Inverse problems for the wave equation using mixed formulations and space-time FEM

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joint work with Arnaud Münch

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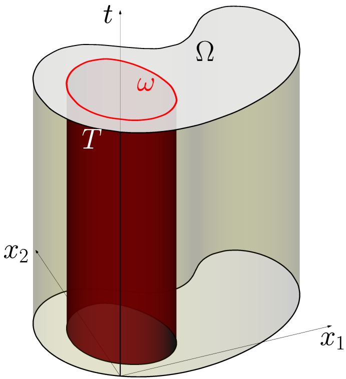
We consider the following wave equation:

$$\begin{cases} w_{tt} - \nabla \cdot (c \nabla w) + dw = f, & \text{in } Q_T \\ w = 0, & \text{on } \Sigma_T \\ w(x,0) = w_0(x), \qquad y_t(x,0) = w_1(x), & x \in \Omega. \end{cases} \tag{1}$$

- $Q_T = \Omega \times (0,T)$;
- $\Sigma_T = \partial\Omega \times (0,T)$;
- $q_T = \omega \times (0,T) \subset Q_T$;
- $(w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega)$.
- $f \in L^2(0,T; H^{-1}(\Omega))$.



Observation:

$$y = w|_{q_T}.$$

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- $\Sigma_T = \partial\Omega \times (0,T)$;
- $\Sigma_T^1 = \Gamma_1 \times (0,T)$;
- $q_T = \omega \times (0,T) \subset Q_T$;
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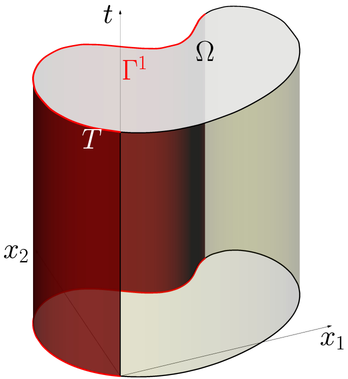
- $f \in L^2(Q_T)$.



Observation:

$$y = \partial_\nu w|_{\Sigma_T^1}.$$

Inverse problems

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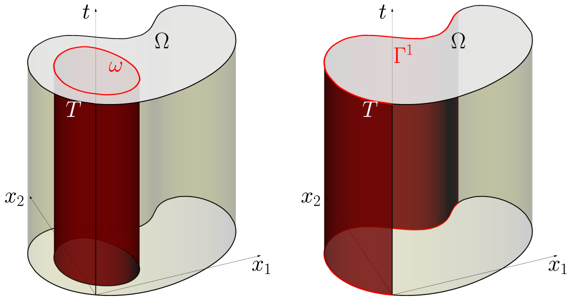
From observations (or measurements) y recover

- initial data (w_0, w_1);
- source term f;
- coefficients c, d...



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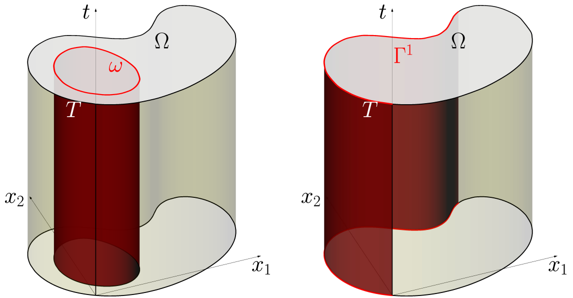
  $c \in C^1(\overline{\Omega})$,
  $c(x) \geq c_0 > 0$
  $d \in L^\infty(Q_T)$.



Questions: existence? unicity? stability? numerical approximation...

Inverse problems using observability/Carleman estimates for PDEs:

- Belina, Klibanov, Imanuvilov, Isakov, Puel, Triggiani, Tucsnak, Yamamoto...
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Least squares method

- optimal control community
- ...

Plan of this talk

Boundary observation

Distributed observation

Time-dependent distributed observation

Regularity of solutions

$$\begin{cases} w_{tt} - \nabla \cdot (c \nabla w) + d w = f, & \text{in } Q_T \\ w = 0, & \text{on } \Sigma_T \\ w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x), & x \in \Omega. \end{cases}$$

If $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(Q_T)$ then solutions of the wave equation have the following regularity:

$$w \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)).$$

We define $\boxed{Z = \left\{ w \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)) \text{ such that } Lw \in L^2(Q_T) \right\}}$

The following *hidden-regularity* property holds

$$\|c(x)\partial_\nu w\|^2_{L^2(\Sigma_T^1)} \leq C_T \left(\|(w(\cdot,0), w_t(\cdot,0))\|^2_{H_0^1 \times L^2} + \|Lw\|^2_{L^2(Q_T)} \right) \qquad (w \in Z).$$

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Generalized observability

Hypothesis:
There exists a constant $C_{obs} > 0$ depending on Γ^1, T, c, d such that the following estimate holds:

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Consequence:

The space Z endowed with the inner product

$$\langle w, \overline{w}\rangle_Z = \langle c\partial_\nu w, c\partial_\nu \overline{w}\rangle_{L^2(\Sigma^1_T)} + \eta\langle Lw, L\overline{w}\rangle_{L^2(Q_T)} \qquad (w, \overline{w} \in Z).$$

is a Hilbert space, for every value of the parameter $\eta > 0$.

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- the measurement $y \in L^2(\Sigma_T^1)$.

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  Without restraining the generality we take $f = 0$.
- the measurement $y \in L^2(\Sigma_T^1)$.

We want to recover:

- the solution $w \in Z$ of

$$\begin{cases} Lw = 0 \\ \partial_\nu w|_{\Sigma_T^1} = y, \end{cases}$$

or, completely equivalent, the initial data (w_0, w_1).

We consider the following minimization problem:

$$\begin{cases} \inf J(w) := \dfrac{1}{2}\,\|c(x)(\partial_\nu w - y)\|^2_{L^2(\Sigma_T^1)} \\ \text{subject to } w \in W := \{w \in Z;\ Lw = 0 \text{ in } L^2(Q_T)\}. \end{cases}$$

This problem is well posed:

- J is continuous over W endowed with the norm of Z
- J is strictly convex
- $J(w) \to \infty$ when $w \in W$ and $\|w\|_Z \to \infty$.

We consider the following relaxed minimization problem:

$$\begin{cases} \inf J_r(w) := \dfrac{1}{2} \|c(x)(\partial_\nu w - y)\|^2_{L^2(\Sigma_T^1)} + \dfrac{r}{2} \|Lw\|^2_{L^2(Q_T)} \\ \text{subject to } w \in Z. \end{cases}$$

We still have that this problem is well posed:

- J is continuous over Z
- J is strictly convex
- $J(w) \to \infty$ when $\|w\|_Z \to \infty$.

A mixed formulation

Find $(w, \lambda) \in Z \times L^2(Q_T)$ solution of

$$\begin{cases} a_r(w, \overline{w}) + b(\overline{w}, \lambda) = l(\overline{w}) & (\overline{w} \in Z) \\ b(w, \overline{\lambda}) = 0 & (\overline{\lambda} \in L^2(Q_T)), \end{cases}$$

where

$$a_r : Z \times Z \to \mathbb{R}, \qquad a_r(w, \overline{w}) = \langle c\partial_\nu w,\ c\partial_\nu \overline{w}\rangle_{L^2(\Sigma_T^1)} + r\langle Lw,\ L\overline{w}\rangle_{L^2(Q_T)}$$

$$b : Z \times L^2(Q_T) \to \mathbb{R}, \qquad b(w, \lambda) = \langle \lambda,\ Lw\rangle_{L^2(Q_T)}$$

$$l : Z \to \mathbb{R}, \qquad l(w) = \langle cy,\ c\partial_\nu w\rangle_{L^2(\Sigma_T^1)}.$$

Theorem (NC, A. Münch)

Under the hypothesis of generalized observability,

1. *The mixed formulation is well-posed.*
2. *The unique solution $(w, \lambda) \in Z \times L^2(Q_T)$ to the mixed formulation is the unique saddle-point of the Lagrangian $\mathcal{L}_r : Z \times L^2(Q_T) \to \mathbb{R}$ defined by*

$$\mathcal{L}_r(w, \lambda) := \frac{1}{2} a_r(w, w) + b(w, \lambda) - l(w).$$

3. *The solution $(w, \lambda) \in Z \times L^2(Q_T)$ satisfies the estimates*

$$\|w\|_Z \leq \|c(x)\, y\|_{L^2(\Sigma_T^1)},$$
$$\|\lambda\|_{L^2(Q_T)} \leq 2\sqrt{C_{\Omega,T} + \eta}\|c(x)\, y\|_{L^2(\Sigma_T^1)}.$$

Idea of the proof

We easily show that:

- the bilinear form a_r is continuous over $Z \times Z$
- the bilinear form b is continuous over $Z \times L^2(Q_T)$
- the linear form l is continuous over Z.

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$$\mathcal{N}(b) = \left\{ w \in Z \;\middle|\; b(w, \lambda) = 0 \text{ for every } \lambda \in L^2(Q_T) \right\}$$

$$\iff \iint_{Q_T} Lw\lambda dxdt = 0 \text{ for every } \lambda \in L^2(Q_T)$$

$$\iff Lw = 0 \text{ in } L^2(Q_T) \qquad \mathcal{N}(b) = W.$$

Since $a_r(w, w) = \|c\partial_\nu w\|^2_{L^2(\Sigma_T^1)} = \|w\|^2_Z$ for every $w \in W$,

a_r is coercive on $\mathcal{N}(b)$.

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- the following inf-sup property holds: there exists $\delta > 0$ such that

$$\inf_{\lambda \in L^2(Q_T)} \sup_{w \in Z} \frac{b(w,\lambda)}{\|w\|_Z \|\lambda\|_{L^2(Q_T)}} \geq \delta.$$

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The conclusion of theorem follows from standard results about mixed formulations.

Some remarks

- The Lagrangian $\mathcal{L}_r$ is an augmentation of the simpler Lagrangian $\mathcal{L}(w,\lambda) = \frac{1}{2}a(w,w) + b(w,\lambda) - l(\lambda)$ where $a(w,\overline{w}) = \langle c\partial_\nu w,\ c\partial_\nu \overline{w}\rangle_{L^2(\Sigma_T^1)}$
- If the solution λ verifies $L\lambda \in L^2(0,T;H^{-1}(\Omega))$ and $(\lambda,\lambda_t)_{|t\in\{0,T\}} \in L^2(\Omega)\times H^{-1}(\Omega)$, the multiplier λ verifies

$$\begin{cases} L\lambda = 0 & \text{in } Q_T \\ \lambda = c(\partial_\nu w - y) & \text{on } \Sigma_T^1 \\ \lambda = 0 & \text{on } \Sigma_T \setminus \Sigma_T^1 \\ \lambda = \lambda_t = 0 & \text{on } \Omega\times\{0,T\}. \end{cases}$$

- If y is the normal derivative of a solution of the wave equation, then the unique multiplier λ must vanish almost everywhere. In this case the saddle point of $\mathcal{L}_r$ is $(w,0)$, where w is the minimum of J_r.

A stabilized mixed formulation

We define the following space Λ

$$\Lambda = \{\lambda \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega)),\\ L\lambda \in L^2([0,T];H^{-1}(\Omega)),\ \lambda(\cdot,0) = \lambda_t(\cdot,0) = 0,\ \lambda_{|\Sigma_T^1} \in L^2(\Sigma_T^1)\}$$

endowed with the inner product $\langle \lambda,\ \overline{\lambda}\rangle_\Lambda := \langle L\lambda, L\overline{\lambda}\rangle_{L^2((0,T);H^{-1}(\Omega))} + \langle c\lambda, c\overline{\lambda}\rangle_{L^2(\Sigma_T^1)}$.
For every $\alpha \in (0,1)$ we define:

$$\begin{aligned}
a_{r,\alpha} : Z \times Z \to \mathbb{R}, &\quad a_{r,\alpha}(w,\overline{w}) = (1-\alpha)\langle c\partial_\nu w, c\partial_\nu \overline{w}\rangle_{L^2(\Sigma_T^1)} + r\langle Lw, L\overline{w}\rangle_{L^2(Q_T)},\\
b_\alpha : Z \times \Lambda \to \mathbb{R}, &\quad b_\alpha(w,\lambda) = \langle Lw, \lambda\rangle_{L^2(Q_T)} - \alpha\langle c\partial_\nu w, c\lambda\rangle_{L^2(\Sigma_T^1)}\\
c_\alpha : \Lambda \times \Lambda \to \mathbb{R}, &\quad c_\alpha(\lambda,\overline{\lambda}) = \alpha\langle L\lambda, L\overline{\lambda}\rangle_{L^2((0,T);H^{-1}(\Omega))} + \alpha\langle c\lambda, c\overline{\lambda}\rangle_{L^2(\Sigma_T^1)}\\
l_{1,\alpha} : Z \to \mathbb{R}, &\quad l_{1,\alpha}(w) = (1-\alpha)\langle c\partial_\nu w, cy\rangle_{L^2(\Sigma_T^1)}\\
l_{2,\alpha} : \Lambda \to \mathbb{R}, &\quad l_{2,\alpha}(\lambda) = -\alpha\langle c\lambda, cy\rangle_{L^2(\Sigma_T^1)}.
\end{aligned}$$

A stabilized mixed formulation

Proposition (NC, A. Münch)

Under the hypothesis of generalized observability, for every $\alpha \in (0,1)$*, the following stabilized mixed formulation*

$$\left\{\begin{array}{rcll} a_{r,\alpha}(w,\overline{w}) + b_\alpha(\overline{w},\lambda) & = & l_{1,\alpha}(\overline{w}), & (\overline{w} \in Z) \\ b_\alpha(w,\overline{\lambda}) - c_\alpha(\lambda,\overline{\lambda}) & = & l_{2,\alpha}(\overline{\lambda}), & (\overline{\lambda} \in \Lambda) \end{array}\right.$$

is well-posed. Moreover, the unique pair $(y,\lambda) \in Z \times \Lambda$ *satisfies*

$$\theta\|w\|_Z^2 + \alpha\|\lambda\|_\Lambda^2 \leq \frac{(1-\alpha)^2 + \alpha\theta}{\theta}\|y\|^2_{L^2(\Sigma_T^1)}$$

with $\theta := \min\left(1-\alpha, r/\eta\right)$.

- the bilinear form $a_{r,\alpha}$ is continuous over $Z \times Z$
- the bilinear form b_α is continuous over $Z \times \Lambda$
- the bilinear form c_α is continuous over $\Lambda \times \Lambda$
- the linear forms $l_{1,\alpha}$ and $l_{2,\alpha}$ are continuous over Z and Λ respectively.

Moreover,

- the bilinear forms $a_{r,\alpha}$ and c_α are coercive:

$$\begin{aligned} a_{r,\alpha}(w,w) &\geq \theta \|w\|_Z^2 \qquad (w \in Z) \\ c_\alpha(\lambda,\lambda) &\geq \alpha \|\lambda\|_\Lambda^2 \qquad (\lambda \in \Lambda). \end{aligned}$$

The conclusion follows applying a result in Boffi, Brezzi and Fortin (2013).

Some more remarks

- the unique solution of the stabilized mixed formulation corresponds to the saddle-point of the following Lagrangian:

$$\mathcal{L}_{r,\alpha}(w,\lambda) = \mathcal{L}_r(w,\lambda) - \frac{\alpha}{2}\|L\lambda\|^2_{L^2(0,T,H^{-1})} - \frac{\alpha}{2}\|c(x)(\lambda - (\partial_\nu w - y))\|^2_{L^2(\Sigma_T^1)}.$$

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- the term $\frac{\alpha}{2}\|L\lambda\|^2_{L^2(0,T,H^{-1}(\Omega))}$ in the Lagrangian is a stabilization term.

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- the term $\dfrac{\alpha}{2}\|L\lambda\|^2_{L^2(0,T,H^{-1}(\Omega))}$ in the Lagrangian is a stabilization term.
- if the solution of the unstabilized mixed formulation is regular enough then it coincides to the solution of the stabilized mixed formulation.

Recovering the source term and the solution?

Without any assumptions on the source f, for a given observation y the couple (w, f) such that

$$\begin{cases} Lw = f & \text{in } Q_T \\ \partial_\nu w = y & \text{on } \Sigma_T^1 \end{cases}$$

is not anymore unique.

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Assume that $f(x,t) = \sigma(t)\mu(x)$ with

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Theorem (Yamamoto, Zhang (2001))

Assume that (Σ_T^1, Q_T) satisfies the geometric optic condition. Let $w = w(\mu) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ be the weak solution of the wave equation $Lw = \sigma\mu$ with $c := 1$ and $(w_0, w_1) = (0,0)$. Then, there exists a positive constant C such that

$$C^{-1}\|\mu\|_{H^{-1}(\Omega)} \leq \|\partial_\nu w\|_{L^2(\Sigma_T^1)} \leq C\|\mu\|_{H^{-1}(\Omega)}, \qquad (\mu \in H^{-1}(\Omega)).$$

A minimization problem

$$\begin{cases} \inf J(w,\mu) := \dfrac{1}{2}\|c(x)(\partial_\nu w - y)\|^2_{L^2(\Sigma_T^1)} + \dfrac{r}{2}\|Lw - \sigma\mu\|^2_{L^2(Q_T)}, \\ \text{subject to} \quad (w,\mu) \in W \end{cases} \qquad (\mathcal{P}_{w,\mu})$$

where W is the space defined by

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- $\partial_\nu w \in L^2(\Sigma_T^1)$.
- W endowed with the norm $\|(w,\mu)\|_W = \|c\partial_\nu w\|^2_{L^2(\Sigma_T^1)}$ is a Hilbert space.

A minimization problem

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$$Lw - \sigma\mu = 0 \text{ in } Q_T,\ w(\cdot,0) = w_t(\cdot,0) = 0\Big\}.$$

- $\partial_\nu w \in L^2(\Sigma^1_T)$.
- W endowed with the norm $\|(w,\mu)\|_W = \|c\partial_\nu w\|^2_{L^2(\Sigma^1_T)}$ is a Hilbert space.
- The extremal problem $(\mathcal{P}_{w,\mu})$ is well-posed.

A minimization problem

$$\begin{cases} \inf J(w,\mu) := \dfrac{1}{2}\|c(x)(\partial_\nu w - y)\|^2_{L^2(\Sigma_T^1)} + \dfrac{r}{2}\|Lw - \sigma\mu\|^2_{L^2(Q_T)}, \\ \text{subject to} \quad (w,\mu) \in W \end{cases} \qquad (\mathcal{P}_{w,\mu})$$

where W is the space defined by

$$W := \Big\{ (w,\mu);\ w \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)),\ \mu \in H^{-1}(\Omega),$$
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- The extremal problem $(\mathcal{P}_{w,\mu})$ is well-posed.
- The solution μ are uniformly bounded in $H^{-1}(\Omega)$.

Hypothesis:

There exists a positive constant $C_{obs} = C(\Sigma_T^1, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that the following estimate holds:

$$\|\mu\|^2_{H^{-1}(\Omega)} \leq C_{obs}\left(\|c(x)\partial_\nu w\|^2_{L^2(\Sigma_T^1)} + \|Lw - \sigma\mu\|^2_{L^2(Q_T)}\right), \qquad (w,\mu) \in Y. \quad (\mathcal{H}_2)$$

where

$$Y := \left\{ \begin{array}{ll} (w,\mu); & w \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)),\ \mu \in H^{-1}(\Omega), \\ & Lw - \sigma\mu \in L^2(Q_T),\ w(\cdot,0) = w_t(\cdot,0) = 0 \end{array} \right\}.$$

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- if the velocity c is constant, then $(\mathcal{H}_2)$ is a consequence of the Yamamoto and Zhang Theorem.
- under the hypothesis $(\mathcal{H}_2)$, the space Y endowed with the following inner product

$$\langle (w,\mu), (\overline{w},\overline{\mu})\rangle_Y := \langle c\partial_\nu w, c\partial_\nu \overline{w}\rangle_{L^2(\Gamma_T)} + \eta\langle Lw - \sigma\mu, L\overline{w} - \sigma\overline{\mu}\rangle_{L^2(Q_T)},$$

  is a Hilbert space.

A mixed formulation in Y

For any $r \geq 0$, we now define the mixed formulation: find $((w,\mu),\lambda) \in Y \times L^2(Q_T)$ solution of

$$\left\{ \begin{array}{rcll} a_r((w,\mu),(\overline{w},\overline{\mu})) + b((\overline{w},\overline{\mu}),\lambda) & = & l(\overline{w},\overline{\mu}), & ((\overline{w},\overline{\mu}) \in Y) \\ b((w,\mu),\overline{\lambda}) & = & 0, & (\overline{\lambda} \in L^2(Q_T)), \end{array} \right. \qquad (MF_2)$$

where

$$a_r : Y \times Y \to \mathbb{R}, \quad a((w,\mu),(\overline{w},\overline{\mu})) := \langle c\partial_\nu w, c\partial_\nu \overline{w}\rangle_{L^2(\Sigma_T^1)} + r\langle Lw - \sigma\mu, L\overline{w} - \sigma\overline{\mu}\rangle_{L^2(Q_T)},$$

$$b : Y \times L^2(Q_T) \to \mathbb{R}, \quad b((w,\mu),\lambda) := \langle \lambda, Lw - \sigma\mu\rangle_{L^2(Q_T)},$$

$$l : Y \to \mathbb{R}, \quad l(w,\mu) := \langle c\partial_\nu w, cy\rangle_{L^2(\Sigma_T^1)}.$$

Theorem (NC, A. Münch)

Let $r \geq 0$. Under the hypothesis $(\mathcal{H}_2)$, the following holds :

1. *The mixed formulation (MF_2) is well-posed.*
2. *The unique solution $((w,\mu),\lambda) \in Y \times L^2(Q_T)$ is the saddle-point of the Lagrangian $\mathcal{L}_r : Y \times L^2(Q_T) \to \mathbb{R}$ defined by*
   $\mathcal{L}_r((w,\mu),\lambda) := \frac{1}{2} a_r((w,\mu),(w,\mu)) + b((w,\mu),\lambda) - l(w,\mu).$
   Moreover, the pair (w,μ) solves the extremal problem $(\mathcal{P}_{w,\mu})$.
3. *The solution $((w,\mu),\lambda)$ satisfies the estimates:*

$$\|(w,\mu)\|_Y = \|c(x)\partial_\nu w\|_{L^2(\Sigma_T^1)} \leq \|c(x)\, y\|_{L^2(\Sigma_T^1)}$$

and

$$\|\lambda\|_{L^2(Q_T)} \leq 2\sqrt{C_{\Omega,T} + \eta}\|c(x)\, y\|_{L^2(\Sigma_T^1)}$$

for some constant $C_{\Omega,T} > 0$.

We consider the following finite dimensional spaces:

- $Z_h \subset Z$
- $\Lambda_h \subset L^2(Q_T)$.

and for any $h > 0$ we introduce the following approximating problems: find $(w_h, \lambda_h) \in Z_h \times \Lambda_h$ solution of

$$\left\{ \begin{array}{rcll} a_r(w_h, \overline{w}_h) + b(\overline{w}_h, \lambda_h) & = & l(\overline{w}_h), & (\overline{w}_h \in Z_h) \\ b(w_h, \overline{\lambda}_h) & = & 0, & (\overline{\lambda}_h \in \Lambda_h). \end{array} \right.$$

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This mixed formulation is well-posed as a consequence of two things:

1. a_r is coercive on $\mathcal{N}_h(b) = \{w_h \in Z_h;\ b(w_h, \lambda_h) = 0 \text{ for every } \lambda_h \in \Lambda_h\}$
2. a discrete inf-sup condition: for every $h > 0$

$$\boxed{\delta_h := \inf_{\lambda_h \in \Lambda_h} \sup_{w_h \in Z_h} \frac{b(w_h, \lambda_h)}{\|\lambda_h\|_{L^2(Q_T)} \|w_h\|_Z} > 0.}$$

Proposition (NC, A. Münch)

Let $h > 0$. Let (w, λ) and (w_h, λ_h) be the solutions of continuous and discrete mixed formulations respectively. Let δ_h be the discrete inf-sup constant. Then,

$$\|w - w_h\|_Z \leq 2\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)d(w, Z_h) + \frac{1}{\sqrt{\eta}}d(\lambda, \Lambda_h),$$

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq \left(2 + \frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h}d(w, Z_h) + \frac{3}{\sqrt{\eta}\delta_h}d(\lambda, \Lambda_h)$$

where $d(\lambda, \Lambda_h) := \inf_{\lambda_h \in \Lambda_h} \|\lambda - \lambda_h\|_{L^2(Q_T)}$ *and* $d(w, Z_h) := \inf_{w_h \in Z_h} \|w - w_h\|_Z$.

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where $d(\lambda,\Lambda_h):=\inf_{\lambda_h\in\Lambda_h}\|\lambda-\lambda_h\|_{L^2(Q_T)}$ and $d(w,Z_h):=\inf_{w_h\in Z_h}\|w-w_h\|_Z$.

Remarks:

1. if $r=0$ the discrete mixed formulation may be not well-posed.

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Remarks:

1. if $r = 0$ the discrete mixed formulation may be not well-posed.
2. what if $\delta_h \to 0$ when $h \to 0$?
3. Z_h must be chosen such that $L w_h \in L^2(Q_T)$ for every $w_h \in Z_h$.

Choice of the discrete spaces

- $\mathcal{T}_h$ a triangulation such that $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$ and $\{\mathcal{T}_h\}_{h>0}$ is a regular family. $h := \max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$
- $Z_h := \{z_h \in C^1(\overline{Q_T}) : z_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h,\ z_h = 0 \text{ on } \Sigma_T\}$, where $\mathbb{P}(K)$ denotes an appropriate space of functions in x and t:
  - the *Bogner-Fox-Schmit* (BFS for short) C^1-element defined for rectangles
  - the reduced *Hsieh-Clough-Tocher* (HCT for short) C^1-element defined for triangles.

We also define the finite dimensional space

$$\Lambda_h := \{\lambda_h \in C^0(\overline{Q_T}), \lambda_h|_K \in \mathbb{Q}(K) \quad \forall K \in \mathcal{T}_h\}$$

where $\mathbb{Q}(K)$ denotes the space of affine functions both in x and t on the element K. Remark that, for any $h > 0$, we have $Z_h \subset Z$ and $\Lambda_h \subset L^2(Q_T)$.

Proposition (BFS element for $N = 1$)

Let $h > 0$, let $k \leq 2$ be a nonnegative integer. Let (w, λ) and (w_h, λ_h) be the solutions of continuous and discrete mixed formulations respectively. If the solution (w, λ) belongs to $H^{k+2}(Q_T) \times H^k(Q_T)$, then there exist positive constants $K_i(\|w\|_{H^{k+2}(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)})$, $i \in \{1, 2, 3\}$ independent of h, such that

$$\|w - w_h\|_Z \leq K_1 \left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right) h^k,$$

$$\|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_2 \left(\left(1 + \frac{1}{\sqrt{\eta}\delta_h}\right)\frac{1}{\delta_h} + \frac{1}{\sqrt{\eta}\delta_h}\right) h^k,$$

$$\|w - w_h\|_{L^2(Q_T)} \leq K_3 \max(1, \frac{2}{\sqrt{\eta}}) \left(1 + \frac{1}{\sqrt{\eta}\delta_h} + \frac{1}{\sqrt{\eta}}\right) h^{k+2}.$$

Evaluating δ_h numerically, we obtain $\delta_h \approx C \dfrac{h}{\sqrt{r}} \quad \text{as} \quad h \to 0^+ \ldots$

- $\alpha \in (0,1)$, $h > 0$
- $\widetilde{\Lambda}_h$ be a closed finite dimensional subspace of Λ such that $L\lambda_h \in L^2(0,T,H^{-1}(\Omega))$ for every $\lambda_h \in \widetilde{\Lambda}_h$. A natural choice is

$$\widetilde{\Lambda}_h = \{\lambda \in Z_h \text{ such that } \lambda(\cdot,0) = \lambda_t(\cdot,0) = 0\}.$$

The discrete version of the stabilized mixed formulation is then the following

$$\left\{ \begin{array}{rcll} a_{r,\alpha}(w_h,\overline{w}_h) + b_\alpha(\overline{w}_h,\lambda_h) & = & l_{1,\alpha}(\overline{w}_h), & (\overline{w}_h \in Z_h) \\ b_\alpha(w_h,\overline{\lambda}_h) - c_\alpha(\lambda_h,\overline{\lambda}_h) & = & l_{2,\alpha}(\overline{\lambda}_h), & \forall \overline{\lambda}_h \in \widetilde{\Lambda}_h. \end{array} \right.$$

- In view of the properties of the forms $a_{r,\alpha}$, c_α, $l_{1,\alpha}$ and $l_{2,\alpha}$, this formulation is well-posed.

Proposition (BFS element for $N = 1$)

Let $h > 0$, let $k \leq 2$ be a positive integer and $\alpha \in (0,1)$. Let (w,λ) and (w_h,λ_h) be the solution of the continuous and discrete stabilized mixed formulations respectively. If (w,λ) belongs to $H^{k+2}(Q_T) \times H^{k+2}(Q_T)$, then there exists $K_i = K_i(\|w\|_{H^{k+2}(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(Q_T)}, \alpha, r, \eta) > 0$, for every $i \in \{1,2,3\}$ and independent of h, such that

$$\|w - w_h\|_Z + \|\lambda - \lambda_h\|_\Lambda \leq K_1 h^k.$$

$$\|w - w_h\|_{L^2(Q_T)} \leq K_2 \frac{h^{k+2}}{\sqrt{\eta}}, \quad \|\lambda - \lambda_h\|_{L^2(Q_T)} \leq K_3 h^k.$$

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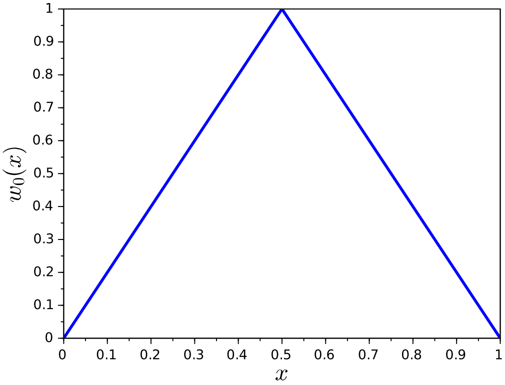
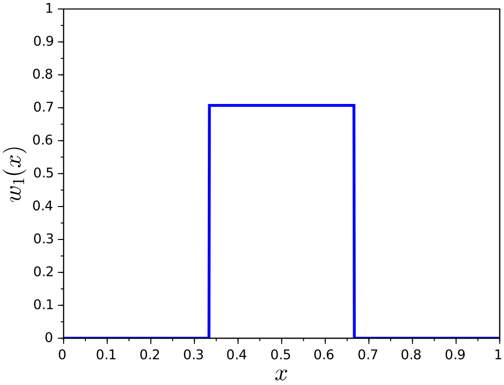
Remarks:

1. these estimates are not depending on a *inf-sup* constant
2. similar estimates can be obtained for the mixed formulation associated to the inverse problem for recovering the source term.

- $\Omega = (0,1)$, $\Sigma_T^1 = \{1\} \times (0,T)$
- $T = 2$, $c \equiv 1$, $d \equiv 0$, $f = 0$.





The corresponding solution is given by

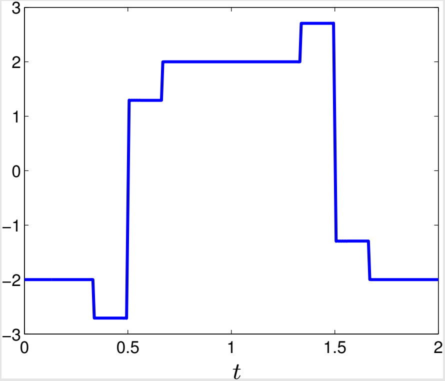
$$w(x,t) = \sum_{k>0} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sqrt{2} \sin(k\pi x)$$

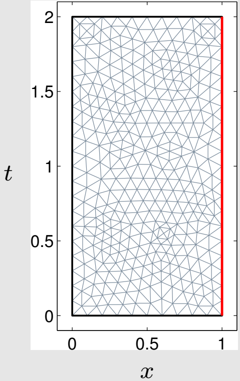
with

$$a_k = \frac{4\sqrt{2}}{\pi^2 k^2} \sin(\pi k/2), \quad b_k = \frac{1}{\pi k} (\cos(\pi k/3) - \cos(2\pi k/3)), \quad k > 0.$$

The observation is then given by:

$$y(t) = \sum_{k>0} (-1)^k k\pi \sqrt{2} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right)$$





$$y(t) = \sum_{k>0} (-1)^k k\pi\sqrt{2}\left(a_k \cos(k\pi t) + \frac{b_k}{k\pi}\sin(k\pi t)\right)$$

Numerical experiments

Reconstruction of the solution – one dimensional case

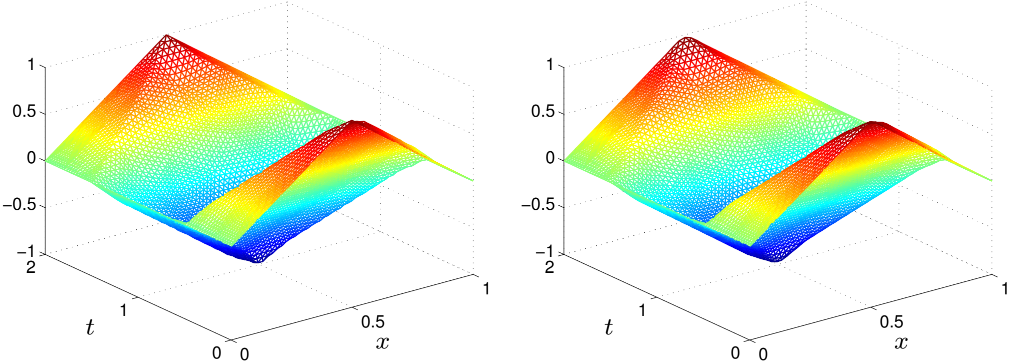


Figure : Exact solution w and approximated solution w_h on the mesh $\sharp\ 3$.

h	7.62×10^{-2}	3.81×10^{-2}	1.91×10^{-2}	9.53×10^{-3}	4.77×10^{-3}
$\frac{\Vert w-w_h\Vert_{L^2(Q_T)}}{\Vert w\Vert_{L^2(Q_T)}}$	3.67×10^{-2}	1.35×10^{-2}	5.99×10^{-3}	2.63×10^{-3}	1.22×10^{-3}
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	2.12×10^{-2}	1.08×10^{-2}	5.45×10^{-3}	2.53×10^{-3}	1.18×10^{-3}
κ_h	2.15×10^{6}	1.11×10^{7}	1.03×10^{8}	8.67×10^{8}	6.94×10^{9}

Table : Example HCT element - $r = h^2$ - $T = 2$.

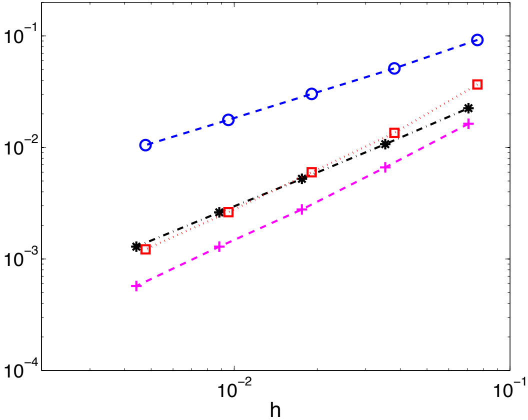


Figure : Example **EX1** -$T = 2$ - Relative error $\|w - w_h\|_{L^2(Q_T)}/\|w\|_{L^2(Q_T)}$ w.r.t. h for the BFS element with $r = h^2$ $(+)$ and $r = 1$ $(\star)$, the HCT element with $r = h^2$ $(\Box)$ and $r = 1$ $(\circ)$

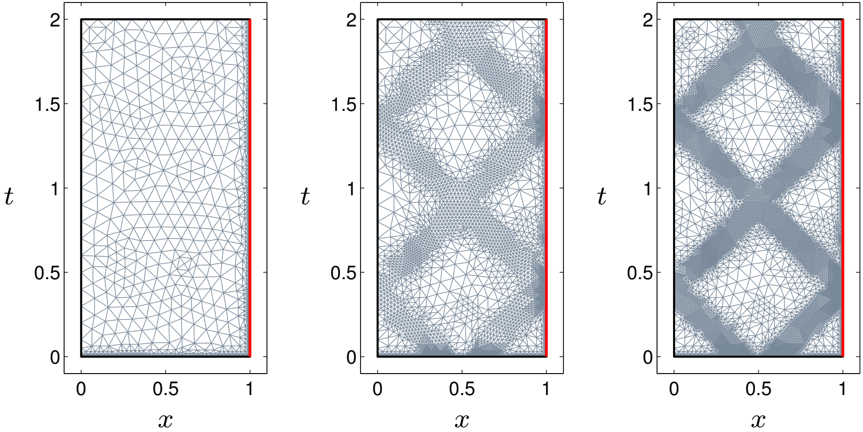
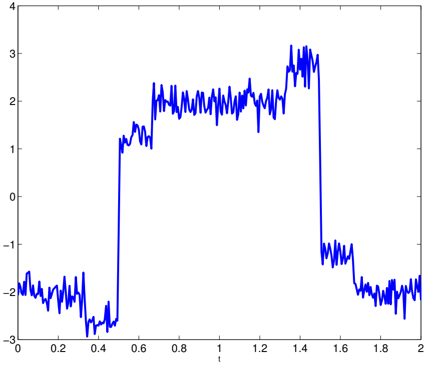
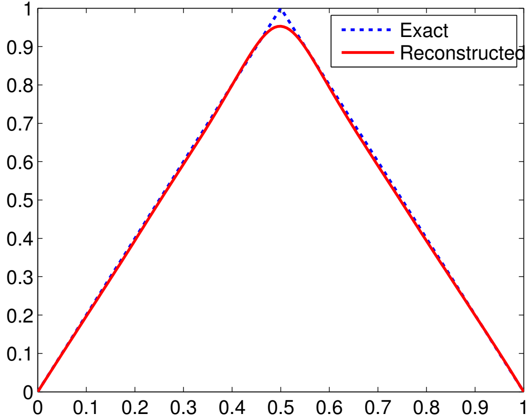


Figure : Iterative refinement of the triangular mesh over Q_T with respect to the variable w_h

We can add noise to the observation

$$\widetilde{y}^{\sigma}(t) = y(t) + N_{\sigma}(t) \qquad (t \in (0, T))$$




Exact
Reconstructed
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
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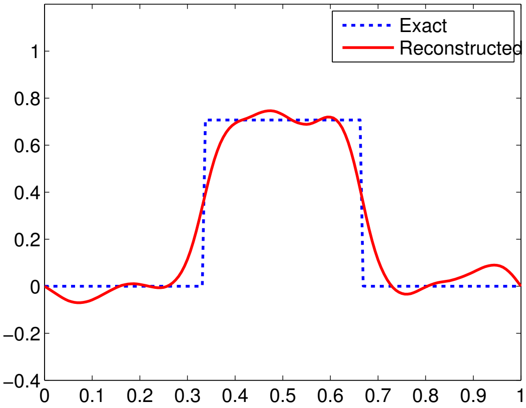
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1


Exact
Reconstructed
1
0.8
0.6
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−0.4
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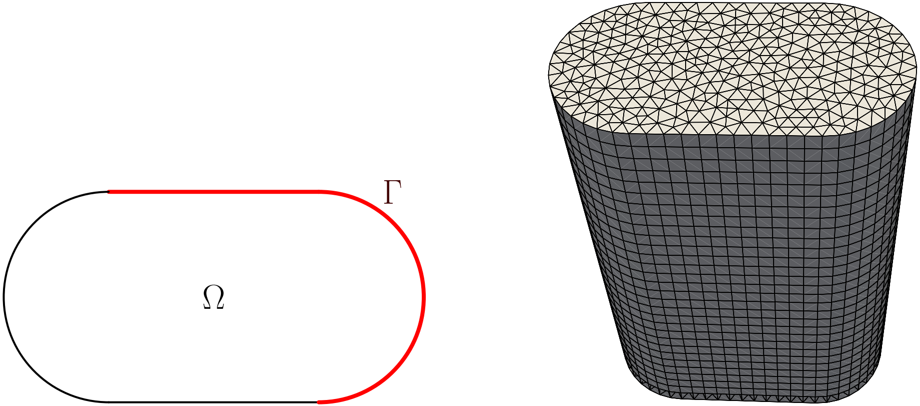


Figure : Bunimovich's stadium and the subset Γ of $\partial\Omega$ on which the observation is available (Left). Example of mesh of the domain Q_T (Right).

Mesh number	1	2	3
Number of elements	1 860	18 060	158 280
Number of nodes	1 216	10 261	84 241
Δx	1.82×10^{-1}	8.2×10^{-2}	3.95×10^{-2}
Δt (Height of elements)	0.2	0.1	0.05
h	2.7×10^{-1}	1.29×10^{-1}	6.37×10^{-2}

Table : Characteristics of the three meshes associated with Q_T.

$$\begin{cases} -\Delta w_0 = 10, & \text{in } \Omega \\ w_0 = 0, & \text{on } \partial\Omega, \end{cases} \qquad w_1 = 0 \quad \text{in } \Omega.$$

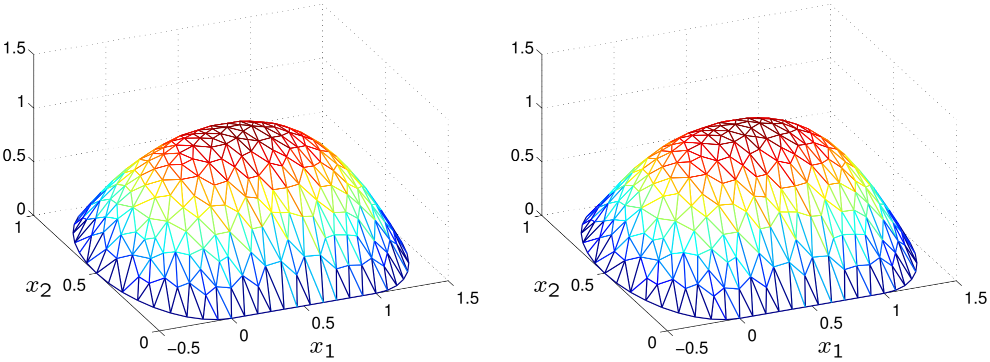
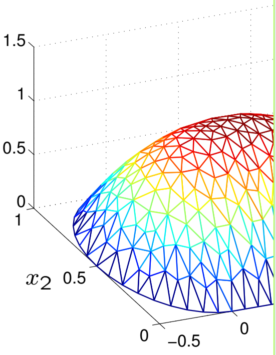
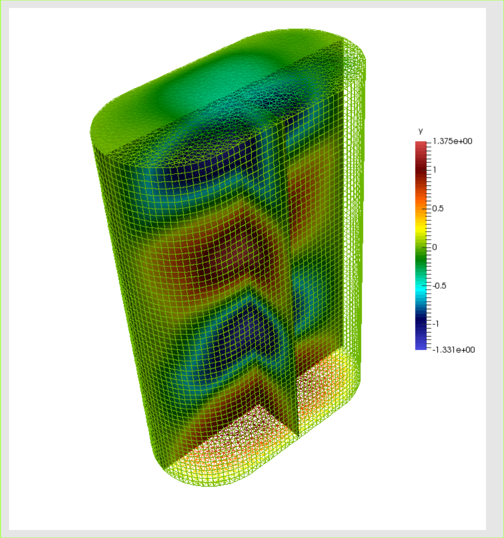


Figure : Initial data w_0 (Left). Reconstructed initial data $w_h(\cdot, 0)$ (Right).





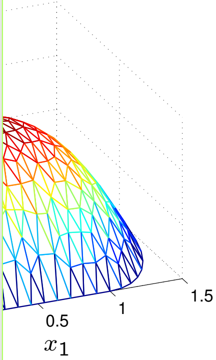


Figure : Initial da $\cdot, 0)$ (Right).

- $\Omega = (0,1)$, $T = 2$, $\Sigma_T^1 = \{1\} \times (0,T)$
- $c = 1$, $d = 0$, $\sigma(t) = 1 + t$

Spatial part of the source we want to reconstruct $\mu \in H^1(0,1)$:

$$\mu(x) = \frac{x}{\theta}\,\mathbb{1}_{[0,\theta]}(x) + \frac{(1-x)}{1-\theta}\,\mathbb{1}_{[\theta,1]}(x), \quad \theta = \frac{1}{3}.$$

In order to get explicit solution, we use that the solution with zero initial conditions can be expanded as follows :

$$\begin{cases} w(x,t) = \displaystyle\sum_{p>0} b_p(t)\sin(p\pi x) \\ b_p(t) := \dfrac{1}{p\pi}\displaystyle\int_0^t \sin(p\pi(t-s))f_p(s)ds, \quad f_p(s) := 2\sigma(s)\displaystyle\int_\Omega \sin(p\pi x)\mu(x)dx. \end{cases}$$

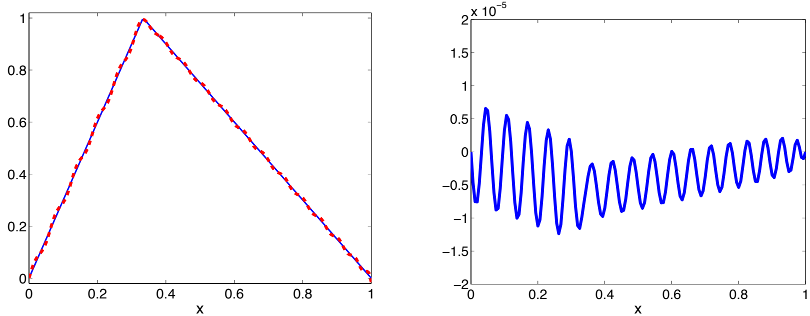


Figure : **Left**: Function μ (full line) and μ_h (dotted line); **Right**: $\frac{(-\Delta)^{-1}(\mu-\mu_h)}{\|(-\Delta)^{-1}\mu\|_{H_0^1(\Omega)}}$ along Ω.

$$\frac{\|w - w_h\|_{L^2(Q_T)}}{\|w\|_{L^2(Q_T)}} = \mathcal{O}(h^{1.9}), \qquad \frac{\|\mu - \mu_h\|_{H^{-1}(\Omega)}}{\|\mu\|_{H^{-1}(\Omega)}} = \mathcal{O}(h^{1.4}).$$

Plan

Boundary observation

Distributed observation

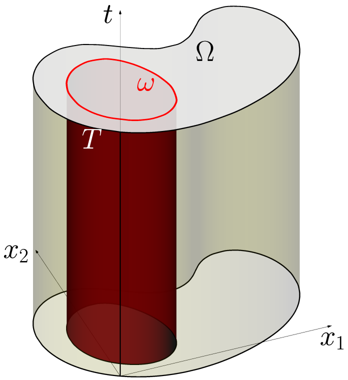
Time-dependent distributed observation

Distributed observation – the inverse problems

$$\begin{cases} w_{tt} - \nabla \cdot (c\nabla w) + dw = f, & \text{in } Q_T \\ w = 0, & \text{on } \Sigma_T \\ w(x,0) = w_0(x), \qquad y_t(x,0) = w_1(x), & x \in \Omega. \end{cases}$$

- $Q_T = \Omega \times (0,T)$;
- $\Sigma_T = \partial\Omega \times (0,T)$;
- $q_T = \omega \times (0,T) \subset Q_T$;
- $(w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega)$.
- $f \in L^2(0,T; H^{-1}(\Omega))$.

Observation: $y = w|_{q_T}$.



$$Z = \left\{ \begin{array}{l} w \in C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)) \text{ such that} \\ Lw \in L^2(0,T,H^{-1}(\Omega)),\ w|_{\Sigma_T} = 0 \end{array} \right\}$$

Hypothesis:

There exists a constant $C_{obs} = C(\omega, T, \|c\|_{C^1(\overline{\Omega})}, \|d\|_{L^\infty(\Omega)})$ such that the following estimate holds :

$$\|(w(\cdot,0), w_t(\cdot,0))\|^2_{L^2\times H^{-1}} \leq C_{obs}\left(\|w\|^2_{L^2(q_T)} + \|Lw\|^2_{L^2(0,T,H^{-1}(\Omega))}\right), \quad w \in Z.$$

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$$\langle w, \overline{w}\rangle_Z := \iint_{q_T} w(t)\,\overline{w}(t)\,dxdt + \eta \int_0^T \langle Lw(t),\, L\overline{w}(t)\rangle_{H^{-1}(\Omega)}\,dt \qquad (w, \overline{w} \in Z).$$

Here, $\langle\cdot,\cdot\rangle_{H^{-1}(\Omega)}$ denotes the inner product in $H^{-1}(\Omega)$ defined by

$$\langle \varphi, \psi\rangle_{H^{-1}(\Omega)} = \int_\Omega \nabla(-\Delta)^{-1}\varphi(x)\cdot\nabla(-\Delta)^{-1}\psi(x)dx, \qquad \forall\varphi,\ \psi \in H^{-1}(\Omega).$$

Distributed observation – mixed formulation

Find $(w, \lambda) \in Z \times L^2(0,T,H_0^1(\Omega))$ solution of

$$\begin{cases} a(w,\overline{w}) + b(\overline{w},\lambda) &= l(\overline{w}), & (\overline{w} \in Z) \\ b(w,\overline{\lambda}) &= 0, & (\overline{\lambda} \in L^2(0,T,H_0^1(\Omega))), \end{cases}$$

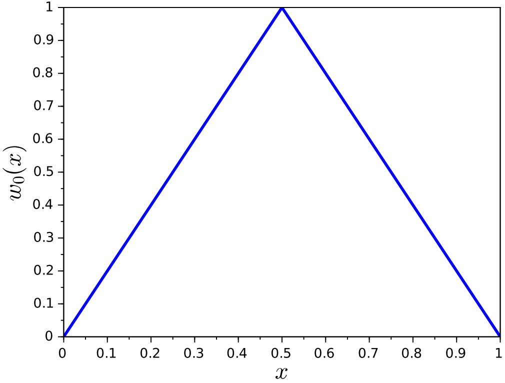
where

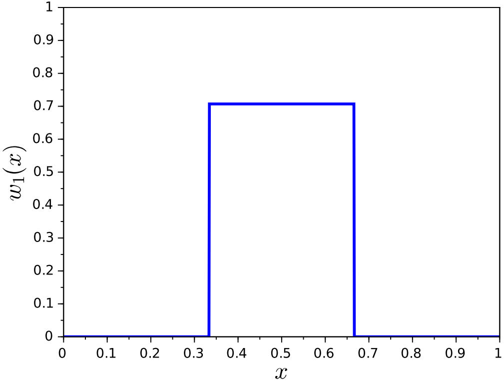
$$a : Z \times Z \to \mathbb{R}, \qquad a(w,\overline{w}) := \iint_{q_T} w\,\overline{w}\,dxdt,$$

$$b : Z \times L^2(0,T,H_0^1(\Omega)) \to \mathbb{R}, \qquad b(w,\lambda) := \int_0^T \langle \lambda(t),\, Lw(t)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} dt,$$

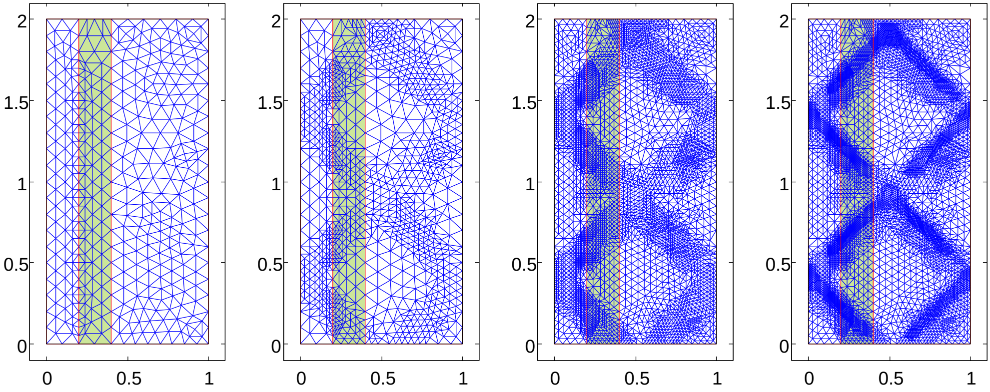
$$l : Z \to \mathbb{R}, \qquad l(w) := \iint_{q_T} y\,w\,dxdt.$$

- $\Omega = (0,1)$, $q_T = (0.1, 0.3) \times (0, T)$
- $T = 2$, $c \equiv 1$, $d \equiv 0$, $f = 0$.





Numerical experiments - 1d



Numerical experiments - 1d

Mesh number	1	2	3	4
$\sharp$ elements	792	2 108	7 902	14 717
$\sharp$ points	429	1 101	4 041	7 462
$\frac{\|w-w_h\|_{L^2(Q_T)}}{\|w\|_{L^2(Q_T)}}$	1.34×10^{-2}	8.69×10^{-3}	6.01×10^{-3}	5.9×10^{-3}
$\|\lambda_h\|_{L^2(Q_T)}$	1.14×10^{-5}	7.99×10^{-6}	5.02×10^{-6}	4.79×10^{-6}

Table : Information concerning the meshes and approximation errors for mesh adaptation strategy.

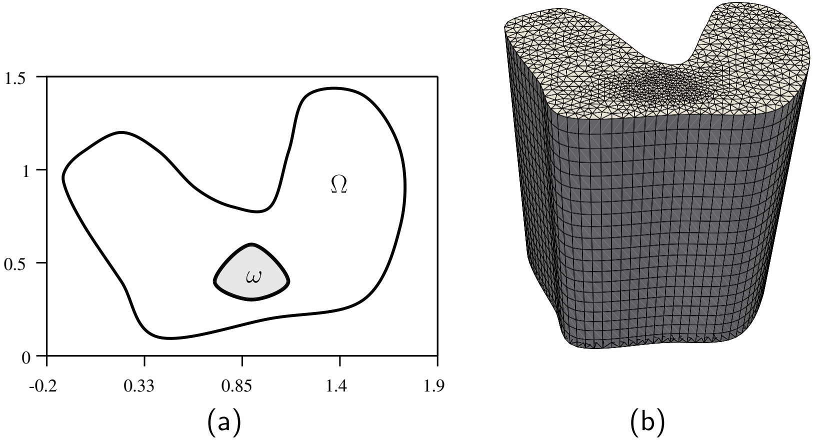


Figure : (a) Example of sets Ω and ω. (b) Example of mesh of the domain Q_T.

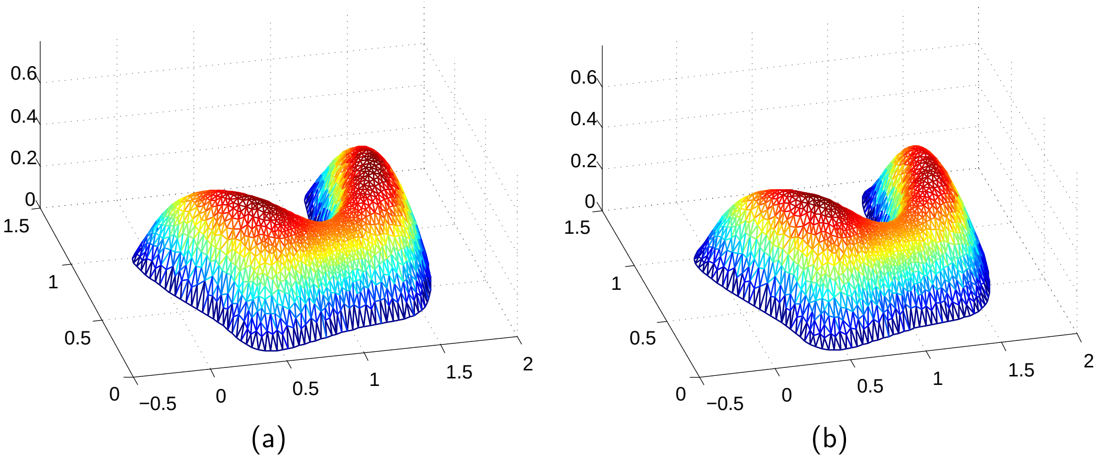


Figure : (a) Initial data w_0. (b) Reconstructed initial data $w_h(\cdot, 0)$.

Mesh number	1	2	3
$\frac{\|\overline{w}_h - w_h\|_{L^2(Q_T)}}{\|\overline{w}_h\|_{L^2(Q_T)}}$	1.88×10^{-1}	8.04×10^{-2}	7.11×10^{-2}

Table : Errors in the reconstructed solution

- w_h is the solution of the mixed formulation
- $\overline{w}_h$ numerical computed solution of the wave equation which was used to simulate the observation y.

Plan

Boundary observation

Distributed observation

Time-dependent distributed observation

Some references concerning the controllability with moving controls

- A. Y. Khapalov, *Controllability of the wave equation with moving point control*, Appl. Math. Optim. (1995).
- L. Cui, X. Liu, H. Gao, *Exact controllability for a one-dimensional wave equation in non-cylindrical domains*, J. Math. Anal. Appl. (2013).
- C. Castro, *Exact controllability of the 1-D wave equation from a moving interior point*, ESAIM COCV (2013).

Proposition (C. Carlos, N.C, A. Münch – 2014)

Assume that $q_T \subset (0,1) \times (0,T)$ is a finite union of connected open sets and satisfies the following hypotheses:
any characteristic line starting at a point $x \in (0,1)$ at time $t=0$ and following the optical geometric laws when reflecting at the boundary Σ_T must meet q_T.
Then, there exists $C>0$ such that the following estimate holds :

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{L^2(0,1)\times H^{-1}(0,1)} \leq C\left(\|\varphi\|^2_{L^2(q_T)} + \|L\varphi\|^2_{L^2(0,T;H^{-1}(0,1))}\right),$$

for every $\varphi \in C([0,T],L^2(0,1)) \cap C^1([0,T],H^{-1}(0,1))$ and satisfying $L\varphi \in L^2(0,T;H^{-1}(0,1))$.

Observability inequality in time-dependent domain case

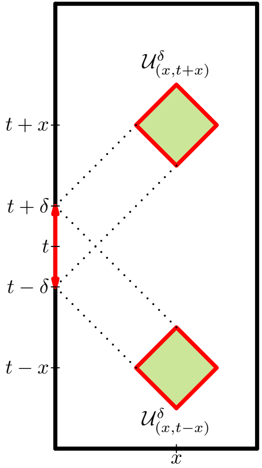
Idea of the proof

We follow the method used by C. Castro in the case of a moving pointwise control:

C. Castro, *Exact controllability of the 1-D wave equation from a moving interior point*, ESAIM COCV., 19 (2013).

Some ingredients of the proof :

- D'Alembert formulae;



$$\int_{t-\delta}^{t+\delta} |\varphi_x(0,s)|^2 ds \leq \frac{1}{\delta} \iint_{\mathcal{U}^\delta_{(x,t+x)}} \left(|\varphi_x|^2 + |\varphi_t|^2\right) dyds$$

$$\int_{t-\delta}^{t+\delta} |\varphi_x(0,s)|^2 ds \leq \frac{1}{\delta} \iint_{\mathcal{U}^\delta_{(x,t-x)}} \left(|\varphi_x|^2 + |\varphi_t|^2\right) dyds$$

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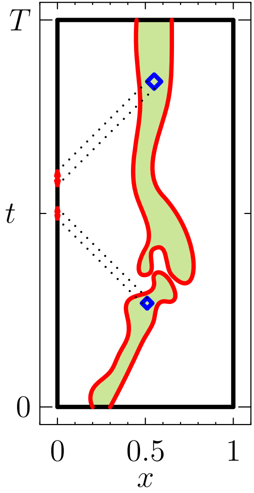
Some ingredients of the proof :

- D'Alembert formulae;
- known observability inequality in the boundary case;

Observability inequality in time-dependent domain case

Idea of the proof



Boundary observability inequality:

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|_H^2 \le C\int_0^T |\varphi_x(0,t)|^2 dt.$$

combined with the previous estimate gives:

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|_V^2 \le C\left(\|\varphi_t\|_{L^2(q_T)}^2 + \|\varphi_x\|_{L^2(q_T)}^2\right)$$

$$H = L^2(0,1)\times H^{-1}(0,1)$$

$$V = H_0^1(0,1)\times L^2(0,1)$$

Observability inequality in time-dependent domain case

Idea of the proof

We follow the method used by C. Castro in the case of a moving pointwise control:

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Some ingredients of the proof :

- D'Alembert formulae;
- known observability inequality in the boundary case;
- equi-repartition of energy.

Remark

The proof of the proposition is specific to the one-dimensional case.

Numerical experiments

Example of observation domains q_T and the associated meshes

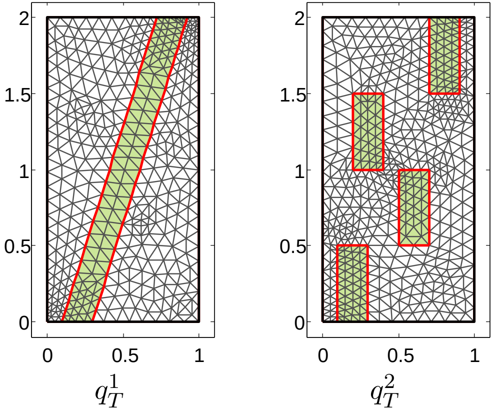

Figure : Domain q_T^1 and domain q_T^2 triangulated using some coarse meshes.

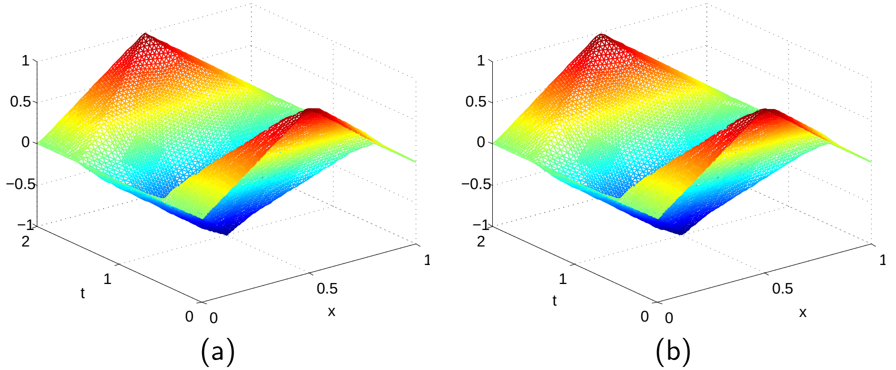

Figure : (a) Reference solution w. (b) Solution reconstructed from the observation $y = w|_{q_T^2}$.

h	6.24×10^{-2}	3.12×10^{-2}	1.56×10^{-2}	7.8×10^{-3}	3.9×10^{-3}
$\frac{\Vert w - w_h \Vert_{L^2(Q_T)}}{\Vert w \Vert_{L^2(Q_T)}}$	1.38×10^{-2}	6.37×10^{-3}	2.64×10^{-3}	1.15×10^{-3}	5.25×10^{-4}
$\Vert \lambda_h \Vert_{L^2(Q_T)}$	6.37×10^{-6}	1.65×10^{-6}	3.88×10^{-7}	9.74×10^{-8}	2.90×10^{-8}
κ	2.02×10^{8}	2.62×10^{9}	2.05×10^{10}	1.61×10^{11}	1.32×10^{12}
$\dim(\{\lambda_h\})$	554	2 135	8 381	33 209	132 209

Table : Observation domain q_T^2 - $r = 1$ - $T = 2$.

Conclusion and perspectives

Conclusion:

- We reduced the inverse problems of reconstruction of solution or of a source term to the resolution of a mixed formulation – for the wave equation.
- Boundary or distributed observation can be used in this method.
- Method is constructive – numerical convergent scheme.

Some perspectives:

- the method can be extended to more general systems
- avoid the use of C^1 finite elements?

- C. Castro, N. Cîndea, A. Münch, *Controllability of the linear 1D wave equation with inner moving forces*, SICON (2014).
- N. Cîndea, A. Münch, *Inverse problems for linear hyperbolic equations using mixed formulations*, Inverse Problems 31 (7), 075-001, 2015.
- N. Cîndea, A. Münch, *Reconstruction of the solution and the source of hyperbolic equations from boundary measurements: mixed formulations*, submitted.

Thank you!