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GLOBAL EXISTENCE RESULTS FOR SOME VISCOELASTIC MODELS WITH AN INTEGRAL CONSTITUTIVE LAW*

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Abstract. We provide a proof of global regularity of solutions of some models of viscoelastic flow with an integral constitutive law in two spatial dimensions and in a periodic domain. Models that are included in these results are classical models for flow memory: for instance, some K-BKZ models, the PSM model, or the Wagner model. The proof is based on the fact that these models naturally give an L^∞-bound on the stress and that they allow one to control the spatial gradient of the stress. The main result does not cover the case of the Oldroyd-B model.

Key words. viscoelastic flow, integral law, K-BKZ model, global existence result

AMS subject classifications. 35A01, 35B45, 35Q35, 76A05, 76A10, 76D05

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1. Introduction.

1.1. Presentation of the result. In this article, we are interested in the global (with respect to time) existence result for models of viscoelastic fluids. Usually, obtaining a global existence result for a highly nonlinear system of PDEs is quite challenging. The models we are interested in here are nonlinear on several levels: The first one is the well-known nonlinearity of the Navier–Stokes equations describing the hydrodynamics; this is the main reason we do not expect to have results in the three-dimensional case. The second level of nonlinearity comes from the rheology that we consider. More precisely, the viscoelasticity is described by the constitutive relation linking the stress and the strain. The framework for our study corresponds to the case where the extra-stress $\boldsymbol{\tau}$ is given, at any time t and at each point $\boldsymbol{x}$, by an integral law of the form

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \int_{-\infty}^{t} \mathcal{F}(t-\sigma, \boldsymbol{F}(\sigma,t,\boldsymbol{x}))\,\mathrm{d}\sigma. \tag{1.1}$$

The tensor $\boldsymbol{F}$ contains all the information of past deformations. It naturally depends on the velocity field of the flow: the relation (1.1) is then strongly coupled with the Navier–Stokes equations. Under assumptions on the behavior of the functional $\mathcal{F}$, we prove that the resulting system admits a global solution in the two-dimensional case and in a periodic domain, but without assuming that the data are small. Note that the results presented here are certainly still accurate in the entire domain $\mathbb{R}^2$, but it would probably need to carefully consider the arguments of proof to extend to the case of bounded domains.

These assumptions on the functional $\mathcal{F}$ (see subsection 3.3 and Remark 3.1) allow us to consider most of the usual integral models: the Wagner model, the PSM model, and some K-BKZ models.

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1.2. Mathematical results on viscoelastic model with an integral constitutive law. The integral models have been extensively studied over the last fifty years. In this regard, we can read the review article written by Mitsoulis [26] for the 50th anniversary of the K-BKZ models. However, there are few mathematical works on such viscoelastic models. The first significant results are probably due to Kim [20], Renardy [27], Hrusa and Renardy [17], Renardy, Hrusa, and Nohel [30, section IV.5]. Kim discusses a situation in which the nonlinearity in the constitutive equation (that is, the functional $\mathcal{F}$ in the relation (1.1)) has a special form. Renardy, Hrusa, and Nohel study spatially periodic three-dimensional motions with a more general nonlinearity (but sufficiently smooth). In all these works the solution is either local in time or global but with small data. Later, Brandon and Hrusa [4] study a one-dimensional model with a singularity in the nonlinearity: they obtain global existence results for sufficiently small data. Very recently (see [6]) some theoretical results were proved for a large family of nonlinearities: local existence, global existence with small data, and uniqueness results.

1.3. Some global existence results for viscoelastic models.

The Oldroyd-B model. There are many ways to describe a flow of viscoelastic fluid. The most famous model is the Oldroyd-B model, for which the question of global existence is still open, even in the two-dimensional case. This model expresses the constitutive relation between $\boldsymbol{\tau}$, the extra-stress, and $D\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + {}^T(\nabla \boldsymbol{u}))$, the deformation tensor, as follows:

$$\lambda \overset{\triangledown}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\mu D\boldsymbol{u}.$$

In this expression, the constants μ and λ, respectively, correspond to a polymeric viscosity and a relaxation time. The notation $\triangledown$ stands for the upper-convective derivative:

$$\overset{\triangledown}{\boldsymbol{\tau}} = \partial_t \boldsymbol{\tau} + \boldsymbol{u} \cdot \nabla \boldsymbol{\tau} - \nabla \boldsymbol{u} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot {}^T(\nabla \boldsymbol{u}).$$

Most of the models of viscoelastic flows can be seen as generalizations of the Oldroyd-B model, and as we shall see, some of these generalizations admit global solutions.

Many objective (frame indifferent) models. The classical way to introduce this Oldroyd-B model is to compare any elementary fluid element to a one-dimensional mechanical system composed of springs and dashpots. The derivative $\overset{\triangledown}{\boldsymbol{\tau}}$ is one way to extrapolate the convected derivative while preserving the invariance under Galilean transformation. There exists a one-parameter family of such models. This parameter is usually denoted by a, and the Oldroyd-B case corresponds to the case $a = 1$.

Such models have been extensively studied. Guillopé and Saut [13, 14, 15, 16] proved the existence of local strong solutions. Fernández-Cara, Guillén, and Ortega [11, 12] proved local well-posedness in Sobolev spaces. In Chemin and Masmoudi [5], local and global well-posedness in critical Besov spaces is given. In these papers, some global existence results hold when assuming small data.

It may be noted that in the case $a = 0$ (namely, the corotational case)—and only for that case—a global existence result of weak solution has been shown; see the result of Lions and Masmoudi [22].

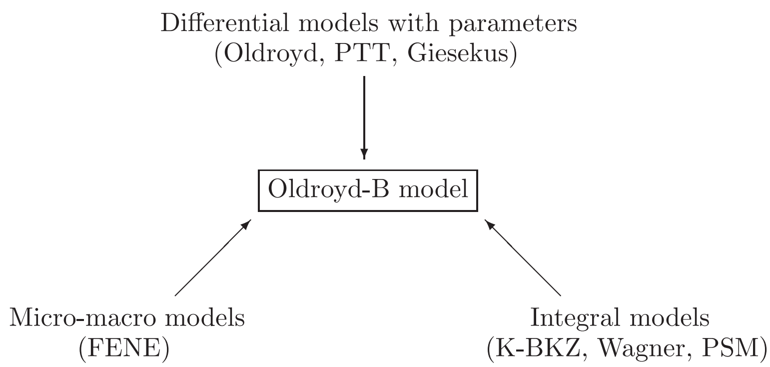
Micro-macro approach. On the other hand, the Oldroyd-B model can be seen as a special case of micro-macro models. This family of models is based on the fact that the constraint can be defined (using the formula of Kramers) from the distribution of the polymer chains. The distribution function is itself a solution of an equation of Fokker–Planck type wherein a spring acts.

The complete model couples the Navier–Stokes equations and this Fokker–Planck equation. A lot of local existence results are proved according to the exact form of the potential spring force; see, for instance, [18, 23, 28, 32]. Note that the Oldroyd-B model corresponds to the case where the spring force is assumed to be a linear Hookean force.

Recently, Masmoudi [24] proved global existence of weak solutions to the FENE (finite extensible nonlinear elastic) dumbbell model. In this model, a polymer is idealized as an elastic dumbbell consisting of two beads joined by a spring whose elongation cannot exceed a limit. The spring force therefore has a very specific shape.

Integral models. Finally, the Oldroyd-B model is a special case of integral-type models. These models are built on the natural remark expressing the fact that the fluid is a memory medium: the stress at a given time depends on all past constraints. The Oldroyd-B model corresponds to a linear case (the influence of the Finger tensor is linear). We show in this paper global existence result for the usual integral models, particularly those including a nonlinear dependence with respect to the Finger tensor.

The situation described above can be represented by the following diagram, where the Oldroyd-B model can be viewed as a particular case of some different approaches:



Monodimensional case: Shear flows. Some global existence results (without assuming that the data are small) already exist for some integral models: they correspond to some special flows which can be viewed as monodimensional cases. Indeed, if the flow is assumed to be sufficiently simple, then the Navier–Stokes equations reduce to more simple equations which automatically imply more results. The Poiseuille flow of a K-BKZ-fluid is then studied in [1]. They especially study the steady flow equation and its stability. More recently, Renardy [29] proved the global existence in time of solutions to time-dependent shear flows for such integral viscoelastic behavior. The essential point in the proof is an a priori estimate for the shear stress which allows one to easily deduce—in this "one"-dimensional case—a bound on the shear velocity.

1.4. Outline of this paper. In the next section (section 2), we present the model coupling the hydrodynamic Navier–Stokes equations and the stress constitutive relation. Section 3 is devoted to the presentation of the mathematical framework. We also give in section 3 the main assumptions on the functional $\mathcal{F}$ introduced in (1.1). In section 4 we give the main result and describe the method for the proof. The last three sections (5, 6, and 7) are devoted to the proof. More exactly, we first give some estimates on the velocity field in section 5. Next we show how to control the extra-stress using the velocity (section 6). We finally conclude the proof in section 7.

2. Governing equations. For a general viscoelastic and incompressible fluid, we start with the following equations for the conservation of momentum and mass:

$$\begin{aligned} &\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p - \eta\,\Delta\boldsymbol{u} = \operatorname{div}\boldsymbol{\tau},\\ &\operatorname{div}\boldsymbol{u} = 0. \end{aligned} \tag{2.1}$$

The two unknowns are the vector velocity field $\boldsymbol{u}$ and the scalar pressure p. The positive real η is the kinematic viscosity of the fluid. This system is closed using a constitutive equation connecting the extra-stress $\boldsymbol{\tau}$ and the velocity gradient $\nabla\boldsymbol{u}$. The role of this additional contribution $\boldsymbol{\tau}$ is to take into account the past history of the fluid. It can be expressed by an integral with respect to all past time:[1]

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \int_{-\infty}^{t} m(t-\sigma)\,\mathcal{S}\big(\boldsymbol{F}(\sigma,t,\boldsymbol{x})\big)\,\mathrm{d}\sigma. \tag{2.2}$$

The scalar function m (the memory) and the tensorial function $\mathcal{S}$ are given by the properties of the fluids we consider, whereas the deformation tensor $\boldsymbol{F}$ is coupled with the velocity field of the flow. More precisely the tensor $\boldsymbol{F}$ satisfies the relation

$$\partial_t\boldsymbol{F} + \boldsymbol{u}\cdot\nabla\boldsymbol{F} = \boldsymbol{F}\cdot\nabla\boldsymbol{u}. \tag{2.3}$$

In this paper we are interested in the two-dimensional periodical case with respect to the spatial variable: $\boldsymbol{x}\in\mathbb{T}^2$. Consequently there is no boundary condition. We must impose the initial conditions. For the velocity, they correspond to a given velocity at $t=0$. For the deformation tensor $\boldsymbol{F}$ we give its initial value at $t=0$. We also note that by definition of the deformation, we must have $\boldsymbol{F}(\sigma,\sigma,\boldsymbol{x}) = \boldsymbol{\delta}$ for all past time σ and for any $\boldsymbol{x}\in\mathbb{T}^2$ (the tensor $\boldsymbol{\delta}$ representing the identity tensor).

It may be more interesting to work with the new variable $s = t-\sigma$, which represents the age instead of the parameter σ. We then introduce $\boldsymbol{G}(s,t,\boldsymbol{x}) = \boldsymbol{F}(t-s,t,\boldsymbol{x})$, and the system reads

$$\begin{cases} \partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p - \eta\Delta\boldsymbol{u} = \operatorname{div}\boldsymbol{\tau} & \text{on } (0,+\infty)\times\mathbb{T}^2,\\ \operatorname{div}\boldsymbol{u} = 0 & \text{on } (0,+\infty)\times\mathbb{T}^2,\\ \boldsymbol{\tau}(t,\boldsymbol{x}) = \displaystyle\int_0^{+\infty} m(s)\,\mathcal{S}\big(\boldsymbol{G}(s,t,\boldsymbol{x})\big)\,\mathrm{d}s & \text{for } (t,\boldsymbol{x})\in(0,+\infty)\times\mathbb{T}^2,\\ \partial_s\boldsymbol{G} + \partial_t\boldsymbol{G} + \boldsymbol{u}\cdot\nabla\boldsymbol{G} = \boldsymbol{G}\cdot\nabla\boldsymbol{u} & \text{on } (0,+\infty)\times(0,+\infty)\times\mathbb{T}^2. \end{cases} \tag{2.4}$$

System (2.4) is closed with the following initial conditions:

$$\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0, \qquad \boldsymbol{G}\big|_{t=0} = \boldsymbol{G}_0, \qquad \boldsymbol{G}\big|_{s=0} = \boldsymbol{\delta}. \tag{2.5}$$

3. Mathematical framework and assumptions.

3.1. Tensorial analysis. In system (2.4), the first equation is a vectorial equation (the velocity $\boldsymbol{u}$ is a function with values in $\mathbb{R}^2$), and the last two equations are

[1] This is the particular case of the separable single-integral model. We can use more general models like those given by (1.1). In this paper, the proofs are written with a separate model (2.2), but they can easily be generalized; see Remark 3.1.

tensorial equations (the stress $\boldsymbol{\tau}$ and the deformation tensor $\boldsymbol{G}$ are functions with values in the set of the 2-tensors). In the following proofs, we need to work with the gradient of such 2-tensors, that is, with 3-tensors, and even with 4-tensors. We introduce here some definitions for tensors of arbitrary order.

DEFINITION 3.1. *Let $\boldsymbol{A}$ be a p-tensor and $\boldsymbol{B}$ a q-tensor. For any $0 \le s \le \min\{p,q\}$ we define the following $(p+q-2s)$-tensor $\boldsymbol{A} \overset{(s)}{:} \boldsymbol{B}$ component by component:*

$$\Big(\boldsymbol{A} \overset{(s)}{:} \boldsymbol{B}\Big)_{i_1,\dots,i_{p-s},j_{s+1},\dots,j_q} = \sum_{k_1,\dots,k_s} a_{i_1,\dots,i_{p-s},k_1,\dots,k_s} b_{k_1,\dots,k_s,j_{s+1},\dots,j_q}.$$

For simplicity, we will denote $\boldsymbol{A} \overset{(0)}{:} \boldsymbol{B} = \boldsymbol{A}\boldsymbol{B}$, $\boldsymbol{A} \overset{(1)}{:} \boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{B}$, and $\boldsymbol{A} \overset{(2)}{:} \boldsymbol{B} = \boldsymbol{A} : \boldsymbol{B}$.

Note also that the product $\overset{(p)}{:}$ is a scalar product on the set of the p-tensors. It allows us to define a generalized Frobenius norm.

DEFINITION 3.2. *The Frobenius norm of a p-tensor $\boldsymbol{A}$ is defined by $\big(\boldsymbol{A} \overset{(p)}{:} \boldsymbol{A}\big)^{1/2}$. It will always be denoted $|\boldsymbol{A}|$ (regardless of the value of p). Using the components of $\boldsymbol{A}$, we have*

$$|\boldsymbol{A}|^2 = \sum_{i_1,\dots,i_p} a^2_{i_1,\dots,i_p}.$$

Finally, the Cauchy–Schwarz inequality can be easily generalized as follows.

PROPOSITION 3.3. *Let $\boldsymbol{A}$ be a p-tensor, $\boldsymbol{B}$ a q-tensor, and $0 \le s \le \min\{p,q\}$. We have*

$$|\boldsymbol{A} \overset{(s)}{:} \boldsymbol{B}| \le |\boldsymbol{A}||\boldsymbol{B}|. \tag{3.1}$$

Note that the norms used in this proposition are not all the same: on the left-hand side of inequality (3.1), it corresponds to the Frobenius norm on the $(p+q-2s)$-tensors, whereas on the right-hand side it corresponds to the Frobenius norm on the p-tensors and on q-tensors.

3.2. Functional spaces. We use the following usual notation.

✓ For all real $s \ge 0$ and every integer $q \ge 1$, the set $W^{s,q}(\mathbb{T}^2)$ corresponds to the Sobolev spaces. We classically denote by $L^q(\mathbb{T}^2) = W^{0,q}(\mathbb{T}^2)$ the associated Lebesgue space. Since we will frequently use functions with values in $\mathbb{R}^2$ or in the space $\mathcal{L}(\mathbb{R}^2)$ of real 2-tensors, the usual notation will be abbreviated. For instance, the space $(W^{1,q}(\mathbb{T}^2))^2$ will be denoted $W^{1,q}(\mathbb{T}^2)$. Moreover, all norms will be denoted by indices, for instance, $\|\boldsymbol{u}\|_{W^{1,q}(\mathbb{T}^2)}$.

✓ The space $D^r_q(\mathbb{T}^2)$ stands for some fractional domain of the Stokes operator A_q in $L^q(\mathbb{T}^2)$ (cf. section 2.3 in [9]). Its norm is defined by

$$\|\boldsymbol{v}\|_{D^r_q(\mathbb{T}^2)} := \|\boldsymbol{v}\|_{L^q(\mathbb{T}^2)} + \Big(\int_0^{+\infty} \|A_q \mathrm{e}^{-tA_q}\boldsymbol{v}\|^r_{L^q(\mathbb{T}^2)}\,\mathrm{d}t\Big)^{1/r}.$$

Roughly, the vector-fields of $D^r_q(\mathbb{T}^2)$ are vectors which have $2-\frac{2}{r}$ derivatives in $L^q(\mathbb{T}^2)$ and are divergence-free. It may be identified with Besov spaces. It also can be viewed as an interpolate space between $L^q(\mathbb{T}^2)$ and the domain of the Stokes operator $D(A_q)$; see [9].

✓ The notation of type $L^r(0,T;W^{1,q}(\mathbb{T}^2))$ denotes the space of r-integrable functions on $(0,T)$, with values in the space $W^{1,q}(\mathbb{T}^2)$. Similarly, an expression like

$g \in L^\infty(\mathbb{R}^+; L^r(0,T; L^q(\mathbb{T}^2)))$ means that

$$\sup_{s\in\mathbb{R}^+} \Big(\int_0^T \|g(s,t,\cdot)\|^r_{L^q(\mathbb{T}^2)} \,\mathrm{d}t \Big)^{\frac{1}{r}} < +\infty.$$

✓ Finally let us denote by $\mathfrak{P}$ the orthogonal projector in $L^2(\mathbb{T}^2)$ onto the set of the divergence-free vectors fields of $L^2(\mathbb{T}^2)$.

3.3. Assumptions. In this section we present the assumptions that we need for the proof. These assumptions concern the functions m and $\mathcal{S}$ introduced in the extra-stress expression (2.2):

(H1) $m : s \in \mathbb{R}^+ \longmapsto m(s) \in \mathbb{R}$ is measurable, decreasing, and positive and $\int_0^\infty m(s)\,\mathrm{d}s = 1$;

(H2) $\mathcal{S} : \boldsymbol{G} \in \mathcal{L}(\mathbb{R}^2) \longmapsto \mathcal{S}(\boldsymbol{G}) \in \mathcal{L}(\mathbb{R}^2)$ is of class $\mathcal{C}^1$ and satisfies the following:
–There exists $\mathcal{S}_\infty \geq 0$ such that for all $\boldsymbol{G} \in \mathcal{L}(\mathbb{R}^2)$ we have $|\mathcal{S}(\boldsymbol{G})| \leq \mathcal{S}_\infty$.
–There exists $\mathcal{S}'_\infty \geq 0$ such that for all $\boldsymbol{G} \in \mathcal{L}(\mathbb{R}^2)$ we have $|\boldsymbol{G}||\mathcal{S}'(\boldsymbol{G})| \leq \mathcal{S}'_\infty$.

As specified above the matricial norms used here correspond to the Frobenius norms. We take care of the fact that the derivative $\mathcal{S}'(\boldsymbol{G})$ may be represented by a tensor of order 4: $(S'(\boldsymbol{G}))_{ijk\ell}$ corresponds to the derivation of $(\mathcal{S}(\boldsymbol{G}))_{k\ell}$ with respect to the component $\boldsymbol{G}_{ij}$.

Notes on the assumptions.

✓ The first assumption (H1) is related to the memory function m. It is linked to the principle of fading memory; see [7]. Usually, the memory function is a combination of exponentially decreasing functions which satisfies assumption (H1). Note that in some cases the memory function is described as an infinite sum of exponentially decreasing functions. This is the case of the Doi–Edwards model; see [10]. Despite the singularity of such a function at 0, it satisfies hypothesis (H1).

✓ The second assumption concerns the function $\mathcal{S}$. In practice much of classical integral models read in the two-dimensional case[2]

$$\mathcal{S}(\boldsymbol{G}) = h(I_1)\, {}^T\boldsymbol{G} \cdot \boldsymbol{G},$$

where $I_1 = \mathrm{Tr}({}^T\boldsymbol{G} \cdot \boldsymbol{G})$ is the only invariant of interest; the other one is given by $\det({}^T\boldsymbol{G} \cdot \boldsymbol{G})$ and is equal to 1 since the flow is assumed to be incompressible. For such a case, assumption (H2) is equivalent to the following:

– There exists $C \geq 0$ such that for all $x \geq 0$ we have $x|h(x)| \leq C$.
– There exists $C' \geq 0$ such that for all $x \geq 0$ we have $x^2|h'(x)| \leq C'$.

Examples. As an example of a popular viscoelastic constitutive equation used in the past 30 years, which possesses enough degree of complexity so as to capture as accurately as possible the complex nature of polymeric liquids, we present the K-BKZ/PSM model (from the initials of Kaye, Bernstein, Kearsley, Zapas and of Papanastasiou, Scriven, Macosko). It is written as the integral law (2.2) where the

[2]These models are usually introduced for tridimensional flows. The strain memory is then a function of the Finger strain tensor $\boldsymbol{B} = {}^T\boldsymbol{G} \cdot \boldsymbol{G}$ and on the Cauchy–Green strain tensor $\boldsymbol{B}^{-1}$. In that case, the function $\mathcal{S}$ is given by (see [2, 3, 19])

$$\mathcal{S}(\boldsymbol{G}) = h_1(I_1, I_2)\boldsymbol{B} + h_2(I_1, I_2)\boldsymbol{B}^{-1},$$

where $I_1 = \mathrm{Tr}(\boldsymbol{B})$ and $I_2 = \mathrm{Tr}(\boldsymbol{B}^{-1})$. For planar incompressible flows, we have $\boldsymbol{B}^{-1} = \mathrm{Tr}(\boldsymbol{B})\boldsymbol{\delta} - \boldsymbol{B}$, and consequently $I_1 = I_2$.

memory function is given by

$$m(s) = \sum_{k=1}^{N} \frac{G_k}{\lambda_k} \mathrm{e}^{-s/\lambda_k},$$

and where the strain-memory function is given by $\mathcal{S}(\boldsymbol{G}) = h(I_1)\, {}^T\boldsymbol{G} \cdot \boldsymbol{G}$ with

$$h(I_1) = \frac{\alpha}{\alpha + I_1 - 3}.$$

The constant α can be determined from shear flow data. The constants λ_k and G_k are the relaxation times and relaxation moduli, respectively, and N is the number of relaxation modes. Typically models use around $N = 8$ modes with, for instance, the following dimensional values (see [25, Table 4.1, p. 136]):

k	λ_k (s)	G_k (Pa)
1	10^{-4}	1.29×10^5
2	10^{-3}	9.48×10^4
3	10^{-2}	5.86×10^4
4	10^{-1}	2.67×10^4
5	10^{0}	9.80×10^3
6	10^{1}	1.89×10^3
7	10^{2}	1.80×10^2
8	10^{3}	1.00×10^0

For such a model, we easily verify that the two assumptions (H1) and (H2) hold (except for the normalization of the memory function $\int_0^\infty m(s)\,\mathrm{d}s = 1$, which obviously depends on the dimensionless procedure). In practice, almost all models of type K-BKZ satisfy assumptions (H1) and (H2) (see the review by Mitsoulis [26] where a lot of models are proposed).

Nevertheless, we note that the Oldroyd-B model, which corresponds to the "simple" case $m(s) = \mathrm{e}^{-s}$ and $\mathcal{S}(\boldsymbol{G}) = {}^T\boldsymbol{G} \cdot \boldsymbol{G} - \boldsymbol{\delta}$, does not satisfy assumption (H2). The study presented here does not cover such Oldroyd models; the global result in this case remains an open question.

Remark 3.1. If we want to use a nonseparable integral law like

$$\boldsymbol{\tau}(t, \boldsymbol{x}) = \int_0^{+\infty} \mathcal{F}\big(s, \boldsymbol{G}(s, t, \boldsymbol{x})\big)\,\mathrm{d}s, \tag{3.2}$$

assumptions (H1) and (H2) become the following:

- There exists $m_1 \in L^1(\mathbb{R}^+)$ such that for all $(s, \boldsymbol{G}) \in \mathbb{R}^+ \times \mathcal{L}(\mathbb{R}^2)$ we have the inequality $|\mathcal{F}\big(s, \boldsymbol{G}\big)| \leq m_1(s)$.
- There exists $m_2 \in L^1(\mathbb{R}^+)$ decreasing such that for $(s, \boldsymbol{G}) \in \mathbb{R}^+ \times \mathcal{L}(\mathbb{R}^2)$ we have $|\boldsymbol{G}||\partial_{\boldsymbol{G}} \mathcal{F}\big(s, \boldsymbol{G}\big)| \leq m_2(s)$.

All proofs remain unchanged.

4. Main result. We now announce the main result of this paper, namely, a global existence theorem for the system (2.4). More precisely, we have the following theorem.

THEOREM 4.1. *Let q and r be two integers such that $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$. We assume that the initial conditions $\boldsymbol{u}_0$ and $\boldsymbol{G}_0$ satisfy*

$$\boldsymbol{u}_0 \in D_q^r(\mathbb{T}^2), \qquad \boldsymbol{G}_0 \in L^\infty(\mathbb{R}^+; W^{1,q}(\mathbb{T}^2)) \cap W^{1,\infty}(\mathbb{R}^+; L^q(\mathbb{T}^2)),$$

and there exists $\mu > 0$ *such that* $\det \boldsymbol{G}_0 \geq \mu$ *on* $\mathbb{R}^+ \times \mathbb{T}^2$. *Let* $\eta > 0$, m *satisfy* (H1), *let* $\mathcal{S}$ *satisfy* (H2), *and let* $T > 0$ *be arbitrary.*

There exists a constant C *depending only on the norm of the initial data* q, r, μ, η, $\mathcal{S}_\infty$, $\mathcal{S}'_\infty$, *and* T *with* C *bounded for bounded* T, *and a unique solution* $(\boldsymbol{u}, p, \boldsymbol{\tau}, \boldsymbol{G})$ *of* (2.4)–(2.5) *such that*

$$\|\nabla^2 \boldsymbol{u}\|_{L^r(0,T;L^q(\mathbb{T}^2))} \leq C, \quad \|\nabla \boldsymbol{u}\|_{L^\infty((0,T)\times\mathbb{T}^2)} \leq C,$$
$$\|\nabla \boldsymbol{\tau}\|_{L^r(0,T;L^q(\mathbb{T}^2))} \leq C, \quad \|\boldsymbol{\tau}\|_{L^\infty((0,T)\times\mathbb{T}^2)} \leq C,$$
$$\text{and} \quad \int_0^T \int_0^\infty m(s) \Big\| \frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|} \Big\|^r_{L^q(\mathbb{T}^2)} (s,t) \, \mathrm{d}s\mathrm{d}t \leq C$$

hold.

Remark 4.1.

✓ The pressure p is a Lagrange multiplier associated to the divergence-free constraint. It can be solved using the Riesz transforms. More precisely, taking the divergence of the first equation of system (2.4), we use the periodic boundary conditions to obtain

$$p = -(-\Delta)^{-1} \operatorname{div} \operatorname{div} (\boldsymbol{\tau} - \boldsymbol{u} \otimes \boldsymbol{u}). \tag{4.1}$$

From Theorem 4.1, the solutions of system (2.4) discussed in this paper have $\boldsymbol{\tau} - \boldsymbol{u} \otimes \boldsymbol{u}$ in $L^\infty(0,T;L^2(\mathbb{T}^2))$. The pressure in the solution of (2.4) is meant to be given by (4.1).

✓ In many applications, the fluid is assumed to be initially quiescent. In that case, we have $\boldsymbol{G}_0 = \boldsymbol{\delta}$ and $\det \boldsymbol{G}_0 = 1$. Moreover, we will see that the quantity $\det \boldsymbol{G}$ is only convected by the flow. If the fluid is assumed to be at rest in the past (that is, for s large enough), then we always have $\det \boldsymbol{G}_0 = 1$. The assumption on the positivity of $\det \boldsymbol{G}_0$, for instance, allows us to consider such cases.

In the following, we will denote by C constants that may depend on the initial conditions, on the viscosity η, on some integers r, q, on the bounds $\mathcal{S}_\infty$ and $\mathcal{S}'_\infty$, on the constant μ, and on the time T. These constants will always be bounded for bounded T.

Sketch of the proof. Using the assumptions given in Theorem 4.1, the local existence is proved in [6]. It is based on a fixed point argument and some estimates. The existence time is small since we need some contraction in the fixed point theorem. Nevertheless, to obtain the local existence we do not need assumption (H2): we only assume that the function $\mathcal{S}$ is of class $\mathcal{C}^1$.

The purpose of this article is to establish additional bounds using the additional assumption (H2). We then consider a solution $(\boldsymbol{u}, p, \boldsymbol{\tau}, \boldsymbol{G})$ to system (2.4)–(2.5) in $[0,T]$ with the regularity proved in [6]:

$$\begin{array}{ll} \boldsymbol{u} \in L^r(0,T;W^{2,q}(\mathbb{T}^2)), & \partial_t \boldsymbol{u} \in L^r(0,T;L^q(\mathbb{T}^2)), \\ \boldsymbol{\tau} \in L^\infty(0,T;W^{1,q}(\mathbb{T}^2)), & \partial_t \boldsymbol{\tau} \in L^r(0,T;L^q(\mathbb{T}^2)), \\ \boldsymbol{G} \in L^\infty(\mathbb{R}^+ \times (0,T);W^{1,q}(\mathbb{T}^2)), & \partial_s \boldsymbol{G},\, \partial_t \boldsymbol{G} \in L^\infty(\mathbb{R}^+;L^r(0,T;L^q(\mathbb{T}^2))). \end{array}$$

The next steps are to obtain estimates on this solution.

Roughly speaking the first part of assumption (H2) implies that the extra-stress $\boldsymbol{\tau}$ is L^∞-bounded. The second part of assumption (H2) gives a control of $\nabla \boldsymbol{\tau}$ with respect to $\frac{\nabla \boldsymbol{G}}{\boldsymbol{G}}$. These controls on the extra-stress will be transformed into controls on the velocity using the Navier–Stokes equations. Finally the equation on $\boldsymbol{G}$ allows us to deduce a bound on $\frac{\nabla \boldsymbol{G}}{\boldsymbol{G}}$.

5. A priori estimates for the spatial gradient of the velocity. The following key result is a direct consequence of assumptions (H1) and (H2) on the stress tensor by means of the function $\mathcal{S}$.

LEMMA 5.1. *We have the following L^∞-bound:*

$$\|\boldsymbol{\tau}\|_{L^\infty((0,T)\times\mathbb{T}^2)} \le \mathcal{S}_\infty. \tag{5.1}$$

We can prove that the velocity field is also bounded.

LEMMA 5.2. *There exists a constant C such that*

$$\|\boldsymbol{u}\|_{L^\infty((0,T)\times\mathbb{T}^2)} \le C. \tag{5.2}$$

Proof. On the one hand, we use the local-in-time result to obtain a bound for $\boldsymbol{u}$ in $(0,T_0)\times\mathbb{T}^2$ for some $T_0>0$. Indeed the local existence result gives a bound for $\boldsymbol{u}$ in $L^r(0,T_0;W^{2,q}(\mathbb{T}^2))$ and a bound for $\partial_t\boldsymbol{u}$ in $L^r(0,T_0;L^q(\mathbb{T}^2))$. For $r\ge 2$, by an Aubin–Simon theorem (see [31]) this implies a bound for $\boldsymbol{u}$ in $\mathcal{C}(0,T_0;W^{1,q}(\mathbb{T}^2))$ which, for $q>2$, provides the $L^\infty((0,T_0)\times\mathbb{T}^2)$ bound on $\boldsymbol{u}$.

On the other hand, a result proved by Constantin and Seregin (see [8, Prop. 2.4]) gives an L^∞-bound for the solution $\boldsymbol{u}$ to the Navier–Stokes equations (2.1) in $(\sigma,T)\times\mathbb{T}^2$ for any $\sigma>0$, as soon as $\boldsymbol{\tau}$ is bounded in some $L^4((0,T)\times\mathbb{T}^2)$.

Taking $\sigma=T_0/2$, this allows us to conclude the proof of Lemma 5.2. ☐

LEMMA 5.3. *For all $1<q,r<+\infty$ there exists a constant C such that*

$$\|\nabla\boldsymbol{u}\|_{L^r(0,T;L^q(\mathbb{T}^2))} \le C. \tag{5.3}$$

Proof. The proof is based on the integral representation of the solution to the Navier–Stokes equation (2.1):

$$\nabla\boldsymbol{u}(t,\boldsymbol{x}) = \mathrm{e}^{\eta t\Delta}\nabla\boldsymbol{u}_0 + \int_0^t \mathrm{e}^{\eta(t-\sigma)\Delta}\mathfrak{P}\Delta(\boldsymbol{\tau}-\boldsymbol{u}\otimes\boldsymbol{u})(\sigma,\boldsymbol{x})\,\mathrm{d}\sigma. \tag{5.4}$$

We use the fact that the linear operator $\mathfrak{T}:\boldsymbol{f}\longmapsto\int_0^t \mathrm{e}^{\eta(t-\sigma)\Delta}\Delta\boldsymbol{f}(\sigma)\,\mathrm{d}\sigma$ is bounded in $L^r(0,T;L^q(\mathbb{T}^2))$ for $1<q,r<+\infty$; see [21, p. 64]. The previous Lemmas 5.1 and 5.2 give estimates for $\boldsymbol{f}=\mathfrak{P}(\boldsymbol{\tau}-\boldsymbol{u}\otimes\boldsymbol{u})$ in $L^r(0,T;L^q(\mathbb{T}^2))$ for any $1<q,r<+\infty$, which completes the proof of Lemma 5.3. ☐

PROPOSITION 5.4. *For $\frac{1}{q}+\frac{1}{r}<\frac{1}{2}$ there exists a constant C such that for all $t\in(0,T)$*

$$\|\nabla\boldsymbol{u}(t,\cdot)\|_{L^\infty(\mathbb{T}^2)} \le C + C\ln(\mathrm{e}+\|\nabla\boldsymbol{\tau}\|_{L^r(0,t;L^q(\mathbb{T}^2))}), \tag{5.5}$$

$$\|\nabla^2\boldsymbol{u}\|_{L^r(0,t;L^q(\mathbb{T}^2))} \le C + C\|\nabla\boldsymbol{\tau}\|_{L^r(0,t;L^q(\mathbb{T}^2))}. \tag{5.6}$$

Proof. The proof is also based on the integral representation (5.4). We will use the following result about the kernel of the heat equation (see [21]): First of all, if $\boldsymbol{f}(\sigma,\cdot)\in L^\infty(\mathbb{T}^2)$ for all $\sigma\in(0,T)$, then we have, for all $\sigma\in(0,T)$,

$$\|\mathrm{e}^{\eta(t-\sigma)\Delta}\Delta\boldsymbol{f}(\sigma,\cdot)\|_{L^\infty(\mathbb{T}^2)} \le C(t-\sigma)^{-1}\|\boldsymbol{f}(\sigma,\cdot)\|_{L^\infty(\mathbb{T}^2)}. \tag{5.7}$$

Next, if $\boldsymbol{f}(\sigma,\cdot)\in L^q(\mathbb{T}^2)$ for all $\sigma\in(0,T)$ and $1<q<\infty$, then we have, for all $\sigma\in(0,T)$,

$$\|\mathrm{e}^{\eta(t-\sigma)\Delta}\Delta\boldsymbol{f}(\sigma,\cdot)\|_{L^\infty(\mathbb{T}^2)} \le C(t-\sigma)^{-\frac{q+2}{2q}}\|\nabla\boldsymbol{f}(\sigma,\cdot)\|_{L^q(\mathbb{T}^2)}. \tag{5.8}$$

Denoting $\boldsymbol{f} = \mathfrak{P}(\boldsymbol{\tau} - \boldsymbol{u} \otimes \boldsymbol{u})$, expression (5.4) reads, for any $0 < t^\star < t$,

$$\nabla \boldsymbol{u}(t, \boldsymbol{x}) = \mathrm{e}^{\eta t \Delta} \nabla \boldsymbol{u}_0 + \int_0^{t-t^\star} \mathrm{e}^{\eta(t-\sigma)\Delta} \Delta \boldsymbol{f}(\sigma, \boldsymbol{x})\, \mathrm{d}\sigma + \int_{t-t^\star}^{t} \mathrm{e}^{\eta(t-\sigma)\Delta} \Delta \boldsymbol{f}(\sigma, \boldsymbol{x})\, \mathrm{d}\sigma. \tag{5.9}$$

We take the L^∞-norm with respect to the spatial variable and use (5.7) and (5.8) to obtain

$$\begin{aligned} \|\nabla \boldsymbol{u}(t, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C + C \int_0^{t-t^\star} (t-\sigma)^{-1} \|\boldsymbol{f}(\sigma, \cdot)\|_{L^\infty(\mathbb{T}^2)}\, \mathrm{d}\sigma \\ + C \int_{t-t^\star}^{t} (t-\sigma)^{-\frac{q+2}{2q}} \|\nabla \boldsymbol{f}(\sigma, \cdot)\|_{L^q(\mathbb{T}^2)}\, \mathrm{d}\sigma. \end{aligned} \tag{5.10}$$

Using the Hölder inequality, we deduce

$$\begin{aligned} \|\nabla \boldsymbol{u}(t, \cdot)\|_{L^\infty(\mathbb{T}^2)} &\leq C + C \ln\left(\frac{t}{t^\star}\right) \|\boldsymbol{f}\|_{L^\infty((0,T)\times\mathbb{T}^2)} \\ &\quad + C\Big(\int_{t-t^\star}^{t} (t-\sigma)^{-\frac{q+2}{2q}\frac{r}{r-1}}\, \mathrm{d}\sigma \Big)^{\frac{r-1}{r}} \|\nabla \boldsymbol{f}\|_{L^r((0,t);L^q(\mathbb{T}^2))} \\ &\leq C + C \ln\left(\frac{t}{t^\star}\right) \|\boldsymbol{f}\|_{L^\infty((0,T)\times\mathbb{T}^2)} + C {t^\star}^{\alpha} \|\nabla \boldsymbol{f}\|_{L^r((0,t);L^q(\mathbb{T}^2))}, \end{aligned} \tag{5.11}$$

where $\alpha = \frac{1}{2} - \frac{1}{q} - \frac{1}{r}$ is positive due to the assumption $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$. According to Lemmas 5.1 and 5.2 we know that $\|\boldsymbol{f}\|_{L^\infty((0,T)\times\mathbb{T}^2)} \leq C$. In the same way, according to Lemmas 5.1, 5.2, and 5.3, we have $\|\nabla \boldsymbol{f}\|_{L^r(0,t;L^q(\mathbb{T}^2))} \leq C + C\|\nabla \boldsymbol{\tau}\|_{L^r(0,t;L^q(\mathbb{T}^2))}$. Inequality (5.11) reads

$$\|\nabla \boldsymbol{u}(t, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C + C \ln\left(\frac{t}{t^\star}\right) + C {t^\star}^{\alpha} \|\nabla \boldsymbol{\tau}\|_{L^r((0,t);L^q(\mathbb{T}^2))}. \tag{5.12}$$

We now choose

$$t^\star = \min\{\mathrm{e}^{-1}, \|\nabla \boldsymbol{\tau}\|_{L^r((0,t);L^q(\mathbb{T}^2))}^{-1/\alpha}\}\, t.$$

Since $\mathrm{e}^{-1} < 1$ we have $0 < t^\star < t$, and since $\alpha < \frac{1}{2}$ we have $\ln\left(\frac{t}{t^\star}\right) \leq \frac{1}{\alpha} \ln\big(\mathrm{e} + \|\nabla \boldsymbol{\tau}\|_{L^r((0,t);L^q(\mathbb{T}^2))}\big)$. Estimate (5.11) gives the first result, (5.5), of Proposition 5.4.

To prove the second inequality, (5.6), of Proposition 5.4, we take the spatial gradient of expression (5.4):

$$\nabla^2 \boldsymbol{u}(t, \boldsymbol{x}) = \mathrm{e}^{\eta t \Delta} \nabla^2 \boldsymbol{u}_0 + \int_0^{t} \mathrm{e}^{\eta(t-\sigma)\Delta} \Delta \nabla \boldsymbol{f}(\sigma, \boldsymbol{x})\, \mathrm{d}\sigma. \tag{5.13}$$

Taking the $L^r(0,t;L^q(\mathbb{T}^2))$-norm, the initial term $\mathrm{e}^{\eta t \Delta} \nabla^2 \boldsymbol{u}_0$ exactly corresponds to the norm of $\boldsymbol{u}_0$ in the space $D_q^r(\mathbb{T}^2)$. The integral term is controlled using the boundedness of the operator $\mathfrak{T}$ introduced in the proof of Lemma 5.3. We note once again the control of $\|\nabla \boldsymbol{f}\|_{L^r(0,t;L^q(\mathbb{T}^2))}$ using $\|\nabla \boldsymbol{\tau}\|_{L^r(0,t;L^q(\mathbb{T}^2))}$. $\square$

6. Control of the stress gradient.

LEMMA 6.1. *There exists a constant C such that for all $(s, t, \boldsymbol{x}) \in \mathbb{R}^+ \times (0, T) \times \mathbb{T}^2$ we have*

$$|\boldsymbol{G}(s, t, \boldsymbol{x})| \geq C > 0. \tag{6.1}$$

Proof. By assumption, for all $(s, \boldsymbol{x}) \in (0, +\infty) \times \mathbb{T}^2$ we have

$$\det(\boldsymbol{G}(s, 0, \boldsymbol{x})) \geq \mu > 0. \tag{6.2}$$

Moreover we have $\boldsymbol{G}|_{s=0} = \boldsymbol{\delta}$ so that for all $(t, \boldsymbol{x}) \in (0, T) \times \mathbb{T}^2$ we have

$$\det(\boldsymbol{G}(0, t, \boldsymbol{x})) = 1. \tag{6.3}$$

A simple calculation shows that the quantity $\det(\boldsymbol{G})$ satisfies

$$\mathcal{D} \det(\boldsymbol{G}) = \operatorname{div} \boldsymbol{u} \det(\boldsymbol{G}) = 0,$$

where $\mathcal{D}$ refers to the one order derivating operator $\mathcal{D} = \partial_s + \partial_t + \boldsymbol{u} \cdot \nabla$. The value $\det(\boldsymbol{G})$ is then constant along the characteristic lines. Since all the characteristic lines start with the lines $\{s = 0\}$ and $\{t = 0\}$, we deduce from (6.2) and (6.3) that $\det(\boldsymbol{G}) \geq \min(\mu, 1)$ on $\mathbb{R}^+ \times (0, T) \times \mathbb{T}^2$.

Due to the inequality of arithmetic and geometric means, we have

$$|\boldsymbol{G}|^2 = \operatorname{Tr}({}^T\boldsymbol{G} \cdot \boldsymbol{G}) \geq 2\sqrt{\det({}^T\boldsymbol{G} \cdot \boldsymbol{G})} = 2|\det(\boldsymbol{G})| \geq 2 \min(\mu, 1),$$

which concludes the proof of Lemma 6.1. ∎

Since $\nabla \boldsymbol{G} \in L^\infty(\mathbb{R}^+ \times (0, T); L^q(\mathbb{T}^2))$, Lemma 6.1 implies $\frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|} \in L^\infty(\mathbb{R}^+ \times (0, T); L^q(\mathbb{T}^2))$. We use this quantity to estimate the gradient of the stress.

LEMMA 6.2. *For $1 < q, r < +\infty$, and for all $t \in (0, T)$ we have*

$$\|\nabla \boldsymbol{\tau}\|^r_{L^r(0,t;L^q(\mathbb{T}^2))} \leq \mathcal{S}'_\infty \int_0^t \int_0^\infty m(s) \Big\| \frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|} \Big\|^r_{L^q(\mathbb{T}^2)} (s, t) \, \mathrm{d}s\mathrm{d}t. \tag{6.4}$$

Proof. To obtain estimate (6.4), we first derivate the stress tensor $\boldsymbol{\tau}$ with respect to the spatial coordinates:

$$\nabla \boldsymbol{\tau}(t, \boldsymbol{x}) = \int_0^\infty m(s) \, \mathcal{S}'(\boldsymbol{G}(s, t, \boldsymbol{x})) : \nabla \boldsymbol{G}(s, t, \boldsymbol{x}) \, \mathrm{d}s. \tag{6.5}$$

Using assumption (H2), we write

$$|\nabla \boldsymbol{\tau}(t, \boldsymbol{x})| \leq \mathcal{S}'_\infty \int_0^\infty m(s) \Big| \frac{\nabla \boldsymbol{G}(s, t, \boldsymbol{x})}{|\boldsymbol{G}(s, t, \boldsymbol{x})|} \Big| \, \mathrm{d}s. \tag{6.6}$$

From the triangular inequality we deduce that

$$\begin{aligned} \|\nabla \boldsymbol{\tau}\|_{L^r(0,t;L^q(\mathbb{T}^2))} &\leq \mathcal{S}'_\infty \int_0^\infty m(s) \, \Big\| \frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|} \Big\|_{L^r(0,t;L^q(\mathbb{T}^2))} (s) \, \mathrm{d}s \\ &\leq \mathcal{S}'_\infty \int_0^\infty m(s) \Big(\int_0^t \Big\| \frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|} \Big\|^r_{L^q(\mathbb{T}^2)} (s, \sigma) \, \mathrm{d}\sigma \Big)^{1/r} \mathrm{d}s. \end{aligned} \tag{6.7}$$

Writing $m(s) = m(s)^{1-\frac{1}{r}} \times m(s)^{\frac{1}{r}}$ we apply Hölder's inequality to deduce estimate (6.4) and conclude the proof of Lemma 6.2. ∎

It is then natural to define, for every time $t \in (0, T)$, the value

$$y(t) = \int_0^t \int_0^\infty m(s) \Big\| \frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|} \Big\|^r_{L^q(\mathbb{T}^2)} (s, \sigma) \, \mathrm{d}s\mathrm{d}\sigma. \tag{6.8}$$

The following lemma gives a differential inequation about this quantity.

PROPOSITION 6.3. *For all integers q, r such that $\frac{1}{q}+\frac{1}{r}<\frac{1}{2}$ and for all $t\in[0,T]$ the quantity $y(t)$ introduced by* (6.8) *satisfies*

$$y'(t)\leq C+y(t)+Cy(t)\|\nabla \boldsymbol{u}\|_{L^\infty((0,T)\times\mathbb{T}^2)}+C\|\nabla^2\boldsymbol{u}\|^r_{L^r(0,t;L^q(\mathbb{T}^2))}. \tag{6.9}$$

Proof. The equation satisfied by $\boldsymbol{G}$ reads

$$\mathcal{D}\boldsymbol{G}=\boldsymbol{G}\cdot\nabla\boldsymbol{u}\qquad\text{on }(0,+\infty)\times(0,T)\times\mathbb{T}^2, \tag{6.10}$$

where we recall that $\mathcal{D}$ corresponds to the operator $\mathcal{D}=\partial_s+\partial_t+\boldsymbol{u}\cdot\nabla$. We take the scalar product of (6.10) by $-q|\nabla\boldsymbol{G}|^q|\boldsymbol{G}|^{-q-2}\boldsymbol{G}$:

$$|\nabla\boldsymbol{G}|^q\mathcal{D}|\boldsymbol{G}|^{-q}=-q|\nabla\boldsymbol{G}|^q|\boldsymbol{G}|^{-q-2}(\boldsymbol{G}\cdot\nabla\boldsymbol{u}):\boldsymbol{G}. \tag{6.11}$$

Using the generalized Cauchy–Schwarz inequality (3.1), we deduce

$$|\nabla\boldsymbol{G}|^q\mathcal{D}|\boldsymbol{G}|^{-q}\leq q|\nabla\boldsymbol{G}|^q|\boldsymbol{G}|^{-q}|\nabla\boldsymbol{u}|. \tag{6.12}$$

Next we take the spatial derivative of (6.10). We obtain the following 3-tensor equation:

$$\mathcal{D}\nabla\boldsymbol{G}=\nabla\boldsymbol{G}\cdot\nabla\boldsymbol{u}+(\boldsymbol{G}\cdot\nabla^2\boldsymbol{u})^\dagger-\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{G}. \tag{6.13}$$

More precisely, the component (i,j,k) of this equation reads

$$\mathcal{D}\partial_i\boldsymbol{G}_{jk}=\partial_i\boldsymbol{G}_{j\ell}\partial_\ell\boldsymbol{u}_k+\boldsymbol{G}_{j\ell}\partial_\ell\partial_i\boldsymbol{u}_k-\partial_i\boldsymbol{u}_\ell\partial_\ell\boldsymbol{G}_{jk}. \tag{6.14}$$

Taking the scalar product of this equation by $q|\boldsymbol{G}|^{-q}|\nabla\boldsymbol{G}|^{q-2}\nabla\boldsymbol{G}$ and using the Cauchy–Schwarz inequality, we deduce

$$|\boldsymbol{G}|^{-q}\mathcal{D}|\nabla\boldsymbol{G}|^q\leq 2q|\nabla\boldsymbol{G}|^q|\boldsymbol{G}|^{-q}|\nabla\boldsymbol{u}|+q|\nabla\boldsymbol{G}|^{q-1}|\boldsymbol{G}|^{-(q-1)}|\nabla^2\boldsymbol{u}|. \tag{6.15}$$

Adding inequality (6.15) to inequality (6.12), we deduce

$$\mathcal{D}\big(|\nabla\boldsymbol{G}|^q|\boldsymbol{G}|^{-q}\big)\leq 3q|\nabla\boldsymbol{G}|^q|\boldsymbol{G}|^{-q}|\nabla\boldsymbol{u}|+q|\nabla\boldsymbol{G}|^{q-1}|\boldsymbol{G}|^{-(q-1)}|\nabla^2\boldsymbol{u}|.$$

Integrating with respect to the spatial variable, we obtain

$$\partial_s\Big\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^q_{L^q(\mathbb{T}^2)}+\partial_t\Big\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^q_{L^q(\mathbb{T}^2)}\leq 3q\int_{\mathbb{T}^2}\Big|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big|^q|\nabla\boldsymbol{u}|+q\int_{\mathbb{T}^2}\Big|\frac{|\nabla\boldsymbol{G}|}{|\boldsymbol{G}|}\Big|^{q-1}|\nabla^2\boldsymbol{u}|.$$

We now use the Hölder inequality to write

$$\begin{aligned}\partial_s\Big\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^q_{L^q(\mathbb{T}^2)}+\partial_t\Big\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^q_{L^q(\mathbb{T}^2)}\leq{}&3q\Big\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^q_{L^q(\mathbb{T}^2)}\|\nabla\boldsymbol{u}\|_{L^\infty(\mathbb{T}^2)}\\&+q\Big\|\frac{|\nabla\boldsymbol{G}|}{|\boldsymbol{G}|}\Big\|^{q-1}_{L^q(\mathbb{T}^2)}\|\nabla^2\boldsymbol{u}\|_{L^q(\mathbb{T}^2)}.\end{aligned} \tag{6.16}$$

We multiply (6.16) by $\frac{r}{q}\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\|^{r-q}_{L^q(\mathbb{T}^2)}$ to get

$$\begin{aligned}\partial_s\Big\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^r_{L^q(\mathbb{T}^2)}+\partial_t\Big\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^r_{L^q(\mathbb{T}^2)}\leq{}&3r\Big\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^r_{L^q(\mathbb{T}^2)}\|\nabla\boldsymbol{u}\|_{L^\infty(\mathbb{T}^2)}\\&+r\Big\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^{r-1}_{L^q(\mathbb{T}^2)}\|\nabla^2\boldsymbol{u}\|_{L^q(\mathbb{T}^2)}.\end{aligned} \tag{6.17}$$

Using the Young inequality, we obtain

$$\begin{aligned}\partial_s\Big\|\frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^r_{L^q(\mathbb{T}^2)}+\partial_t\Big\|\frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^r_{L^q(\mathbb{T}^2)} \le 3r\Big\|\frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^r_{L^q(\mathbb{T}^2)}\|\nabla \boldsymbol{u}\|_{L^\infty(\mathbb{T}^2)} \\ +\Big\|\frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|}\Big\|^r_{L^q(\mathbb{T}^2)}+(r-1)^{r-1}\|\nabla^2\boldsymbol{u}\|^r_{L^q(\mathbb{T}^2)}.\end{aligned} \tag{6.18}$$

We multiply by $m(s)$ and integrate for $s\in(0,+\infty)$. Assuming (H1) we deduce that the first term is nonnegative (we also recall that $\boldsymbol{G}\big|_{s=0}=\boldsymbol{\delta}$), and we obtain

$$y''\le 3r\,y'\|\nabla\boldsymbol{u}\|_{L^\infty(\mathbb{T}^2)}+y'+(r-1)^{r-1}\|\nabla^2\boldsymbol{u}\|^r_{L^q(\mathbb{T}^2)}. \tag{6.19}$$

Integrating now with respect to time in $(0,t)$, with $0\le t\le T$ we deduce

$$y'(t)\le 3r\,y(t)\|\nabla\boldsymbol{u}\|_{L^\infty((0,T)\times\mathbb{T}^2)}+y'(0)+y(t)+(r-1)^{r-1}\|\nabla^2\boldsymbol{u}\|^r_{L^r(0,t;L^q(\mathbb{T}^2))}. \tag{6.20}$$

The value of $y'(0)$ is given with respect to the initial condition $\boldsymbol{G}_{\rm old}$:

$$y'(0)=\int_0^\infty m(s)\Big\|\frac{\nabla\boldsymbol{G}_0}{|\boldsymbol{G}_0|}\Big\|^r_{L^q(\mathbb{T}^2)}(s)\,\mathrm{d}s.$$

We will note that $y'(0)$ is bounded since $\boldsymbol{G}_0\in L^\infty(\mathbb{R}^+;W^{1,q}(\mathbb{T}^2))$ and $|\boldsymbol{G}_0|\ge\sqrt{2\min(\mu,1)}$ on $\mathbb{T}^2$:

$$y'(0)\le\frac{1}{(2\min(\mu,1))^{r/2}}\|\boldsymbol{G}_0\|_{L^\infty(\mathbb{R}^+;W^{1,q}(\mathbb{T}^2))}.$$

Estimate (6.20) takes the form required in Proposition 6.3. ∎

7. Conclusion: Proof of Theorem 4.1.

Proposition 7.1. *The function y introduced by* (6.8) *satisfies the following inequality on* $(0,T)$*:*

$$y'\le C(\mathrm{e}+y)\ln(\mathrm{e}+y). \tag{7.1}$$

This implies $y\le C$ on $(0,T)$.

Proof. In terms of function y, Lemma 6.2 reads as follows: for all $t\in(0,T)$ we have

$$\|\nabla\boldsymbol{\tau}\|^r_{L^r(0,t;L^q(\mathbb{T}^2))}\le Cy(t). \tag{7.2}$$

Consequently, estimates (5.5) and (5.6) of Proposition 5.4 can be written as

$$\|\nabla\boldsymbol{u}(t,\cdot)\|_{L^\infty(\mathbb{T}^2)}\le C+C\ln(\mathrm{e}+y(t)), \tag{7.3}$$

$$\|\nabla^2\boldsymbol{u}\|^r_{L^r(0,t;L^q(\mathbb{T}^2))}\le C+Cy(t). \tag{7.4}$$

Using Proposition 6.3 we deduce that the function y satisfies the following inequality on $(0,T)$:

$$y'\le C+Cy+Cy\ln(\mathrm{e}+y),$$

which we can rewrite, up to a change of constants C, as (7.1).

Since all the solutions of (7.1) are bounded for finite time,

$$y(t)\le \mathrm{e}^{\mathrm{e}^{Ct}}\qquad\text{for all } t\in(0,T),$$

the proof is complete. ∎

REFERENCES

[1] A. C. T. Aarts and A. A. F. van de Ven, *Transient behaviour and stability points of the Poiseuille flow of a KBKZ-fluid*, J. Engrg. Math., 29 (1995), pp. 371–392.

[2] B. Bernstein, E. A. Kearsley, and L. J. Zapas, *A study of stress relaxation with finite strain*, Trans. Soc. Rheol., 12 (1959), pp. 623–727.

[3] B. Bernstein, E. A. Kearsley, and L. J. Zapas, *A study of stress relaxation with finite strain*, Trans. Soc. Rheol., 17 (1964), pp. 35–92.

[4] D. Brandon and W. J. Hrusa, *Global existence of smooth shearing motions of a nonlinear viscoelastic fluid*, J. Integral Equations Appl., 2 (1990), pp. 333–351.

[5] J.-Y. Chemin and N. Masmoudi, *About lifespan of regular solutions of equations related to viscoelastic fluids*, SIAM J. Math. Anal., 33 (2001), pp. 84–112.

[6] L. Chupin, *Existence results for the flow of viscoelastic fluids with an integral constitutive law*, J. Math. Fluid Mech., 15 (2013), pp. 783–806.

[7] B. D. Coleman and V. J. Mizel, Arch. Ration. Mech. Anal., 29 (1968), pp. 18–31.

[8] P. Constantin and G. Seregin, *Hölder continuity of solutions of 2D Navier-Stokes equations with singular forcing*, in Nonlinear Partial Differential Equations and Related Topics, Amer. Math. Soc. Transl. Ser. 2 229, AMS, Providence, RI, 2010, pp. 87–95.

[9] R. Danchin, *Density-dependent incompressible fluids in bounded domains*, J. Math. Fluid Mech., 8 (2006), pp. 333–381.

[10] M. Doi and S. F. Edwards, *The Theory of Polymer Dynamics*, Oxford University Press, 1988.

[11] E. Fernández-Cara, F. Guillén, and R. R. Ortega, *Existence et unicité de solution forte locale en temps pour des fluides non newtoniens de type Oldroyd (version L^s–L^r)*, C. R. Acad. Sci. Paris Sér. I Math., 319 (1994), pp. 411–416.

[12] E. Fernández-Cara, F. Guillén, and R. R. Ortega, *Some theoretical results concerning non-Newtonian fluids of the Oldroyd kind*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 26 (1998), pp. 1–29.

[13] C. Guillopé and J.-C. Saut, *Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type*, RAIRO Modél. Math. Anal. Numér., 24 (1990), pp. 369–401.

[14] C. Guillopé and J. C. Saut, *Résultats d'existence pour des fluides viscoélastiques à loi de comportement de type différentiel*, C. R. Acad. Sci. Paris Sér. I Math., 305 (1987), pp. 489–492.

[15] C. Guillopé and J. C. Saut, *Existence results for the flow of viscoelastic fluids with a differential constitutive law*, Nonlinear Anal., 15 (1990), pp. 849–869.

[16] C. Guillopé and J. C. Saut, *Mathematical problems arising in differential models for viscoelastic fluids*, in Mathematical Topics in Fluid Mechanics (Lisbon, 1991), Pitman Res. Notes Math. Ser. 274, Longman Sci. Tech., Harlow, UK, 1992, pp. 64–92.

[17] W. J. Hrusa and M. Renardy, *A model equation for viscoelasticity with a strongly singular kernel*, SIAM J. Math. Anal., 19 (1988), pp. 257–269.

[18] B. Jourdain, T. Lelièvre, and C. Le Bris, *Existence of solution for a micro-macro model of polymeric fluid: The FENE model*, J. Funct. Anal., 209 (2004), pp. 162–193.

[19] A. Kaye, *Non-Newtonian Flow in Incompressible Fluids*, Technical Report 134, Cranfield College of Aeronautics, 1962.

[20] J. U. Kim, *Global smooth solutions of the equations of motion of a nonlinear fluid with fading memory*, Arch. Rational Mech. Anal., 79 (1982), pp. 97–130.

[21] P. G. Lemarié-Rieusset, *Recent Developments in the Navier-Stokes Problem*, Chapman Hall/CRC Res. Notes Math. 431, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[22] P. L. Lions and N. Masmoudi, *Global solutions for some Oldroyd models of non-Newtonian flows*, Chinese Ann. Math. Ser. B, 21 (2000), pp. 131–146.

[23] N. Masmoudi, *Well-posedness for the FENE dumbbell model of polymeric flows*, Comm. Pure Appl. Math., 61 (2008), pp. 1685–1714.

[24] N. Masmoudi, *Global existence of weak solutions to the FENE dumbbell model of polymeric flows*, Invent. Math., 191 (2013), pp. 427–500.

[25] E. Mitsoulis, *Computational polymer processing*, in Modeling and Simulation in Polymers, P. D. Gujrati and A. I. Leonov, eds., Wiley-VCH, Weinheim, Germany, 2010.

[26] E. Mitsoulis, 50 *years of the K-BKZ constitutive relation for polymers*, ISRN Polymer Sci., 2013 (2013), 952379.

[27] M. Renardy, *Existence of slow steady flows of viscoelastic fluids of integral type*, Z. Angew. Math. Mech., 68 (1988), pp. T40–T44.

[28] M. Renardy, *An existence theorem for model equations resulting from kinetic theories of polymer solutions*, SIAM J. Math. Anal., 22 (1991), pp. 313–327.

[29] M. Renardy, *Global existence of solutions for shear flow of certain viscoelastic fluids*, J. Math. Fluid Mech., 11 (2009), pp. 91–99.

[30] M. Renardy, W. J. Hrusa, and J. A. Nohel, *Mathematical Problems in Viscoelasticity*, Pitman Monogr. Surv. Pure Appl. Math. 35, Longman Scientific & Technical, Harlow, UK, 1987.

[31] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.

[32] H. Zhang and P. Zhang, *Local existence for the FENE-dumbbell model of polymeric fluids*, Arch. Ration. Mech. Anal., 181 (2006), pp. 373–400.