

# Roughness effect on the Neumann boundary condition

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## Abstract

We study the effect of a periodic roughness on a Neumann boundary condition. We show that, as in the case of a Dirichlet boundary condition, it is possible to approach this condition by a more complex law on a domain without rugosity, called wall law. This approach is however different from that usually used in Dirichlet case. In particular, we show that this wall law can be explicitly written using an energy developed in the roughness boundary layer. The first part deals with the case of a Laplace operator in a simple domain but many more general results are next given: when the domain or the operator are more complex, or with Robin-Fourier boundary conditions. Some numerical illustrations are used to obtain magnitudes for the coefficients appearing in the new wall laws. Finally, these wall laws can be interpreted using a fictive boundary without rugosity. That allows to give an application to the water waves equation.

**Key words:** Neumann boundary condition, Asymptotic development, Rough boundaries, Hight order approximation, Wall laws, Laplace equation, Water wave, Dirichlet-Neumann operator.

## 1 Introduction

The understanding of roughness induced effects has been the topic of many recent papers like [1, 2, 4, 6, 7, 13, 14, 15, 18]. In all these papers, the main goal is to understand how to approach a Dirichlet boundary condition on a rough surface using a new boundary condition on a smooth equivalent surface. From a physical point of view, these studies allow to justify the Navier law for fluid flow modeled by the Navier-Stokes equations.

Concerning the Neumann boundary conditions there exist far fewer results. Some results, however, treat such boundary conditions in slightly different contexts. For instance in [5], the authors are interested in the presence of small inclusions which modifies the solution of the Laplace equation, the boundary condition on the inclusion being of Neumann type. In [12], the Neumann problem in a two-dimensional domain with a narrow slit is studied. In these two articles [5] and [12] the geometry of the perturbation is really different from the problem of the present paper. In the framework of fluid mechanic and the Navier-Stokes equations, the natural alternative to Dirichlet (no-slip) condition is the Navier (slip) condition. Starting from the Navier condition on a rough surface, Anne-Laure Dalibard and David Gérard-Varet [11] recently give error estimates for the homogenized no-slip condition and provide an accurate effective boundary condition of Navier type again. The problem studied in the present paper is different because, for example, the solution to a Stokes system

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on a half-space with a Navier boundary condition does not behave like a solution to the Laplace system on a half-space with a Neumann boundary condition (for instance, the solution is defined up to an additive constant in the latter case).

In [21], the authors show that it is possible to replace Dirichlet or Neumann conditions on a embossed surface by approximate effective conditions on a smooth surface. The authors give explicit expressions for the effective boundary condition in the “dilute limit”, that is when the area fraction covered by the bosses are small. For the Neumann problem, they find that the normal derivative on a smooth surface equals a suitable combination of first- and second-order derivatives (the exact formulae is given by equation (128), p. 446, or by equation (74), p. 436, for “simple” bosses). Their method consists in represent the solution to the Laplace equation in the embossed surface as a sum of solution to the Laplace equation in the smooth domain and in each bosses. Finally, in the last part of their article, p. 447, the authors note that it would be appropriate to consider the finite-concentration case (and note only the dilute limit). In some sense, the present paper answers this question.

More precisely, this paper is organized as follows.

- In the section 2, we study the roughness induced effects on the solution of the Laplace equation in a flat domain with roughness, taking into account the homogeneous Neumann boundary condition on the rough part of the boundary. The beginning of this section 2 provides an opportunity to present the notations used, to recall the results concerning the case of Dirichlet boundary conditions and to present the main results of this section (subsection 2.1). We give in the subsection 2.2 the main proofs of this section. The method follows the main ideas developed in [9] for thin film approximation with roughness-induced correctors. The subsection 2.3 shows that the study can be generalized to more complex boundary conditions such as the Robin-Fourier type boundary conditions. Finally numerical results (subsection 2.4) give some quantitative informations on the new wall law and validate this effective boundary condition.

- The section 3 is devoted to obtain the same kind of result for more complex domains (that is for a non-flat domain with roughness). The principle of this part is to make a change of variables to reduce to the case of a flat domain. Of course the Laplace equation in this new domain becomes an equation whose coefficients reflect the initial geometry. The purpose of the subsection 3.3 is to show that the study of the section 2 remains valid in such cases. In the last section 3.4, we show how our results can be used in the context of water waves. In particular we show the effect of roughness on an operator of Dirichlet-Neumann type. This allows to give some answers to the following question: what is the impact of boulders at the bottom of the ocean on the form of surface waves ?

## 2 Laplace equation in a flat domain with roughness

### 2.1 Notations and main results

#### 2.1.1 Notations

In this section, we are interested in the behavior of the solution to the Laplace equation on a rough simple domain  $\Omega_\varepsilon$ . More precisely, the domain  $\Omega_\varepsilon$  is defined by (see Figure 1)

$$\Omega_\varepsilon = \left\{ (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} ; -\varepsilon h\left(\frac{x}{\varepsilon}\right) < y < 1 \right\},$$

and its boundaries are noted  $\Gamma^+$  and  $\Gamma_\varepsilon^-$  :

$$\begin{aligned} \Gamma^+ &= \left\{ (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} ; y = 1 \right\}, \\ \Gamma_\varepsilon^- &= \left\{ (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} ; y = -\varepsilon h\left(\frac{x}{\varepsilon}\right) \right\}. \end{aligned}$$

Throughout this paper, the function  $h$  defining the roughness at the bottom of the domain is assumed to be periodic (with period equals to 1). Moreover, all constants not depending on the parameter  $\varepsilon$  will be considered harmless, and we shall denote by  $A \lesssim B$  any inequality  $A \leq CB$  where  $C$  is such a constant. In the sequel we assume for sake of simplicity that  $d = 2$  but all the results are valid for any dimension  $d \geq 2$ . In the same way the results presented here assume that  $x \in \mathbb{R}^{d-1}$  but same results are valid for  $x \in \mathbb{T}^{d-1}$ , that is for periodic conditions with respect to the horizontal variable.

The case of a Laplace equation with Dirichlet type boundary conditions has been extensively studied (we also recall the main results in the next section). The purpose of this paper is to propose a similar approach in the case of Neumann boundary conditions, which is, as we shall see, really different from what has been done before. More exactly, we are interested in the following problem

$$\begin{cases} \Delta\varphi = 0 & \text{on } \Omega_\varepsilon, \\ \varphi = f & \text{on } \Gamma^+, \\ \partial_n\varphi = 0 & \text{on } \Gamma_\varepsilon^-, \end{cases} \quad (1)$$

where the function  $f$  is a data and where the notation  $\partial_n$  corresponds to the exterior normal derivative. It is more usual to study this Laplace problem with a source term in the domain  $\Omega_\varepsilon$  rather than a non-homogeneous boundary condition. We chose to use this problem since the original goal was to understand the effect of roughness on the Dirichlet-Neumann operator (see the result of Part 3.4). Nevertheless, the study made here can be easily adapted to the case of the presence of a source term, see for instance the subsection 2.3.

#### 2.1.2 About the Dirichlet boundary conditions

The same kind of problem with Dirichlet boundary condition has been much studied recently, see for instance [1, 2, 6, 7, 15, 18]. It corresponds, for the more

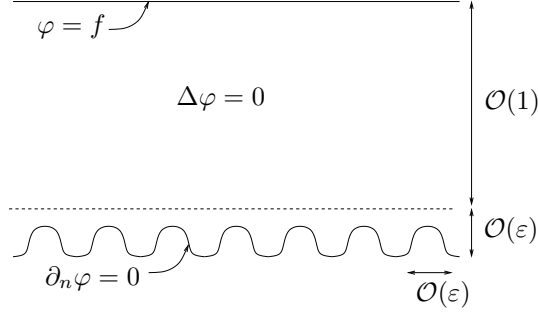


Figure 1: Domain  $\Omega_\varepsilon$  and the boundary conditions on  $\varphi$ .

simple cases, to the following equations

$$\begin{cases} \Delta\varphi = 1 & \text{on } \Omega_\varepsilon, \\ \varphi = 0 & \text{on } \Gamma^+, \\ \varphi = 0 & \text{on } \Gamma_\varepsilon^-. \end{cases}$$

The behavior of the solution for small values of the parameter  $\varepsilon$  is described by the following development

$$\varphi(x, y) = \varphi_0(x, y) + \varepsilon \tilde{\varphi}_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) + \varepsilon \alpha \varphi_1(x, y) + \mathcal{O}(\varepsilon^2).$$

The first term  $\varphi_0$  corresponds to the case without roughness. It is first defined in  $\Omega_0$  and next extended for  $y < 0$ . This extension results in an error of order  $\varepsilon$  in the boundary layer, which is corrected by the term  $\varepsilon \tilde{\varphi}_1$ . This contribution  $\tilde{\varphi}_1$  is defined in an infinite cell and we can prove that it does not vanish at infinity, but tends to a constant  $\alpha$ . This constant generates a new error on the top of the domain which is corrected by the term  $\varepsilon \alpha \varphi_1(x, y)$ . We can rigorously justify this development, and derive a wall law of order 1:

$$\varphi = \varepsilon \alpha \partial_y \varphi + \mathcal{O}(\varepsilon^2) \quad \text{on } \Gamma_0^-.$$

### 2.1.3 Results for the Neumann boundary conditions

In the case of Neumann type condition on the roughness, that is to say considering the system (1), we will justify the same kind of development. More precisely, we will get a development in the following form:

$$\varphi(x, y) = \varphi_0(x, y) + \varepsilon \partial_x \varphi_0(x, 0) \tilde{\varphi}_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) + \varepsilon \beta \varphi_1(x, y) + \mathcal{O}(\varepsilon^2). \quad (2)$$

As in the Dirichlet case, the main term  $\varphi_0$  of this development corresponds to the case without roughness. It satisfies

$$\begin{cases} \Delta\varphi_0 = 0 & \text{on } \Omega_0, \\ \varphi_0 = f & \text{on } \Gamma^+, \\ \partial_y \varphi_0 = 0 & \text{on } \Gamma_0^-. \end{cases} \quad (3)$$

Besides the fact that the corrector depends on the slow variable  $x$  (since the source term  $f$  depends on  $x$ , otherwise the solution  $\varphi$  is trivial), the result

seems similar:  $\varphi_0$  correspond to the solution without roughness, the contribution  $\varepsilon \tilde{\varphi}_1$  corrects the boundary layer error. But unlike the Dirichlet case, we can construct  $\tilde{\varphi}_1$  so it is negligible outside the boundary layer. Indeed the additional term  $\varepsilon \beta \varphi_1$  compensates for the next terms of the development. The role of  $\beta$  seems quite different from that of  $\alpha$  in the Dirichlet case. Yet it may be interpreted as an excess energy created in the boundary layer:

$$\beta = \int_{\mathbb{T}} h(X) dX - \int_{\omega} |\nabla \tilde{\varphi}_1|^2.$$

The constant  $\beta$  which appear in the development only depend on the geometry of the roughness. Some example of numerical values for this coefficient are given in the subsection 2.4 devoted to numerical simulations.

From the development (2) we prove the following results:

**Theorem 2.1** *Let  $\varphi$  be the solution of the Laplace equation (1) on the rough domain  $\Omega_\varepsilon$ , and  $\varphi_0$  be the solution on the Laplace equation (3) on the smooth domain  $\Omega_0$ . We have*

$$\|\varphi - \varphi_0\|_{L^2(\Omega_0)} = \mathcal{O}(\varepsilon).$$

If we want to control more regular norm like Sobolev norms, we use the exponential decreasing of  $\tilde{\varphi}_1(X, Y)$  when  $Y$  goes to  $+\infty$ . We obtain

**Theorem 2.2** *Let  $\varphi$  be the solution of the Laplace equation (1) on the rough domain  $\Omega_\varepsilon$ , and  $\varphi_0$  be the solution on the Laplace equation (3) on the smooth domain  $\Omega_0$ . For all  $s > 0$  and for any  $\delta > 0$  we have*

$$\|\varphi - \varphi_0\|_{H^s(\Omega_0^\delta)} = \mathcal{O}(\varepsilon),$$

where  $\Omega_0^\delta$  corresponds to the part of the domain  $\Omega_\varepsilon$  far from the roughness:

$$\Omega_0^\delta = \left\{ (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} ; \delta < y < 1 \right\}. \quad (4)$$

Moreover, we can deduce using the boundary conditions satisfied by all the contributions that

$$\partial_y \varphi = \varepsilon \beta \partial_y^2 \varphi + \mathcal{O}(\varepsilon^2) \quad \text{on } \Gamma_0^-. \quad (5)$$

Consequently, to obtain an approximation of the solution to the Laplace system (1) at order  $\varepsilon^2$ , we can study the same problem on a not rough domain  $\Omega_0$  provided replace the classical Neumann boundary condition by the wall law  $\partial_y \varphi = \varepsilon \beta \partial_y^2 \varphi$ . More precisely if we consider the system

$$\begin{cases} \Delta \varphi_{\text{app}} = 0 & \text{on } \Omega_0, \\ \varphi_{\text{app}} = f & \text{on } \Gamma^+, \\ \partial_y \varphi_{\text{app}} = \varepsilon \beta \partial_y^2 \varphi_{\text{app}} & \text{on } \Gamma_0^-, \end{cases} \quad (6)$$

then we prove

**Theorem 2.3** *Let  $\varphi$  be the solution on the Laplace equation (1) on the rough domain  $\Omega_\varepsilon$ . There exists a constant  $\beta$  only depending on the roughness such that the solution  $\varphi_{\text{app}}$  of the Laplace equation (6) on the smooth domain  $\Omega_0$  satisfies, for all  $s \geq 0$  and for any  $\delta > 0$ , the relations*

$$\|\varphi - \varphi_{\text{app}}\|_{H^s(\Omega_0^\delta)} = \mathcal{O}(\varepsilon^2),$$

where  $\Omega_0^\delta$  corresponds to the part of the domain  $\Omega_\varepsilon$  far from the roughness (see its definition with equation (4)).

Using the Taylor formula, it is possible to interpret the wall law as follows. To obtain an approximation of the solution to the Laplace system (1) at order  $\varepsilon^2$ , we can study the same problem with Neumann boundary condition on a flat domain slightly different from  $\Omega_0$ :

$$\bar{\Omega} = \left\{ (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} ; -\varepsilon\beta < y < 1 \right\}.$$

In other words, if we consider the system

$$\begin{cases} \Delta \bar{\varphi}_{\text{app}} = 0 & \text{on } \bar{\Omega}, \\ \bar{\varphi}_{\text{app}} = f & \text{on } \Gamma^+, \\ \partial_y \bar{\varphi}_{\text{app}} = 0 & \text{on } \bar{\Gamma}^-, \end{cases} \quad (7)$$

the boundary  $\bar{\Gamma}^-$  being the bottom boundary of the domain  $\bar{\Omega}$  then we prove

**Theorem 2.4** *Let  $\varphi$  be the solution on the Laplace equation (1) on the rough domain  $\Omega_\varepsilon$ . There exists a constant  $\beta$  only depending on the roughness such that the solution  $\bar{\varphi}_{\text{app}}$  of the Laplace equation (7) on the smooth domain  $\bar{\Omega}$  satisfies, for all  $s \geq 0$  and for any  $\delta > 0$ , the relations*

$$\|\varphi - \bar{\varphi}_{\text{app}}\|_{H^s(\Omega_0^\delta)} = \mathcal{O}(\varepsilon^2),$$

where  $\Omega_0^\delta$  corresponds to the part of the domain  $\Omega_\varepsilon$  far from the roughness (see its definition with equation (4)).

In the subsection 2.4, numerical simulations confirm the fact the flat domain  $\bar{\Omega}$  give better approximations than the flat domain  $\Omega_0$ .

## 2.2 Asymptotic development and proofs

### 2.2.1 Proposal for an ansatz

In this subsection we will determine more accurately how to correct errors made by each new term in the development of the solution  $\varphi$  of the system (1) with respect to  $\varepsilon$ . As we noted in the previous part, the corrector of order 1 can not be justified using terms order 1 only. Moreover, using the method of this paper, we need correctors of order 3 to justify terms of order 2. For these reasons, we propose an ansatz to all orders. To be educational, we first present how we obtain the ansatz. This ansatz is fully justified in the subsection 2.2.3.

**Step 1: main order term** At first we see the roughness as a perturbation of the flat case. We then approach the solution  $\varphi$  by the function  $\varphi_0$  which satisfied the equation (3) in a domain without roughness. Indeed, the zeroth order of the development can be easily understood using a weak formulation of the Laplace equation (1): for all  $\psi \in H^1(\Omega_\varepsilon)$  such that  $\psi = 0$  on  $\Gamma^+$  we have

$$\int_{\Omega_\varepsilon} \nabla \varphi \cdot \nabla \psi = - \int_{\Omega_\varepsilon} F \psi, \quad (8)$$

the function  $F$  being a lift of the non homogeneous Dirichlet boundary condition on  $\Gamma^+$ . With this formulation, passing to the limit  $\varepsilon \rightarrow 0$ , we deduce that the first term of the development is the solution  $\varphi_0$  of the system (3). We note that the existence and uniqueness in  $H^1(\Omega_0)$  for such a problem is well known (using for instance the Lax-Milgram theorem on a weak formulation). Moreover, we have the following regularity result.

**Lemma 2.1** *If  $f \in L^2(\mathbb{R})$  then the solution  $\varphi$  of the system (3) is regular in  $\Omega_0$  and its trace at the bottom satisfies  $\varphi(\cdot, 0) \in C^\infty(\mathbb{R})$ .*

**Proof of lemma 2.1.** Using the Fourier transform of the system (3), we have

$$\begin{cases} -|\xi|^2 \widehat{\varphi}(\xi, y) + \partial_y^2 \widehat{\varphi}(\xi, y) = 0, \\ \widehat{\varphi}(\xi, 1) = \widehat{f}(\xi), \\ \partial_y \widehat{\varphi}(\xi, 0) = 0. \end{cases}$$

We can easily solve this ordinary differential equation. We obtain

$$\widehat{\varphi}(\xi, y) = \frac{\widehat{f}(\xi)}{\text{ch}(|\xi|)} \text{ch}(|\xi|y).$$

In particular, using the fact that  $f \in L^2(\mathbb{R})$ , that is  $\widehat{f} \in L^2(\mathbb{R})$ , the function  $\widehat{\varphi}(\xi, 0)$  exponentially decreases to 0 when  $|\xi|$  tends to  $+\infty$ , that imply the regularity of the function  $x \in \mathbb{R} \mapsto \varphi(x, 0)$ .

In the case where the problem is defined in a bounded domain with periodical condition, we have  $f \in L^2(\mathbb{T})$  and the lemma 2.1 is proved using the Fourier series instead of the Fourier transform.  $\square$

This solution  $\varphi_0$  can be defined on  $\Omega_\varepsilon$ , that is for  $y < 0$ , using for instance the following Taylor formulae: for all  $y$  such that  $|y| < 1$  we have

$$\varphi_0(x, y) = \sum_{j=0}^{+\infty} \frac{y^j}{j!} \partial_y^j \varphi_0(x, 0).$$

The function  $\varphi_0$  is clearly not a solution of the initial problem (1) since it does not satisfy the bottom boundary condition  $\partial_n \varphi_0 = 0$  on  $\Gamma_\varepsilon^-$ . More precisely, we have for all  $x \in \mathbb{R}$

$$\begin{aligned} \partial_n \varphi_0(x, -\varepsilon h(x/\varepsilon)) &= n \cdot \nabla \varphi_0(x, -\varepsilon h(x/\varepsilon)) \\ &= \frac{1}{\sqrt{1+h'(x/\varepsilon)^2}} \begin{pmatrix} -h'(x/\varepsilon) \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \partial_x \varphi_0(x, 0) + \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon) \end{pmatrix} \\ &= \frac{-h'(x/\varepsilon)}{\sqrt{1+h'(x/\varepsilon)^2}} \partial_x \varphi_0(x, 0) + \mathcal{O}(\varepsilon). \end{aligned}$$

**Step 2: boundary layer corrector** This error caused by  $\varphi_0$  and the condition at the bottom can be compensated by a term acting primarily in the boundary layer. To zoom in on this boundary layer, we introduce the following rescaled variables

$$X = \frac{x}{\varepsilon} \quad \text{and} \quad Y = \frac{y}{\varepsilon}.$$

The domain and its bound which are considered for these variables are defined by (see Figure 2)

$$\begin{aligned}\omega &= \{(X, Y) \in \mathbb{R}^{d-1} \times \mathbb{R} ; -h(X) < Y\}, \\ \gamma &= \{(X, Y) \in \mathbb{R}^{d-1} \times \mathbb{R} ; Y = -h(X)\}.\end{aligned}$$

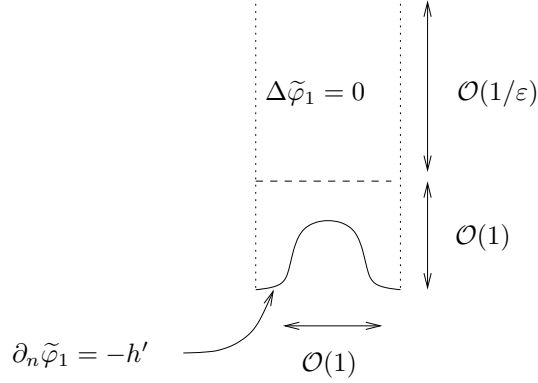


Figure 2: The domain  $\omega$  and the boundary condition for the first corrector  $\tilde{\varphi}_1$ .

Since the main order of  $\partial_n \varphi_0(x, -\varepsilon h(x/\varepsilon))$  is proportional to  $\partial_x \varphi_0(x, 0)$ , the first correction is proportional to  $\partial_x \varphi_0(x, 0)$ . It is written  $\varepsilon \partial_x \varphi_0(x, 0) \tilde{\varphi}_1(X, Y)$  where the function  $\tilde{\varphi}_1$ , which only depends on the variables  $X$  and  $Y$ , obeys<sup>2</sup>

$$\begin{cases} \Delta \tilde{\varphi}_1 = 0 & \text{on } \omega, \\ h' \partial_X \tilde{\varphi}_1 + \partial_Y \tilde{\varphi}_1 = -h' & \text{on } \gamma. \end{cases} \quad (9)$$

Notice that the boundary condition is a Neumann condition since it reads  $\partial_n \tilde{\varphi}_1 = \frac{h'}{\sqrt{1+h'^2}}$ ,  $n$  being the exterior normal derivative to the domain  $\omega$ . The function  $\tilde{\varphi}_1$  is then defined up to an additive constant. In fact, we can choose this constant so that the solution  $\tilde{\varphi}_1$  decays exponentially fast to 0 as  $Y$  tends to  $+\infty$ . More generally we have the following result about an equation on the form

$$\begin{cases} \Delta \psi = f & \text{on } \omega \\ h' \partial_X \psi + \partial_Y \psi = g & \text{on } \gamma \\ \psi \text{ is periodic w.r.t. } X, \end{cases} \quad (10)$$

where the source terms  $f$  and  $g$  satisfying

$$\int_{\mathbb{T}} g(X) dX + \int_{\{Y < 0\}} f(X, Y) dX dY = 0. \quad (11)$$

In this relation, the set  $\{Y < 0\}$  is a subset of  $\omega$  defined by  $\{(X, Y) \in \omega ; Y < 0\}$ . We denote by  $L^2_{\#}(\omega)$  the set of functions  $L^2(\omega)$  and periodic with respect to the variable  $X$ , and by  $L^2_{\#}(\mathbb{T})$  the  $L^2$  periodic functions on the torus  $\mathbb{T}$ .

<sup>2</sup>Since there will be no ambiguity in this part, the differential operators  $\Delta$ ,  $\text{div}$  and  $\nabla$  will be act on the variables  $(X, Y)$  for the functions like  $\tilde{\varphi}_1$  and sometimes on variables  $(x, y)$  for the functions like  $\varphi_0$ .



**Lemma 2.2** *Let  $f \in L^2_{\sharp}(\omega)$ . We denote by  $f_k \in L^2(0, +\infty)$  its Fourier coefficients. Let  $g \in L^2_{\sharp}(\mathbb{T})$  such that the relation (11) holds. If  $f_0 = 0$  and  $f_k(Y) = P_k(Y) e^{-|k|Y}$  where  $P_k$  is a polynomial then there exists a unique solution  $\psi \in L^2_{\sharp}(\omega)$  to the system (10). Moreover, the Fourier coefficients  $\psi_k$  of the function  $\psi$  satisfy  $\psi_0 = 0$  and  $\psi_k(Y) = Q_k(Y) e^{-|k|Y}$  where  $Q_k$  is a polynomial.*

**Proof of lemma 2.2.**

- The existence and uniqueness result in the space  $L^2_{\sharp}(\omega)$  is classical (using for instance the weak formulation and the Lax-Milgram theorem).
- Using the Stokes formula we obtain

$$\begin{aligned} \int_{\{Y < 0\}} f(X, Y) dX dY &= \int_{\{Y < 0\}} \Delta \psi(X, Y) dX dY \\ &= - \int_{\mathbb{T}} g(X) dX + \int_{\mathbb{T}} \partial_Y \psi(X, 0) dX. \end{aligned}$$

Since the assumption (11) holds, this relation reads in term of Fourier coefficients:  $\psi'_0(0) = 0$ .

- Taking the Fourier transform of the Laplace equation (10), we have for all  $Y \geq 0$  and for all  $k \in \mathbb{Z}$  :

$$-k^2 \psi_k(Y) + \psi_k''(Y) = f_k(Y).$$

For  $k = 0$ , we use the assumption  $f_0 = 0$ , the relation  $\psi'_0(0) = 0$  and the fact that  $\psi_0 \in L^2(0, +\infty)$  to deduce  $\psi_0 = 0$ . For  $k \neq 0$ , we note that the solutions of the ordinary differential equation

$$-k^2 \psi_k(Y) + \psi_k''(Y) = P_k(Y) e^{-|k|Y}$$

are on the form  $\psi_k(Y) = Q_k(Y) e^{-|k|Y} + \tilde{Q}_k(Y) e^{|k|Y}$  where  $Q_k$  and  $\tilde{Q}_k$  are two polynomials. Since  $\psi_k \in L^2(0, +\infty)$  we necessarily have  $\tilde{Q}_k = 0$  and we obtain the result of the lemma.  $\square$

Consequently, the solution  $\psi$  of the system (10) has the following Fourier decomposition, for all  $(X, Y) \in \mathbb{T} \times [0, +\infty[$ :

$$\psi(X, Y) = \sum_{k \neq 0} Q_k(Y) e^{-ikX - |k|Y}.$$

We deduce the following result:

**Corollary 2.1** *Under the assumption of the lemma 2.2, the solution  $\psi \in L^2_{\sharp}(\omega)$  the system (10) satisfy: for any  $\delta < 1$ , for all  $(a, b) \in \mathbb{N}^2$ , there exists  $Y_0 > 0$  such that for all  $Y > Y_0$  and for all  $X \in \mathbb{T}$  we have*

$$|\partial_X^a \partial_Y^b \psi(X, Y)| \leq e^{-\delta Y}.$$

We apply this corollary for the system (9), that is with  $f = 0$  and  $g = h'$  (the assumption (11) is satisfied). As announced, we deduce that the solution  $\tilde{\varphi}_1$  decays exponentially fast to 0 as  $Y$  tends to  $+\infty$ . We note that this term does not really affect the condition on the top boundary  $\Gamma^+$ , contrary to what happens in the Dirichlet case.

**Step 3: Next boundary layer corrector** If you put the proposed development  $\varphi_0(x, y) + \varepsilon \partial_x \varphi_0(x, 0) \tilde{\varphi}_1(X, Y)$  in the original Laplace problem (1) then you realize that there are terms you do not know control. For instance, evaluating the laplacian  $\partial_x \varphi_0(x, 0) \tilde{\varphi}_1(X, Y)$  we make appear the term  $2\partial_x^2 \varphi_0(x, 0) \partial_X \tilde{\varphi}_1(X, Y)$ . It is then necessary to involve another correction, proportional to  $\partial_x^2 \varphi_0(x, 0)$ , namely  $\varepsilon^2 \partial_x^2 \varphi_0(x, 0) \tilde{\varphi}_2(X, Y)$  where the function  $\tilde{\varphi}_2$ , which only depends on the variables  $X$  and  $Y$ , obeys

$$\begin{cases} \Delta \tilde{\varphi}_2 = -2\partial_X \tilde{\varphi}_1 & \text{on } \omega, \\ h' \partial_X \tilde{\varphi}_2 + \partial_Y \tilde{\varphi}_2 = -h - h' \tilde{\varphi}_1 + \beta_2 & \text{on } \gamma. \end{cases}$$

Contrary to the case of  $\tilde{\varphi}_1$ , the solution of this system (which is defined up to an additive constant again) can not vanish when  $Y$  to  $+\infty$  except for a good choice of an additive constant in the boundary value, denoted  $\beta_2$ . Indeed, using the lemma 2.2 we prove that the solution  $\tilde{\varphi}_2$  exponentially decays to 0 if we choose

$$\beta_2 = \int_{\mathbb{T}} h(X) dX + \int_{\mathbb{T}} h'(X) \tilde{\varphi}_1(X, -h(X)) dX + 2 \int_{\{Y < 0\}} \partial_X \tilde{\varphi}_1(X, Y) dX dY.$$

We can simplify this expression noting that, first we have by  $X$ -periodicity:

$$\begin{aligned} \int_{\{Y < 0\}} \partial_X \tilde{\varphi}_1(X, Y) dX dY &= \int_{\mathbb{T}} \left( \int_{-h(X)}^0 \partial_X \tilde{\varphi}_1(X, Y) dY \right) dX \\ &= - \int_{\mathbb{T}} h'(X) \tilde{\varphi}_1(X, -h(X)) dX, \end{aligned}$$

and next, by the Stokes formula:

$$\begin{aligned} 0 &= - \int_{\omega} \Delta \tilde{\varphi}_1 \tilde{\varphi}_1 = \int_{\omega} |\nabla \tilde{\varphi}_1|^2 - \int_{\gamma} \partial_n \tilde{\varphi}_1 \tilde{\varphi}_1 \\ &= \int_{\omega} |\nabla \tilde{\varphi}_1|^2 - \int_{\mathbb{T}} h'(X) \tilde{\varphi}_1(X, -h(X)) dX. \end{aligned}$$

So, the constant  $\beta_2$  can be written making appear the average of the height  $m(h)$  and the energy  $E(h)$  which are two quantities only depending on the height  $h$ :

$$m(h) := \int_{\mathbb{T}} h(X) dX \quad \text{and} \quad E(h) := \int_{\omega} |\nabla \tilde{\varphi}_1|^2. \quad (12)$$

Finally, the solution  $\tilde{\varphi}_2$  decays exponentially fast to 0 as  $Y$  tends to  $+\infty$  if (and only if) we choose

$$\beta_2 = m(h) - E(h). \quad (13)$$

**Step 4: First corrector in the main domain** The introduction of this constant  $\beta_2$  in the boundary condition must be compensate using a new corrector  $\varepsilon \varphi_1(x, y)$  in the main domain  $\Omega_0$  satisfying

$$\begin{cases} \Delta \varphi_1 = 0 & \text{on } \Omega^+, \\ \varphi_1 = 0 & \text{on } \Gamma^+, \\ \partial_y \varphi_1 = -\partial_x^2 \varphi_0(x, 0) \beta_2 & \text{on } \Gamma^0. \end{cases} \quad (14)$$

**Remark 2.1** *The solution  $\varphi_1$  of the system (14) can be obtained using a formulation with the same type of boundary conditions than for the main Laplace system (1). Indeed, we let  $\varphi_1 = \psi_1 + \beta_2 \partial_y \varphi_0$  and  $\psi_1$  satisfies (provided  $f$  is regular enough)*

$$\begin{cases} \Delta \psi_1 = 0 & \text{on } \Omega^+, \\ \psi_1 = -\beta_2 \partial_y \varphi_0 & \text{on } \Gamma^+, \\ \partial_y \psi_1 = 0 & \text{on } \Gamma^0. \end{cases}$$

**Step 5: Final proposition for the ansatz** As for the main term  $\varphi_0$ , the corrector term  $\varphi_1$  should be extended for  $y < 0$ , thus generating a new error... We will see that we need a lot of terms to justify the development (with the method used in this paper). We then directly proposed a development up to the order  $N \in \mathbb{N}$ :

$$\varphi(x, y) = \sum_{k=0}^{N-1} \varepsilon^k \varphi_k(x, y) + \sum_{k=1}^N \varepsilon^k \left( \sum_{j=1}^k \partial_x^j \varphi_{k-j}(x, 0) \tilde{\varphi}_j(X, Y) \right) + R(x, y). \quad (15)$$

The system verified by each term of this development will be specified in the next section.

### 2.2.2 System for each profile

In this section, we put the ansatz (15) into the equation (1). By identifying the different powers of  $\varepsilon$ , we deduce that the system must meet all profiles  $\varphi_k$ ,  $\tilde{\varphi}_k$  as well as the residue  $R$ .

#### Step 1: Laplace equation

We compute the laplacian of  $\varphi$  given by (15) as follows

$$\begin{aligned} \Delta \varphi(x, y) &= \sum_{k=0}^{N-1} \varepsilon^k \Delta \varphi_k(x, y) + \sum_{k=1}^N \varepsilon^k \left( \sum_{j=1}^k \partial_x^{j+2} \varphi_{k-j}(x, 0) \tilde{\varphi}_j(X, Y) \right) \\ &\quad + 2 \sum_{k=1}^N \varepsilon^{k-1} \left( \sum_{j=1}^k \partial_x^{j+1} \varphi_{k-j}(x, 0) \partial_X \tilde{\varphi}_j(X, Y) \right) \\ &\quad + \sum_{k=1}^N \varepsilon^{k-2} \left( \sum_{j=1}^k \partial_x^j \varphi_{k-j}(x, 0) \Delta \tilde{\varphi}_j(X, Y) \right) + \Delta R(x, y). \end{aligned}$$

Ordering the powers of  $\varepsilon$  we write

$$\begin{aligned}
\Delta\varphi(x, y) &= \sum_{k=0}^{N-1} \varepsilon^k \Delta\varphi_k(x, y) \\
&+ \sum_{k=1}^N \varepsilon^{k-2} \left[ \sum_{j=1}^k \partial_x^j \varphi_{k-j}(x, 0) \left( \Delta\tilde{\varphi}_j(X, Y) + 2\partial_X \tilde{\varphi}_{j-1}(X, Y) + \tilde{\varphi}_{j-2}(X, Y) \right) \right] \\
&+ \varepsilon^{N-1} \left[ \sum_{j=1}^N \partial_x^{j+1} \varphi_{N-j}(x, 0) \left( \tilde{\varphi}_{j-1}(X, Y) + 2\partial_X \tilde{\varphi}_j(X, Y) \right) \right] \\
&+ \varepsilon^N \left[ \sum_{j=1}^N \partial_x^{j+2} \varphi_{N-j}(x, 0) \tilde{\varphi}_j(X, Y) \right] + \Delta R(x, y),
\end{aligned} \tag{16}$$

where we use the convention  $\tilde{\varphi}_0 = \tilde{\varphi}_{-1} = 0$  and  $\varphi_{-1} = 0$ . We then choose to impose the following conditions on the profiles

$$\begin{cases} \Delta\varphi_k = 0 & \text{in } \Omega_0 \text{ for all } k \in \{0, \dots, N-1\} \\ \Delta\tilde{\varphi}_k = -2\partial_X \tilde{\varphi}_{k-1} - \tilde{\varphi}_{k-2} & \text{in } \omega \text{ for all } k \in \{1, \dots, N\}. \end{cases} \tag{17}$$

The residue  $R$  is chosen such that the two last lines of the equality (16) vanish. It satisfies  $\Delta R = r_0$  where  $r_0$  is on the following form  $r_0(x, y) = \varepsilon^{N-1} \mathcal{F}(x, x/\varepsilon, y/\varepsilon) + \varepsilon^N \mathcal{G}(x, x/\varepsilon, y/\varepsilon)$ , the functions  $\mathcal{F}$  and  $\mathcal{G}$  doing not depend on  $\varepsilon$ .

### Step 2: Bottom boundary condition

The Neumann homogeneous boundary condition on  $\Gamma_\varepsilon^-$  for the function  $\varphi$  is equivalent to the following relation, for all  $x \in \mathbb{R}$ ,

$$h'(X) \partial_x \varphi(x, -\varepsilon h(X)) + \partial_y \varphi(x, -\varepsilon h(X)) = 0.$$

The functions  $\varphi_k$  defined on  $\Omega_0$  are extended on  $\Omega_\varepsilon$  (that is for small negative values of the coordinate  $y$ ) using the formula of Taylor-Young. We deduce that for all integers  $a$  and  $b$ , and for  $\varepsilon$  small enough, we have

$$\partial_x^a \partial_y^b \varphi_i(x, -\varepsilon h(X)) = \sum_{\ell=0}^{+\infty} \frac{(-\varepsilon h(X))^\ell}{\ell!} \partial_x^a \partial_y^{\ell+b} \varphi_i(x, 0).$$

Using the ansatz (15), the first derivatives of the function  $\varphi$  can be written as

$$\begin{aligned}
\partial_x \varphi(x, -\varepsilon h(X)) &= \sum_{p=0}^{N-1} \varepsilon^p \left[ \sum_{j=0}^p \frac{(-h(X))^j}{j!} \partial_x \partial_y^j \varphi_{p-j}(x, 0) \right. \\
&+ \sum_{j=0}^p \partial_X \tilde{\varphi}_{j+1}(X, -h(X)) \partial_x^{j+1} \varphi_{p-j}(x, 0) \\
&\left. + \sum_{k=0}^p \tilde{\varphi}_k(X, -h(X)) \partial_x^{k+1} \varphi_{p-k}(x, 0) - \partial_x \varphi_p(x, 0) \right] + \mathcal{O}(\varepsilon^N),
\end{aligned}$$

$$\begin{aligned} \partial_y \varphi(x, -\varepsilon h(X)) &= \sum_{p=0}^{N-1} \varepsilon^p \left[ \sum_{j=0}^p \frac{(-h(X))^j}{j!} \partial_y^{j+1} \varphi_{p-j}(x, 0) \right. \\ &\quad \left. + \sum_{j=0}^p \partial_Y \tilde{\varphi}_{j+1}(X, -h(X)) \partial_x^{j+1} \varphi_{p-j}(x, 0) \right] + \mathcal{O}(\varepsilon^N). \end{aligned}$$

Recall that in the subsection 2.2 we will see that the boundary condition on  $\varphi_0$  at the bottom is written  $\partial_y \varphi_0(x, 0) = 0$  and that we need a constant  $\beta_2$  for the boundary condition on  $\varphi_1$ , that is  $\partial_y \varphi_1(x, 0) = -\beta_2 \partial_x^2 \varphi_0(x, 0)$ . More generally, we propose the following boundary condition of Neumann type for  $\varphi_k$ ,  $k \in \{1, \dots, N-1\}$ :

$$\left| \partial_y \varphi_k(x, 0) = - \sum_{j=0}^{k-1} \beta_{k+1-j} \partial_x^{k+1-j} \varphi_j(x, 0). \right. \quad (18)$$

The constants  $\beta_2, \beta_3, \dots, \beta_N$  will be precise later (these constants will be used to assure that the boundary correctors  $\tilde{\varphi}_k$  have no influence in the main domain  $\Omega_0$ ).

Moreover, we recall that from the equation (17) we have the relations  $\Delta \varphi_k = 0$  for any  $k \in \{0, \dots, N-1\}$ . Consequently it is possible to express all the derivatives  $\partial_x^a \partial_y^b \varphi_k(x, 0)$  using the  $x$ -derivatives only. For instance we obtain

$$\partial_x \partial_y^j \varphi_{p-j}(x, 0) = \begin{cases} (-1)^{\frac{j}{2}} \partial_x^{j+1} \varphi_{p-j}(x, 0) & \text{if } j \text{ is even,} \\ (-1)^{\frac{j+1}{2}} \sum_{k=0}^{p-j} \beta_{p-j+1-k} \partial_x^{p-k+1} \varphi_k(x, 0) & \text{if } j \text{ is odd,} \end{cases}$$

and

$$\partial_y^{j+1} \varphi_{p-j}(x, 0) = \begin{cases} (-1)^{\frac{j+1}{2}} \partial_x^{j+1} \varphi_{p-j}(x, 0) & \text{if } j \text{ is odd,} \\ (-1)^{\frac{j+2}{2}} \sum_{k=0}^{p-j} \beta_{p-j+1-k} \partial_x^{p-k+1} \varphi_k(x, 0) & \text{if } j \text{ is even.} \end{cases}$$

We then made a few manipulations on the index to rewrite the term  $\Theta = \sum_{j=0}^p \frac{(-h(X))^j}{j!} \partial_x \partial_y^j \varphi_{p-j}(x, 0)$  which appear in the derivative  $\partial_x \varphi(x, -\varepsilon h(x/\varepsilon))$ :

$$\begin{aligned} \Theta &= \sum_{\substack{j=0 \\ j \text{ even}}}^p \frac{(-h(X))^j}{j!} (-1)^{\frac{j}{2}} \partial_x^{j+1} \varphi_{p-j}(x, 0) \\ &\quad + \underbrace{\sum_{\substack{j=0 \\ j \text{ odd}}}^p \frac{(-h(X))^j}{j!} (-1)^{\frac{j+1}{2}} \sum_{k=0}^{p-j} \beta_{p-j+1-k} \partial_x^{p-k+1} \varphi_k(x, 0)}_{= \Theta_1} \end{aligned}$$

where the contribution  $\Theta_1$  reads

$$\begin{aligned}\Theta_1 &= \sum_{k=0}^p \left( \sum_{\substack{j=0 \\ j \text{ odd}}}^{p-k} \frac{(-h(X))^j}{j!} (-1)^{\frac{j+1}{2}} \beta_{p-j+1-k} \right) \partial_x^{p-k+1} \varphi_k(x, 0) \\ &= \sum_{i=0}^p \left( \sum_{\substack{j=0 \\ j \text{ odd}}}^i \frac{(-h(X))^j}{j!} (-1)^{\frac{j+1}{2}} \beta_{i-j+1} \right) \partial_x^{i+1} \varphi_{p-i}(x, 0)\end{aligned}$$

In the same spirit, we write the term  $\Psi = \sum_{j=0}^p \frac{(-h(X))^j}{j!} \partial_y^{j+1} \varphi_{p-j}(x, 0)$  which appear in the derivative  $\partial_y \varphi(x, -\varepsilon h(x/\varepsilon))$  as follows

$$\begin{aligned}\Psi &= \sum_{\substack{j=0 \\ j \text{ odd}}}^p \frac{(-h(X))^j}{j!} (-1)^{\frac{j+1}{2}} \partial_x^{j+1} \varphi_{p-j}(x, 0) \\ &\quad + \sum_{i=0}^p \left( \sum_{\substack{j=0 \\ j \text{ even}}}^i \frac{(-h(X))^j}{j!} (-1)^{\frac{j+2}{2}} \beta_{i-j+1} \right) \partial_x^{i+1} \varphi_{p-i}(x, 0).\end{aligned}$$

We can therefore write the derivatives of  $\varphi$  using only derivatives with respect to the variable  $x$ :

$$\begin{aligned}\partial_x \varphi(x, -\varepsilon h(x/\varepsilon)) &= \sum_{p=0}^{N-1} \varepsilon^p \sum_{j=0}^p \partial_x^{j+1} \varphi_{p-j}(x, 0) U_j + \mathcal{O}(\varepsilon^N), \\ \partial_y \varphi(x, -\varepsilon h(x/\varepsilon)) &= \sum_{p=0}^{N-1} \varepsilon^p \sum_{j=0}^p \partial_x^{j+1} \varphi_{p-j}(x, 0) V_j + \mathcal{O}(\varepsilon^N),\end{aligned}$$

where we have defined

$$\begin{cases} U_{2j} = \partial_X \tilde{\varphi}_{2j+1} + \tilde{\varphi}_{2j} + \frac{(-1)^j}{(2j)!} h^{2j} + \sum_{s=0}^{j-1} (-1)^s \frac{h^{2s+1}}{(2s+1)!} \beta_{2j-2s} & \text{for all } j \in \mathbb{N}, \\ U_{2j-1} = \partial_X \tilde{\varphi}_{2j} + \tilde{\varphi}_{2j-1} + \sum_{s=0}^{j-1} (-1)^{s+1} \frac{h^{2s+1}}{(2s+1)!} \beta_{2j-2s-1} & \text{for all } j \in \mathbb{N}^*, \end{cases}$$

and

$$\begin{cases} V_{2j} = \partial_Y \tilde{\varphi}_{2j+1} + \sum_{s=0}^{j-1} (-1)^{s+1} \frac{h^{2s}}{(2s)!} \beta_{2j-2s+1} & \text{for all } j \in \mathbb{N}, \\ V_{2j-1} = \partial_Y \tilde{\varphi}_{2j} - \frac{(-1)^j}{(2j-1)!} h^{2j-1} + \sum_{s=0}^{j-1} (-1)^{s+1} \frac{h^{2s}}{(2s)!} \beta_{2j-2s} & \text{for all } j \in \mathbb{N}^*. \end{cases}$$

Cancel the normal derivative at the bottom can be done by imposing  $h'U_j + V_j = 0$  for all  $j \in \mathbb{N}$ , so by imposing the following condition on the boundary  $\gamma$ : for

all  $j \in \mathbb{N}^*$

$$\left| \begin{aligned} h' \partial_X \tilde{\varphi}_{2j} + \partial_Y \tilde{\varphi}_{2j} &= -h' \tilde{\varphi}_{2j-1} + \frac{(-1)^j}{(2j-1)!} h^{2j-1} \\ &+ \sum_{s=0}^{j-1} (-1)^s \frac{h^{2s}}{(2s)!} \left( \frac{h'h}{2s+1} \beta_{2j-2s-1} + \beta_{2j-2s} \right), \end{aligned} \right. \quad (19)$$

and for all  $j \in \mathbb{N}$

$$\left| \begin{aligned} h' \partial_X \tilde{\varphi}_{2j+1} + \partial_Y \tilde{\varphi}_{2j+1} &= -h' \tilde{\varphi}_{2j} - \frac{(-1)^j}{(2j)!} h' h^{2j} \\ &+ \sum_{s=0}^{j-1} (-1)^{s+1} \frac{h^{2s}}{(2s)!} \left( \frac{h'h}{2s+1} \beta_{2j-2s} - \beta_{2j-2s+1} \right), \end{aligned} \right. \quad (20)$$

where we recall that by convention  $\tilde{\varphi}_0 = 0$ . Note that the residue  $R$  obeys  $\partial_n R|_{\Gamma_\varepsilon^-} = r_-$  where  $r_- = \mathcal{O}(\varepsilon^N)$ .

### Step 3: Top boundary condition

Using the ansatz (15) with  $y = 1$  we obtain

$$\varphi(x, 1) = \sum_{k=0}^{N-1} \varepsilon^k \varphi_k(x, 1) + \sum_{k=1}^N \varepsilon^k \left( \sum_{j=1}^k \partial_x^j \varphi_{k-j}(x, 0) \tilde{\varphi}_j(X, 1/\varepsilon) \right) + R(x, 1).$$

We will see that for a good choice of the constants  $\beta_k$ , the functions  $\tilde{\varphi}_k(X, Y)$  exponentially decreases to 0 when  $Y$  goes to infinity. We then naturally impose

$$\left| \begin{aligned} \varphi_0(x, 1) &= f(x), \\ \varphi_k(x, 1) &= 0 \quad \text{for all } k \in \{1, \dots, N-1\}, \end{aligned} \right.$$

so that the residue  $R$  satisfies the following top boundary condition  $R|_{\Gamma^+} = r_+$  where  $r_+ = \mathcal{O}(e^{-1/\varepsilon})$ .

### 2.2.3 Study of the different profiles

In this section, we recall the systems satisfied by each term of the ansatz (15), including the residue. We first note that the system satisfied by the boundary layer terms  $\tilde{\varphi}_k$  and by the the “main domain” terms  $\varphi_k$  are only coupled *via* the constants  $\beta_k$ . In fact, this is due to the very specific form of the ansatz. We will see that this “separation of variables” does not valid in more complex cases (see for instance part 3.3.2).

**Profiles defined on the main domain** The equation satisfied by the main order term of the ansatz, that is  $\varphi_0$ , corresponds to the case  $\varepsilon = 0$ : it is given by the system (3) and the regularity of the solution is given by the Lemma 2.1. The other terms of the ansatz (15) which are defined in the main domain  $\Omega_0$  are the profiles  $\varphi_k$ ,  $k \geq 1$ . They satisfy the same type of problem:

$$\left\{ \begin{aligned} \Delta \varphi_k &= 0 \quad \text{on } \Omega_0, \\ \varphi_k &= 0 \quad \text{on } \Gamma^+, \\ \partial_y \varphi_k &= - \sum_{j=0}^{k-1} \beta_{k+1-j} \partial_x^{k+1-j} \varphi_j(x, 0) \quad \text{on } \Gamma_0^-. \end{aligned} \right.$$

Thus, knowing the solutions  $\varphi_\ell$  for all  $\ell < k$  (which are regular at the bottom, that is for  $y = 0$ ) we deduce the solution  $\varphi_k$ . A simple induction implies that all these problems are well posed. We have the following result:

**Proposition 2.1** *For any choice of the constants  $\beta_j$ ,  $j \in \{2, \dots, N\}$ , all the main terms  $\varphi_k$ ,  $k \in \{0, \dots, N-1\}$ , are well defined in  $L^2(\Omega)$ .*

**Remark 2.2** *Let  $\varphi_k = \psi_k + \sum_{j=0}^{k-1} \beta_{k+1-j} \partial_x^{k-1-j} \partial_y \varphi_j$ . We obtain the solution  $\varphi_k$  using a formulation with the same type of boundary conditions than for the main Laplace system (1) for  $\psi_k$  (provided  $f$  is regular enough):*

$$\begin{cases} \Delta \psi_k = 0 & \text{on } \Omega_0, \\ \psi_k = - \sum_{j=0}^{k-1} \beta_{k+1-j} \partial_x^{k-1-j} \partial_y \varphi_j & \text{on } \Gamma^+, \\ \partial_y \psi_k = 0 & \text{on } \Gamma_0^-. \end{cases}$$

**Profiles defined on the boundary layer** At this stage, the correctors  $\tilde{\varphi}_k$  satisfy equations of the form

$$\begin{cases} \Delta \tilde{\varphi}_k = f_k & \text{on } \omega \\ h' \partial_X \tilde{\varphi}_k + \partial_Y \tilde{\varphi}_k = g_k & \text{on } \gamma, \end{cases} \quad (21)$$

the sources terms  $f_k$  and  $g_k$  being defined by formula (17) and (19)-(20) respectively (we note that the terms  $g_k$  depend on the choice of the constants  $\beta_2, \dots, \beta_k$ ).

We prove (see lemma 2.2) that there exists a solution  $\tilde{\varphi}_k$  to system (21) which exponentially decreases to 0 as  $Y$  tends to  $+\infty$  if we have the ‘‘compatibility’’ relation:

$$\int_{\mathbb{T}} g_k(X) dX + \int_{\{Y < 0\}} f_k(X, Y) dX dY = 0.$$

We can determine the constants  $\beta_2, \dots, \beta_N$  inductively by requiring that every solution  $\tilde{\varphi}_k$  exponentially decays to 0 (the example corresponding to the case of the computation of the constant  $\beta_2$  is given in the previous part, see page 10). Using the corollary 2.1, we prove by induction on  $k$  the following result about the boundary layer problems:

**Proposition 2.2** *All the boundary layer corrector  $\tilde{\varphi}_k$  are well defined in  $L^2_{\sharp}(\omega)$ . They satisfy: for any  $\delta < 1$ , for all  $(a, b) \in \mathbb{N}^2$ , there exists  $Y_0 > 0$  such that for all  $Y > Y_0$  and for all  $X \in \mathbb{T}$  we have*

$$|\partial_X^a \partial_Y^b \tilde{\varphi}_k(X, Y)| \leq e^{-\delta Y}.$$

**Residue estimate** The equation on the rest  $R$  is written as follows:

$$\begin{cases} \Delta R = r_0 & \text{on } \Omega_\varepsilon, \\ R = r_+ & \text{on } \Gamma^+, \\ \partial_n R = r_- & \text{on } \Gamma_\varepsilon^-. \end{cases} \quad (22)$$



The existence and uniqueness of a smooth solution to this problem is classical (as soon as the source terms and the domain are regular). What is important here is to obtain estimates and to know their dependence on the parameter  $\varepsilon$ . In practice, since we propose an expansion of the solution  $\varphi$  in powers of  $\varepsilon$  for all order, it suffices to show that there exists estimates of the residue  $R$  on the form  $\varepsilon^{-M}$ , the power  $M \in \mathbb{N}$  being not really important.

From the expressions of the source terms  $r_0$ ,  $r_+$  and  $r_-$ , we know that they satisfy, for all  $s \geq 0$ ,

$$\|r_0\|_{H^s(\Omega_\varepsilon)} \lesssim \varepsilon^{N-1-s}, \quad \|r_+\|_{H^s(\Gamma^+)} \lesssim e^{-1/\varepsilon} \quad \text{and} \quad \|r_-\|_{H^s(\Gamma_\varepsilon^-)} \lesssim \varepsilon^N.$$

To obtain a bound on the solution  $R$  of the system (22) we define the following lift for all  $(x, y) \in \Omega_\varepsilon$ :

$$R^-(x, y) = \mathcal{F}\left(\frac{d^-}{\varepsilon\alpha}\right) r_-(p^-) d^- \quad \text{and} \quad R^+(x, y) = \mathcal{F}(1-y) r_+(x),$$

where  $\mathcal{F} \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^+)$  satisfies  $\mathcal{F}(0) = 1$ ,  $\mathcal{F}'(0) = 0$  and  $\mathcal{F}(d) = 0$  for  $d \geq 1$ , where  $\alpha$  represents the reach of the curve  $Y = -h(X)$ ,  $d^- = \text{dist}((x, y), \Gamma_\varepsilon^-)$  and  $p^- = \text{proj}((x, y), \Gamma_\varepsilon^-)$ . We note that the function  $R^-$  vanished far from the boundary  $\Gamma_\varepsilon^-$  and satisfies  $\partial_n R^- = r_-$  on  $\Gamma_\varepsilon^-$ , whereas function  $R^+$  vanished far from the boundary  $\Gamma^+$  and satisfies  $R^+ = r_+$  on  $\Gamma^+$ . Consequently, introducing the function  $\tilde{R} = R - R^- - R^+$  we have

$$\begin{cases} \Delta \tilde{R} = \tilde{r}_0 & \text{on } \Omega_\varepsilon, \\ \tilde{R} = 0 & \text{on } \Gamma^+, \\ \partial_n \tilde{R} = 0 & \text{on } \Gamma_\varepsilon^-, \end{cases}$$

where  $\tilde{r}_0 = r_0 - \Delta R^- - \Delta R^+$ . For the solution  $\tilde{R}$  we have the classical bounds  $\|\tilde{R}\|_{H^s(\Omega_\varepsilon)} \leq \|\tilde{r}_0\|_{H^{s-2}(\Omega_\varepsilon)}$  for all  $s \geq 0$ . From the bounds on  $r_0$ ,  $R^-$  and  $R^+$  (that is from the bounds on  $r_0$ ,  $r_-$ ,  $r_+$  and the bound on the height  $h$ ), we can deduce bounds for the residue  $R$  in the norm  $H^s(\Omega_\varepsilon)$ .

**Proposition 2.3** *For all  $s \geq 0$ , there exist  $C \in \mathbb{R}$  and  $M \in \mathbb{N}$  only depending on the functions  $h$ ,  $f$  and on the real  $s$  such that the solution of the system (22) satisfies*

$$\|R\|_{H^s(\Omega_\varepsilon)} \leq C \varepsilon^{-M}.$$

#### 2.2.4 Proofs of Theorems 2.1, 2.2, 2.3 and 2.4

- To prove the Theorem 2.1 it suffices to give a bound in  $L^2(\Omega_0)$  with respect to the parameter  $\varepsilon$  for the term  $A = \partial_x \varphi_0(x, 0) \tilde{\varphi}_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ . Using the change of variable  $y = \varepsilon Y$  we have:

$$\begin{aligned} \|A\|_{L^2(\Omega_0)}^2 &= \int_{\mathbb{R}} \int_0^1 \left| \partial_x \varphi_0(x, 0) \tilde{\varphi}_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \right|^2 dy dx \\ &= \varepsilon \int_{\mathbb{R}} \int_0^{1/\varepsilon} \left| \partial_x \varphi_0(x, 0) \tilde{\varphi}_1\left(\frac{x}{\varepsilon}, Y\right) \right|^2 dY dx \\ &\leq \varepsilon \int_{\mathbb{R}} \int_0^{+\infty} \left| \partial_x \varphi_0(x, 0) \tilde{\varphi}_1\left(\frac{x}{\varepsilon}, Y\right) \right|^2 dY dx. \end{aligned}$$

Using the two-scale convergence introduced by G. Nguetseng in [19] we know that

$$\begin{aligned} \|A\|_{L^2(\Omega_0)}^2 &\lesssim \varepsilon \int_{\mathbb{R}} \int_{\mathbb{T}} \int_0^{+\infty} |\partial_x \varphi_0(x, 0) \tilde{\varphi}_1(X, Y)|^2 dY dX dx \\ &\lesssim \varepsilon \left( \int_{\mathbb{R}} |\partial_x \varphi_0(x, 0)|^2 dx \right) \left( \int_{\mathbb{T}} \int_0^{+\infty} |\tilde{\varphi}_1(X, Y)|^2 dY dX \right). \end{aligned}$$

From the lemmas 2.1 and 2.2 we know that  $\partial_x \varphi_0(\cdot, 0)$  is bounded in  $L^2(\mathbb{R})$  and that  $\tilde{\varphi}_1$  is bounded in  $L^2(\omega)$ . We deduce that  $\|A\|_{L^2(\Omega_0)}^2 \lesssim \varepsilon$  that implies the result of the Theorem 2.1.  $\square$

- Let us now turn to the proof of Theorem 2.2. We must control the derivatives of  $A$ . Clearly, for each derivative we lose a power of  $\varepsilon$ . But, the corollary 2.1 implies that each derivative of  $\tilde{\varphi}_1$  is exponentially decreasing far from the bottom boundary  $Y = 0$ , so that we can retrieve this power as soon we are not on the boundary  $Y = 0$ . As example, for the  $y$ -derivatives, using the same method that to control  $\|A\|_{L^2(\Omega_0)}$  we have, for all  $k \geq 0$  and for any  $\delta > 0$  the following estimate:

$$\|\partial_y^k A\|_{L^2(\Omega_0^\delta)}^2 \lesssim \varepsilon^{1-k} \int_{\mathbb{T}} \int_{\delta/\varepsilon}^{+\infty} |\partial_Y \tilde{\varphi}_1(X, Y)|^2 dY dX$$

From the corollary 2.1 we deduce  $\|\partial_y^k A\|_{L^2(\Omega_0^\delta)}^2 \lesssim \varepsilon^{1-k} e^{-\delta/\varepsilon} \lesssim \varepsilon$ .  $\square$

- The Theorem 2.3 comes from the two following remarks: First, using the development of  $\varphi$  with respect to  $\varepsilon$ , and using the bounds on the oscillating terms  $\tilde{\varphi}_k$  (see for instance the bounds on  $A$  just before) we have

$$\|\varphi - (\varphi_0 + \varepsilon \varphi_1)\|_{H^s(\Omega_0^\delta)} = \mathcal{O}(\varepsilon^2). \quad (23)$$

Next, using the system satisfied by  $\varphi_0$  and  $\varphi_1$  we can deduce that the combination  $\varphi_0 + \varepsilon \varphi_1$  satisfies the same system than  $\varphi_{\text{app}}$ , up to a term of order  $\varepsilon^2$ . We have

$$\|(\varphi_0 + \varepsilon \varphi_1) - \varphi_{\text{app}}\|_{H^s(\Omega_0^\delta)} = \mathcal{O}(\varepsilon^2). \quad (24)$$

Thus, the Theorem 2.3 is an immediate consequence of these two estimates (23) and (24).  $\square$

- The Theorem 2.4 is proved using  $\partial_y \overline{\varphi}_{\text{app}}(x, 0) - \varepsilon \beta \partial_y^2 \overline{\varphi}_{\text{app}}(x, 0) = \partial_y \overline{\varphi}_{\text{app}}(x, -\varepsilon \beta) + \mathcal{O}(\varepsilon^2)$ . We deduce that  $\overline{\varphi}_{\text{app}}$  satisfies the same system than  $\varphi_{\text{app}}$ , up to a term of order  $\varepsilon^2$ . That implies

$$\|\varphi_{\text{app}} - \overline{\varphi}_{\text{app}}\|_{H^s(\Omega_0^\delta)} = \mathcal{O}(\varepsilon^2),$$

and that conclude the proof of the Theorem 2.4.  $\square$

### 2.3 Some other cases relative to the Laplace equation

A simple case corresponds to the case where the top boundary condition  $\varphi(x, 1) = f$  does not depend on  $x$ . Indeed, the development proposed in the ansatz (which make appear a lot of derivatives of  $\varphi_0$  with respect to the  $x$  variable, which become zero) is exact at any order since the exact solution to the Laplace problem (1) in that case is the constant solution  $\varphi(x, y) = f$ .

Moreover, the method presented in the previous part can be easily adapted to more complex cases. We give here two possible generalizations (we will see in the subsection 3.3.2 another possible extension with a more complex operator):

### 2.3.1 Additional source terms

A first interesting case corresponds to the following Laplace problem

$$\begin{cases} \Delta\varphi = F & \text{on } \Omega_\varepsilon, \\ \varphi = 0 & \text{on } \Gamma^+, \\ \partial_n\varphi = 0 & \text{on } \Gamma_\varepsilon^-, \end{cases}$$

where the function  $F$  is a data. For such a problem, the same method that those presented in this paper allows us to obtain the same results and a development of the solution (provided that the function  $F$  is regular because the proposed development uses the trace  $\varphi_0(x, 0)$  of the solution  $\varphi_0$  to the Laplace problem  $\Delta\varphi_0 = F$ ).

**Example -** We consider the previous case with  $F = 1$ . That corresponds to the case which is generally studied with the homogeneous Dirichlet boundary condition. Moreover, in some contexts it is possible to show that this case is representative of cases where  $F$  is not constant, see for instance [8]. The first term of the development is explicit and is given by

$$\varphi_0(x, y) = \frac{1}{2}(y^2 - 1).$$

This solution satisfies  $h'\partial_x\varphi_0(x, -\varepsilon h) + \partial_y\varphi_0(x, -\varepsilon h) = -\varepsilon h$ . We then introduce a corrector  $\varepsilon^2\tilde{\varphi}_2(X, Y)$ , solution of the following problem

$$\begin{cases} \Delta\tilde{\varphi}_2 = 0 & \text{on } \omega, \\ \partial_n\tilde{\varphi}_2 = h - m & \text{on } \gamma, \end{cases}$$

where the constant  $m$  is chosen such that the solution  $\tilde{\varphi}_2(X, Y)$  exponentially decreases to 0 when  $y$  goes to  $+\infty$ : that is  $m = \int_{\mathbb{T}} h(X) dX$ . This constant brings a new error at the bottom, which is corrected by an additive term  $\varepsilon\varphi_1(x, y)$ . The function  $\varphi_1$  is solution of the following Laplace problem

$$\begin{cases} \Delta\varphi_1 = 0 & \text{on } \Omega_\varepsilon, \\ \varphi_1 = 0 & \text{on } \Gamma^+, \\ \partial_y\varphi_1 = m & \text{on } \Gamma_0^-. \end{cases}$$

The solution is explicit too:  $\varphi_1(x, y) = m(y - 1)$  and we deduce the following development:

$$\varphi(x, y) = \frac{1}{2}(y^2 - 1) + \varepsilon m(y - 1) + \varepsilon^2\tilde{\varphi}_2\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) + \mathcal{O}(e^{-1/\varepsilon}).$$

**Remark 2.3** *Clearly, it is possible to consider a Laplace problem with source terms  $F$  in the main domain  $\Omega_\varepsilon$  and a source term  $f$  on the top boundary  $\Gamma^+$ .*

### 2.3.2 Robin-Fourier boundary conditions

Another possible generalization concerns the type of boundary conditions imposed at the bottom of the domain. Indeed, conditions of type Robin-Fourier

can also be considered. This case might seem a mixture of Dirichlet and Neumann cases. In reality, it is really different since the approximation order 0 is not so obvious as in the case Dirichlet or Neumann case (see for example the justification in the Neumann case, p. 6). Indeed, consider the following Laplace equation with Robin-Fourier condition with a coefficient  $\lambda \in \mathbb{R}$ :

$$\begin{cases} \Delta\varphi = F & \text{on } \Omega_\varepsilon, \\ \varphi = 0 & \text{on } \Gamma^+, \\ \partial_n\varphi = \lambda\varphi & \text{on } \Gamma_\varepsilon^-. \end{cases}$$

Its weak formulation reads: for all  $\psi \in H^1(\Omega_\varepsilon)$  such that  $\psi = 0$  on  $\Gamma^+$  we have

$$\int_{\Omega_\varepsilon} \nabla\varphi \cdot \nabla\psi - \lambda \int_{\Gamma_\varepsilon^-} \varphi\psi = - \int_{\Omega_\varepsilon} F\psi.$$

The passage to the limit  $\varepsilon \rightarrow 0$  in the boundary terms is not obvious. Because the length of  $\Gamma_\varepsilon^-$  does not generally tends to the length of  $\Gamma_0^-$ , it is clear that the limit equation is not intuitive. Such phenomena are well known and are studied in other contexts (see for instance the book [8]).

Nevertheless, using exactly the same approach that those presented in the Neumann case, it is possible to recover the result. We present here the main ideas without going into details that are similar to the Neumann case.

- First, we look for the solution  $\varphi$  as  $\varphi(x, y) = \varphi_0(x, y) + \varepsilon\tilde{\varphi}_1(x, X, Y) + \dots$
- Next, we put this development into the Laplace equation and separate the powers of  $\varepsilon$ . We obtain (in fact using the next corrector for the order 0 as in the Neumann case)  $\Delta_{X,Y}\tilde{\varphi}_1 = 0$  and  $\Delta\varphi_0 = F$ .
- We now put the development into the Robin-Fourier boundary condition (after approaching  $\varphi(x, -\varepsilon h(x/\varepsilon))$  by  $\varphi(x, 0)$ ). We obtain, for  $Y = -h(X)$

$$\begin{aligned} & h'(X)\partial_x\varphi_0(x, 0) + h'(X)\partial_X\tilde{\varphi}_1(X, Y) + \partial_y\varphi_0(x, 0) \\ & + \partial_Y\tilde{\varphi}_1(X, Y) - \lambda\sqrt{1 + h'(X)^2}\varphi_0(x, 0) = \mathcal{O}(\varepsilon). \end{aligned}$$

- From this relation, we want to deduce two boundary conditions: one for the main order term  $\varphi_0$  and another for the boundary layer corrector  $\tilde{\varphi}_1$ . Separating the variables, these two relations can be written making appear a Robin-Fourier condition for the main order term ( $\mu$  being a constant we will choose later)

$$\partial_y\varphi_0(x, 0) = \lambda\mu\varphi_0(x, 0),$$

$$h'\partial_X\tilde{\varphi}_1 + \partial_Y\tilde{\varphi}_1 = -h'\partial_x\varphi_0(x, 0) + \lambda\left(\sqrt{1 + h'(X)^2} - \mu\right)\varphi_0(x, 0).$$

- Moreover, the boundary layer corrector, written as  $h'\partial_X\tilde{\varphi}_1 + \partial_Y\tilde{\varphi}_1 = g$ , must not disturb the solution away from the boundary  $\Gamma_\varepsilon^+$ . In other words, we must necessarily have the relation  $\int_{\mathbb{T}} g dX = 0$  (which corresponds to the assumption (11) to apply the lemma 2.2). That impose the value of the constant  $\mu$ :

$$\mu = \int_{\mathbb{T}} \sqrt{1 + h'(X)^2}. \quad (25)$$

- Finally, we put the development into the top boundary condition to deduce that the main order term satisfies

$$\begin{cases} \Delta\varphi_0 = F & \text{on } \Omega_0, \\ \varphi_0(x, 1) = 0, \\ \partial_y\varphi_0(x, 0) = \lambda\mu\varphi_0(x, 0). \end{cases}$$

**Remark 2.4** Note that the method presented here can give an estimation of the error and can propose a wall law in the case of Robin-Fourier law. Moreover, for more general (for example nonlinear) law like

$$\partial_n \varphi = G(x, \varphi) \quad \text{on } \Gamma_\varepsilon^-,$$

we can easily prove that this law becomes the following wall law when  $\varepsilon$  tends to 0:

$$\partial_y \varphi_0 = \mu G(x, \varphi_0) \quad \text{on } \Gamma_0^-.$$

The coefficient  $\mu$  is given by the relation (25) and does not depend on the non-linearity  $G$ .

**Remark 2.5** In [3], J. Arrieta and S. Bruschib justify this first order law. We find in particular the law they are proposing (see the example 2.3 p. 6 of [3]). Moreover, using our formulation, we can give a geometrical formulation for the constant  $\mu$  (see also [20] about homogenization in a climatization problem). It can be interpreted as the length of a roughness (defined as the graph of the application  $h : X \in [0, 1] \mapsto h(X)$ ).

## 2.4 Numerical validation

### 2.4.1 Some examples for the value for the coefficient $\beta_2$

We can compute the value of the coefficient  $\beta_2$  for some “classical” geometries. Recall that this coefficient is linked to the geometry  $h$  via the formulae (see page 10)

$$\beta_2 = \int_{\mathbb{T}} h(X) dX - \int_{\omega} |\nabla \tilde{\varphi}_1|^2.$$

In this equality, the function  $\tilde{\varphi}_1$  is the solution of the Laplace problem

$$\begin{cases} \Delta \tilde{\varphi}_1 = 0 & \text{on } \omega, \\ h' \partial_X \tilde{\varphi}_1 + \partial_Y \tilde{\varphi}_1 = -h' & \text{on } \gamma, \end{cases} \quad (26)$$

which tends exponentially fast to 0 as  $Y$  tends to  $+\infty$ .

To compute the coefficient  $\beta_2$  for a given geometry  $h$ , we first solve this problem (26) and then compute the associated energy. All the computations were performed with the FreeFem++ program<sup>3</sup>.

Taking for instance the sinusoidal boundary  $h(X) = \sin(2\pi X)$ . The value of the corresponding coefficient satisfies  $\beta_2 \approx 0.839$ . For other examples of patterns of roughness, the results are given with the Figure 3.

### 2.4.2 Comparison between average-bottom and better-flat approximations

From the asymptotic development of the solution  $\varphi$  of the Laplace equation (1) on the rough domain  $\Omega_\varepsilon$ , we first deduce that theoretically  $\varphi_0$  (which is the solution of the Laplace equation in the flat domain  $\Omega_0$ ) satisfies  $\|\varphi - \varphi_0\|_{L^2(\Omega_0^\delta)} =$

<sup>3</sup>This software, see <http://www.freefem.org/ff++> is based on weak formulation of the problem and finite elements method.

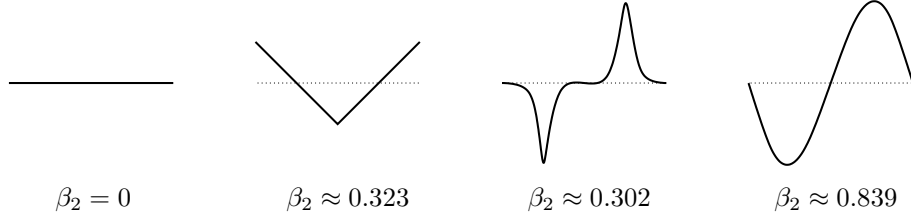


Figure 3: Different patterns of roughness and their coefficient  $\beta_2$ .

$\mathcal{O}(\varepsilon)$  (see the Theorem 2.1). Moreover, the next terms of the asymptotic development implies that if we consider the Laplace equation in the flat domain slightly different  $\bar{\Omega} = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} ; -\varepsilon\beta_2 < y < 1\}$  then the solution  $\bar{\varphi}_{\text{app}}$  satisfies  $\|\varphi - \bar{\varphi}_{\text{app}}\|_{L^2(\Omega_\varepsilon^*)} = \mathcal{O}(\varepsilon^2)$  (see the Theorem 2.4).

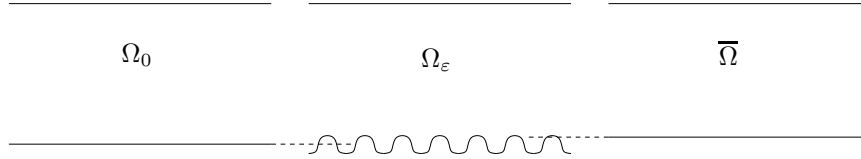


Figure 4: The three domains:  $\Omega_\varepsilon$  with roughness (middle),  $\Omega_0$  flat with average of the roughness (left),  $\bar{\Omega}$  flat taking to account the first order roughness effects.

We propose in this subsection to verify these properties numerically. We consider the 1-periodical source term  $f(x) = \cos(2\pi x)$  so that it is easy to have an explicit expression for the solutions  $\varphi_0$  and  $\bar{\varphi}$ :

$$\varphi_0(x, y) = \frac{\text{ch}(2\pi y)}{\text{ch}(2\pi)} \cos(2\pi x) \quad \text{and} \quad \bar{\varphi}(x, y) = \frac{\text{ch}(2\pi(y - \beta_2\varepsilon))}{\text{ch}(2\pi(1 - \beta_2\varepsilon))} \cos(2\pi x).$$

In this example, the roughness is describe using the form function  $h(X) = \sin(X)$  for which the “roughness coefficient” is approximated by  $\beta_2 \approx 0.839$  (see the previous numerical simulations in the subsection 2.4.1). We numerically compute some solutions  $\varphi$  for different values of the parameter  $\varepsilon$  and evaluate the errors  $\|\varphi - \varphi_0\|_{L^2}$  and  $\|\varphi - \bar{\varphi}\|_{L^2}$ . The results are given on the table-figure 5. These results confirm that use the “ideal” flat domain  $\bar{\Omega}$  give better estimates than use the “mean-oscillation” flat domain  $\Omega_0$ .

### 3 Influence of the domain

#### 3.1 Notations and main results

In this section, we are interested in the behavior of the solution to the Laplace equation on a rough domain  $\Omega_\varepsilon^{T,B}$  much more complicated than the domain  $\Omega_\varepsilon$  introduced in the section 2. More precisely, the domain  $\Omega_\varepsilon^{T,B}$  is defined by (see Figure 3.1)

$$\Omega_\varepsilon^{T,B} = \left\{ (x, z) \in \mathbb{R}^{d-1} \times \mathbb{R} ; B(x) - \varepsilon h\left(\frac{z}{\varepsilon}\right) < y < T(x) \right\},$$

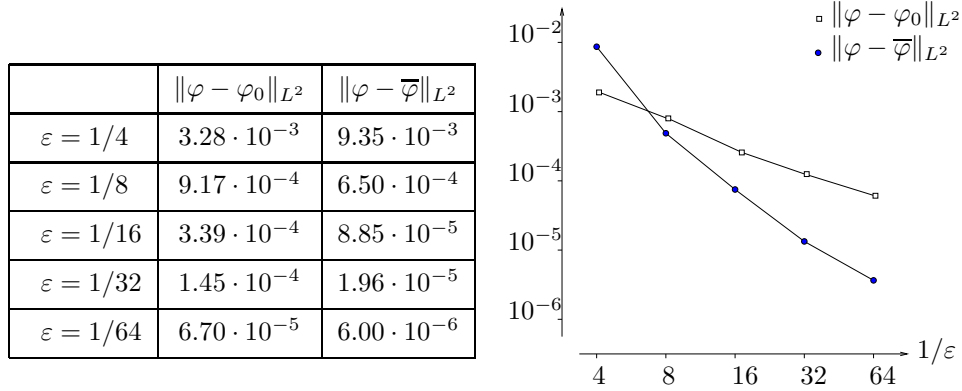


Figure 5: Error  $\|\varphi - \varphi_0\|_{L^2}$  and  $\|\varphi - \bar{\varphi}\|_{L^2}$  with respect to  $\varepsilon$ .

where the functions  $T$  and  $B$  are supposed to be regular and satisfied the relation  $T(x) - B(x) \geq H_{\min} > 0$  for all  $x \in \mathbb{R}$ . The boundaries are naturally denoted by  $\Gamma^{T,B,+}$  and  $\Gamma_\varepsilon^{T,B,-}$ . We are thus interested in the following system

$$\begin{cases} \Delta\varphi = 0 & \text{on } \Omega_\varepsilon^{T,B}, \\ \varphi = f & \text{on } \Gamma^{T,B,+}, \\ \partial_n\varphi = 0 & \text{on } \Gamma_\varepsilon^{T,B,-}, \end{cases} \quad (27)$$

where the function  $f$  is a data and where the notation  $\partial_n$  correspond to the exterior normal derivative.

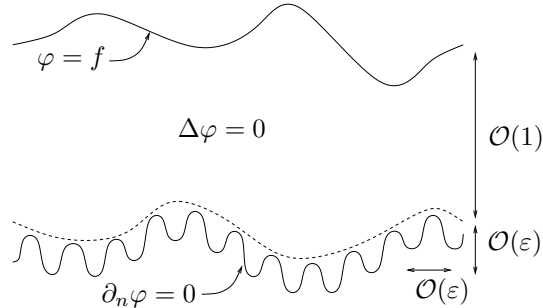


Figure 6: A more general domain  $\Omega_\varepsilon^{T,B}$ .

As in the simple case corresponding to  $T = 1$  and  $B = 0$  (see section 2) we will see that the solution  $\varphi$  to this Laplace equation (27) tends to the solution  $\varphi_0$  of the following system:

$$\begin{cases} \Delta\varphi_0 = 0 & \text{on } \Omega_0^{T,B}, \\ \varphi_0 = f & \text{on } \Gamma^{T,B,+}, \\ \partial_n\varphi_0 = 0 & \text{on } \Gamma_0^{T,B,-}, \end{cases} \quad (28)$$

More precisely we prove:

**Theorem 3.1** *Let  $\varphi$  be the solution of the Laplace equation (27) on the rough domain  $\Omega_\varepsilon^{T,B}$ , and  $\varphi_0$  be the solution on the Laplace equation (28) on the smooth domain  $\Omega_0^{T,B}$ . For all  $s > 0$  and for any  $\delta > 0$  we have*

$$\|\varphi - \varphi_0\|_{H^s(\Omega_0^{T,B+\delta})} = \mathcal{O}(\varepsilon).$$

Moreover, as in the section 2 again, we will give two possible interpretations to the next terms of the development of  $\varphi$  with respect to the powers of  $\varepsilon$ .

- The first correspond to an approximation of  $\varphi$  at order 2 in the domain without roughness, but using a accurate wall law.

**Theorem 3.2** *Let  $\varphi$  be the solution on the Laplace equation (27) on the rough domain  $\Omega_\varepsilon^{T,B}$ . There exists an operator  $\mathcal{R}$  of order 2 whose the coefficients only depend on the roughness  $h$  and on the topography  $B$  such that the solution  $\varphi_{app}$  of the Laplace equation*

$$\begin{cases} \Delta\varphi_{app} = 0 & \text{on } \Omega_0^{T,B}, \\ \varphi_{app} = f & \text{on } \Gamma^{T,B,+}, \\ \partial_n\varphi_{app} = \varepsilon\mathcal{R}\varphi_{app} & \text{on } \Gamma_0^{T,B,-}, \end{cases} \quad (29)$$

*satisfies the relation*

$$\|\varphi - \varphi_{app}\|_{H^s(\Omega_0^{T,B+\delta})} = \mathcal{O}(\varepsilon^2),$$

*for all  $s \geq 0$  and for any  $\delta > 0$ .*

Note that this operator  $\mathcal{R}$  will be explicitly defined using the classical normal and tangential derivatives  $\partial_n$  and  $\partial_\tau$  (see equation (48), page 31).

- The second possible interpretation corresponds to the existence of a not rough boundary close to the bottom boundary  $\Gamma_0^{T,B,-}$  which allow to give an approximation of  $\varphi$  at order 2 with Neumann boundary condition.

**Theorem 3.3** *Let  $\varphi$  be the solution on the Laplace equation (27) on the rough domain  $\Omega_\varepsilon^{T,B}$ . There exists a function  $C$  only depending on the roughness  $h$  and on the topography  $B$  such that the solution  $\bar{\varphi}_{app}$  of the Laplace equation*

$$\begin{cases} \Delta\bar{\varphi}_{app} = 0 & \text{on } \Omega_0^{T,B+\varepsilon C}, \\ \bar{\varphi}_{app} = f & \text{on } \Gamma^{T,B+\varepsilon C,+}, \\ \partial_n\bar{\varphi}_{app} = 0 & \text{on } \Gamma_0^{T,B+\varepsilon C,-}, \end{cases} \quad (30)$$

*satisfies the relation*

$$\|\varphi - \bar{\varphi}_{app}\|_{H^1(\Omega_0^{T,B+\delta})} = \mathcal{O}(\varepsilon^2),$$

*for all  $s \geq 0$  and for any  $\delta > 0$ .*

Note that in a next paragraph p. 32 the “optimal” boundary given by the graph of the function  $B + \varepsilon C$  will be numerically computed for an example of roughness  $h$  and for a given bottom  $B$ .



### 3.2 Influence of the top boundary

In this subsection, we show that the study of the section 2 may simply adapt to the case where the upper boundary of the domain  $\Omega_\varepsilon$  is not flat. With the notations introduced in the previous subsection, we are interested here in the Laplace equation (27) in the domain  $\Omega_\varepsilon^{T,0}$  where  $T$  is a regular function such that for all  $x \in \mathbb{R}$  we have  $T(x) \geq T_{\min} > 0$ .

We can do exactly the same ansatz as in the flat case:

$$\varphi(x, y) = \sum_{k=0}^{N-1} \varepsilon^k \varphi_k(x, y) + \sum_{k=1}^N \varepsilon^k \left( \sum_{j=1}^k \partial_x^j \varphi_{k-j}(x, 0) \tilde{\varphi}_j(X, Y) \right) + R(x, y).$$

The main profiles  $\varphi_k$  satisfy the same system as in the flat case (using  $\Omega_0^{T,0}$  instead of  $\Omega_0$ ) and the boundary layer profiles  $\tilde{\varphi}_k$  exactly satisfy the same system as in the flat case. The only difference stems from the top boundary condition satisfied by the residue  $R$  since we have, using  $h = T(x)$  in the ansatz:

$$\begin{aligned} \varphi(x, T(x)) &= \sum_{k=0}^{N-1} \varepsilon^k \varphi_k(x, T(x)) + \sum_{k=1}^N \varepsilon^k \left( \sum_{j=1}^k \partial_x^j \varphi_{k-j}(x, 0) \tilde{\varphi}_j(X, T(x)/\varepsilon) \right) \\ &\quad + R(x, T(x)). \end{aligned}$$

Using the top boundary condition for the profiles  $\varphi_k$ , that is  $\varphi_0(x, T(x)) = f(x)$  and  $\varphi_k(x, T(x)) = 0$  for all  $k \in \{1, \dots, N-1\}$ , we estimate the error of the remainder using, for all  $j \in \{1, \dots, N\}$

$$\tilde{\varphi}_j(X, T(x)/\varepsilon) \leq e^{-T(x)/\varepsilon} \leq e^{-T_{\min}/\varepsilon}.$$

In particular, we deduce from this study that the shape of the upper boundary does not affect the conditions at the lower boundary. For example, the wall law remains the same:  $\partial_y \varphi(x, 0) = -\varepsilon \beta \partial_y^2 \varphi(x, 0) + \mathcal{O}(\varepsilon^2)$ , the coefficient  $\beta$  only depending on the geometry of the roughness (see the subsection 2.4 for some examples of the values for this coefficient).

### 3.3 Influence of the bottom boundary

In the case of a more general domain  $\Omega_\varepsilon^{T,B}$  where the bottom (regardless of roughness) is not flat, the result shown above no seems so simple.

#### 3.3.1 Diffeomorphism and transformation of the domain

We use a diffeomorphism, denoted  $\Phi$ , that transforms this domain  $\Omega_\varepsilon^{T,B}$  into the domain  $\Omega_\varepsilon$  studied in the first part. Using the results of the subsection 3.2, we know that the form of the top boundary  $T$  has no influence in the description of the boundary layer effects. Consequently, we assume for sake of simplicity that  $T = B + 1$ , and we simply denoted by  $\Omega_\varepsilon^B$  the domain  $\Omega_\varepsilon^{B+1,B}$ .

The simple diffeomorphism that we consider is the following

$$\Phi : (x, z) \in \Omega_\varepsilon^B \longmapsto (x, y = z - B(x)) \in \Omega_\varepsilon.$$

Taking into account this change of variable, we can transform the Laplace system (1) for  $\varphi$  in a system for  $\psi = \varphi \circ \Phi^{-1}$ . Indeed, the laplacian with respect to

the variables  $(x, z) \in \Omega_\varepsilon^B$  becomes a linear operator with respect to the variables  $(x, y) \in \Omega_\varepsilon$ :

$$\Delta\varphi = \operatorname{div}(A \cdot \nabla\psi),$$

where the matrix  $A$  is related to the change of variable  $\Phi$ :

$$A = \begin{pmatrix} 1 & -B' \\ -B' & 1 + B'^2 \end{pmatrix}. \quad (31)$$

In the same way, the normal derivative on the bottom boundary of  $\Omega_\varepsilon^B$ , denoted  $\Gamma_\varepsilon^B$ , reads

$$\partial_n\varphi = \frac{\sqrt{1+h'^2}}{\sqrt{1+(B'-h')^2}}(A \cdot \nabla\psi) \cdot n. \quad (32)$$

The normal  $n$  appearing in the left hand side member denotes the unitary exterior normal vector to the domain  $\Omega_\varepsilon^B$  at the bottom, whereas the normal, always denoted  $n$ , appearing in the right hand side member denotes the unitary exterior normal vector to the domain  $\Omega_\varepsilon$ . Similarly, the operator acting on the function  $\varphi$  are operators with respect to the variable  $(x, z)$  whereas the operator acting on the function  $\psi$  are operators with respect to the variable  $(x, y)$ .

Now, the Laplace system (1) is written as follows:

$$\begin{cases} \operatorname{div}(A \cdot \nabla\psi) = 0 & \text{on } \Omega_\varepsilon, \\ \psi = f & \text{on } \Gamma^+, \\ (A \cdot \nabla\psi) \cdot n = 0 & \text{on } \Gamma_\varepsilon^-. \end{cases} \quad (33)$$

### 3.3.2 Elliptic operator

In this subsection, we determine the first terms in the expansion of the solution of a system of type (33) with respect to the parameter  $\varepsilon$ . The principle is similar to the case of the Laplace equation (1) studied in the first part. The main difference is that we can not write the correction terms (in the boundary layer) as products of terms oscillating terms and “slow” terms. Indeed, the development take the following form

$$\psi(x, y) = \sum_{k=0}^{N-1} \varepsilon^k \psi_k(x, y) + \sum_{k=1}^N \varepsilon^k \bar{\psi}_k(x, X, Y) + R(x, y). \quad (34)$$

We present here the equations satisfied by the first terms of development only, the other terms are treating the same way (see for instance the case studied in the first part).

#### Step 1: Elliptic equation

For a generic function  $\bar{\psi}$  depending on the slow variable  $x$  and on the variables  $(X = \frac{x}{\varepsilon}, Y = \frac{y}{\varepsilon})$ , we have

$$\begin{aligned} \operatorname{div}\left(A \cdot \nabla\left(\bar{\psi}\left(x, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right)\right) &= \frac{1}{\varepsilon^2} \widetilde{\operatorname{div}}(A \cdot \widetilde{\nabla}(\bar{\psi})) + \frac{1}{\varepsilon} \widetilde{\operatorname{div}}(A \cdot \nabla(\bar{\psi})) \\ &\quad + \frac{1}{\varepsilon} \operatorname{div}(A \cdot \widetilde{\nabla}(\bar{\psi})) + \operatorname{div}(A \cdot \nabla(\bar{\psi})), \end{aligned}$$

where the notations  $\widetilde{\text{div}}$  and  $\widetilde{\nabla}$  correspond to the operators with respect to the variables  $(X, Y)$ , whereas the classical notations  $\text{div}$  and  $\nabla$  correspond to the operators with respect to the variables  $(x, y)$ . We can now use the ansatz (34) into the first equation of the system (33). We conclude, after separating the different powers of  $\varepsilon$ , and after separating fast and slow variables:

$$\begin{cases} \text{div}(A \cdot \nabla \psi_0) = 0 & \text{on } \Omega_0, \\ \text{div}(A \cdot \nabla \psi_1) = 0 & \text{on } \Omega_0, \\ \widetilde{\text{div}}(A \cdot \widetilde{\nabla}(\overline{\psi}_1)) = 0 & \text{on } \omega, \\ \widetilde{\text{div}}(A \cdot \widetilde{\nabla}(\overline{\psi}_2)) = f & \text{on } \omega, \end{cases}$$

where the source term is given by  $f = -\widetilde{\text{div}}(A \cdot \nabla(\overline{\psi}_1)) - \text{div}(A \cdot \widetilde{\nabla}(\overline{\psi}_1))$ .

**Step 2: Bottom boundary condition**

On the bottom boundary  $\Gamma_\varepsilon^-$ , the Neumann boundary condition is written (up to a normalization)

$$-(A \cdot \nabla \psi) \cdot n = h'[A \cdot \nabla \psi]_1 + [A \cdot \nabla \psi]_2 = 0.$$

We use the Taylor developments for the functions  $A \cdot \nabla \psi_k$  to deduce  $A \cdot \nabla \psi_k|_{\Gamma_\varepsilon^-}$  as an approximation of  $A \cdot \nabla \psi_k|_{\Gamma_0^-} = (A \cdot \nabla \psi_k)^o$  where for sake of simplicity we denote by the ‘‘power’’  $^o$  a function evaluated for  $(x, y) = (x, 0)$ . We plug the ansatz into the bottom boundary condition. For the first orders with respect to the power of  $\varepsilon$ , we obtain

$$\begin{aligned} & h'[A \cdot \nabla \psi_0]_1^o + [A \cdot \nabla \psi_0]_2^o + h'[A^o \cdot \widetilde{\nabla} \overline{\psi}_1]_1 + [A^o \cdot \widetilde{\nabla} \overline{\psi}_1]_2 \\ & + \varepsilon \left( -hh' \partial_y [A \cdot \nabla \psi_0]_1^o - h \partial_y [A \cdot \nabla \psi_0]_2^o - hh' [\partial_y A^o \cdot \widetilde{\nabla} \overline{\psi}_1]_1 \right. \\ & \quad - h [\partial_y A^o \cdot \widetilde{\nabla} \overline{\psi}_1]_2 + h'[A^o \cdot \widetilde{\nabla} \overline{\psi}_2]_1 + [A^o \cdot \widetilde{\nabla} \overline{\psi}_2]_2 \\ & \quad \left. + h'[A \cdot \nabla \psi_1]_1^o + [A \cdot \nabla \psi_1]_2^o + h'[A^o \cdot \nabla \overline{\psi}_1]_1 + [A^o \cdot \nabla \overline{\psi}_1]_2 \right) = \mathcal{O}(\varepsilon^2). \end{aligned}$$

Separating the fast and slow variables, we obtain:

$$\begin{cases} [A \cdot \nabla \psi_0]_2 = 0 & \text{on } \Gamma_0^-, \\ [A \cdot \nabla \psi_1]_2 = -\beta_2 & \text{on } \Gamma_0^-, \\ h'[A^o \cdot \widetilde{\nabla} \overline{\psi}_1]_1 + [A^o \cdot \widetilde{\nabla} \overline{\psi}_1]_2 = -h'[A \cdot \nabla \psi_0]_1^o & \text{on } \gamma, \\ h'[A^o \cdot \widetilde{\nabla} \overline{\psi}_2]_1 + [A^o \cdot \widetilde{\nabla} \overline{\psi}_2]_2 = g + \beta_2 & \text{on } \gamma, \end{cases}$$

where the source term is given by

$$\begin{aligned} g &= hh' \partial_y [A \cdot \nabla \psi_0]_1^o + h \partial_y [A \cdot \nabla \psi_0]_2^o + hh' [\partial_y A^o \cdot \widetilde{\nabla} \overline{\psi}_1]_1 \\ & \quad + h [\partial_y A^o \cdot \widetilde{\nabla} \overline{\psi}_1]_2 - h'[A \cdot \nabla \psi_1]_1^o - h'[A^o \cdot \nabla \overline{\psi}_1]_1 - [A^o \cdot \nabla \overline{\psi}_1]_2, \end{aligned}$$

and where the coefficient  $\beta_2$  (which depends on  $x$ ) will be specified later so that the solution  $\overline{\psi}_2$  has no influence in the main domain  $\Omega_0$ .

**Step 3: Systems satisfied by the first profiles**

We conclude that the system must check the first profiles  $\psi_0$ ,  $\psi_1$ ,  $\bar{\psi}_1$  and  $\bar{\psi}_2$ . The main term  $\psi_0$  satisfies the system without roughness:

$$\begin{cases} \operatorname{div}(A \cdot \nabla \psi_0) = 0 & \text{on } \Omega_0, \\ \psi_0 = f & \text{on } \Gamma^+, \\ [A \cdot \nabla \psi_0]_2 = 0 & \text{on } \Gamma_0^-. \end{cases} \quad (35)$$

This term creates an error in the rough layer, which is corrected using  $\bar{\psi}_1$ :

$$\begin{cases} \widetilde{\operatorname{div}}(A^\circ \cdot \widetilde{\nabla}(\bar{\psi}_1)) = 0 & \text{on } \omega, \\ h'[A^\circ \cdot \widetilde{\nabla} \bar{\psi}_1]_1 + [A^\circ \cdot \widetilde{\nabla} \bar{\psi}_1]_2 = -h'[A \cdot \nabla \psi_0]_1^\circ & \text{on } \gamma. \end{cases} \quad (36)$$

This corrector depends on the rescaled variables  $(X, Y)$  and rapidly decreases in the main domain (that is when  $Y$  tend to  $+\infty$ ). Moreover, it depends on the variable  $x$  and must be corrected using  $\bar{\psi}_2$ :

$$\begin{cases} \widetilde{\operatorname{div}}(A^\circ \cdot \widetilde{\nabla}(\bar{\psi}_2)) = f & \text{on } \omega, \\ h'[A^\circ \cdot \widetilde{\nabla} \bar{\psi}_2]_1 + [A^\circ \cdot \widetilde{\nabla} \bar{\psi}_2]_2 = \beta_2 + g & \text{on } \gamma. \end{cases}$$

The choice of the function  $\beta_2$  (only depending on  $x$ ) allow us to ensures that this correction rapidly tends to 0 in the main domain. Using the result of the Lemma 2.2 (which is valid for the operator  $\widetilde{\operatorname{div}}(A^\circ \cdot \widetilde{\nabla})$  - the proof is similar to those of the Lemma 2.2), we deduce that

$$\beta_2(x) = - \int_{\{Y < 0\}} f(x, X, Y) dX dY - \int_{\mathbb{T}} g(x, X) dX. \quad (37)$$

Finally, the next corrector  $\psi_1$  take into account this coefficient:

$$\begin{cases} \operatorname{div}(A \cdot \nabla \psi_1) = 0 & \text{on } \Omega_0, \\ \psi_1 = 0 & \text{on } \Gamma^+, \\ [A \cdot \nabla \psi_1]_2 = -\beta_2 & \text{on } \Gamma_0^-. \end{cases}$$

### 3.3.3 Asymptotic development in the non flat case

In the interesting case where the matrix  $A$  is given with respect to the bottom topography  $B$ , see equation (31), the first corrector  $\bar{\psi}_1$  satisfies

$$\begin{cases} \partial_X^2 \bar{\psi}_1 - 2B' \partial_{XY}^2 \bar{\psi}_1 + (1 + B'^2) \partial_Y^2 \bar{\psi}_1 = 0 & \text{on } \omega, \\ h'(\partial_X \bar{\psi}_1 - B' \partial_Y \bar{\psi}_1) + (-B' \partial_X \bar{\psi}_1 + (1 + B'^2) \partial_Y \bar{\psi}_1) = -h'[A \cdot \nabla \psi_0]_1^\circ & \text{on } \gamma. \end{cases} \quad (38)$$

We will note however that this case presented to the peculiarity of having a matrix does not depend on the variable  $y$ , that simplify some calculus. For instance the contribution  $\bar{\psi}_2(x, X, Y)$  satisfies

$$\begin{cases} \partial_X^2 \bar{\psi}_2 - 2B' \partial_{XY}^2 \bar{\psi}_2 + (1 + B'^2) \partial_Y^2 \bar{\psi}_2 = f & \text{on } \omega, \\ h'(\partial_X \bar{\psi}_2 - B' \partial_Y \bar{\psi}_2) + (-B' \partial_X \bar{\psi}_2 + (1 + B'^2) \partial_Y \bar{\psi}_2) = \beta_2 + g & \text{on } \gamma. \end{cases} \quad (39)$$

where the source terms are given by

$$\begin{aligned} f &= -2\partial_{xX}^2 \bar{\psi}_1 + 2B' \partial_{xY}^2 \bar{\psi}_1 + B'' \partial_Y \bar{\psi}_1 & \text{and} \\ g &= -(h' - B') \partial_x \bar{\psi}_1 + h h' \partial_y [A \cdot \nabla \psi_0]_1^\circ + h \partial_y [A \cdot \nabla \psi_0]_2^\circ - h' [A \cdot \nabla \psi_1]_1^\circ. \end{aligned}$$

**Remark 3.1** Using the change of variable  $Z = Y + B'(x)X$ , the contributions of the boundary layer terms as  $\overline{\psi}_1(x, X, Y)$  reads  $\overline{\psi}_1(x, X, Y) = u_1(x, X, Z)$  where the function  $u_1$  solves

$$\begin{cases} \Delta u_1 = 0 & \text{on } \omega^B, \\ (h'(X) - B'(x))\partial_X u_1 + \partial_Z u_1 = -h' [A \cdot \nabla \psi_0]_1(x, 0) & \text{on } \gamma^B, \end{cases}$$

where  $\omega^B := \{Z > B'(x)X - h(X)\}$  and  $\gamma^B := \{Z = B'(x)X - h(X)\}$ .

The coefficient  $\beta_2$ , that is to say that which occurs at the main order in the wall law, can be calculated from the averaging of the source terms  $f$  and  $g$  (using the formulation (37)).

- The average of the source term  $g$  is given by:

$$\begin{aligned} \int_{\mathbb{T}} g(x, X) dX &= -\partial_x \left( \int_{\mathbb{T}} h'(X) \overline{\psi}_1(x, X, -h(X)) dX \right) \\ &\quad + B'(x) \partial_x \left( \int_{\mathbb{T}} \overline{\psi}_1(x, X, -h(X)) dX \right) + m(h) \partial_y [A \cdot \nabla \psi_0]_2^o, \end{aligned}$$

where we recall that the notation  $m(h)$  stands for the average  $m(h) = \int_{\mathbb{T}} h(X) dX$ . To simplify this expression, we proceed as follows. First, we multiply the first equation of the system (38) by  $Y$  and integrate over the set  $\{Y < 0\}$ . Using integrations by parts and the boundary condition corresponding to the second equation of the system (38) we obtain

$$B'(x) \int_{\{Y < 0\}} \partial_X \overline{\psi}_1(x, X, Y) dX dY = (1 + B'(x)^2) \int_{\{Y < 0\}} \partial_Y \overline{\psi}_1(x, X, Y) dX dY. \quad (40)$$

Next, as in the flat case we note that

$$\int_{\{Y < 0\}} \partial_X \overline{\psi}_1(x, X, Y) dX dY = - \int_{\mathbb{T}} h'(X) \overline{\psi}_1(x, X, -h(X)) dX. \quad (41)$$

In the same way we deduce that (recalling that from the Fourier analysis, we have  $\int_{\mathbb{T}} \overline{\psi}_1(x, X, 0) dX = 0$ )

$$\int_{\{Y < 0\}} \partial_Y \overline{\psi}_1(x, X, Y) dX dY = - \int_{\mathbb{T}} \overline{\psi}_1(x, X, -h(X)) dX. \quad (42)$$

The equality (40) is then written

$$\int_{\mathbb{T}} \overline{\psi}_1(x, X, -h(X)) dX = \frac{B'(x)}{1 + B'(x)^2} \int_{\mathbb{T}} h'(X) \overline{\psi}_1(x, X, -h(X)) dX.$$

As in the flat case again, this quantities can be connected with an energy:

$$E(h, x) := \int_{\omega} (A \cdot \widetilde{\nabla} v) \cdot \widetilde{\nabla} v, \quad (43)$$

where  $v(x, X, Y)$  is the solution of the following system

$$\begin{cases} \widetilde{\text{div}}(A \cdot \widetilde{\nabla} v) = 0 & \text{on } \omega, \\ h' [A^o \cdot \widetilde{\nabla} v]_1 + [A^o \cdot \widetilde{\nabla} v]_2 = -h' & \text{on } \gamma. \end{cases} \quad (44)$$

Since  $\bar{\psi}_1 = [A \cdot \nabla \psi_0]_1^o v$ , we obtain

$$\int_{\mathbb{T}} h'(X) \bar{\psi}_1(x, X, -h(X)) dX = [A \cdot \nabla \psi_0]_1^o E(h, x).$$

- The average of the function  $f$  is defined by

$$\begin{aligned} \int_{\{Y < 0\}} f(x, X, Y) dX dY &= -2\partial_x \left( \int_{\{Y < 0\}} \partial_X \bar{\psi}_1(x, X, Y) dX dY \right) \\ &\quad + 2B'(x) \partial_x \left( \int_{\{Y < 0\}} \partial_Y \bar{\psi}_1(x, X, Y) dX dY \right) \\ &\quad + B''(x) \int_{\{Y < 0\}} \partial_Y \bar{\psi}_1(x, X, Y) dX dY. \end{aligned}$$

Using the relation (41) and (42) we can add the averages of the functions  $g$  and  $f$ . We deduce an expression for the coefficient  $\beta_2$ :

$$\beta_2(x) = -m(h) \partial_y [A \cdot \nabla \psi_0]_2^o - \partial_x \left( \mathcal{E}(h, B') [A \cdot \nabla \psi_0]_1^o \right),$$

where the quantity  $\mathcal{E}$  is defined by  $\mathcal{E}(h, B') = \frac{E(h, x)}{1 + B'(x)^2}$ .

Since the main profile  $\psi_0$  satisfies the equation (35), we have  $\partial_x [A \cdot \nabla \psi_0]_1 + \partial_y [A \cdot \nabla \psi_0]_2 = 0$ . We can define  $\beta_2$  as follows

$$\beta_2(x) = (\mathcal{E}(h, B') - m(h)) \partial_y [A \cdot \nabla \psi_0]_2^o - \partial_x (\mathcal{E}(h, B')) [A \cdot \nabla \psi_0]_1^o. \quad (45)$$

### 3.3.4 Wall law taking into account the geometry

In this subsection, we will see that the asymptotic development (34) and the boundary conditions satisfied by each profile allow to obtain an effective boundary condition at order 1. We then prove the Theorem 3.2.

More precisely, using the boundary conditions satisfied by the first term of the development of  $\psi$ , we have

$$[A \cdot \nabla \psi]_2^o = -\varepsilon \beta_2 + \mathcal{O}(\varepsilon^2). \quad (46)$$

This relationship between derivatives with respect to variables  $(x, y)$  of the function  $\psi$  at the boundary  $\{y = 0\}$  can be interpreted using a relation between the derivatives of the function  $\varphi$  with respect to the variables  $(x, z)$  at the boundary  $\{z = B(x)\}$  and next with respect to the variables  $(n, \tau)$  corresponding to the normal and tangential coordinates.

For instance, we have the following equalities, holds on the boundary  $\{y = 0\} = \{z = B(x)\}$ :

$$\begin{aligned} \partial_n \varphi &= \frac{1}{\sqrt{1 + B'^2}} (B' \partial_x \varphi - \partial_z \varphi) = \frac{-1}{\sqrt{1 + B'^2}} [A \cdot \nabla \psi]_2^o, \\ \partial_\tau \varphi &= \frac{1}{\sqrt{1 + B'^2}} (\partial_x \varphi + B' \partial_z \varphi) = \frac{1}{\sqrt{1 + B'^2}} \partial_x \psi. \end{aligned}$$

Similarly, we can obtain the following formulae of “change of variables”:

$$\begin{aligned}\partial_n^2 \varphi - B' \partial_{n\tau}^2 \varphi - \frac{B' B''}{(1 + B'^2)^{3/2}} \partial_\tau \varphi &= \partial_y [A \cdot \nabla \psi]_2, \\ \partial_\tau \varphi + B' \partial_n \varphi &= \sqrt{1 + B'^2} [A \cdot \nabla \psi]_1.\end{aligned}$$

Finally, the relation (46) is written

$$\partial_n \varphi = \varepsilon \mathcal{R} \varphi + \mathcal{O}(\varepsilon^2) \quad (47)$$

where the operator  $\mathcal{R}$  is defined by

$$\mathcal{R} \varphi = \frac{\mathcal{E} - m}{\sqrt{1 + B'^2}} \left( \partial_n^2 \varphi - B' \partial_{n\tau}^2 \varphi - \frac{B' B''}{(1 + B'^2)^{3/2}} \partial_\tau \varphi \right) - \frac{\partial_x \mathcal{E}}{1 + B'^2} \left( \partial_\tau \varphi + B' \partial_n \varphi \right). \quad (48)$$

**Remark 3.2** *This wall law is a generalization of the flat case ( $B' = 0$ ) for which we find the law (5). This general law naturally appears the slope of the bottom (that is the quantity  $B'$ ) and the curvature of the bottom (that is  $B''/(1 + B'^2)^{3/2}$ ).*

### 3.4 Application to the water waves

In this subsection we give an example of application and in the same time we prove the Theorem 3.3.

The Neumann boundary conditions are very frequently used. For instance, in fluid mechanics such a condition is natural if we consider that the velocity field  $\mathbf{u}$  is tangential to the boundary:  $\mathbf{u} \cdot \mathbf{n} = 0$ , and it derives from a potential:  $\mathbf{u} = \nabla \varphi$ . This approach allows us to “simply” write a flow of an incompressible and irrotational fluid with free surface.

More precisely, the water wave equation describes the evolution of the water surface, parameterized by the function  $T$ . It reads

$$\begin{cases} \partial_t T = G(T, B) f \\ \partial_t f + g T + \frac{1}{2} (\partial_x f)^2 - \frac{(G(T, B) f + \partial_x T \partial_x f)^2}{2(1 + (\partial_x T)^2)} = 0. \end{cases} \quad (49)$$

In this system, the function  $B$  describes the bathymetry and the operator  $G(T, B)$  corresponds to the following Dirichlet-Neumann operator

$$G(T, B) f = \sqrt{1 + (\partial_x T)^2} \partial_n \varphi$$

where  $\varphi$  is the solution of the following Laplace problem

$$\begin{cases} -\Delta \varphi = 0 & \text{on } \Omega^{T, B}, \\ \varphi = f & \text{on } \Gamma^{T, B, +}, \\ \partial_n \varphi = 0 & \text{on } \Gamma^{T, B, -}. \end{cases} \quad (50)$$

The formulation (49) of the water waves equation corresponds to the Hamiltonian exhibited by Zakharov [22] and was written first under this form by Craig and Sulem [10]. Moreover, first existence results for such a system is given by David Lannes in [16].

**Fictitious bathymetry** The goal of the paragraph is to show that if the bottom of the domain is rough (with periodical roughness of size  $\varepsilon$  as studied in the previous sections) then you can use a virtual not rough bottom to obtain an error of order  $\varepsilon^2$  on the height of the water.

We are thus first interested in the system (27), that is the system (50) with a rough domain  $\Omega_\varepsilon^{T,B}$ . For any regular function  $C$ , consider the following surface defined as a perturbation of the bottom surface  $\Gamma_0^{T,B,-}$ :

$$\Sigma = \{(x, z) \in \mathbb{R}^{d-1} \times \mathbb{R} ; z = B(x) + \varepsilon C(x)\}.$$

The normal derivative of  $\varphi$  along this surface is written

$$\partial_n \varphi \Big|_\Sigma = \frac{1}{\sqrt{1 + (B' + \varepsilon C')^2}} \left( (B' + \varepsilon C') \partial_x \varphi(x, B + \varepsilon C) - \partial_z \varphi(x, B + \varepsilon C) \right).$$

Using the Taylor development, we obtain

$$\begin{aligned} \partial_n \varphi \Big|_\Sigma &= \partial_n \varphi \Big|_{\Gamma_0^{T,B,-}} + \frac{\varepsilon}{\sqrt{1 + B'^2}} \left( B' C \partial_{xz}^2 \varphi + C' \partial_x \varphi - C \partial_z^2 \varphi \right) \\ &\quad - \frac{\varepsilon B' C'}{1 + B'^2} \partial_n \varphi \Big|_{\Gamma_0^{T,B,-}} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Using the relation (46) we have  $\partial_n \varphi \Big|_{\Gamma_0^{T,B,-}} = \frac{\varepsilon \beta_2}{\sqrt{1 + B'^2}} + \mathcal{O}(\varepsilon^2)$ . We obtain

$$\partial_n \varphi \Big|_\Sigma = \frac{\varepsilon}{\sqrt{1 + B'^2}} \left( \beta_2 + C(B' \partial_{xz}^2 \varphi - \partial_z^2 \varphi) + C' \partial_x \varphi \right) + \mathcal{O}(\varepsilon^2).$$

In term of variables  $(x, y)$ , that is using the function  $\psi$  we have

$$\partial_n \varphi \Big|_\Sigma = \frac{\varepsilon}{\sqrt{1 + B'^2}} \left( \beta_2 - C \partial_y [A \cdot \nabla \psi]_2^o + C' [A \cdot \nabla \psi]_1^o \right) + \mathcal{O}(\varepsilon^2).$$

Now, according to the expression of  $\beta_2$  given by the formula (45) if we take  $C = \mathcal{E} - m$  then  $\partial_n \varphi \Big|_\Sigma = \mathcal{O}(\varepsilon^2)$ . Finally, to obtain an approximation of the solution to the Laplace equation in  $\Omega_\varepsilon^{T,B}$  at order  $\varepsilon^2$ , we can study the same problem with the classical Neumann boundary condition on a not rough domain slightly different from  $\Omega_0^{T,B}$ :

$$\Omega_0^{T,B+\varepsilon C} = \{(x, z) \in \mathbb{R}^{d-1} \times \mathbb{R} ; B(x) - \varepsilon(m(h) - \mathcal{E}(h, B')) < z < T(x)\}.$$

In conclusion, the Dirichlet-Neumann operator  $G[T, B - \varepsilon h(\cdot/\varepsilon)]$  used in the formulation of the water wave equation can be approximated to order  $\varepsilon^2$  by the Dirichlet-Neumann operator  $G[T, B - \varepsilon(m(h) - \mathcal{E}(h, B'))]$ . Note that some similar developments of the Dirichlet-Neumann operator are given in [17]. The authors prove that using such developments it is possible to derive the Green-Naghdi equations (also called Serre or fully nonlinear Boussinesq equations).

**Some numerical experiments** The above result indicates that we can significantly improve the approximation using the domain  $\overline{\Omega}^B$  instead of using the



domain  $\Omega_0$ . To really determine the domain  $\overline{\Omega}^B$ , we must know how to calculate the energy  $\mathcal{E}$  from the data of the shape of the roughness  $h$  and the average shape of the bottom  $B$ .

To calculate this energy  $\mathcal{E}$ , we must first solve the equation (44) and then estimate  $E$  defined by the relation (43). The graphic 7 provides results of numerical simulations in the case of sinusoidal roughness (precisely  $h(X) = \sin(2\pi X)$ ) and for different slopes  $B'$ .

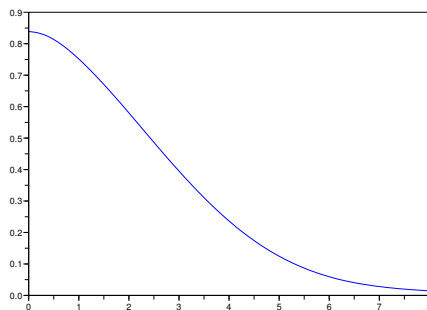


Figure 7: Function  $B' \mapsto \mathcal{E}$  given the energy with respect to the slope of the bottom, the form of the roughness being fixed.

Knowing the energy  $\mathcal{E}$  as a function of the slope  $B'$ , it is easy to determine the domain  $\overline{\Omega}^B$ . The figure 8 provides an example of a rough bottom, approximated by its average value  $z = B(x)$  and approximated by the “ideal” surface  $z = B(x) - \varepsilon(m(h) - \mathcal{E}(h, B'(x)))$ .

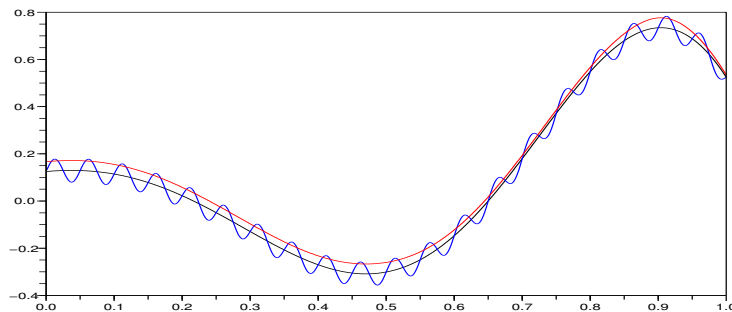


Figure 8: Example of real bottom (oscillating - without error), mean-average approximation (error of order  $\varepsilon$ ) and virtual boundary (red line - error of order  $\varepsilon^2$ ).

## 4 Conclusion

In this paper we have considered a Laplace system with a Neumann condition on a rough boundary. We obtained a development of the solution to any order when the frequency and amplitude of the roughness becomes smaller. The first orders are interpreted in two ways.

- We first show that we can approximate (to order 2) the “rough” solution through a “smooth” solution provided to introduce a peculiar boundary condition on the smooth boundary.
- We also show that we can make such an approximation by imposing still a Neumann condition on the smooth boundary, even slightly modify this smooth boundary (without adding roughness).

Several points indicate the originality of these approximations. On the one hand, they are explicit in the sense that it is possible to calculate precisely the wall law depending on the shape of the roughness (numerical simulations confirm this point). On the other hand, these results are fundamentally different from those obtained for Dirichlet conditions: In the Neumann case, the wall law absorb the excess of energy created by the roughness.

Several prospects in this work may be considered, in particular concerning the assumption of roughness. In the present paper, they are supposed to be periodic and defined as the graph of a regular function. It seems interesting to study the random roughness case, as it has been recently studied for the Dirichlet boundary conditions, see [4, 13, 14]. Another interesting case is the case where the application  $h$  which defines the roughness is not regular. For example when we wish to model roughness in the form of pulses (the function  $h$  is only  $L^\infty$ ). It does not seem obvious to adapt the arguments used in the present article.

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