) \max\{\varepsilon, \varepsilon^2 h^{-1} \Delta t\}, \qquad k \in \mathbb{N}.$$

Main result

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Suppose that (A, B) is exactly observable. With some technical assumptions on

- *the projector* $\Pi_h : \mathcal{D}(A_0^{\frac{1}{2}}) \times H \to V_h \times V_h$
-

and
$C(x$
hold

1. $\|\pi_h \varphi\|_{\mathcal{D}(A_0^{\frac{1}{2}})} \leq C_0 \|\varphi\|_{\mathcal{D}(A_0^{\frac{1}{2}})}, \qquad \forall \varphi \in \mathcal{D}(A_0^{\frac{1}{2}}),$
2. $\|\pi_h \varphi - \varphi\|_{\mathcal{D}(A_0^{\frac{1}{2}})} \leq C_0 h^\theta \|\varphi\|_{\mathcal{D}(A_0)}, \qquad \forall \varphi \in \mathcal{D}(A_0),$
3. $\|\widetilde{\pi}_h \varphi - \varphi\| \leq C_0 h^\theta \|\varphi\|_{\mathcal{D}(A_0^{\frac{1}{2}})}, \qquad \forall \varphi \in \mathcal{D}(A_0^{\frac{1}{2}}).$

Main result

Theorem (D. Chapelle, N.C., P. Moireau (2012))

Supp... that (A, B) is exactly observable. With some technical assu...

and
$C(x$...
hold...

1. $\mathcal{V}_\varepsilon$ is a self-adjoint, negative definite operator.
2. The orthogonal projector $\gamma_{1/\varepsilon}$ from X onto $(\mathcal{C}_h(1/\varepsilon))^2$ and $\mathcal{V}_\varepsilon$ commute.
3. There exist $c > 0$ and $C > 0$ such that

$$\begin{cases} \sqrt{\varepsilon}\|(-\mathcal{V}_\varepsilon)^{\frac{1}{2}} z\|_h \leq C\|z\|_h, & \forall z \in (\mathcal{C}_h(1/\varepsilon))^2, \\ \sqrt{\varepsilon}\|(-\mathcal{V}_\varepsilon)^{\frac{1}{2}} z\|_h \geq c\|z\|_h, & \forall z \in \left(\mathcal{C}_h(1/\varepsilon)^{\perp}\right)^2, \end{cases}$$

uniformly with respect to ε.

Main result

Theorem (D. Chapelle, N.C., P. Moireau (2012))

Suppose that (A, B) is exactly observable. With some technical assumptions on

- *the projector* $\Pi_h : \mathcal{D}(A_0^{\frac{1}{2}}) \times H \to V_h \times V_h$
- *the viscosity operator* $\mathcal{V}_\varepsilon \in \mathcal{L}(V_h \times V_h)$

and choosing $\varepsilon = \max\{\Delta t, h^\theta\}$, *there exists a positive constant* $C(x_0)$ *depending on* $x_0 \in \mathcal{D}(A)$ *such that the following estimate holds*

$$\|E_h^k\|_X \leq C(x_0) \max\{\varepsilon, \varepsilon^2 h^{-1} \Delta t\}, \qquad k \in \mathbb{N}.$$

Example of viscosity operator

- A first viscosity operator

$$\mathcal{V}_\varepsilon = A_h^2 = \begin{pmatrix} -A_{0h} & 0_h \\ 0_h & -A_{0h} \end{pmatrix}.$$

- Another possible viscosity operator

$$\mathcal{V}_\varepsilon = A_h^2 \left(I - \varepsilon A_h^2\right)^{-1}.$$

- ...

Idea of the proof (1)

The error verifies

$$\boxed{\begin{aligned}&\frac{\widetilde{E}_h^{k+1}-E_h^k}{\Delta t}=(A_h-B_h^*B_h)\frac{E_h^k+\widetilde{E}_h^{k+1}}{2}+G_h^k\\&\frac{E_h^{k+1}-\widetilde{E}_h^{k+1}}{\Delta t}=\varepsilon\mathcal{V}_\varepsilon E_h^{k+1}\\&E_h^0=0,\end{aligned}}$$

where

$$G_h^k=(-\frac{\Delta t}{2}A_h+\frac{\Delta t\gamma}{2}B_h^*B_h-1)\varepsilon\mathcal{V}_\varepsilon\Pi_h x((k+1)\Delta t)+\Delta t^2C(x)$$
$$+(A_h\Pi_h-\Pi_h A)\frac{x(k\Delta t)+x((k+1)\Delta t)}{2}.$$

Idea of the proof (2)

A homogeneous dissipative system

$$\boxed{\begin{array}{l} \dfrac{\widetilde{F}_h^{k+1} - F_h^k}{\Delta t} = (A_h - B_h^* B_h)\dfrac{F_h^k + \widetilde{F}_h^{k+1}}{2} \\ \dfrac{F_h^{k+1} - \widetilde{F}_h^{k+1}}{\Delta t} = \varepsilon \mathcal{V}_\varepsilon F_h^{k+1} \end{array}}$$

We prove that there exist $M, \mu > 0$ such that

$$\|F_h^k\|_X \leq M e^{-\mu k \Delta t} \|F_h^0\|_X. \tag{1}$$

Therefore, $$\|E_h^k\|_X \leq \frac{M \Delta t}{1 - e^{-\mu k \Delta t}} \max_{0 \leq i \leq k} \|G_h^k\|_X.$$

To prove (1) is enough to prove an observability inequality for the *low frequencies* of the corresponding space semi-discretized system.

Idea of the proof (3)

Consider the following semi-discrete system

$$\begin{cases} \ddot{w}_h(t) + A_{0h} w_h(t) = 0, & t > 0 \\ w_h(0) = w_{0h}, & \dot{w}_h(0) = w_{h1}. \end{cases}$$

Proposition

With the assumptions of main theorem, there exists a time $T^* > 0$, an observability constant $k^* > 0$ and an $\eta > 0$ such that

$$\int_0^{T^*} \|B_{0h} w_h(t)\|_Z^2 \, dt \geq k^* \left(\|A_{0h}^{\frac{1}{2}} w_{0h}\|^2 + \|w_{1h}\|^2 \right).$$

for every $(w_{0h}, w_{1h}) \in \left(\mathcal{C}_h(\eta/h^\theta)\right)^2$, *where*

$$\mathcal{C}_h(\beta) = \textit{span} \left\{ \phi_j^h \textit{ such that } \lambda_j^h \leq \beta \right\}.$$

Sketch of the proof of the Proposition

Hypothesis (exact observability of the continuous system) :

$$\int_0^T \|B_0 w(t)\|_Z^2 \, \mathrm{d}t \geq k_T \left(\|A_0^{\frac{1}{2}} w_0\|^2 + \|w_1\|^2 \right).$$

Ervedoza(2009) $\Updownarrow$ Miller(2005), Ramdani et al.(2005)

Resolvent estimate : there exist $M, m > 0$ such that for all $\phi \in \mathcal{D}(A_0)$ the following estimate holds

$$\|A_0^{\frac{1}{2}} \phi\|^2 \leq M^2 \|(A_0 - \lambda I)\phi\|^2 + m^2 \|B_0 \phi\|^2, \qquad \lambda \in I(A_0).$$

$$\Downarrow \quad ?$$

Discrete resolvent estimate : there exist $M_*, m_* > 0$ such that for every $\phi_h \in \mathcal{C}_h(\alpha/h^\theta)$

$$\|A_{0h}^{\frac{1}{2}} \phi_h\|^2 \leq M_*^2 \|(A_{0h} - \lambda I)\phi_h\|^2 + m_*^2 \|B_{0h} \phi_h\|^2, \qquad \lambda \in [0, \alpha/h^\theta].$$

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$$\|A_{0h}^{\frac{1}{2}} \phi_h\|^2 \leq M_*^2 \|(A_{0h} - \lambda I)\phi_h\|^2 + m_*^2 \|B_{0h} \phi_h\|^2, \qquad \lambda \in [0, \alpha/h^\theta].$$

Sketch of the proof of the Proposition (2)

We put Φ_h, solution of $A_0\Phi_h = A_{0h}\phi_h$, in the resolvent estimate :

$$\|A_0^{\frac{1}{2}}\Phi_h\|^2 \leq M^2\|(A_0-\lambda I)\Phi_h\|^2 + m^2\|B_0\Phi_h\|^2, \qquad \lambda \in I(A_0).$$

We prove that Φ_h is "close" to ϕ_h :

$$\begin{cases} \|A_{0h}^{\frac{1}{2}}\phi_h\|^2 \leq \|A_0^{\frac{1}{2}}\Phi_h\|^2 + C\alpha^{\frac{1}{2}}h^{\frac{\theta}{2}}\|A_{0h}^{\frac{1}{2}}\phi_h\|^2 \\ \|(A_0-\lambda I)\Phi_h\|^2 \leq 2\|(A_{0h}-\lambda I)\phi_h\|^2 + C\alpha^2\|A_{0h}^{\frac{1}{2}}\phi_h\|^2 \\ \|B_0\Phi_h\|_Z^2 \leq 2\|B_{0h}\phi_h\|_Z^2 + C\alpha h^{\theta}\|A_{0h}^{\frac{1}{2}}\phi_h\|^2. \end{cases}$$

We have

$$\left(1-C(\alpha^{\frac{1}{2}}+\alpha+\alpha^2)\right)\|A_{0h}^{\frac{1}{2}}\phi_h\|^2 \leq 2M^2\|(A_{0h}-\lambda I)\phi_h\|^2 + 2m^2\|B_{0h}\phi_h\|_Z^2.$$

Sketch of the proof of the Proposition (3)

Discrete resolvent estimate : there exist $M_*, m_* > 0$ such that for every $\phi_h \in \mathcal{C}_h(\alpha/h^\theta)$

$$\|A_{0h}^{\frac{1}{2}}\phi_h\|^2 \leq M_*^2\|(A_{0h}-\lambda I)\phi_h\|^2 + m_*^2\|B_{0h}\phi_h\|^2, \qquad \lambda \in [0, \alpha/h^\theta].$$

$$\Updownarrow$$

$$\int_0^{T^*} \|B_{0h}w_h(t)\|_Z^2 \,\mathrm{d}t \geq k^* \left(\|A_{0h}^{\frac{1}{2}}w_{0h}\|^2 + \|w_{1h}\|^2\right).$$

for every $(w_{0h}, w_{1h}) \in \left(\mathcal{C}_h(\eta/h^\theta)\right)^2$.

Back to the wave equation

$$\begin{cases} \ddot{w}(t,x) - \Delta w(t,x) = 0, & x \in \Omega, \quad t > 0 \\ w(t,x) = 0, & x \in \partial\Omega, \quad t > 0 \\ w(0,x) = w_0(x), \quad \dot{w}(0,x) = w_1(x), & x \in \Omega \end{cases}$$

with the observation $z(t) = w(t,\cdot)|_\omega$.

- $\mathcal{D}(A_0) = H^2(\Omega) \cap H_0^1(\Omega)$, $A_0 : \mathcal{D}(A_0) \to H = L^2(\Omega)$,

$$A_0\varphi = -\Delta\varphi, \qquad \forall \varphi \in \mathcal{D}(A_0).$$

- $\mathcal{D}(A_0^{\frac{1}{2}}) = H_0^1(\Omega)$, $B_0 \in \mathcal{L}(H_0^1(\Omega), H^1(\omega))$.
- $B_0^* : H^1(\omega) \to H_0^1(\Omega)$, $B_0^*\phi = \psi$, whith

$$\begin{cases} \Delta\psi = 0, & \text{in } \Omega \setminus \omega \\ \psi = 0, & \text{on } \partial\Omega \\ \psi = \phi, & \text{in } \overline{\omega}. \end{cases}$$

Back to the wave equation
Exact observability

Proposition (D. Chapelle, N.C., M. De Buhan, P. Moireau, 2012)

Assume that the geometric control condition of Bardos, Lebeau and Rauch is satisfied for some $\breve{\omega}$ *strict subset of* ω *and some* $T > 0$*. Then the following observability condition holds for every time* $T^* > T$

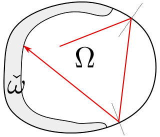
$$\int_0^{T^*} \|w(\cdot,t)\|^2_{H^1(\omega)}\,dt \geq C\left(\|w_0\|^2_{H^1(\Omega)} + \|w_1\|^2_{L^2(\Omega)}\right).$$

Exact observability

Idea of the proof

C. Bardos, G. Lebeau, J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*. SICON 30 (1992) 1024-1065.

It is well known that if $\breve{\omega}$ and T verify the Bardos, Lebeau and Rauch geometric optics condition the following observability inequality holds :



$$\int_0^T \|\dot{w}(t,\cdot)\|^2_{L^2(\breve{\omega})}\,\mathrm{d}t \geq k_T\left(\|w_0\|^2_{H^1_0(\Omega)} + \|w_1\|^2_{L^2(\Omega)}\right).$$

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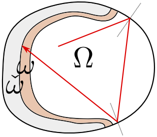


$$\int_0^T \|\dot{w}(t,\cdot)\|^2_{L^2(\breve{\omega})}\,\mathrm{d}t \geq k_T\left(\|w_0\|^2_{H^1_0(\Omega)} + \|w_1\|^2_{L^2(\Omega)}\right).$$

We prove that for $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$C\int_0^T \|\dot{w}(t+\varepsilon,\cdot)\|^2_{L^2(\breve{\omega})}\,\mathrm{d}t \leq \int_0^{T+2\varepsilon} \|w(t,\cdot)\|^2_{H^1(\omega)}\,\mathrm{d}t.$$

Exact observability

Idea of the proof - Details

K. Liu, *Locally distributed control and damping for the conservative systems*. SICON 35 (1997) 1574-1590.

Consider the following cutoff functions :

- $\psi \in C_c^\infty(\overline{\Omega})$, $\quad \psi(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus \omega \\ 1, & \text{if } x \in \breve{\omega} \end{cases}$ and $0 \leq \psi(x) \leq 1$ for every $x \in \overline{\Omega}$.
- $\phi(t) = t^2(T-t)^2$.

Wave equation

$$\begin{cases} \ddot{w}(t,x) - \Delta w(t,x) = 0, & x \in \Omega, \quad t > 0 \\ w(t,x) = 0, & x \in \partial\Omega, \quad t > 0 \\ w(0,x) = w_0(x), \quad \dot{w}(0,x) = w_1(x), & x \in \Omega \end{cases}$$

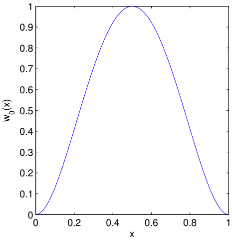
We multiply by $\phi\psi w$ and we integrate by parts.

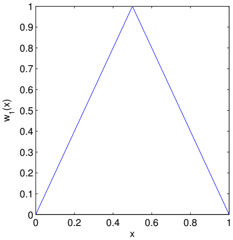
Numerical simulations

One dimensional wave equation - first example

$$\begin{cases} \ddot{w}(t,x) - w_{xx}(t,x) = 0, & t > 0, x \in (0,1) \\ w(t,0) = w(t,1) = 0 & t > 0 \\ w(0,x) = w_0(x), \qquad \dot{w}(0,x) = w_1(x), & x \in (0,1). \end{cases}$$

- P_1 finite elements in space, midpoint Newmark in time.
- $h = 0.005$, $\Delta t = 1.3h$, $\omega = (0.1, 0.3)$, $\theta = 1$.

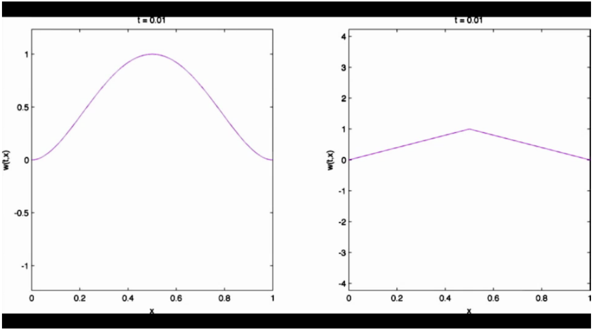




Numerical simulations

One dimensional wave equation - first example

- black - exact solution
- blue - non-viscous observer
- red - viscous observer
- magenta - standard discretization



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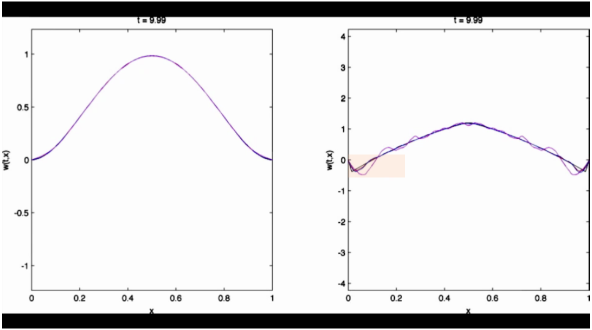

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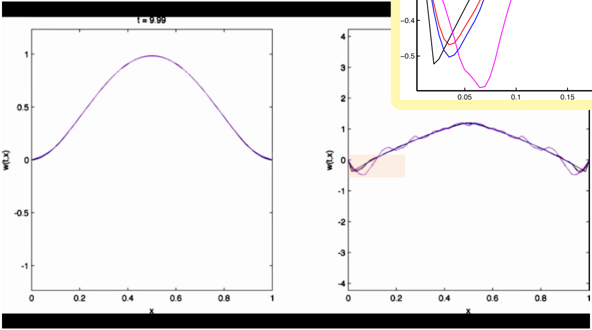
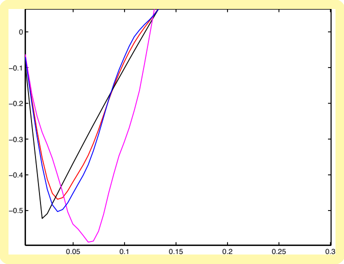
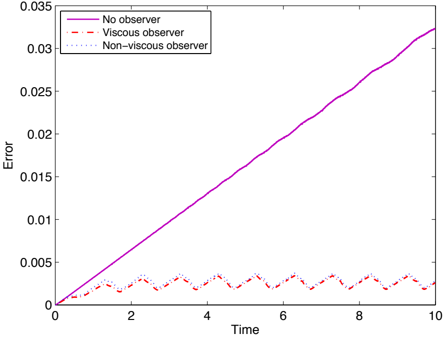




Numerical simulations

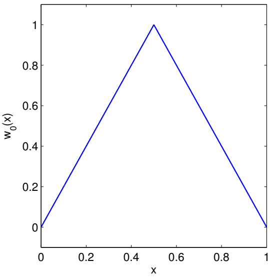
One dimensional wave equation - first example

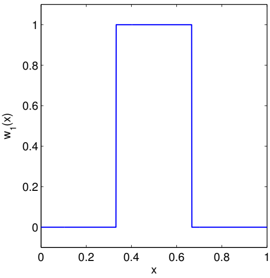
Relative error



Numerical simulations

One dimensional wave equation - second example





- P_1 finite elements in space, midpoint Newmark scheme in time.
- $h = 0.005$, $\Delta t = h$, $\omega = (0.1, 0.3)$.

Numerical simulations

One dimensional wave equation - second example

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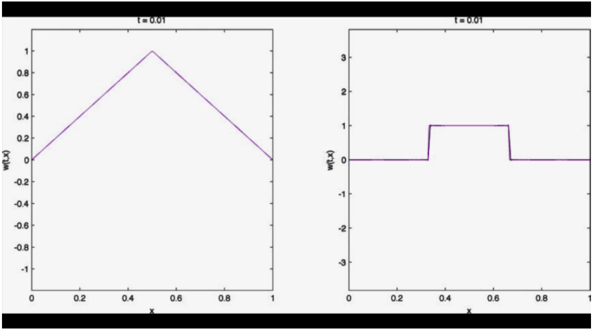

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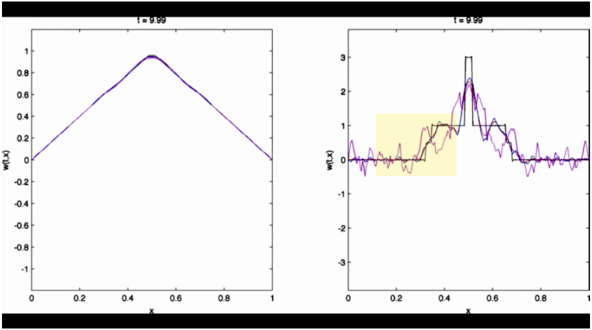

Numerical simulations

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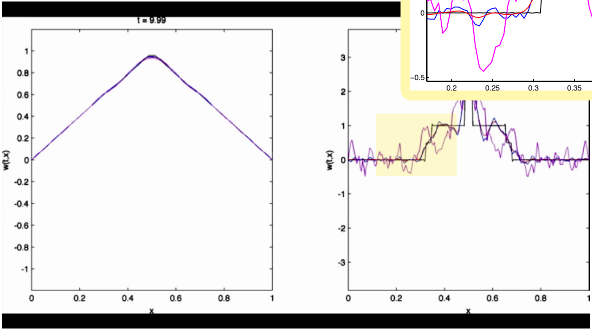
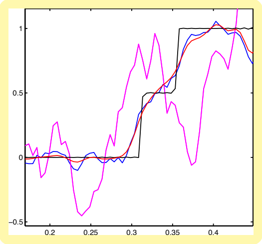
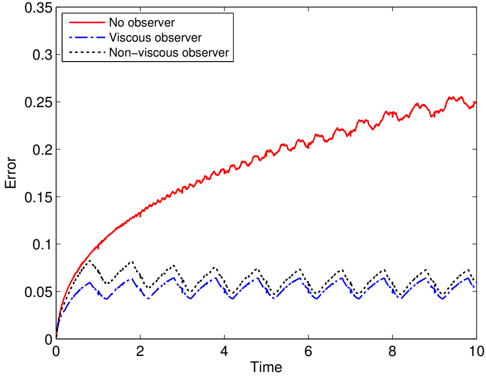




Numerical simulations

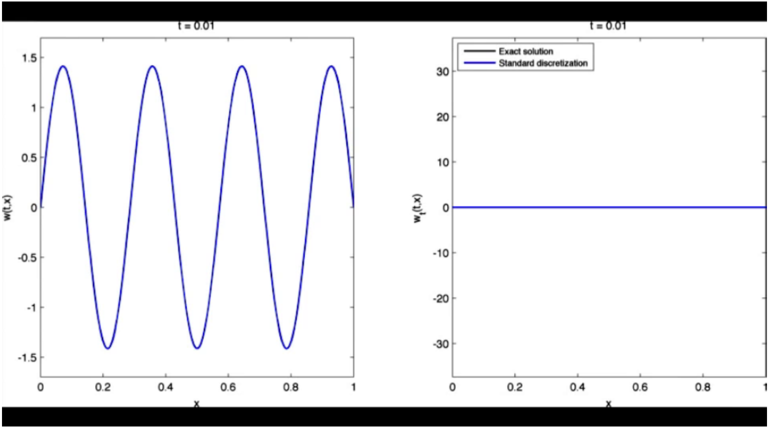
One dimensional wave equation - second example

Relative error



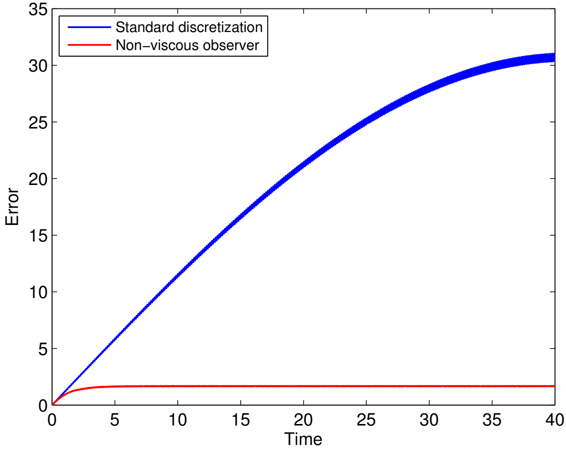
Numerical simulations

Back to the first slide example



Numerical simulations

Back to the first slide example



Numerical simulations

Two dimensional wave equation in a square

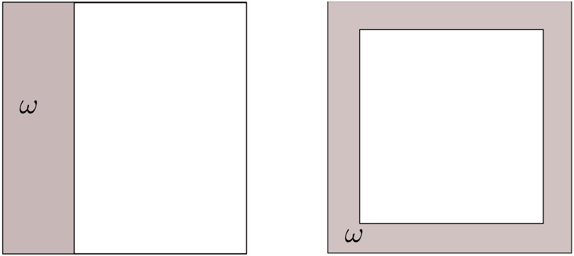


FIGURE : Domain and observation sets

- uniform mesh with $N = 50$ discretization points on each direction.
- $\Delta t = h = \frac{1}{N-1}$.

Numerical simulations

Two dimensional wave equation in a square

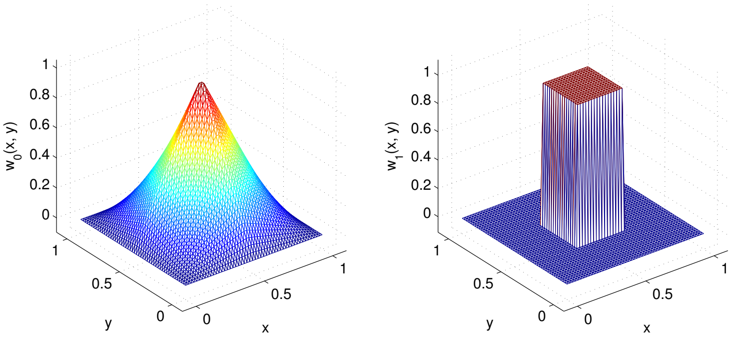


FIGURE : Initial data.

Numerical simulations

Two dimensional wave equation in a square

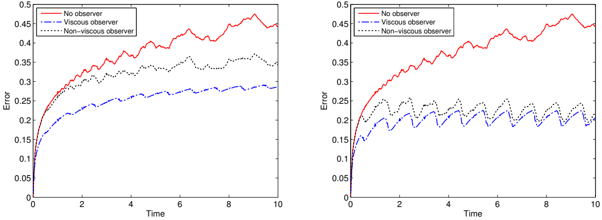


FIGURE : Relative Error.

Eigenvalues for the space semi-discrete system

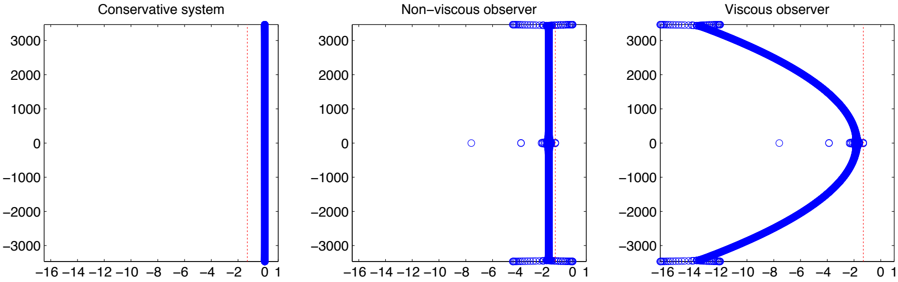


FIGURE : Eigenvalues for space semi-discrete systems

Spectral abscissa versus N

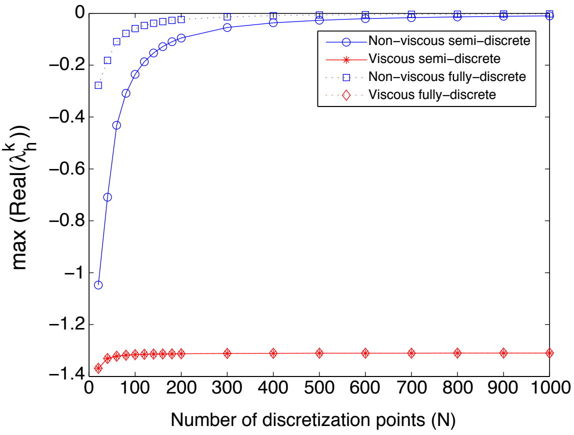


FIGURE : Maximum of $\mathrm{Re}(\lambda_k^h)$ (semi-discrete) and Re $\log(\lambda_{h,\Delta t}^k)/\Delta t$ (fully discrete) when varying N – Gain value $\gamma = 9$

Spectral abscissa versus γ

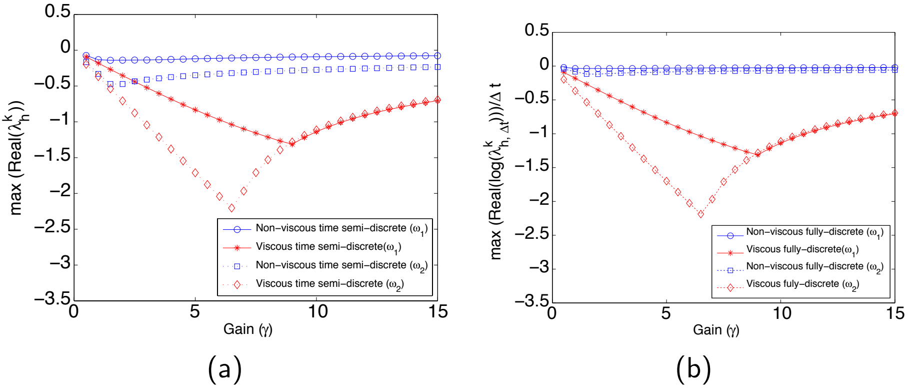


FIGURE : Effect of gain parameter and observation set : (a) Max. of Re(λ_k^h) – (b) Max. of Re $\log(\lambda^k_{h,\Delta t})/\Delta t$

Euler-Bernoulli equation

$$\begin{cases} \ddot{w}(t,x) + \Delta^2 w(t,x) = 0, & (t,x) \in (0,T) \times \Omega \\ w(t,x) = \Delta w(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega \\ w(0,x) = w_0(x), \quad \dot{w}(0,x) = w_1(x), & x \in \Omega. \end{cases}$$

- $H = L^2(\Omega)$ and $A_0 : \mathcal{D}(A_0) \to H$ is defined by

$$\mathcal{D}(A_0) = \{\varphi \in H^4(\Omega) \mid \varphi = \Delta\varphi = 0 \text{ on } \partial\Omega\}, \qquad A_0 = \Delta^2.$$

- A_0 is a self-adjoint positive definite operator with compact resolvents.
- $A_0^{\frac{1}{2}} : \mathcal{D}(A_0^{\frac{1}{2}}) \to H$ is given by

$$\mathcal{D}(A_0^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega), \qquad A_0^{\frac{1}{2}}\varphi = -\Delta\varphi \text{ for all } \varphi \in \mathcal{D}(A_0^{\frac{1}{2}}).$$

- Known data : $z(t) = w(t,\cdot)|_\omega$, with $\omega \subset \Omega$.

Euler-Bernoulli equation

Exact observability

Proposition (D. Chapelle, N.C., M. De Buhan, P. Moireau, 2012)

Assume that $\omega \subset \Omega$ *and* $T > 0$ *are such that*

$$\int_0^T \int_\omega |\dot{w}(t,x)|^2 \, dx \, dt \geq k_T(\|w_0\|^2_{\mathcal{D}(A_0^{\frac{1}{2}})} + \|w_1\|^2_H).$$

Therefore, for any $\check{T} > T$ *and any open set* $\check{\omega}$ *with* $\omega \subset \check{\omega} \subset \Omega$ *we have*

$$\int_0^{\check{T}} \|w(t,\cdot)\|^2_{H^2(\check{\omega})} \, dt \geq k_T^2(\|w_0\|^2_{\mathcal{D}(A_0^{\frac{1}{2}})} + \|w_1\|^2_H).$$

Numerical simulations

- Spatial discretization :
  - 1d : Hermite type finite elements.
  - 2d : HCT finite elements (TO DO).
- Temporal discretization :
  - midpoint finite differences scheme.

Numerical simulations

Euler-Bernoulli beam equation

$$\begin{cases} \ddot{w}(t,x) + w_{xxxx}(t,x) = 0, & (t,x) \in (0,T) \times (0,1) \\ w(t,x) = w_{xx}(t,x) = 0, & (t,x) \in (0,T) \times 0,1 \\ w(0,x) = w_0(x), \quad \dot{w}(0,x) = w_1(x), & x \in (0,1). \end{cases}$$

$$w_0(x) = \alpha x^7 (1-x)^7, \qquad w_1(x) = 0, \qquad x \in (0,1).$$

- Hermite type finite elements
- Finite-differences midpoint scheme in time.

Numerical simulations
Euler-Bernoulli beam equation

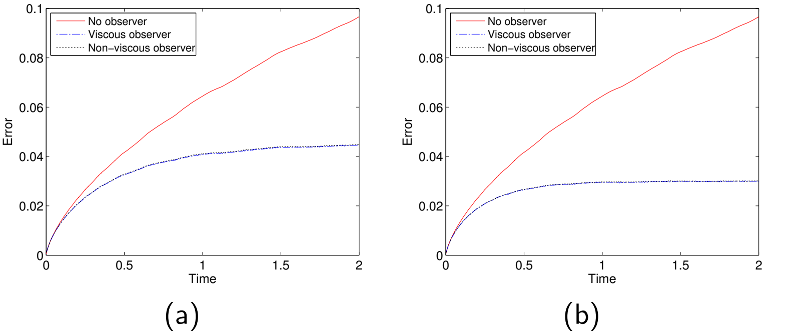


FIGURE : Relative errors for beam equation with $N = 100$ discretization points. Gain values : (a) $\gamma = 5$; (b) $\gamma = 10$.

Outline

1. Introduction
   - An abstract framework
   - Luenberger observers
2. Measurements continuously available in time
   - A viscous observer
   - Uniform observability of a space semi-discrete system
   - Numerical simulations
3. Under-sampled in time measurements
   - An on/off switch observer
   - An observer using interpolated data
   - Numerical simulations

Observers using under-sampled data

We consider the following system :

$$\boxed{\begin{cases} \dot{x}(t) = Ax(t), & t > 0 \\ x(0) = x_0 \end{cases} \qquad z_n = Bx(nN\Delta t), n > 0}$$

- $A : \mathcal{D}(A) \to X$ is a skew-adjoint operator.
- B is a bounded operator from X to Y.
- $N \in \mathbb{N}^*$ is a natural number.

Aim of the talk

Propose semi-discrete in time observers, with discretization time-step Δt which will use only the observations $(z_n)_n$.

Two possibilities :

- time interpolation.
- intermittent corrections.

Observer using under-sampled data

Continuous discrete Luenberger observer :

$$\begin{cases} \dot{\widehat{x}}(t) = A\widehat{x}(t) + \gamma B^*(z(t) - B\widehat{x}(t)) \\ \widetilde{x}(0) = \widetilde{x}_0 \end{cases}$$

Discrete observer with under-sampled data :

$$\begin{cases} \dfrac{\widehat{x}_-^{n+1} - \widehat{x}_+^n}{\Delta t} = A\dfrac{\widehat{x}_-^{n+1} + \widehat{x}_+^n}{2} \\ \dfrac{\widehat{x}_+^{n+1} - \widehat{x}_-^{n+1}}{\Delta t} = \delta^{n+1}\gamma B^*(d^{n+1} - B\widehat{x}_+^{n+1}) + \nu_{\Delta t}A^2\widehat{x}_+^{n+1} \end{cases}$$

On/off switch

$$\delta^n = \begin{cases} 0 \\ 1 \end{cases} \qquad d^n = \begin{cases} z^n & \text{if available} \\ 0 & \text{otherwise} \end{cases}$$

Observer with interpolation

$\delta_n = 1$
$d^n =$ interpolated data.

On/off switch observer

Error system

We define the error by

$$\widetilde{x}^n_+ = x(n\Delta t) - \widehat{x}^n_+, \qquad \widetilde{x}^n_- = x(n\Delta t) - \widehat{x}^n_-.$$

Proposition

Assuming that $x_0 \in \mathcal{D}(A^3)$, the error satisfy the following discrete dynamical system

$$\left\{ \begin{array}{l} \dfrac{\widetilde{x}^{n+1}_- - \widetilde{x}^n_+}{\Delta t} = A\dfrac{\widetilde{x}^{n+1}_- + \widetilde{x}^n_+}{2} + \varepsilon^{n+1}, \\ \dfrac{\widetilde{x}^{n+1}_+ - \widetilde{x}^{n+1}_-}{\Delta t} = -\delta^{n+1}\gamma B^* B\widetilde{x}^{n+1}_+ + \nu_{\Delta t}A^2\widetilde{x}^{n+1}_+ + \varepsilon^{n+1}_\nu, \end{array} \right.$$

where the consistency terms are

$$\varepsilon^{n+1} = \frac{\Delta t^2}{2}A^3\big(\frac{1}{3}x(\mathrm{t}_n) - \frac{1}{2}x(\mathrm{r}_n)\big), \qquad \text{with } \mathrm{t}_n, \mathrm{r}_n \in [n\Delta t; (n+1)\Delta t],$$

$$\varepsilon^{n+1}_\nu = -\nu_{\Delta t}A^2 x((n+1)\Delta t).$$

On/off switch observer

Error estimate

Theorem (N.C., A. Imperiale, P. Moireau)

Assuming that (A, B) is exactly observable and let $N \in \mathbb{N}$. There exist positive constants $M_0, \mu_0(N), C_1$ and C_2 such that the error $\tilde{x}_+^n$ satisfies

$$\|\widetilde{x}_+^n\| \leq M_0 e^{-\mu_0 \lfloor \frac{n}{N} \rfloor \Delta t} \|\widetilde{x}_0\| + \frac{\Delta t}{1 - e^{-\mu_0 \frac{1}{n} \lfloor \frac{n}{N} \rfloor \Delta t}} (\Delta t^2 C_1 + \nu_{\Delta t} C_2),$$

where

$$C_1 = \frac{5}{12} \|A^3 x_0\| \qquad \text{and} \qquad C_2 = \|A^2 x_0\|.$$

Idea of the proof

We thus consider the following dynamical system

$$\begin{cases} \dfrac{\widetilde{x}_-^{n,k+1} - \widetilde{x}_+^{n,k}}{\Delta t} = A\Big(\dfrac{\widetilde{x}_-^{n,k+1} + \widetilde{x}_+^{n,k}}{2}\Big), \qquad n \geq 0, \qquad 0 \leq k \leq N-1 \\ \dfrac{\widetilde{x}_+^{n,k+1} - \widetilde{x}_-^{n,k+1}}{\Delta t} = \nu_{\Delta t} A^2 \widetilde{x}_+^{n,k+1} - \delta_{k,N-1}\gamma B^* B \widetilde{x}_+^{n,k+1}, \qquad n \geq 0, \\ \widetilde{x}_+^{n+1,0} = \widetilde{x}_+^{n,N}, \qquad \widetilde{x}_-^{n+1,0} = \widetilde{x}_-^{n,N}, \end{cases}$$

where

$$\delta_{k,j} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{otherwise.} \end{cases}$$

We denote $\widetilde{E}^{n,k} = \dfrac{1}{2}\left\|\widetilde{x}_+^{n,k}\right\|^2$.

Idea of the proof (2)

We prove the following energy estimate :

$$\widetilde{E}^{n_2,k_2} + \gamma\Delta t \sum_{j=n_1+\delta_{k_1,0}}^{n_2} \left\| B\widetilde{x}_+^{j,0} \right\|^2 + \Delta \mathrm{t}\nu_{\Delta \mathrm{t}} \sum\sum_{[i,j]=[k_1,n_1]}^{[k_2,n_2]} \left\| A\widetilde{x}_+^{j,i} \right\|^2$$
$$+\frac{\Delta t}{4} \sum\sum_{[i,j]=[k_1,n_1]}^{[k_2,n_2]} \Delta \mathrm{t}\nu_{\Delta \mathrm{t}}^2 \left\| A^2\widetilde{x}_+^{j,i} \right\|^2 \leq \widetilde{E}^{n_1,k_1}.$$

and then we prove the following observability inequality

$$k_{T,\delta} \left\| \widetilde{x}_+^{0,0} \right\|^2 \leq \Delta t \sum_{n\Delta T \in [0,T]} \left\| B\widetilde{x}_+^{n,0} \right\|^2, \qquad \widetilde{x}_+^{0,0} \in \mathcal{C}_{\delta/\Delta T}.$$

Observer using interpolated data
Error system

Proposition

Assuming that $x_0 \in \mathcal{D}(A^3)$ then the error $\widetilde{x}_+^n$ satisfies the following dynamical system

$$\left\{ \begin{aligned} \frac{\widetilde{x}_-^{n+1} - \widetilde{x}_+^n}{\Delta t} &= A\frac{\widetilde{x}_-^{n+1} + \widetilde{x}_+^n}{2} + \varepsilon^{n+1}, \\ \frac{\widetilde{x}_+^{n+1} - \widetilde{x}_-^{n+1}}{\Delta t} &= -\gamma B^* B \widetilde{x}_+^{n+1} + \nu_{\Delta t} A^2 \widetilde{x}_+^{n+1} + \varepsilon_\nu^{n+1} + \gamma B^* \varepsilon_d^{n+1}, \end{aligned} \right.$$

where ε^{n+1} and ε_ν^{n+1} are as for the on/off switch and

$$\varepsilon_d^{n+1} = Bx((n+1)\Delta t) - d^{n+1}.$$

Observer using interpolated data

Error estimate

Theorem (N.C., A. Imperiale, P. Moireau)

Assuming that (A, B) is exactly observable and denoting

$$\varepsilon_d = \max_{1 \leq i \leq n} \|\varepsilon_d^i\|,$$

there exist positive constants M_0, μ_0, C_1, C_2 and C_3 independent of Δt and n such that

$$\|\widetilde{x}_+^n\| \leq M_0 e^{-\mu_0 n \Delta t} \|\widetilde{x}_0\| + \frac{\Delta t}{1 - e^{-\mu_0 \Delta t}} (\Delta^2 C_1 + \nu_{\Delta t} C_2 + \gamma C_3 |\varepsilon_d|)$$

Numerical simulations

No error on the initial data

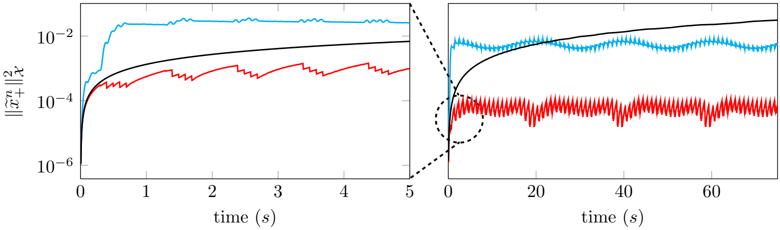


FIGURE : Estimation error with $\frac{\Delta T}{\Delta t} = 20$, $\alpha = 0$. In **(black)** is the simulation without correction, in (cyan) is the observer with linear data interpolation and in (red) is on/off observer

Numerical simulations

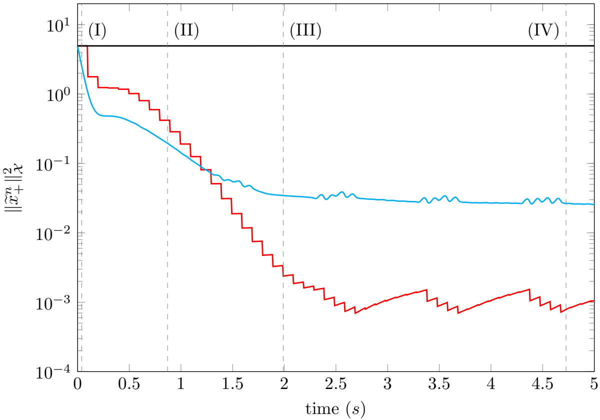


FIGURE : Numerical results with $\frac{\Delta T}{\Delta t} = 20$, $\alpha = 1$ and $\delta\varphi(s) = \sin(\pi s)$. In (green) is the exact solution without perturbation, in **(black)** is the simulation without correction, in (cyan) is the observer with linear data interpolation and in (red) is on/off observer.

Numerical simulations

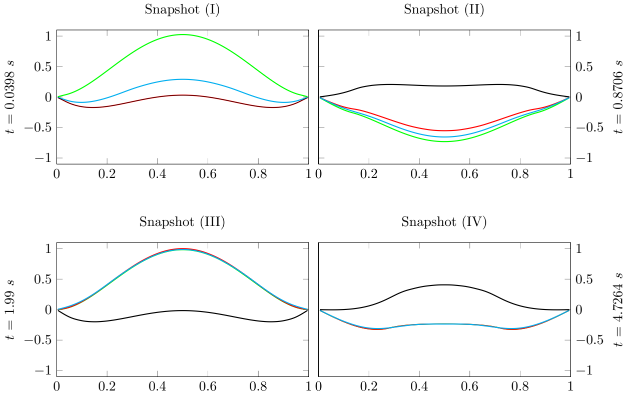


FIGURE : Numerical results with $\frac{\Delta T}{\Delta t} = 20$, $\alpha = 1$ and $\delta\varphi(s) = \sin(\pi s)$. In (green) is the exact solution without perturbation, in **(black)** is the simulation without correction, in (cyan) is the observer with linear data interpolation and in (red) is on/off observer.

Numerical simulations

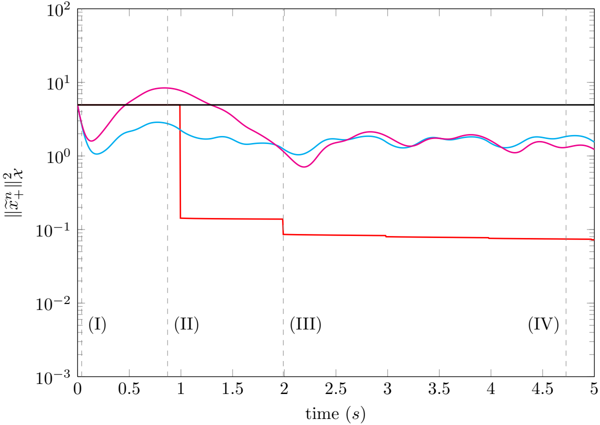


FIGURE : Numerical results with $\frac{\Delta T}{\Delta t} = 200$, $\alpha = 1$ and $\delta\varphi(s) = \sin(\pi s)$. In (green) is the exact solution without perturbation, in **(black)** is the simulation without correction, in (cyan) is the observer with linear data, in (purple) is the observer with cubic data interpolation and in (red) is on/off observer.

Numerical simulations

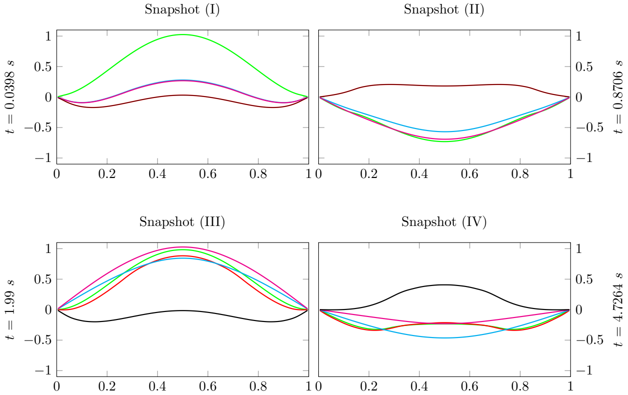


FIGURE : Numerical results with $\frac{\Delta T}{\Delta t} = 200$, $\alpha = 1$ and $\delta\varphi(s) = \sin(\pi s)$. In (green) is the exact solution without perturbation, in **(black)** is the simulation without correction, in (cyan) is the observer with linear data, in (purple) is the observer with cubic data interpolation and in (red) is on/off observer.

Spectral analysis

Comparison between on/off switch and interpolation

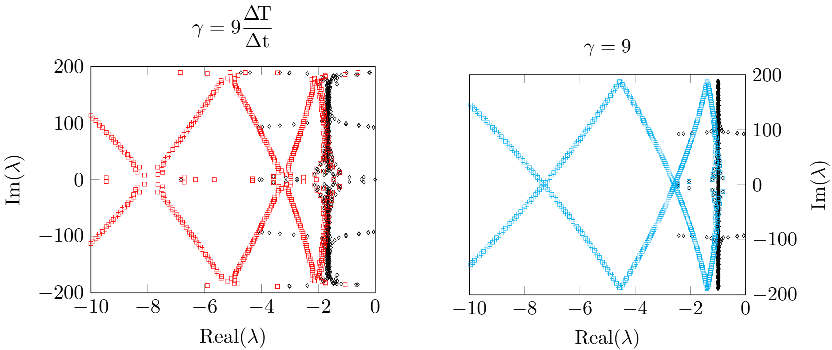


FIGURE : Comparison between the time-discrete on/off observer (in (red)) and the time-discrete observer using interpolated data (in (cyan)) with $\Delta t = h^2$ and $\Delta T = 5h$.

Some references

- D. Chapelle, N. Cîndea, P. Moireau Improving convergence in numerical analysis using observers - The wave-like equation case. Mathematical Models and Methods in Applied Sciences (M3AS), Vol. 22, No. 12 (2012).
- D. Chapelle, N. Cîndea, M. de Buhan, P. Moireau. Exponential convergence of an observer based on partial field measurements for the wave equation Mathematical Problems in Engineering (2012).
- N. Cîndea, A. Imperiale, P. Moireau. Numerical convergence of a time semi-discrete under-sampled observer - to appear in ESAIM COCV.

Thank you !