

Rigorous derivation of the thin film approximation with roughness-induced correctors*

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Abstract

We derive the thin film approximation including roughness-induced correctors. This corresponds to the description of a confined Stokes flow whose thickness is of order ε (designed to be small) ; but we also take into account the roughness patterns of the boundary that are described at order ε^2 , leading to a perturbation of the classical Reynolds approximation. The asymptotic expansion leading to the description of the scale effects is rigorously derived, through a sequence of Reynolds-type problems and Stokes-type (boundary layer) problems. Well-posedness of the related problems and estimates in suitable functional spaces are proved, at any order of the expansion. In particular, we show that the micro-/macro-scale coupling effects may be analysed as the consequence of two features: the interaction between the macroscopic scale (order 1) of the flow and the microscopic scale (order ε of the thin film) is perturbed by the interaction with a microscopic scale of order ε^2 related to the roughness patterns (as expected through the classical Reynolds approximation) ; moreover, the converging-diverging profile of the confined flow, which is typical in lubrication theory (note that the case of a constant cross-section channel has no interest) provides additional micro-macro-scales coupling effects.

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1 Introduction

1.1 General framework

Lubricated flows are very present in today's world: from the journal bearings to the computer disk drives through the microfluid or in biofluid mechanics. The first relevant model for such thin flows was proposed by O. Reynolds in 1886, see [22]. From a mathematical point of view, the rigorous justification of the Reynolds equation from the Stokes equation is due to G. Bayada and M. Chambat [4]. Other studies have further refined this result, especially those of S. Nazarov [21] and more recently J. Wilkening [24].

From another point of view, many studies investigate the effect of wall roughness on Newtonian flows. In 1827, Navier [20] was one of the first scientists to note that the roughness could drag a fluid. Since then, numerous studies attempted to prove mathematical results in this direction, see for instance the works of W. Jäger and A. Mikelić [17], Y. Amirat *et al.* [1, 2] and more recently the works of D. Bresch and V. Milisic [10, 10]. Note that all these works formulate the roughness using a periodic function (whose amplitude and period are supposed to be small). In a context of more general “roughness” patterns, there exists similar recent results, see [3, 14].

Numerous works focus on the combination of the two phenomena: lubrication and roughness. This is for example the case in [5, 7] in which the size of the roughness is assumed to be at least of the same order as the thickness of the fluid considered. In [9], the author consider the case where the roughness is assumed small compared to the thickness of the flow (which is the case of the present paper) but they show a convergence in a rescaled domain, that focuses on the roughness effect. Recently, in [11], J. Casado-Díaz and co-authors proposed a relatively general study (in terms of orders of magnitude of roughness and thickness of the domain). However, their article is entirely focused on the wall laws which is not our point of view in this paper.

In this paper, we focus on flow in a thin domain (with thickness $\varepsilon \ll 1$), lubricated and rough. The size of the roughness was assumed to be of size ε^2 which is physically realistic, see for example [18]. By separating the effects due to lubrication and those due to the roughness, we present and rigorously justify an asymptotic expansion according to ε . The development is done at any order, so that we are guaranteed to be optimal with respect to the truncation error. We also highlight the particular effects of roughness (with respect to the smooth case), and the multiscale coupling effects of the curvature of the macroscopic domain (which cannot be neglected in lubricated devices).

Several relevant questions are not addressed in this article. First, concerning the choice of orders of magnitude for the thickness of the fluid and the roughness (ε and ε^2 in this article). It seems fairly sensible to believe that the proposed method can be adapted to cases where the thickness of the fluid is of order ε , while the roughness is of the order ε^α , with $\alpha > 1$. Nevertheless, the *ansatz* will be different, depending on α . Second, recent works on random roughness,

see [3, 14], could make us think that our results can be extended to more general cases of roughness. In fact, the construction of our development strongly depends on the behavior of solutions of the Stokes equation on a half-space, whose lower boundary is periodic. The behavior of such solutions must be sufficiently decreasing at infinity to justify our development. Unfortunately, it seems that this decrease is only logarithmic in the case of a random boundary (while it is exponential in our periodic case). Besides, another task related to the regularity of the roughness patterns is not addressed in this paper: what is the behavior of the solution when the patterns are not Lipschitz continuous? In particular, what is the influence of crenel patterns over the flow?

1.2 Mathematical formulation

We consider the flow of a viscous incompressible fluid in a domain Ω of \mathbb{R}^{d+1} , $d \in \mathbb{N}^*$. The domain is assumed to be periodic in the \mathbf{x} -direction and upper/lower-bounded by two rough boundaries Γ_ε^+ and Γ_ε^- . In the lubrication context or in microfluidic studies, flows are confined between two very close surfaces. Moreover, it seems natural that, on this scale, the effects of roughnesses cannot be neglected. Mathematically, we take into account this containment by introducing a small parameter $\varepsilon > 0$ and defining

$$\Omega_\varepsilon(t) = \left\{ (\mathbf{x}, y) \in \mathbb{T}^d \times \mathbb{R}, \quad -\varepsilon^2 h^- \left(\frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2} \right) < y < \varepsilon h^+(\mathbf{x}) \right\}$$

where $h^+ \in \mathcal{C}^\infty(\mathbb{T}^d)$ and $h^- \in \mathcal{C}_{per}^\infty(]0, 1[^d)$ are two positive functions, $\mathbf{s} \in \mathbb{R}^d$ denotes the shear velocity of the device (velocity of the lower rigid surface). A typical situation describing the scaling orders (including the thin film assumption and the rough boundary) is illustrated on FIG. 1. Without loss of generality, we have assumed that there is no oscillation at the upper boundary, as the main feature in lubrication theory only deals with the relative distance between the two close surfaces in relative motion.

Stokes equations express, in particular, the momentum conservation connecting the velocity field $\mathbf{U} = (\mathbf{u}, v)$ to the pressure p . These equations must be supplemented by boundary conditions. A well-accepted hypothesis in the fluid dynamics is that if the boundary of the physical domain is impervious, then the viscous fluid completely adheres to it. Thus, no-slip conditions are imposed to the walls, the upper wall being fixed whereas the lower wall is animated with a horizontal shear velocity $(\mathbf{s}, 0)$:

$$\begin{cases} -\Delta \mathbf{U} + \nabla p = 0, & \text{in } \Omega_\varepsilon(t), \\ \operatorname{div} \mathbf{U} = 0, & \text{in } \Omega_\varepsilon(t), \\ \mathbf{U} = \mathbf{0}, & \text{on } \Gamma_\varepsilon^+(t), \\ \mathbf{U} = (\mathbf{s}, 0), & \text{on } \Gamma_\varepsilon^-(t). \end{cases}$$

In the sequel, for the sake of simplicity, we may omit the variable t although the domain depends on time as a parameter.

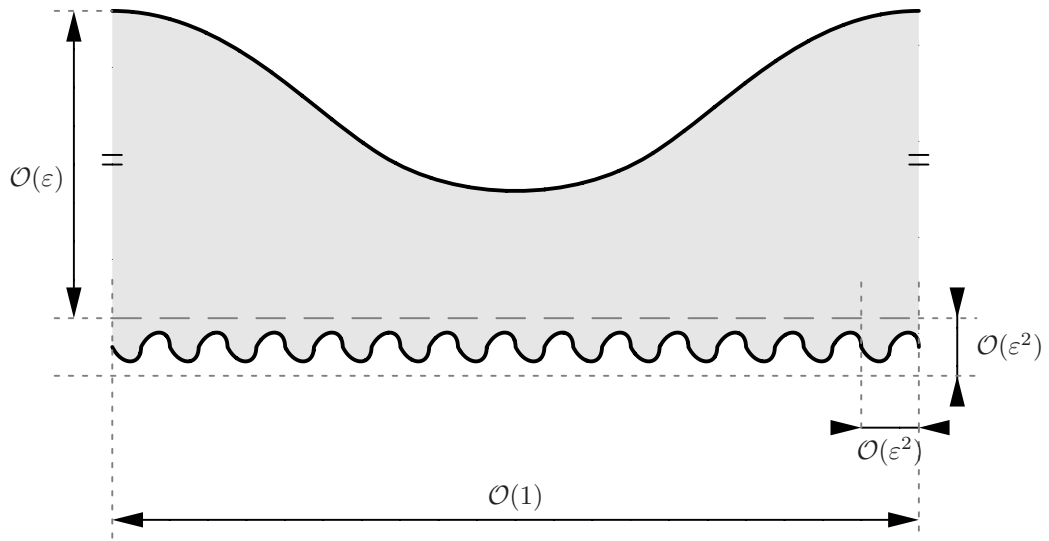


Figure 1: Geometry of the Stokes flow: thin film assumption and roughness patterns.

It is well-known (see [4] or more recently [24]) that the solutions of the Stokes system in a thin confined domain with a flat bottom are approached by those of the Reynolds equation. More precisely, under the thin film assumption, assuming that the bottom is flat, i. e. $h^- = 0$, the flow is governed by:

$$\begin{aligned} \mathbf{u}^+(\mathbf{x}, y) &= \mathbf{u}_0\left(\mathbf{x}, \frac{y}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2), \\ v^+(\mathbf{x}, y) &= \varepsilon v_0\left(\mathbf{x}, \frac{y}{\varepsilon}\right) + \mathcal{O}(\varepsilon^3), \\ p^+(\mathbf{x}, y) &= \frac{p_0(\mathbf{x})}{\varepsilon^2} + \mathcal{O}(1), \end{aligned}$$

where \mathbf{u}_0 , v_0 and p_0 correspond to the rescaled velocity field and pressure at main order. It can be shown that p_0 (which only depends on the variable \mathbf{x}) is the unique solution (defined up to an additive constant) of the Reynolds equation ; besides, the velocity field can be deduced from the pressure gradient by means of a straightforward integration.

In this context, we aim at describing the corresponding correction due to the the rough boundary at main order. More generally, we emphasize that the asymptotic expansion which is classically derived in the context of a flat bottom has to be enriched at any order by a sequence of suitable functions. Thus we derive the general asymptotic expansion that is valid with or without roughness patterns and propose a numerical procedure that is accurate enough to compute the approximation of the solution at any order, by means of an additive procedure of elementary solutions of Stokes or Reynolds-type problems.

Numerical experiments, see Fig. 2 to 5, highlight the differences in terms of computational costs: taking into account the boundary layer due to the roughness patterns leads to the definition of a mesh with a large number of degrees of freedom. In order to avoid this complexity, a possible answer is to derive a procedure based on the computation of simple solutions defined on regular domain: this is the role of the *ansatz*. Numerical results give a preview of the perturbations due to the roughness patterns compared to a smooth boundary, which justifies an insight into the boundary layer structure and its interaction with the main flow.

1.3 Main ideas

In this subsection, we want to present, without particular mathematical developments, the main ideas related to our purpose. When dealing with roughness patterns at the bottom, several difficulties arise. To approach the solution of the Stokes problem in all the domain $\Omega_\varepsilon(t)$, \mathbf{u}^+ , v^+ and p^+ should be also defined on $\Omega_\varepsilon^-(t)$ so that we have, at least, to extend the values of \mathbf{u}_0 , v_0 and p_0 for negative values of $Z := y/\varepsilon$. Then, the influence of the scale effects induced by the roughness patterns on the average flow has to be included in a suitable way. This leads to the definition of an asymptotic expansion based on a sequence of problems defined not only in the classical thin film Reynolds domain but also in the additional boundary layer modelling the roughness patterns.

More precisely, the main idea relies on the following procedure:

- i) *Reynolds flow at main order for $Z > 0$.* The main flow is described by the Reynolds solution (in the rescaled domain $\{0 < Z < h^+(\mathbf{x})\}$ due to the scaling process of the thin film domain), with the classical no-slip boundary condition located at the fictitious boundary $Z = 0$.
- ii) *Extension of the Reynolds flow at main order in the boundary layer $Z < 0$.* This solution is extended to $Z < 0$ by means of a polynomial function which satisfies the Stokes system, thus guaranteeing that the extended function in real variables satisfies the Stokes system in the whole domain. Unfortunately, the no-slip boundary condition is not satisfied at the bottom (as it has been imposed on $y = 0$ instead) but we can check that it is approached at an order $\mathcal{O}(\varepsilon)$. This is why we need to define an additive corrector.
- iii) *Corrective Stokes flow.* We define the solution of a Stokes-type problem (in the rescaled domain $\{Y > -h^-(\mathbf{X})\}$ due to the scaling process of the boundary layer) which counterbalances the deviation of the boundary condition at the bottom, so that the sum of the initial Reynolds solution and the corrective Stokes-type solution, expressed in real variables, does satisfy the boundary condition at the bottom.

To this point, we emphasize that this procedure leads to the definition of a solution which satisfies both the Stokes system and the boundary condition at the bottom. Of course, we have to check whether the no-slip boundary condition is satisfied at the top of the domain or not.

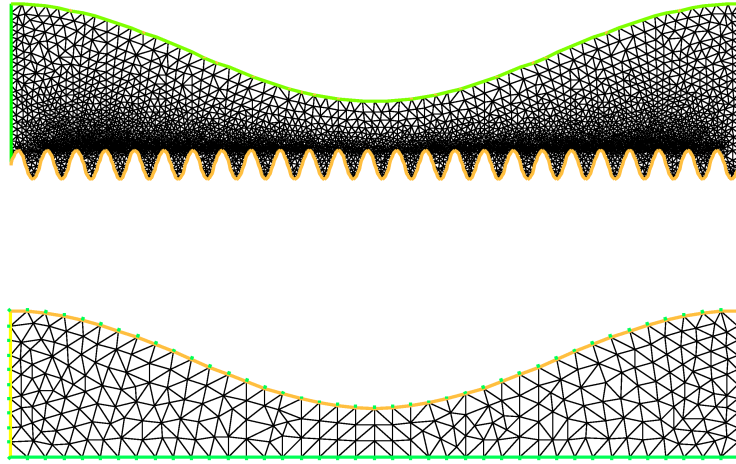


Figure 2: Mesh used for the computations of the solution of the Stokes system (approximation with the (P_2, P_2, P_1) finite element approximation). Domain with roughness patterns or with smooth boundary.

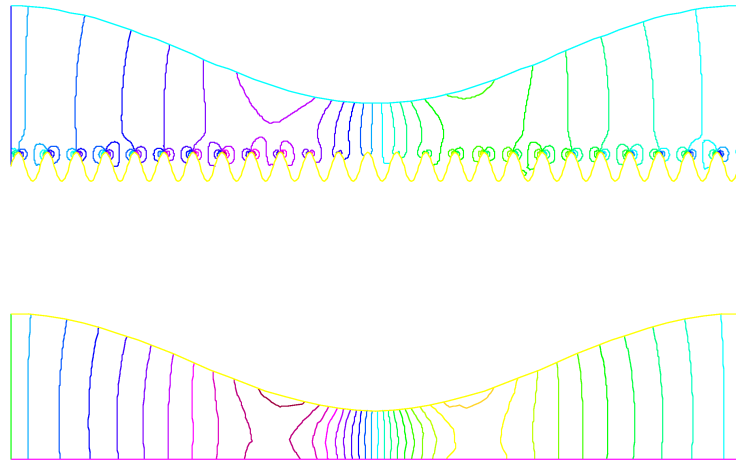


Figure 3: Pressure distribution in the domain with/without roughness patterns.

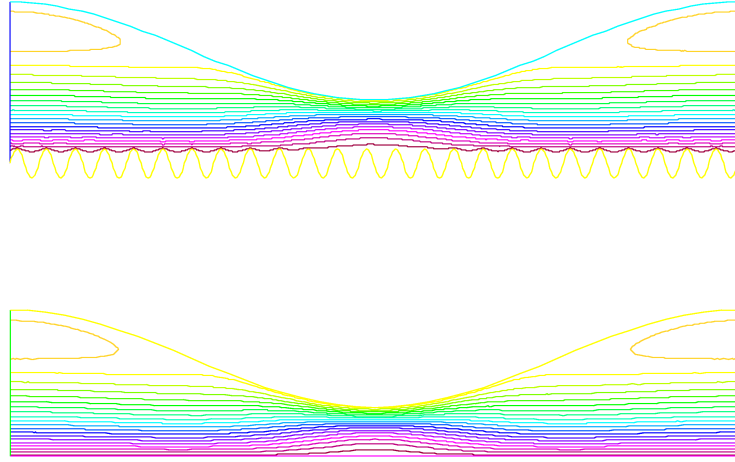


Figure 4: Horizontal velocity distribution in the domain with/without roughness patterns.

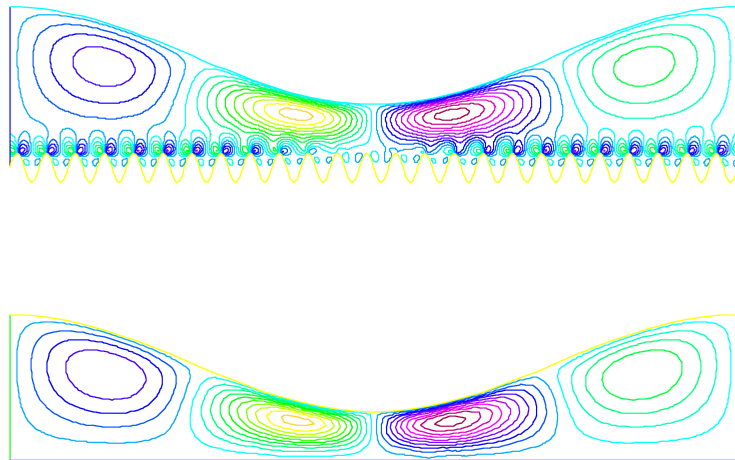


Figure 5: Vertical velocity distribution in the domain with/without roughness patterns.

iv) *Identification of the top boundary condition deviation: towards an iterative procedure.* As we will see, since the Reynolds solution has been made compatible with the boundary condition at the top, the question only relies on the value of the corrector at the top. We will prove that the value of the corrector on the top boundary is a non-zero value ; more precisely, the corrector Stokes solution exponentially decreases (as $Y \rightarrow +\infty$) to a constant which can be identified to this non-zero value. This justifies the renewal of the procedure: we define the solution of the Reynolds equation with the previous non-zero value on the top $Z = h^+(\mathbf{x})$, zero value at the bottom $Z = 0$, extended for $Z < 0$. In this way, the sum of the previous functions satisfy the Stokes system, the no-slip boundary condition at the top. Unfortunately, it does not satisfy the no-slip boundary condition at the bottom, because of the extension process, but we will see that the boundary condition is now satisfied with an order $\mathcal{O}(\varepsilon^2)$. Thus, the Stokes correction procedure can be repeated easily by defining a suitable Stokes solution whose behaviour at infinity will be analyzed in order to counterbalance the perturbation effects on the boundary conditions at the top.

To summarize the procedure, one may say that, to each Reynolds flow, one may associate a corrective Stokes flow, whose property relies on the correction of the no-slip boundary condition at the bottom. In return, the behaviour of the corrective Stokes solution at infinity has a perturbation impact on the no-slip boundary condition on the top, thus leading to the suitable definition of the Reynolds flow at the next order.

After defining the sequence of corrective problems (which, in practice, is not so obvious), we will focus on the behaviour of the related solutions. To be more precise, this analysis cannot be decoupled from the definition of the corrective problems, as constants have to be chosen carefully in order to work with well-posed problems and induce a suitable behaviour of the elementary solutions.

1.4 Organisation of the paper

The paper is organized as follows:

In Section 2, we present the formal asymptotic expansion, based on a sequence of functions which alternatively satisfy a Reynolds-type problem and a corrective Stokes problem (defined in a semi-infinite boundary layer domain). The main ideas leading to the consideration of such an asymptotic expansion are also presented.

In Section 3, we prove the well-posedness of the intermediate problems and analyse the behaviour of the solutions. Moreover, we establish an algorithm related to the computation of the approximation of the solution, at any order. In particular, we show that each problem only depends on the previous ones, although this property is not clear at first glance !

Section 4 is devoted to the error analysis which, in the end, rigorously justifies the asymptotic expansion. As the remainder satisfies a Stokes problem with source terms and non-homogeneous boundary conditions, we first recall classical Stokes estimates, based on Bogovskii formulae, and establish control inequalities on these source terms. As the domain does depend on the small parameter, we then establish the estimates in adapted spaces which, in particular, do not depend on ε .

In Section 5, we focus on the coupling scale effects: we present quantitative comparison results related to the order of convergence of the asymptotic expansion with or without roughness correction, illustrating the degradation of the convergence procedure when omitting the roughness correction. Then we focus on another scales-coupling effect related to the converging-diverging profile of the lubricated space: in particular, we show how this situation is much more complicated than the analysis of the constant cross-section channel (which is not relevant in the lubrication framework) commonly done in the boundary layer analysis of the Stokes problem.

2 Asymptotic expansion: *ansatz* and intermediate problems

2.1 Notations

With the sight of the different scales in the domain, we split the domain $\Omega_\varepsilon(t)$ into three parts : $\Omega_\varepsilon(t) = \Omega_\varepsilon^-(t) \cup \Gamma \cup \Omega_\varepsilon^+(t)$ where $\Omega_\varepsilon^-(t)$ and $\Omega_\varepsilon^+(t)$ are defined by

$$\begin{aligned}\Omega_\varepsilon^+ &= \left\{ (\mathbf{x}, y) \in \mathbb{T}^d \times \mathbb{R}, \quad 0 < y < \varepsilon h^+(\mathbf{x}) \right\}, \\ \Omega_\varepsilon^-(t) &= \left\{ (\mathbf{x}, y) \in \mathbb{T}^d \times \mathbb{R}, \quad -\varepsilon^2 h^- \left(\frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2} \right) < y < 0 \right\},\end{aligned}$$

and the boundary Γ connecting the two subdomains is defined by

$$\Gamma = \mathbb{T}^d \times \{0\}.$$

The first step of the constuction of the *ansatz* is to notice that the flow is controlled by that in the domain Ω_ε^+ which is of “order ε ” with respect to the vertical coordinate, and that the flow in the domain $\Omega_\varepsilon^-(t)$, which is of “order ε^2 ” in both horizontal and vertical directions, can induce a correction. Roughly speaking, the flow is mainly governed by a Reynolds flow in the domain Ω_ε^+ corresponding to the classical thin film assumption. But, due to the roughness patterns, one must add corrective terms which consist in a Stokes flow at scale ε^2 , located in a boundary layer domain.

Let us define the two rescaled subdomains. As a matter of fact, the main flow is governed by the Reynolds thin film flow, based on the changes of variables

$$Z := \frac{y}{\varepsilon}.$$

Due to the consideration of the roughness patterns, the boundary layer is rescaled by the homothetic transformation

$$\mathbf{X} := \frac{\mathbf{x}}{\varepsilon^2}, \quad Y := \frac{y}{\varepsilon^2}, \quad T := \frac{t}{\varepsilon^2}.$$

Definition 2.1 We define the following rescaled domains:

- The Reynolds domain is defined by

$$\omega_R := \{(\mathbf{x}, Z) \in \mathbb{T}^d \times \mathbb{R}, \quad 0 < Z < h^+(\mathbf{x})\},$$

with the following upper/lower boundaries:

$$\gamma_+ = \{(\mathbf{x}, Z) \in \mathbb{T}^d \times \mathbb{R}, \quad Z = h^+(\mathbf{x})\}, \quad \gamma_0 = \{(\mathbf{x}, Z) \in \mathbb{T}^d \times \mathbb{R}, \quad Z = 0\}.$$

- The boundary layer domain is defined by

$$\omega_{bl}(T) = \{(\mathbf{X}, Y) \in]0, 1[^d \times \mathbb{R}, \quad -h^-(\mathbf{X} - \mathbf{s}T) < Y\},$$

with the following lower boundary

$$\gamma_{bl}(T) = \{(\mathbf{X}, Y) \in]0, 1[^d \times \mathbb{R}, \quad Y = -h^-(\mathbf{X} - \mathbf{s}T)\}.$$

Notice that the boundary layer does depend on time, as this subdomain has a moving boundary which immerges from the roughness patterns of the lower surface and the shear velocity of this surface. Actually, time-dependant boundary conditions leads us to define a more suitable rescaled variable which takes into account the shear effects and the adhering conditions that relate time and space variables. Notice that time-dependency of the boundary layer is taken into account in the space variable as a simple parameter, which allows us to insist on the instantaneity of the Stokes system, even at this rescaled level.

2.2 Ansatz

We propose the following asymptotic expansion

$$\begin{pmatrix} \mathbf{u} \\ v \\ p \end{pmatrix} := \begin{pmatrix} \mathbf{u}^{(N)} \\ v^{(N)} \\ p^{(N)} \end{pmatrix} + \begin{pmatrix} \mathcal{R}^{(N)} \\ \mathcal{S}^{(N)} \\ \mathcal{Q}^{(N)} \end{pmatrix} \quad (1)$$

with the following partial sums:

$$\begin{aligned} \mathbf{u}^{(N)}(\mathbf{x}, y, t) &= \sum_{j=0}^N \varepsilon^j \left[\mathbf{u}_j \left(\mathbf{x}, \frac{y}{\varepsilon} \right) + \varepsilon \tilde{\mathbf{u}}_{j+1} \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2}, \frac{y}{\varepsilon^2} \right) \right], \\ v^{(N)}(\mathbf{x}, y, t) &= \sum_{j=0}^N \varepsilon^{j+1} \left[v_j \left(x, \frac{y}{\varepsilon} \right) + \tilde{v}_{j+1} \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2}, \frac{y}{\varepsilon^2} \right) \right], \\ p^{(N)}(\mathbf{x}, y, t) &= \sum_{j=0}^N \varepsilon^{j-2} \left[p_j \left(\mathbf{x}, \frac{y}{\varepsilon} \right) + \varepsilon \tilde{p}_{j+1} \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2}, \frac{y}{\varepsilon^2} \right) \right]. \end{aligned}$$

Each term of this expansion corresponds to the solution of a Reynolds problem or Stokes problem (which will be further discussed). More precisely, we will see that (\mathbf{u}_j, v_j, p_j) is the solution of a Reynolds-type problem. This solution being extended in the boundary layer, this leads to a perturbation of the no-slip boundary condition on the shearing (bottom) surface. Thus, the exact boundary condition is not satisfied and we have to impose a correction ; this is the role of $(\tilde{\mathbf{u}}_{j+1}, \tilde{v}_{j+1}, \tilde{p}_{j+1})$ which is the solution of a Stokes problem in an unbounded (semi-infinite) domain. As a consequence, the behaviour of the Stokes solution, as $Y \rightarrow +\infty$, is such that it defines a perturbation of the zero no-slip boundary condition at the top of the domain and, thus, this will be taken into account in the definition of the elementary solution at next order, in order to balance all the effects related to the successive perturbations of the flow and boundary conditions.

Let us mention that the expansion includes the definition of a remainder

$$(\mathcal{R}^{(N)}, \mathcal{S}^{(N)}, \mathcal{Q}^{(N)})$$

which, by means of subtraction, is proven to satisfy a Stokes problem (with source terms) in the “physical” domain. A major task of this work is to derive some bounds on the remainder (with respect to ε) in order to prove in a rigorous way that the asymptotic expansion is valid.

Before describing the systems satisfied by the previous terms, let us highlight that difficulties are twofold: not only well-posedness of the elementary problems is a major task, but also suitable definition of these elementary problems is crucial: in the range of difficulty, it can be viewed as the most important point of the analysis, as it enhances to include all the corrective properties of the expansion by keeping the well-posedness properties of the elementary problems and feasibility of a numerical procedure (algorithm) for the computation of the solution.

2.3 Order 0 and first correction at order 1

We first describe the systems satisfied by the main contributions of the flow. We put the *ansatz*, see Eq. (1), into the Stokes system.

- *Horizontal components of the velocity field.* For the first equation of the initial system, we obtain the following expression with respect to the ε powers:

$$\begin{aligned} \mathbf{0} &= -\partial_x^2 \mathbf{u} - \partial_y^2 \mathbf{u} + \partial_x p \\ &= \varepsilon^{-3} \left(-\Delta_{\mathbf{X}} \tilde{\mathbf{u}}_1 - \partial_Y^2 \tilde{\mathbf{u}}_1 + \nabla_{\mathbf{X}} \tilde{p}_1 \right) \\ &\quad + \varepsilon^{-2} \left(-\partial_Z^2 \mathbf{u}_0 + \nabla_{\mathbf{x}} p_0 - \Delta_{\mathbf{X}} \tilde{\mathbf{u}}_2 - \partial_Y^2 \tilde{\mathbf{u}}_2 + \nabla_{\mathbf{X}} \tilde{p}_2 \right) + \mathcal{O}(\varepsilon^{-1}). \end{aligned}$$

This decomposition allows us to propose the following equation

$$-\Delta_{\mathbf{X}} \tilde{\mathbf{u}}_1 - \partial_Y^2 \tilde{\mathbf{u}}_1 + \nabla_{\mathbf{X}} \tilde{p}_1 = \mathbf{0},$$

and separating the variables $(\mathbf{x}, \mathbf{X}, Y)$ and the variables (\mathbf{x}, Z) , we propose the following equations

$$-\partial_Z^2 \mathbf{u}_0 + \nabla_{\mathbf{x}} p_0 = -\mathbf{A}_0 \quad \text{and} \quad -\Delta_{\mathbf{x}} \tilde{\mathbf{u}}_2 - \partial_Y^2 \tilde{\mathbf{u}}_2 + \nabla_{\mathbf{x}} \tilde{p}_2 = \mathbf{A}_0,$$

where function \mathbf{A}_0 may only depend on common variable \mathbf{x} .

- *Vertical component of the velocity field.* For the second equation of the initial system, we obtain the following expression with respect to the ε powers:

$$0 = -\partial_x^2 v - \partial_y^2 v + \partial_y p = \varepsilon^{-3} \left(-\Delta_{\mathbf{x}} \tilde{v}_1 - \partial_Y^2 \tilde{v}_1 + \partial_Y \tilde{p}_1 + \partial_Z p_0 \right) + \mathcal{O}(\varepsilon^{-2}).$$

Separating the variables $(\mathbf{x}, \mathbf{X}, Y)$ and the variables (\mathbf{x}, Z) again, we obtain

$$\partial_Z p_0 = -B_0 \quad \text{and} \quad -\Delta_{\mathbf{x}} \tilde{v}_1 - \partial_Y^2 \tilde{v}_1 + \partial_Y \tilde{p}_1 = B_0,$$

where function B_0 may only depend on common variable \mathbf{x} .

- *Free divergence equation.* The third equation of the initial system reads

$$\begin{aligned} 0 &= \operatorname{div}_{\mathbf{x}} \mathbf{u} + \partial_Z v \\ &= \varepsilon^{-1} \left(\operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_1 + \partial_Y \tilde{v}_1 \right) + \varepsilon^0 \left(\operatorname{div}_{\mathbf{x}} \mathbf{u}_0 + \partial_Z v_0 + \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_2 + \partial_Y \tilde{v}_2 \right) + \mathcal{O}(\varepsilon^1). \end{aligned}$$

This justifies the following equation

$$\operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_1 + \partial_Y \tilde{v}_1 = 0,$$

and the definition of a function C_0 which only depends on variable \mathbf{x} such that

$$\operatorname{div}_{\mathbf{x}} \mathbf{u}_0 + \partial_Z v_0 = -C_0 \quad \text{and} \quad \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_2 + \partial_Y \tilde{v}_2 = C_0.$$

- *Boundary conditions on Γ_{ε}^+ .* We transcribe the *ansatz* for $z = \varepsilon h^+(\mathbf{x})$. For the horizontal velocity \mathbf{u} we obtain

$$\mathbf{0} = \mathbf{u}_0(\mathbf{x}, h^+(\mathbf{x})) + \varepsilon \tilde{\mathbf{u}}_1 \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2}, \frac{h^+(\mathbf{x})}{\varepsilon} \right) + \varepsilon \mathbf{u}_1(\mathbf{x}, h^+(\mathbf{x})) + \mathcal{O}(\varepsilon^2).$$

This computation leads to

$$\mathbf{u}_0(\mathbf{x}, h^+(\mathbf{x})) = \mathbf{0} \quad \text{and} \quad \mathbf{u}_1(\mathbf{x}, h^+(\mathbf{x})) = -\lim_{\varepsilon \rightarrow 0} \tilde{\mathbf{u}}_1 \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2}, \frac{h^+(\mathbf{x})}{\varepsilon} \right).$$

Actually, the boundary condition will be analyzed in a more explicit way, as we will impose, in fact, the following approximate boundary condition

$$\mathbf{u}_1(\mathbf{x}, h^+(\mathbf{x})) = -\lim_{Y \rightarrow \infty} \int_{\mathbb{T}^d} \tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{X} - \mathbf{s}T, Y) \, d\mathbf{X},$$

which corresponds to the previous boundary condition, up to the scaling procedure. In the same way, for the vertical velocity component v , we obtain

$$0 = \varepsilon v_0(\mathbf{x}, h^+(\mathbf{x})) + \varepsilon \tilde{v}_1 \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2}, \frac{h^+(\mathbf{x})}{\varepsilon} \right) + \mathcal{O}(\varepsilon^2).$$

This computation leads to

$$v_0(\mathbf{x}, h^+(\mathbf{x})) = - \lim_{\varepsilon \rightarrow 0} \tilde{v}_1 \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2}, \frac{h^+(\mathbf{x})}{\varepsilon} \right),$$

which will be translated into

$$v_0(\mathbf{x}, h^+(\mathbf{x})) = - \lim_{Y \rightarrow \infty} \int_{\mathbb{T}^d} \tilde{v}_1(\mathbf{x}, \mathbf{X} - \mathbf{s}T, Y) \, d\mathbf{X}.$$

- *Boundary conditions on Γ_ε^- .* We extend the solution (\mathbf{u}_0, v_0, p_0) in the boundary layer Ω_ε^- of the domain Ω_ε . Indeed, the natural boundary conditions at the bottom for the velocity (\mathbf{u}_0, v_0) are given on γ_0 corresponding to the fictitious boundary $\{Z = 0\}$:

$$\mathbf{u}_0 = \mathbf{s} \quad \text{and} \quad v_0 = 0 \quad \text{on } \gamma_0.$$

Next, noticing that the Reynolds solution (\mathbf{u}_0, v_0, p_0) is polynomial with respect to the vertical variable Z , we can consider that it is defined and regular on $\omega_R \cup (\mathbb{T} \times \mathbb{R}^-)$. More precisely, we have (using for instance a Taylor formula and the degree of the polynomials), for all $(\mathbf{x}, Z) \in \mathbb{T} \times \mathbb{R}^-$

$$\begin{aligned} \mathbf{u}_0(\mathbf{x}, Z) &= \mathbf{s} + Z \partial_Z \mathbf{u}_0(\mathbf{x}, 0) + \frac{Z^2}{2} \partial_Z^2 \mathbf{u}_0(\mathbf{x}, 0) + \frac{Z^3}{3!} \partial_Z^3 \mathbf{u}_0(\mathbf{x}, 0), \\ v_0(\mathbf{x}, Z) &= Z \partial_Z v_0(\mathbf{x}, 0) + \frac{Z^2}{2} \partial_Z^2 v_0(\mathbf{x}, 0) + \frac{Z^3}{3!} \partial_Z^3 v_0(\mathbf{x}, 0) + \frac{Z^4}{4!} \partial_Z^4 v_0(\mathbf{x}, 0). \end{aligned}$$

In this way, the extended function satisfies the Stokes system in the whole domain. Besides, we deduce that, at the boundary Γ_ε^- , we obtain

$$\begin{aligned} \mathbf{u}_0 \left(\mathbf{x}, -\varepsilon h^- \left(\frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2} \right) \right) &= \mathbf{s} - \varepsilon h^- \left(\frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2} \right) \partial_Z \mathbf{u}_0(\mathbf{x}, 0) + \mathcal{O}(\varepsilon^2), \\ v_0 \left(\mathbf{x}, -\varepsilon h^- \left(\frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2} \right) \right) &= -\varepsilon h^- \left(\frac{\mathbf{x} - \mathbf{s}t}{\varepsilon^2} \right) \partial_Z v_0(\mathbf{x}, 0) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Remark 2.1 *It is important to notice that the next order term in this development with respect to ε , that is term of order ε^2 , will be offset in the boundary layer by the next terms such as $\tilde{\mathbf{u}}_2$ (so that we should not forget those terms later in the development). In practice, we will show that the constant (w.r.t. Z) \mathbf{A}_0 , B_0 and C_0 are zero so that the horizontal velocity \mathbf{u}_0 is a polynomial of degree 2 in the variable Z , and the vertical velocity v_0 is a polynomial of degree 3.*

In the same way, we will build u_1 as the solution of a Reynolds-type problem satisfying the boundary conditions

$$\mathbf{u}_1(\mathbf{x}, h^+(\mathbf{x})) = - \lim_{Y \rightarrow \infty} \int_{\mathbb{T}^d} \tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{X} - sT, Y) d\mathbf{X}, \quad \mathbf{u}_1(\mathbf{x}, 0) = \mathbf{0}.$$

The combination of the homogeneous Dirichlet condition on γ_0 and the extension of u_1 to Ω_ε^- leads to the following property:

$$\mathbf{u}_1\left(\mathbf{x}, -\varepsilon h^-\left(\frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}\right)\right) = \mathcal{O}(\varepsilon).$$

Plugging this development in the *ansatz* on Γ_ε^- , we obtain

$$\begin{aligned} \mathbf{s} &= \mathbf{u}_0\left(\mathbf{x}, -\varepsilon h^-\left(\frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}\right)\right) + \varepsilon \tilde{\mathbf{u}}_1\left(\mathbf{x}, \frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}, -h^-\left(\frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}\right)\right) \\ &\quad + \varepsilon \mathbf{u}_1\left(\mathbf{x}, -\varepsilon h^-\left(\frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}\right)\right) + \mathcal{O}(\varepsilon^2) \\ &= \mathbf{s} - \varepsilon h^-\left(\frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}\right) \partial_Z \mathbf{u}_0(\mathbf{x}, 0) + \varepsilon \tilde{\mathbf{u}}_1\left(\mathbf{x}, \frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}, -h^-\left(\frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}\right)\right) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Thus, we impose the boundary condition on $\gamma_{bl}(0)$:

$$\tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{X}, -h^-(\mathbf{X})) = h^-(\mathbf{X}) \partial_Z \mathbf{u}_0(\mathbf{x}, 0).$$

Concerning the boundary conditions on Γ_ε^- written for the vertical velocity v , we obtain

$$\begin{aligned} 0 &= \varepsilon v_0\left(\mathbf{x}, -\varepsilon h^-\left(\frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}\right)\right) + \varepsilon \tilde{v}_1\left(\mathbf{x}, \frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}, -h^-\left(\frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}\right)\right) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon \tilde{v}_1\left(\mathbf{x}, \frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}, -h^-\left(\frac{\mathbf{x} - s\mathbf{t}}{\varepsilon^2}\right)\right) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

We then impose the boundary condition on $\gamma_{bl}(0)$:

$$\tilde{v}_1(\mathbf{x}, \mathbf{X}, -h^-(\mathbf{X})) = 0.$$

Summarizing the previous decomposition procedure, we deduce the equations and the boundary conditions satisfied by the first terms of the *ansatz*:

- **Main flow at scale ε^0 : Reynolds flow in the thin film domain.**
The functions (\mathbf{u}_0, v_0, p_0) satisfy the classical Reynolds problem

$$(\mathbf{R}^{(0)}) \left\{ \begin{array}{ll} -\partial_Z^2 \mathbf{u}_0 + \nabla_{\mathbf{x}} p_0 &= -\mathbf{A}_0, \quad \text{on } \omega_R, \\ \partial_Z p_0 &= -B_0, \quad \text{on } \omega_R, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u}_0 + \partial_Z v_0 &= -C_0, \quad \text{on } \omega_R, \\ \mathbf{u}_0 &= \mathbf{s}, \quad \text{on } \gamma_0, \\ v_0 &= 0, \quad \text{on } \gamma_0, \\ \mathbf{u}_0 &= \mathbf{0}, \quad \text{on } \gamma_+, \\ v_0 &= -\beta_1, \quad \text{on } \gamma_+, \end{array} \right.$$

where the functions \mathbf{A}_0 , B_0 and C_0 only depend on the variable \mathbf{x} . We will see, *a posteriori*, that $\mathbf{A}_0 = \mathbf{0}$, $B_0 = 0$ and $C_0 = 0$. Coefficient β_1 will be related to the corrective procedure (although it will be proven to be independent from the corrective procedure), see Remark 2.6.

- **First correction at scale ε^1 : Stokes flow in the boundary layer.**
We first set $T = 0$ (the boundary problem can be defined in a similar way for any time $T \neq 0$). The functions $(\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1)$ satisfy a Stokes problem:

$$(S^{(1)}) \left\{ \begin{array}{l} -\Delta_{\mathbf{x}} \tilde{\mathbf{u}}_1 - \partial_Y^2 \tilde{\mathbf{u}}_1 + \nabla_{\mathbf{x}} \tilde{p}_1 = \mathbf{0}, \quad \text{on } \omega_{bl}(0), \\ -\Delta_{\mathbf{x}} \tilde{v}_1 - \partial_Y^2 \tilde{v}_1 + \partial_Y \tilde{p}_1 = B_0, \quad \text{on } \omega_{bl}(0), \\ \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_1 + \partial_Y \tilde{v}_1 = 0, \quad \text{on } \omega_{bl}(0), \\ \tilde{\mathbf{u}}_1 = \tilde{\mathbf{U}}_1, \quad \text{on } \gamma_{bl}(0), \\ \tilde{v}_1 = \tilde{V}_1, \quad \text{on } \gamma_{bl}(0), \\ (\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1) \text{ is } \mathbf{X}\text{-periodic,} \end{array} \right.$$

where the source term in the boundary condition should be read as

$$\tilde{\mathbf{U}}_1 : \mathbf{X} \rightarrow h^-(\mathbf{X}) \partial_Z \mathbf{u}_0(\mathbf{x}, 0) \quad \text{and} \quad \tilde{V}_1 \equiv 0.$$

The value of $\tilde{\mathbf{U}}_1$ is chosen as follows: the solution of the Reynolds problem ($R^{(0)}$) being initially defined for $Z > 0$, it is naturally defined on $Z < 0$ by means of the polynomial extension (as the solution of Problem ($R^{(0)}$) is polynomial in the Z variable). In this way, the Reynolds solution expressed in real variables satisfies the Stokes system in the whole real domain. Then the value of $\tilde{\mathbf{U}}_1$ corresponds to the value of (the extension on $Z < 0$ of) \mathbf{u}_0 , in rescaled variables.

Moreover, we will prove that the solutions $\tilde{\mathbf{u}}_1$ and \tilde{v}_1 of ($S^{(1)}$) satisfy

$$\begin{aligned} \lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} \quad \text{exists; it is denoted } \boldsymbol{\alpha}_1(\mathbf{x}), \\ \lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{v}_1(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} \quad \text{exists; it is denoted } \beta_1(\mathbf{x}). \end{aligned}$$

Remark 2.2 Since (\mathbf{u}_0, v_0, p_0) is the solution of the classical Reynolds equation (see the system ($R^{(0)}$) with $\mathbf{A}_0 = \mathbf{0}$, $B_0 = 0$ and $C_0 = 0$), we easily compute

$$\tilde{\mathbf{U}}_1 : \mathbf{X} \mapsto -h^-(\mathbf{X}) \left(\frac{h^+(\mathbf{x})}{2} \nabla_{\mathbf{x}} p_0(\mathbf{x}) + \frac{\mathbf{s}}{h^+(\mathbf{x})} \right).$$

In the same way, we will see that $\beta_1 = 0$ whereas $\boldsymbol{\alpha}_1 \neq \mathbf{0}$.

Remark 2.3 Notice in particular that the variables \mathbf{x} only plays the role of a parameter in this Stokes problem ($S^{(1)}$). This remark will be common to all the Stokes problems written in this part.

Remark 2.4 Rigorously, the boundary layer problem is time-dependent and should be defined as:

$$\left\{ \begin{array}{ll} -\Delta_{\mathbf{X}} \tilde{\mathbf{u}}_{[T]} - \partial_Y^2 \tilde{\mathbf{u}}_{[T]} + \nabla_{\mathbf{X}} \tilde{p}_{[T]} = \mathbf{0}, & \text{on } \omega_{bl}(T), \\ -\Delta_{\mathbf{X}} \tilde{v}_{[T]} - \partial_Y^2 \tilde{v}_{[T]} + \partial_Y \tilde{p}_{[T]} = B_0, & \text{on } \omega_{bl}(T), \\ \operatorname{div}_{\mathbf{X}} \tilde{\mathbf{u}}_{[T]} + \partial_Y \tilde{v}_{[T]} = 0, & \text{on } \omega_{bl}(T), \\ \tilde{\mathbf{u}}_{[T]} = \tilde{\mathbf{U}}_1(\cdot - \mathbf{s}T), & \text{on } \gamma_{bl}(T), \\ \tilde{v}_{[T]} = \tilde{V}_1(\cdot - \mathbf{s}T), & \text{on } \gamma_{bl}(T), \\ (\tilde{\mathbf{u}}_{[T]}, \tilde{v}_{[T]}, \tilde{p}_{[T]}) \text{ is } \mathbf{X}\text{-periodic.} \end{array} \right.$$

Notice that the so-called “initial boundary corrector” $(\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1)$ does not depend on time T , unlikely to the boundary corrector solution. But we now argue that the “general boundary corrector” $(\tilde{\mathbf{u}}_{[T]}, \tilde{v}_{[T]}, \tilde{p}_{[T]})$ (defined at time T) can be deduced from the “initial boundary corrector” thanks to the periodic structure of the bottom function h^- :

$$\left(\tilde{\mathbf{u}}_{[T]}, \tilde{v}_{[T]}, \tilde{p}_{[T]} \right)(\cdot, \mathbf{X}, Y) = \left(\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1 \right)(\cdot, \mathbf{X} - \mathbf{s}T, Y).$$

A similar remark can be made on all issues discussed later.

Remark 2.5 On the same way, the rigorous definition of α_1 (or of β_1), in the construction of the corrector system, should be the following one:

$$\alpha_1(\mathbf{x}) = \lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{X} - \mathbf{s}T, Y) d\mathbf{X}$$

which, actually, does not depend on T since $\tilde{\mathbf{u}}_1$ is periodic with respect to \mathbf{X} (using the change of variable $\mathbf{X}' = \mathbf{X} - \mathbf{s}T$).

Remark 2.6 The behaviour at infinity of the solution of the Stokes problem is such that

$$\lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{v}_1(\mathbf{x}, \mathbf{X}, Y) d\mathbf{X} := \beta_1(\mathbf{x}).$$

This limit value is exactly the one that has to be imposed in the definition of the previous Reynolds problem, namely $(\mathbf{R}^{(1)})$. At first glance, one may think that problems $(\mathbf{R}^{(1)})$ and $(\mathbf{S}^{(1)})$ are strongly coupled through the constant β_1 :

- constant β_1 is necessary to define the Reynolds problem ;
- constant β_1 results from the behaviour of the solution of the Stokes problem, whose data highly depend on the solution of the Reynolds problem.

As will be proven later, not only is each problem well-posed but also - this might be surprising - the two problems are NOT coupled, as β_1 will be proven to be independent from the Stokes problem ! See in particular Proposition 3.7 and also Subsection 3.3 (algorithm).

Due to the non-zero limit of the integral quantity $\int_{]0,1[^d} \tilde{\mathbf{u}}_1(\mathbf{x}, \mathbf{X}, Y) d\mathbf{X}$ as Y tends to $+\infty$, the contribution of $\tilde{\mathbf{u}}_1$ in the asymptotic development brings an error at the top boundary. That is corrected by the following contribution:

- **Second correction at scale ε^1 : Reynolds flow in the thin film domain.** The functions (\mathbf{u}_1, v_1, p_1) satisfy the Reynolds problem:

$$(R^{(1)}) \left\{ \begin{array}{l} -\partial_Z^2 \mathbf{u}_1 + \nabla_{\mathbf{x}} p_1 = -\mathbf{A}_1, \quad \text{on } \omega_R, \\ \partial_Z p_1 = -B_1, \quad \text{on } \omega_R, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u}_1 + \partial_Z v_1 = -C_1, \quad \text{on } \omega_R, \\ \mathbf{u}_1 = \mathbf{0}, \quad \text{on } \gamma_0, \\ v_1 = 0, \quad \text{on } \gamma_0, \\ \mathbf{u}_1 = -\boldsymbol{\alpha}_1, \quad \text{on } \gamma_+, \\ v_1 = -\beta_2, \quad \text{on } \gamma_+. \end{array} \right.$$

where the two functions \mathbf{A}_1 , B_1 and C_1 only depend on the variable \mathbf{x} and will be precise later (indeed, $\mathbf{A}_1 = \mathbf{0}$, $B_1 = 0$ and C_1 is given by Eq. (7)).

Now, we present the systems satisfied by the corrective terms in the asymptotic expansions: these contributions allow a better description of the initial Stokes flow by increasing the order of the approximation. Notice that the following system highly depends on the previous solutions as source terms.

2.4 Higher orders of the asymptotic expansion

Each order of precision is obtained using a Reynolds flow corresponding to the next order of the thin film assumption ; then the corrections due to the roughness patterns have to be taken into account. Notice that the solutions of the previous systems may play the role of source terms in the proposed corrections.

- **Correction at scale ε^j : Stokes flow in the boundary layer.**

For $2 \leq j \leq N + 1$, the functions $(\tilde{\mathbf{u}}_j, \tilde{v}_j, \tilde{p}_j)$ satisfy the classical Stokes problem:

$$(S^{(j)}) \left\{ \begin{array}{l} -\Delta_{\mathbf{x}} \tilde{\mathbf{u}}_j - \partial_Y^2 \tilde{\mathbf{u}}_j + \nabla_{\mathbf{x}} \tilde{p}_j = \tilde{\mathcal{F}}_j, \quad \text{on } \omega_{bl}(0), \\ -\Delta_{\mathbf{x}} \tilde{v}_j - \partial_Y^2 \tilde{v}_j + \partial_Y \tilde{p}_j = \tilde{\mathcal{G}}_j, \quad \text{on } \omega_{bl}(0), \\ \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_j + \partial_Y \tilde{v}_j = \tilde{\mathcal{H}}_j, \quad \text{on } \omega_{bl}(0), \\ \tilde{\mathbf{u}}_j = \tilde{\mathcal{U}}_j, \quad \text{on } \gamma_{bl}(0), \\ \tilde{v}_j = \tilde{\mathcal{V}}_j, \quad \text{on } \gamma_{bl}(0), \\ (\tilde{\mathbf{u}}_j, \tilde{v}_j, \tilde{p}_j) \text{ is } \mathbf{X}\text{-periodic,} \end{array} \right.$$

where the boundary conditions are related to

$$\begin{aligned} \tilde{\mathcal{U}}_j : \mathbf{X} &\rightarrow - \sum_{k=1}^{[\frac{j+1}{2}]+1} \frac{(-1)^k h^-(\mathbf{X})^k}{k!} \partial_Z^k \mathbf{u}_{j-k}(\mathbf{x}, 0), \\ \tilde{\mathcal{V}}_j : \mathbf{X} &\rightarrow -h^-(\mathbf{X}) C_{j-2}(\mathbf{x}) + \sum_{k=2}^{[\frac{j}{2}]+2} \frac{(-1)^k h^-(\mathbf{X})^k}{k!} \operatorname{div}_{\mathbf{x}} \partial_Z^{k-1} \mathbf{u}_{j-k-1}(\mathbf{x}, 0). \end{aligned}$$

The source terms are defined by

$$\begin{aligned}\tilde{\mathcal{F}}_j &: (\mathbf{X}, Y) \rightarrow \mathbf{A}_{j-2} + (2\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \tilde{\mathbf{u}}_{j-2} + \Delta_{\mathbf{x}} \tilde{\mathbf{u}}_{j-4} - \nabla_{\mathbf{x}} \tilde{p}_{j-2})(\cdot, \mathbf{X}, Y), \\ \tilde{\mathcal{G}}_j &: (\mathbf{X}, Y) \rightarrow B_{j-1} + (2\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \tilde{v}_{j-2} + \Delta_{\mathbf{x}} \tilde{v}_{j-4})(\cdot, \mathbf{X}, Y), \\ \tilde{\mathcal{H}}_j &: (\mathbf{X}, Y) \rightarrow C_{j-2} - \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_{j-2}(\cdot, \mathbf{X}, Y).\end{aligned}$$

The value of $\tilde{\mathbf{u}}_j$ and \tilde{v}_j is chosen as follows: the solution of the Reynolds problem $(\mathbf{R}^{(j-1)})$ being initially defined for $Z > 0$, it is naturally defined on $Z < 0$ by means of the polynomial extension (as the solution of Problem $(\mathbf{R}^{(j-1)})$ is polynomial in the Z variable). In this way, the Reynolds solution expressed in real variables satisfies the Stokes system in the whole real domain. Then the value of $\tilde{\mathbf{u}}_j$ and \tilde{v}_j corresponds to the value of (the extension on $Z < 0$ of) \mathbf{u}_{j-1} and v_{j-1} , in rescaled variables.

Moreover, we will prove that, using a good choice for the values \mathbf{A}_{j-2} , B_{j-1} and C_{j-2} (see Eq. (5)–(7)), the solutions $\tilde{\mathbf{u}}_j$ and \tilde{v}_j of $(\mathbf{S}^{(j)})$ satisfy

$$\begin{aligned}\lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{\mathbf{u}}_j(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} &\text{ exists; it is denoted } \boldsymbol{\alpha}_j(\mathbf{x}), \\ \lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{v}_j(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} &\text{ exists; it is denoted } \beta_j(\mathbf{x}).\end{aligned}$$

Remark 2.7 Note that for small values of the integer j , the expressions of the source terms are lightly different. In fact, by convention we must read 0 when a term is not defined. For example, $\tilde{\mathcal{F}}_2 = \mathbf{A}_0$ since $\tilde{\mathbf{u}}_0 = \mathbf{0}$, $\tilde{\mathbf{u}}_{-2} = \mathbf{0}$ and $\tilde{p}_0 = 0$.

Remark 2.8 The boundary terms given by $\tilde{\mathbf{u}}_j$ and \tilde{v}_j come from to the error due to the extension of all the previous interior terms \mathbf{u}_k and v_k , $k < j$, see their extensions (14) on page 31.

◦ **Main flow at scale ε^j : Reynolds flow in the thin film domain.**

For $2 \leq j \leq N$, the functions (\mathbf{u}_j, v_j, p_j) satisfy the Reynolds problem:

$$(\mathbf{R}^{(j)}) \left\{ \begin{array}{ll} -\partial_Z^2 \mathbf{u}_j + \nabla_{\mathbf{x}} p_j = \mathcal{F}_j, & \text{on } \omega_R, \\ \partial_Z p_j = \mathcal{G}_j, & \text{on } \omega_R, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u}_j + \partial_Z v_j = \mathcal{H}_j, & \text{on } \omega_R, \\ \mathbf{u}_j = \mathbf{0}, & \text{on } \gamma_0, \\ v_j = 0, & \text{on } \gamma_0, \\ \mathbf{u}_j = -\boldsymbol{\alpha}_j, & \text{on } \gamma_+, \\ v_j = -\beta_{j+1}, & \text{on } \gamma_+, \end{array} \right.$$

with the general source terms

$$\begin{aligned}\mathcal{F}_j &: (\mathbf{x}, Z) \rightarrow -\mathbf{A}_j(\mathbf{x}) + \Delta_{\mathbf{x}} \mathbf{u}_{j-2}(\mathbf{x}, Z), \\ \mathcal{G}_j &: (\mathbf{x}, Z) \rightarrow -B_j(\mathbf{x}) + \partial_Z^2 v_{j-2}(\mathbf{x}, Z) + \Delta_{\mathbf{x}} v_{j-4}(\mathbf{x}, Z), \\ \mathcal{H}_j &: \mathbf{x} \rightarrow -C_j(\mathbf{x}).\end{aligned}$$

Remark 2.9 As previously, note that for small values of the integer j , the expressions of the source terms are slightly different. By convention we must read 0 when a term is not defined. For example $\mathcal{G}_2 = -B_2 + \partial_Z^2 v_0$ since $v_{-2} = 0$.

Remark 2.10 Coefficient β_{j+1} should be related to the corrective procedure (although it will be proven to be independent from the corrective procedure) at the next step. More precisely, the behaviour at infinity of the solution of the Stokes problem $(S^{(j)})$ is such that

$$\lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{v}_{j+1}(\mathbf{x}, \mathbf{X}, Y) d\mathbf{X} := \beta_{j+1}(\mathbf{x}).$$

This limit value is exactly the one that has to be imposed in the definition of the Reynolds problem $(R^{(j)})$. At first glance, one may think that, by means of construction, problems $(R^{(j+1)})$ and $(S^{(j+1)})$ are strongly coupled through the constant β_{j+1} . As will be proven later, not only is each problem well-posed but also - this might be surprising - the two problems are NOT coupled, as β_{j+1} will be proven to be independent from the Stokes problem! See in particular Proposition 3.7 and also Subsection 3.3 (algorithm).

Then, subtracting the asymptotic expansion from the initial solution, we easily find that the remainder should satisfy a Stokes system in the initial domain, with source terms which highly depend on the above solutions, see part 4. In order to make the previous asymptotic expansion rigorous, we will have to control the remainder. Before entering into the details of the definition and control of the remainder, let us describe the mathematical properties of the Reynolds-type and Stokes-type problems which have been presented in this section.

3 Mathematical results related to the different scale problems

3.1 Stokes problems: well-posedness and behaviour of the solutions

In this section, we show that the Stokes problems $(S^{(j)})$ introduced above are well posed. Moreover, we prove that for a suitable choice of the “constants” \mathbf{A}_j , B_j and C_j , the limits

$$\lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{\mathbf{u}}_j(\mathbf{x}, \mathbf{X}, Y) d\mathbf{X} \quad \text{and} \quad \lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{v}_j(\mathbf{x}, \mathbf{X}, Y) d\mathbf{X},$$

do exist.

We present a result (see Proposition 3.2) whose interest is twofold: i) it allows us to obtain a well-posedness result on the Stokes problems $(S^{(j)})$ by using a

lift procedure and classical results on Stokes systems in semi-infinite domains ;
ii) it allows us to explain how the boundary conditions and the source term in the divergence equation can be translated into boundary conditions on the plane $Y = 0$.

Definition 3.1 Let $\tilde{\mathcal{H}} \in \mathcal{C}_{per}(\omega_{bl}, \mathbb{R})$, $\tilde{\mathbf{u}} \in \mathcal{C}_{per}([0, 1]^d, \mathbb{R}^d)$ and $\tilde{\mathcal{V}} \in \mathcal{C}_{per}([0, 1]^d, \mathbb{R})$. Consider a solution $(\tilde{\mathbf{u}}, \tilde{v})$ of the following problem

$$\left\{ \begin{array}{ll} \operatorname{div}_{\mathbf{X}} \tilde{\mathbf{u}} + \partial_Y \tilde{v} = \tilde{\mathcal{H}}, & \text{on } \omega_{bl}, \\ \tilde{\mathbf{u}} = \tilde{\mathbf{u}}, & \text{on } \gamma_{bl}, \\ \tilde{v} = \tilde{\mathcal{V}}, & \text{on } \gamma_{bl}, \\ (\tilde{\mathbf{u}}, \tilde{v}) \text{ is } \mathbf{X}\text{-periodic.} \end{array} \right. \quad (2)$$

Then we define the linear operators

$$\begin{aligned} L_u : (\tilde{\mathcal{H}}, \tilde{\mathbf{u}}, \tilde{\mathcal{V}}) &\longmapsto \int_{]0, 1[^d} \tilde{\mathbf{u}}(\mathbf{X}, 0) \, d\mathbf{X} \in \mathbb{R}^d, \\ L_v : (\tilde{\mathcal{H}}, \tilde{\mathbf{u}}, \tilde{\mathcal{V}}) &\longmapsto \int_{]0, 1[^d} \tilde{v}(\mathbf{X}, 0) \, d\mathbf{X} \in \mathbb{R}. \end{aligned}$$

Remark 3.1 The existence of such a solution to equation (2) immediately follows from the fact that for all function $\tilde{\mathbf{u}}$ on ω_{bl} such that $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}$ on $Y = -h^-(\mathbf{X})$, the couple

$$\left(\tilde{\mathbf{u}}(\mathbf{X}, Y), \tilde{\mathcal{V}}(\mathbf{X}) + \int_{-h^-(\mathbf{X})}^Y (\tilde{\mathcal{H}} - \operatorname{div}_{\mathbf{X}} \tilde{\mathbf{u}})(\mathbf{X}, \zeta) \, d\zeta \right)$$

defines a solution of (2).

We will see that for practical cases, the velocity imposed at the bottom are peculiar form. We will use the following proposition.

Proposition 3.2 Let $\mathbf{f} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d)$, $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and $\tilde{\mathcal{H}} \in \mathcal{C}_{per}(\omega_{bl}, \mathbb{R})$. We have

$$L_v \left(\tilde{\mathcal{H}}, \mathbf{f}(h^-(\mathbf{X})), f(h^-(\mathbf{X})) \right) = - \int_{\{Y < 0\}} \tilde{\mathcal{H}}(\mathbf{X}, Y) \, d\mathbf{X} dY - \int_{]0, 1[^d} f(h^-(\mathbf{X})) \, d\mathbf{X}.$$

Proof. We apply the Green's formula :

$$\begin{aligned} \int_{\{Y < 0\}} \tilde{\mathcal{H}}(\mathbf{X}, Y) \, d\mathbf{X} dY &= \int_{\{Y < 0\}} (\operatorname{div}_{\mathbf{X}} \tilde{\mathbf{u}} + \partial_Y \tilde{v}) \\ &= \int_{]0, 1[^d} \begin{pmatrix} \tilde{\mathbf{u}}(\mathbf{X}, -h^-(\mathbf{X})) \\ \tilde{v}(\mathbf{X}, -h^-(\mathbf{X})) \end{pmatrix} \cdot \begin{pmatrix} -\nabla_{\mathbf{X}} h^-(\mathbf{X}) \\ -1 \end{pmatrix} \, d\mathbf{X} \\ &\quad + \int_{]0, 1[^d} \begin{pmatrix} \tilde{\mathbf{u}}(\mathbf{X}, 0) \\ \tilde{v}(\mathbf{X}, 0) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} (-d\mathbf{X}). \end{aligned}$$

As $\tilde{\mathbf{u}}$ and \tilde{v} have a particular shape on the bottom boundary, a simple computation leads to

$$\int_{\{Y < 0\}} \tilde{\mathcal{H}}(\mathbf{X}, Y) \, d\mathbf{X} \, dY = - \int_{]0, 1[^d} f(h^-(\mathbf{X})) \, d\mathbf{X} - \int_{\gamma_{bl}} \mathbf{f}(h^-(\mathbf{X})) \cdot \nabla_{\mathbf{X}} h^-(\mathbf{X}) \, d\mathbf{X} - \int_{]0, 1[^d} \tilde{v}(\mathbf{X}, 0) \, d\mathbf{X}$$

By periodicity of function h^- , we have

$$\int_{\gamma_{bl}} \mathbf{f}(h^-(\mathbf{X})) \cdot \nabla_{\mathbf{X}} h^-(\mathbf{X}) \, d\mathbf{X} = \int_{\gamma_{bl}} \nabla_{\mathbf{X}} \mathbf{F}(h^-(\mathbf{X})) \, d\mathbf{X} = 0,$$

where \mathbf{F} is a primitive of \mathbf{f} . That concludes the proof. @blacksquare

As a consequence of Proposition 3.2, it is possible to define a lift procedure, so that problem $(S^{(j)})$ reduces to an associated Stokes problem with free-divergence and homogeneous boundary conditions. Well-posedness of such a Stokes problem is well-known (see [15, 16, 17, 23] which provide the functional framework).

In the sequel, we focus on the behaviour at infinity of the solution of problem $(S^{(j)})$. This analysis relies on an iterative process.

3.1.1 Initialization step: analysis of problem $(S^{(1)})$

We properly define the Stokes problem $(S^{(1)})$ introduced on page 16 so that it is well-posed and the behaviour of the solution is controlled as $Y \rightarrow +\infty$.

Lemma 3.3 *There exist source term B_0 (in fact $B_0 = 0$) such that the system $(S^{(1)})$ admits a solution which is written, for all $(\mathbf{X}, Y) \in]0, 1[^d \times]0, +\infty[$,*

$$\begin{cases} \tilde{\mathbf{u}}_1(\mathbf{X}, Y) = L_u(0, \tilde{\mathbf{u}}_1, \tilde{\mathcal{V}}_1) + \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \mathbf{P}_{\mathbf{k}}^{(1)}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}} \\ \tilde{v}_1(\mathbf{X}, Y) = L_v(0, \tilde{\mathbf{u}}_1, \tilde{\mathcal{V}}_1) + \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} Q_{\mathbf{k}}^{(1)}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}} \\ \tilde{p}_1(\mathbf{X}, Y) = \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} R_{\mathbf{k}}^{(1)}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}} \end{cases}$$

where $\mathbf{P}_{\mathbf{k}}^{(1)}$, $Q_{\mathbf{k}}^{(1)}$ and $R_{\mathbf{k}}^{(1)}$ are affine functions.

It is important to notice that the \mathbf{X} -average on $\tilde{\mathbf{u}}_1$ (resp. v_1) does not depend on Y . We deduce that its limit when Y tends to $+\infty$, denoted α_1 (resp. β_1), satisfies

$$\alpha_1 = L_u(0, \tilde{\mathbf{u}}_1, \tilde{\mathcal{V}}_1) \quad (\text{resp. } \beta_1 = L_v(0, \tilde{\mathbf{u}}_1, \tilde{\mathcal{V}}_1)).$$

As a straightforward consequence, we get the following property (notice that, for the pressure, polynomial functions $R_1^{(1)}$ and $R_{-1}^{(1)}$ will be considered as constant, in the proof):

Corollary 3.4 *The solution of $(S^{(1)})$ satisfies:*

$$\begin{aligned} \|\tilde{\mathbf{u}}_1(\mathbf{X}, Y) - \boldsymbol{\alpha}_1\| &\leq C(\delta) e^{-\delta Y} & Y > 0, \quad \forall \delta < 2\pi, \\ |\tilde{v}_1(\mathbf{X}, Y) - \beta_1| &\leq C(\delta) e^{-\delta Y} & Y > 0, \quad \forall \delta < 2\pi, \\ |\tilde{p}_1(\mathbf{X}, Y)| &\leq C e^{-2\pi Y} & Y > 0, \end{aligned}$$

where the constant $C(\delta)$ only depends on δ .

Proof.(of Lemma 3.3) The existence of a solution to the Stokes problem $(S^{(1)})$ as follows is usual, see for instance [1, 17]. Moreover, if $(\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1)$ is a solution of $(S^{(1)})$ satisfying $\nabla \tilde{\mathbf{u}}_1, \nabla \tilde{v}_1, \tilde{p}_1 \in L^2(\]0, 1[^d \times (0, +\infty))$, then it is also a solution of:

$$\left\{ \begin{array}{ll} -\Delta_{\mathbf{X}} \tilde{\mathbf{u}}_1 - \partial_Y^2 \tilde{\mathbf{u}}_1 + \nabla_{\mathbf{X}} \tilde{p}_1 = \mathbf{0}, & \text{on } \omega_{bl} \cap \{Y > 0\}, \\ -\Delta_{\mathbf{X}} \tilde{v}_1 - \partial_Y^2 \tilde{v}_1 + \partial_Y \tilde{p}_1 = B_0, & \text{on } \omega_{bl} \cap \{Y > 0\}, \\ \operatorname{div}_{\mathbf{X}} \tilde{\mathbf{u}}_1 + \partial_Y \tilde{v}_1 = 0, & \text{on } \omega_{bl} \cap \{Y > 0\}, \\ \int_{]0, 1[^d} \tilde{\mathbf{u}}_1(\mathbf{X}, 0) \, d\mathbf{X} = L_u(0, \tilde{\mathbf{U}}_1, \tilde{V}_1), & \\ \int_{]0, 1[^d} \tilde{v}_1(\mathbf{X}, 0) \, d\mathbf{X} = L_v(0, \tilde{\mathbf{U}}_1, \tilde{V}_1), & \\ \int_{]0, 1[^d \times (0, +\infty)} \tilde{p}_1(\mathbf{X}, Y) \, d\mathbf{X} \, dY = 0, & \\ (\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1) \text{ is } \mathbf{X}\text{-periodic.} & \end{array} \right.$$

Remark 3.2 *In all the Stokes problems which appear in this paper, the pressures are given up to an additive constant. This constant is chosen here such that*

$$\int_{]0, 1[^d \times (0, +\infty)} \tilde{p}(\mathbf{X}, Y) \, d\mathbf{X} \, dY = 0.$$

This allows us to pass to the Fourier transform with respect to \mathbf{X} :

$$\begin{aligned} \tilde{\mathbf{u}}_1(\mathbf{X}, Y) &= \sum_{\mathbf{k} \in \mathbb{Z}^d} \widehat{\mathbf{u}}_{\mathbf{k}}(Y) e^{2\pi i \mathbf{k} \cdot \mathbf{X}}, & \tilde{v}_1(\mathbf{X}, Y) &= \sum_{\mathbf{k} \in \mathbb{Z}^d} \widehat{v}_{\mathbf{k}}(Y) e^{2\pi i \mathbf{k} \cdot \mathbf{X}}, \\ \tilde{p}_1(\mathbf{X}, Y) &= \sum_{\mathbf{k} \in \mathbb{Z}^d} \widehat{p}_{\mathbf{k}}(Y) e^{2\pi i \mathbf{k} \cdot \mathbf{X}}. \end{aligned}$$

The previous system on $(\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1)$ is translated into

$$\left\{ \begin{array}{ll} (2\pi)^2 \|\mathbf{k}\|^2 \widehat{\mathbf{u}}_{\mathbf{k}} - \widehat{\mathbf{u}}_{\mathbf{k}}'' + 2\pi i \mathbf{k} \widehat{p}_{\mathbf{k}} = \mathbf{0} & \text{on } \{Y > 0\} \quad \forall \mathbf{k} \in \mathbb{Z}^d \\ (2\pi)^2 \|\mathbf{k}\|^2 \widehat{v}_{\mathbf{k}} - \widehat{v}_{\mathbf{k}}'' + \widehat{p}_{\mathbf{k}}' = B_0 \delta_{\mathbf{k}=\mathbf{0}} & \text{on } \{Y > 0\} \quad \forall \mathbf{k} \in \mathbb{Z}^d \\ 2\pi i \mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}} + \widehat{v}_{\mathbf{k}}' = 0 & \text{on } \{Y > 0\} \quad \forall \mathbf{k} \in \mathbb{Z}^d \\ \widehat{\mathbf{u}}_{\mathbf{0}}(0) = L_u(0, \tilde{\mathbf{U}}_1, \tilde{V}_1) & \\ \widehat{v}_{\mathbf{0}}(0) = L_v(0, \tilde{\mathbf{U}}_1, \tilde{V}_1) & \\ \int_0^{+\infty} \widehat{p}_{\mathbf{0}}(Y) \, dY = 0 & \end{array} \right. \quad (3)$$

where $\widehat{\mathbf{u}}_{\mathbf{k}}', \widehat{v}_{\mathbf{k}}'$ and $\widehat{p}_{\mathbf{k}}$ belong to $L^2(0, +\infty)$. Now we solve the Fourier problem and describe the behaviour of the solution of the Stokes problem.

- For $\mathbf{k} = \mathbf{0}$, the system reduces to

$$\left\{ \begin{array}{ll} -\widehat{\mathbf{u}}_0'' = \mathbf{0} & \text{on } \{Y > 0\} \\ -\widehat{v}_0'' + \widehat{p}_0' = B_0 & \text{on } \{Y > 0\} \\ \widehat{v}_0' = 0 & \text{on } \{Y > 0\} \\ \widehat{\mathbf{u}}_0(0) = L_u(0, \widetilde{\mathcal{U}}_1, \widetilde{\mathcal{V}}_1) & \\ \widehat{v}_0(0) = L_v(0, \widetilde{\mathcal{U}}_1, \widetilde{\mathcal{V}}_1) & \\ \int_0^{+\infty} \widehat{p}_0(Y) dY = 0 & \end{array} \right.$$

Then, as we look for a solution in a suitable space, namely

$$\widehat{\mathbf{u}}_0' \in L^2(0, +\infty), \quad \widehat{v}_0' \in L^2(0, +\infty), \quad \widehat{p}_0 \in L^2(0, +\infty),$$

this leads us to the following equalities $\widehat{\mathbf{u}}_0 = L_u(0, \widetilde{\mathcal{U}}_1, \widetilde{\mathcal{V}}_1)$, $\widehat{v}_0 = L_v(0, \widetilde{\mathcal{U}}_1, \widetilde{\mathcal{V}}_1)$, $\widehat{p}_0 = 0$ with the choice

$$B_0 = 0. \quad (4)$$

- For $\mathbf{k} \neq \mathbf{0}$, we proceed as follows. The idea is to decompose $\widehat{\mathbf{u}}_{\mathbf{k}}$ as the sum

$$\widehat{\mathbf{u}}_{\mathbf{k}} = (\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}}) \mathbf{k} + (\mathbf{k}^\perp \cdot \widehat{\mathbf{u}}_{\mathbf{k}}) \mathbf{k}^\perp.$$

- Taking the scalar product with \mathbf{k}^\perp of the first equation of the system (3), we immediately deduce that

$$(2\pi)^2 \|\mathbf{k}\|^2 (\mathbf{k}^\perp \cdot \widehat{\mathbf{u}}_{\mathbf{k}}) - (\mathbf{k}^\perp \cdot \widehat{\mathbf{u}}_{\mathbf{k}})'' = \mathbf{0} \quad \text{on } \{Y > 0\} \quad \forall \mathbf{k} \in \mathbb{Z}^d.$$

Since we have $\mathbf{k}^\perp \cdot \widehat{\mathbf{u}}_{\mathbf{k}} \in L^2(0, +\infty)$ then it takes the form $a_{\mathbf{k}} e^{-2\pi \|\mathbf{k}\| Y}$ with $a_{\mathbf{k}} \in \mathbb{R}$.

- Now, taking the scalar product with \mathbf{k} of the first equation of the system (3), we obtain the pressure with respect to the quantity $\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}}$:

$$\widehat{p}_{\mathbf{k}} = 2\pi i (\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}}) - \frac{i}{2\pi \|\mathbf{k}\|^2} (\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}})''.$$

Moreover, using the third equation of the system (3) we express $\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}}$ as a function of $\widehat{v}_{\mathbf{k}}$:

$$\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}} = (2\pi)^{-1} i \widehat{v}_{\mathbf{k}}',$$

and then the second equation of the system (3) corresponds to the following homogeneous linear differential equation for the quantity $\widehat{v}_{\mathbf{k}}$:

$$\frac{1}{(2\pi)^2 \|\mathbf{k}\|^2} \widehat{v}_{\mathbf{k}}'''' - 2\widehat{v}_{\mathbf{k}}'' + (2\pi)^2 \|\mathbf{k}\|^2 \widehat{v}_{\mathbf{k}} = 0.$$

The solutions of this ODE take the form

$$\widehat{v}_{\mathbf{k}}(Y) = (a_{\mathbf{k}} Y + b_{\mathbf{k}}) e^{2\pi \|\mathbf{k}\| Y} + (c_{\mathbf{k}} Y + d_{\mathbf{k}}) e^{-2\pi \|\mathbf{k}\| Y},$$

with $(a_{\mathbf{k}}, b_{\mathbf{k}}, c_{\mathbf{k}}, d_{\mathbf{k}}) \in \mathbb{R}^4$. Since we have $\widehat{v}_{\mathbf{k}}' \in L^2(0, +\infty)$, then we necessarily obtain $a_{\mathbf{k}} = b_{\mathbf{k}} = 0$. Finally, $\widehat{v}_{\mathbf{k}}$ takes the form

$$\widehat{v}_{\mathbf{k}}(Y) = (c_{\mathbf{k}} Y + d_{\mathbf{k}}) e^{-2\pi \|\mathbf{k}\| Y}, \quad (c_{\mathbf{k}}, d_{\mathbf{k}}) \in \mathbb{R}^2.$$

By using the expression of $\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}}$ and $\widehat{p}_{\mathbf{k}}$ as a function of $\widehat{v}_{\mathbf{k}}$, we successively get

$$\begin{aligned} \mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}}(Y) &= (2\pi)^{-1} i (c_{\mathbf{k}} - 2\pi \|\mathbf{k}\| d_{\mathbf{k}} - \|\mathbf{k}\| c_{\mathbf{k}} Y / L) e^{-2\pi \|\mathbf{k}\| Y}, \\ \widehat{p}_{\mathbf{k}}(Y) &= c_{\mathbf{k}} e^{-2\pi \|\mathbf{k}\| Y}. \end{aligned}$$

Finally, we obtain the contribution $\widehat{\mathbf{u}}_{\mathbf{k}}$ using the results for $\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}}$ and $\mathbf{k}^{\perp} \cdot \widehat{\mathbf{u}}_{\mathbf{k}}$.

Defining the following affine functions

$$\begin{aligned} \mathbf{P}_{\mathbf{k}}^{(0)}(Y) &= (2\pi)^{-1} i ((c_{\mathbf{k}} - 2\pi \|\mathbf{k}\| d_{\mathbf{k}}) - 2\pi \|\mathbf{k}\| c_{\mathbf{k}} Y) \mathbf{k} + a_{\mathbf{k}} \mathbf{k}^{\perp}, \\ Q_{\mathbf{k}}^{(0)}(Y) &= c_{\mathbf{k}} Y + d_{\mathbf{k}} \quad \text{and} \quad R_{\mathbf{k}}^{(0)}(Y) = c_{\mathbf{k}}, \end{aligned}$$

the proof is concluded. @blacksquare

3.1.2 Induction step: analysis of problem $(S^{(j)})$ for $j \geq 2$

We show the following result about the solution of the problem $(S^{(j)})$ introduced page 18:

Lemma 3.5 *Let $j \geq 2$. There exist source terms \mathbf{A}_{j-2} , B_{j-1} and C_{j-2} such that*

$$\int_{]0,1[^d} \widetilde{\mathcal{F}}_j(\mathbf{X}, \cdot) d\mathbf{X} = \mathbf{0}, \quad \int_{]0,1[^d} \widetilde{\mathcal{G}}_j(\mathbf{X}, \cdot) d\mathbf{X} = 0, \quad \int_{]0,1[^d} \widetilde{\mathcal{H}}_j(\mathbf{X}, \cdot) d\mathbf{X} = 0.$$

For such a choice, the solution of the system $(S^{(j)})$ is written, for all $(\mathbf{X}, Y) \in]0, 1[^d \times]0, +\infty[$,

$$\begin{cases} \widetilde{\mathbf{u}}_j(\mathbf{X}, Y) = L_u(\widetilde{\mathcal{H}}_j, \widetilde{\mathcal{U}}_j, \widetilde{\mathcal{V}}_j) + \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \mathbf{P}_{\mathbf{k}}^{(j)}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}} \\ \widetilde{v}_j(\mathbf{X}, Y) = L_v(\widetilde{\mathcal{H}}_j, \widetilde{\mathcal{U}}_j, \widetilde{\mathcal{V}}_j) + \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} Q_{\mathbf{k}}^{(j)}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}} \\ \widetilde{p}_j(\mathbf{X}, Y) = \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} R_{\mathbf{k}}^{(j)}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}} \end{cases}$$

where $\mathbf{P}_{\mathbf{k}}^{(j)}$, $Q_{\mathbf{k}}^{(j)}$ and $R_{\mathbf{k}}^{(j)}$ are polynomial functions.

We deduce that the \mathbf{X} -average of $\widetilde{\mathbf{u}}_j$ and \widetilde{v}_j does not depend on Y , so that

$$\boldsymbol{\alpha}_j = L_u(\widetilde{\mathcal{H}}_j, \widetilde{\mathcal{U}}_j, \widetilde{\mathcal{V}}_j) \quad \text{and} \quad \beta_j = L_v(\widetilde{\mathcal{H}}_j, \widetilde{\mathcal{U}}_j, \widetilde{\mathcal{V}}_j).$$

As a straightforward consequence, we get the following property:

Corollary 3.6 *For all $j \geq 2$, the solution of $(S^{(j)})$ satisfies:*

$$\begin{aligned} \|\widetilde{\mathbf{u}}_j(\mathbf{X}, Y) - \boldsymbol{\alpha}_j\| &\leq C(\delta) e^{-\delta Y} \quad Y > 0, \quad \forall \delta < 2\pi, \\ |\widetilde{v}_j(\mathbf{X}, Y) - \beta_j| &\leq C(\delta) e^{-\delta Y} \quad Y > 0, \quad \forall \delta < 2\pi, \\ |\widetilde{p}_j(\mathbf{X}, Y)| &\leq C(\delta) e^{-\delta Y} \quad Y > 0, \quad \forall \delta < 2\pi, \end{aligned}$$

where $C(\delta)$ only depends on δ .

Proof.(of Lemma 3.5) It is based on the induction.

- *Initialization* ($j = 2$). Recall that

$$\tilde{\mathcal{F}}_2 = \mathbf{A}_0, \quad \tilde{\mathcal{G}}_2 = B_1 \quad \text{and} \quad \tilde{\mathcal{H}}_2 = C_0.$$

In order to ensure that the averages with respect to the variable \mathbf{X} are null, since \mathbf{A}_0 , B_1 and C_0 only depend on the variable \mathbf{x} , we have to choose

$$\mathbf{A}_0 = \mathbf{0}, \quad B_1 = 0 \quad \text{and} \quad C_0 = 0.$$

Consequently the source terms are null and we can apply exactly the same procedure that for the proof of the lemma 3.3. We obtain

$$\begin{cases} \tilde{\mathbf{u}}_2(\mathbf{X}, Y) = L_u(0, \tilde{\mathcal{U}}_2, \tilde{\mathcal{V}}_2) + \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \mathbf{P}_{\mathbf{k}}^{(2)}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}} \\ \tilde{v}_2(\mathbf{X}, Y) = L_v(0, \tilde{\mathcal{U}}_2, \tilde{\mathcal{V}}_2) + \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} Q_{\mathbf{k}}^{(2)}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}} \\ \tilde{p}_2(\mathbf{X}, Y) = \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} R_{\mathbf{k}}^{(2)}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}} \end{cases}$$

where $\mathbf{P}_{\mathbf{k}}^{(2)}$, $Q_{\mathbf{k}}^{(2)}$ and $R_{\mathbf{k}}^{(2)}$ are affine functions.

- *Induction.* Let $j \geq 2$ and suppose that lemma 3.5 holds for any index $k < j$ and let us prove that it still holds for $k = j$.

First, we have to show that it is possible to choose \mathbf{A}_{j-2} , B_{j-1} and C_{j-2} (which do not depend on the Stokes variables (\mathbf{X}, Y)) such that the source terms are free average with respect to \mathbf{X} . Recalling that

$$\tilde{\mathcal{F}}_j(\mathbf{X}, Y) = \mathbf{A}_{j-2} + (2\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}} \tilde{\mathbf{u}}_{j-2} + \Delta_{\mathbf{x}} \tilde{\mathbf{u}}_{j-4} - \nabla_{\mathbf{x}} \tilde{p}_{j-2})(\cdot, \mathbf{X}, Y).$$

Since $\tilde{\mathbf{u}}_{j-2}$ is \mathbf{X} -periodic, and since the \mathbf{X} -average of \tilde{p}_{j-2} is zero by induction assumption, it is sufficient to impose

$$\mathbf{A}_{j-2} = -\Delta_{\mathbf{x}} \left(\int_{]0,1[^d} \tilde{\mathbf{u}}_{j-4}(\cdot, \mathbf{X}, Y) d\mathbf{X} \right),$$

that is

$$\mathbf{A}_{j-2} = -\Delta_{\mathbf{x}} \boldsymbol{\alpha}_{j-4}. \quad (5)$$

It is important to notice that \mathbf{A}_{j-2} does not depend on Y . In the same way, using the definition of $\tilde{\mathcal{G}}_j$: $\tilde{\mathcal{G}}_j(\mathbf{X}, Y) = B_{j-1} + (2\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}} \tilde{v}_{j-2} + \Delta_{\mathbf{x}} \tilde{v}_{j-4})(\cdot, \mathbf{X}, Y)$, we naturally impose

$$B_{j-1} = -\Delta_{\mathbf{x}} \beta_{j-4}. \quad (6)$$

Finally, using the following definition $\tilde{\mathcal{H}}_j(\mathbf{X}, Y) = C_{j-2} - \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_{j-2}(\cdot, \mathbf{X}, Y)$, we impose

$$C_{j-2} = \operatorname{div}_{\mathbf{x}} \boldsymbol{\alpha}_{j-2}. \quad (7)$$

With these choices for A_{j-2} , B_{j-1} and C_{j-2} , the source terms $\tilde{\mathcal{F}}_j$, $\tilde{\mathcal{G}}_j$ and $\tilde{\mathcal{H}}_j$ are periodic and free-average with respect to the \mathbf{X} variable. Moreover, thanks to the induction assumption they take the following form

$$\begin{aligned}\tilde{\mathcal{F}}_j(\mathbf{X}, Y) &= \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}} \tilde{\mathbf{P}}_{\mathbf{k}}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}}, \\ \tilde{\mathcal{G}}_j(\mathbf{X}, Y) &= \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}} \tilde{Q}_{\mathbf{k}}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}}, \\ \tilde{\mathcal{H}}_j(\mathbf{X}, Y) &= \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}} \tilde{R}_{\mathbf{k}}(Y) e^{-2\pi \|\mathbf{k}\| Y + 2\pi i \mathbf{k} \cdot \mathbf{X}},\end{aligned}$$

where $\tilde{\mathbf{P}}_{\mathbf{k}}$, $\tilde{Q}_{\mathbf{k}}$ and $\tilde{R}_{\mathbf{k}}$ are polynomial. Using the Fourier transform of the system (S^(j)) we deduce an equivalent system on the Fourier coefficients $(\widehat{\mathbf{u}}_{\mathbf{k}}, \widehat{v}_{\mathbf{k}}, \widehat{p}_{\mathbf{k}})$:

$$\left\{ \begin{array}{l} (2\pi)^2 \|\mathbf{k}\|^2 \widehat{\mathbf{u}}_{\mathbf{k}} - \widehat{\mathbf{u}}_{\mathbf{k}}'' + 2\pi i \mathbf{k} \widehat{p}_{\mathbf{k}} = \tilde{\mathbf{P}}_{\mathbf{k}} e^{-2\pi \|\mathbf{k}\| Y} \quad \text{on } \{Y > 0\} \quad \forall \mathbf{k} \in \mathbb{Z}^d \\ (2\pi)^2 \|\mathbf{k}\|^2 \widehat{v}_{\mathbf{k}} - \widehat{v}_{\mathbf{k}}'' + \widehat{p}_{\mathbf{k}}' = \tilde{Q}_{\mathbf{k}} e^{-2\pi \|\mathbf{k}\| Y} \quad \text{on } \{Y > 0\} \quad \forall \mathbf{k} \in \mathbb{Z}^d \\ 2\pi i \mathbf{k} \widehat{\mathbf{u}}_{\mathbf{k}} + \widehat{v}_{\mathbf{k}}' = \tilde{R}_{\mathbf{k}} e^{-2\pi \|\mathbf{k}\| Y} \quad \text{on } \{Y > 0\} \quad \forall \mathbf{k} \in \mathbb{Z}^d \\ \widehat{\mathbf{u}}_{\mathbf{0}}(0) = L_u(\tilde{\mathcal{H}}_j, \tilde{\mathcal{U}}_j, \tilde{\mathcal{V}}_j) \\ \widehat{v}_{\mathbf{0}}(0) = L_v(\tilde{\mathcal{H}}_j, \tilde{\mathcal{U}}_j, \tilde{\mathcal{V}}_j) \\ \int_0^{+\infty} \widehat{p}_{\mathbf{0}}(Y) dY = 0 \end{array} \right.$$

where $\widehat{\mathbf{u}}_{\mathbf{k}}'$, $\widehat{v}_{\mathbf{k}}'$ and $\widehat{p}_{\mathbf{k}}$ belong to $L^2(0, +\infty)$.

• For $\mathbf{k} = \mathbf{0}$, since $\tilde{\mathbf{P}}_{\mathbf{0}} = \mathbf{0}$, $\tilde{Q}_{\mathbf{0}} = 0$ and $\tilde{R}_{\mathbf{0}} = 0$ we deduce that (see the proof of Lemma 3.5 for the same kind of calculations):

$$\widehat{\mathbf{u}}_{\mathbf{0}} = L_u(\tilde{\mathcal{H}}_j, \tilde{\mathcal{U}}_j, \tilde{\mathcal{V}}_j), \quad \widehat{v}_{\mathbf{0}} = L_v(\tilde{\mathcal{H}}_j, \tilde{\mathcal{U}}_j, \tilde{\mathcal{V}}_j) \quad \text{and} \quad \widehat{p}_{\mathbf{0}} = 0.$$

• For $\mathbf{k} \neq \mathbf{0}$, using the same method as previously (see the proof of Lemma 3.5), we first obtain a linear differential equation on the function defined by $f(Y) = \mathbf{k}^\perp \cdot \widehat{\mathbf{u}}_{\mathbf{k}}(Y)$:

$$(2\pi)^2 \|\mathbf{k}\|^2 f(Y) - f''(Y) = \mathbf{k}^\perp \cdot \tilde{\mathbf{P}}_{\mathbf{k}} e^{-2\pi \|\mathbf{k}\| Y}.$$

Since $\tilde{\mathbf{P}}_{\mathbf{k}}$ is a polynomial, we know that the solutions of this linear differential equation take the following form:

$$\mathbf{k}^\perp \cdot \widehat{\mathbf{u}}_{\mathbf{k}}(Y) = \mathcal{P}_{\mathbf{k}}(Y) e^{-2\pi \|\mathbf{k}\| Y},$$

where $\mathcal{P}_{\mathbf{k}}$ a polynomial. Next we obtain

$$\begin{aligned}\widehat{p}_{\mathbf{k}} &= 2\pi i (\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}}) - \frac{i}{2\pi \|\mathbf{k}\|^2} (\mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}})'' - \frac{i}{2\pi \|\mathbf{k}\|^2} (\mathbf{k} \cdot \tilde{\mathbf{P}}_{\mathbf{k}}) e^{-2\pi \|\mathbf{k}\| Y}, \\ \mathbf{k} \cdot \widehat{\mathbf{u}}_{\mathbf{k}} &= \frac{i}{2\pi} \left(\widehat{v}_{\mathbf{k}}' - \tilde{R}_{\mathbf{k}} e^{-2\pi \|\mathbf{k}\| Y} \right).\end{aligned}$$

Moreover $\widehat{v}_{\mathbf{k}}$ satisfies the non-homogeneous linear differential equation

$$\begin{aligned} \frac{1}{(2\pi)^2 \|\mathbf{k}\|^2} \widehat{v}_{\mathbf{k}}'''' - 2\widehat{v}_{\mathbf{k}}'' + (2\pi)^2 \|\mathbf{k}\|^2 \widehat{v}_{\mathbf{k}} &= \left(\widetilde{Q}_{\mathbf{k}} + \frac{i}{2\pi \|\mathbf{k}\|^2} (\mathbf{k} \cdot \widetilde{\mathbf{P}}_{\mathbf{k}}') - \frac{i}{\|\mathbf{k}\|} (\mathbf{k} \cdot \widetilde{\mathbf{P}}_{\mathbf{k}}) \right) e^{-2\pi \|\mathbf{k}\| Y}. \end{aligned} \quad (8)$$

The solutions of this ODE are the sum of *i*) the solution of the homogeneous equation (which has been solved before) and *ii*) a particular solution which, due to the form of the right-hand side, can be obtained under the form $Q_{\mathbf{k},1}^{(1)}(Y)e^{-2\pi \|\mathbf{k}\| Y}$, the polynomial function $Q_{\mathbf{k},1}^{(1)}$. From this, we deduce that the solutions of Equation (8) can be written as

$$\begin{aligned} \widehat{v}_{\mathbf{k}}(Y) &= (a_{\mathbf{k}}Y + b_{\mathbf{k}})e^{2\pi \|\mathbf{k}\| Y} + \underbrace{(Q_{\mathbf{k},1}^{(1)}(Y) + c_{\mathbf{k}}Y + d_{\mathbf{k}})}_{= Q_{\mathbf{k}}^{(1)}(Y)} e^{-2\pi \|\mathbf{k}\| Y}. \end{aligned}$$

As before, since $\widehat{v}_{\mathbf{k}}$ belongs to $L^2(0, +\infty)$, we have to keep terms in $Q_{\mathbf{k}}^{(1)}(Y)e^{-2\pi \|\mathbf{k}\| Y}$ only. Finally, we obtain expressions for $\widehat{\mathbf{u}}_{\mathbf{k}}$ and $\widehat{p}_{\mathbf{k}}$, which concludes the proof.

@blacksquare

3.2 Well-posedness of the Reynolds problem

In this part, we show that the Reynolds-type problems $(R^{(j)})$ are well posed as soon as the “constants” \mathbf{A}_j , B_j and C_j are chosen as previously.

In particular, due to the fact that $C_j = \operatorname{div}_{\mathbf{x}} \boldsymbol{\alpha}_j$, system $(R^{(j)})$ implies that $\mathbf{u}_j + \boldsymbol{\alpha}_j$, $v_j + \beta_{j+1}$ and p_j satisfy the following Reynolds-type problems on ω_R :

$$(R) \begin{cases} -\partial_Z^2 \mathbf{u} + \nabla_{\mathbf{x}} p &= \mathcal{F}, & \text{on } \omega_R, \\ \partial_Z p &= \mathcal{G}, & \text{on } \omega_R, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u} + \partial_Z v &= 0, & \text{on } \omega_R, \\ (\mathbf{u}, v) &= (\mathcal{U}^0, \mathcal{V}^0), & \text{on } \gamma_0, \\ (\mathbf{u}, v) &= (\mathbf{0}, 0), & \text{on } \gamma_+. \end{cases}$$

Here, we assume that the data satisfy some regularity assumptions, i. e. $\mathcal{U}^0, \mathcal{V}^0 \in C^\infty(\mathbb{T}^d)^d$, $\mathcal{F} \in C^\infty(\omega_R)^d$ and $\mathcal{G} \in C^\infty(\omega_R)$. Moreover, we assume that

$$\int_{\mathbb{T}^d} \mathcal{V}^0(\mathbf{x}) \, d\mathbf{x} = 0, \quad (9)$$

which correspond to a compatibility condition for the system (R).

Now let us highlight two crucial properties:

- we first show that β_{j+1} only depends on the solution of the Stokes problem $(S^{(j-1)})$, as will be proved in Proposition 3.7 ;

- as a consequence, we show that Assumption (9) is always satisfied for the Reynolds problems $(\mathbf{R}^{(j)})$, as will be stated in Remark 3.3

Proposition 3.7 *Coefficient β_{j+1} which couples problems $(\mathbf{R}^{(j)})$ and $(\mathbf{S}^{(j+1)})$ only depends on the solution of problems $(\mathbf{S}^{(j-1)})$ and $(\mathbf{R}^{(k)})$ for $k \leq j - 2$. More precisely, we have*

$$\begin{aligned} \beta_{j+1}(\mathbf{x}) &= \operatorname{div}_{\mathbf{x}} \left(\int_{\{Y < 0\}} \tilde{\mathbf{u}}_{j-1}(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} dY \right) \\ &\quad - \operatorname{div}_{\mathbf{x}} \left(\sum_{k=2}^{\lfloor \frac{j+1}{2} \rfloor + 2} \frac{(-1)^k}{k!} \left(\int_{]0,1[^d} h^-(\mathbf{X})^k \, d\mathbf{X} \right) \partial_Z^{k-1} \mathbf{u}_{j-k}(\mathbf{x}, 0) \right). \end{aligned}$$

Proof. We recall that from the Fourier analysis we have $\beta_{j+1} = \mathbf{L}_v(\tilde{\mathcal{H}}_{j+1}, \tilde{\mathbf{U}}_{j+1}, \tilde{\mathcal{V}}_{j+1})$, where $\tilde{\mathbf{U}}_{j+1}$ can be viewed as a polynomial function of $h^-(\mathbf{X})$ and where $\tilde{\mathcal{V}}_{j+1}$, which is also a polynomial function of $h^-(\mathbf{X})$, takes the following form

$$\tilde{\mathcal{V}}_{j+1}(\mathbf{X}) = -h^-(\mathbf{X})C_{j-1}(\mathbf{x}) + \sum_{k=2}^{\lfloor \frac{j+1}{2} \rfloor + 2} \frac{(-1)^k h^-(\mathbf{X})^k}{k!} \operatorname{div}_{\mathbf{x}} \partial_Z^{k-1} \mathbf{u}_{j-k-1}(\mathbf{x}, 0),$$

Applying Proposition 3.2, we obtain

$$\begin{aligned} \beta_{j+1}(\mathbf{x}) &= - \int_{\{Y < 0\}} \tilde{\mathcal{H}}_{j+1}(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} dY + C_{j-1}(\mathbf{x}) \int_{]0,1[^d} h^-(\mathbf{X}) \, d\mathbf{X} \\ &\quad - \left(\sum_{k=2}^{\lfloor \frac{j+1}{2} \rfloor + 2} \frac{(-1)^k h^-(\mathbf{X})^k}{k!} \operatorname{div}_{\mathbf{x}} \partial_Z^{k-1} \mathbf{u}_{j-k-1}(\mathbf{x}, 0) \, d\mathbf{X} \right). \end{aligned}$$

Let us rewrite the right-hand side: first, by definition $\tilde{\mathcal{H}}_{j+1} = C_{j-1} - \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_{j-1}$ so that

$$\begin{aligned} \int_{\{Y < 0\}} \tilde{\mathcal{H}}_{j+1}(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} dY &= C_{j-1}(\mathbf{x}) \int_{\{Y < 0\}} 1 \, d\mathbf{X} dY - \int_{\{Y < 0\}} \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_{j-1}(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} dY \\ &= C_{j-1}(\mathbf{x}) \left(\int_{]0,1[^d} h^-(\mathbf{X}) \, d\mathbf{X} \right) - \operatorname{div}_{\mathbf{x}} \left(\int_{\{Y < 0\}} \tilde{\mathbf{u}}_{j-1}(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} dY \right). \end{aligned}$$

Then, the last term in the right-hand side is simply treated by putting the $\operatorname{div}_{\mathbf{x}}$ operator out of the partial sum. Thus, we get

$$\begin{aligned} \beta_{j+1}(\mathbf{x}) &= \operatorname{div}_{\mathbf{x}} \left(\int_{\{Y < 0\}} \tilde{\mathbf{u}}_{j-1}(\mathbf{x}, \mathbf{X}, Y) \, d\mathbf{X} dY \right) \\ &\quad - \operatorname{div}_{\mathbf{x}} \left(\sum_{k=2}^{\lfloor \frac{j+1}{2} \rfloor + 2} \frac{(-1)^k}{k!} \left(\int_{]0,1[^d} h^-(\mathbf{X})^k \, d\mathbf{X} \right) \partial_Z^{k-1} \mathbf{u}_{j-k-1}(\mathbf{x}, 0) \right). \end{aligned}$$

Thus, the proof is concluded.

@blacksquare

Remark 3.3 *It is important to notice that for the Reynolds problems $(R^{(j)})$, Assumption (9) is always satisfied since $\mathcal{V}^0 = \beta_{j+1}$. From Proposition 3.7, we deduce that β_{j+1} is a \mathbf{x} -divergence term which implies, due to the periodicity, that*

$$\int_{\mathbb{T}^d} \beta_{j+1}(\mathbf{x}) \, d\mathbf{x} = 0.$$

This corresponds to Assumption (9).

To study the Reynolds system (R), we first use some algebraic transformations: Integrating the pressure equation gives

$$p(\cdot, Z) = \bar{p} + \int_0^Z \mathcal{G}(\cdot, \zeta) \, d\zeta, \quad (10)$$

where $\mathbf{x} \mapsto \bar{p}(\mathbf{x})$ is a function to be determined (called “reduced pressure”). Then, putting the above equality into the (\mathbf{u}, p) relationship gives

$$-\partial_Z^2 \mathbf{u} + \nabla_{\mathbf{x}} \bar{p} = \mathcal{F} - \int_0^Z \nabla_{\mathbf{x}} \mathcal{G}(\cdot, \mu) \, d\mu.$$

Again, integrating twice in the Z -variable, we obtain:

$$\begin{aligned} \mathbf{u}(\cdot, Z) &= \frac{Z(Z - h^+)}{2} \nabla_{\mathbf{x}} \bar{p} + \frac{h^+ - Z}{h^+} \mathbf{u}^0 \\ &+ \int_0^Z \int_0^\eta \left\{ \mathcal{F}(\cdot, \zeta) - \int_0^\zeta \nabla_{\mathbf{x}} \mathcal{G}(\cdot, \mu) \, d\mu \right\} \, d\zeta \, d\eta \\ &- \frac{y}{h^+} \int_0^{h^+} \int_0^\eta \left\{ \mathcal{F}(\cdot, \zeta) - \int_0^\zeta \nabla_{\mathbf{x}} \mathcal{G}(\cdot, \mu) \, d\mu \right\} \, d\zeta \, d\eta, \end{aligned} \quad (11)$$

and the vertical velocity field is given by

$$v(\cdot, Z) = \mathcal{V}^0 - \int_0^Z \operatorname{div}_{\mathbf{x}} \mathbf{u}(\cdot, \zeta) \, d\zeta. \quad (12)$$

Integrating between 0 and h^+ the divergence equation of (R), we get

$$\begin{aligned} \operatorname{div}_{\mathbf{x}} \left(\frac{h^{+3}}{12} \nabla_{\mathbf{x}} \bar{p} \right) &= \operatorname{div}_{\mathbf{x}} \left(\frac{h^+}{2} \mathbf{u}^0 \right) - \mathcal{V}^0 \\ &+ \operatorname{div}_{\mathbf{x}} \left(\int_0^{h^+} \int_0^y \int_0^\eta \left\{ \mathcal{F}(\cdot, \zeta) - \int_0^\zeta \nabla_{\mathbf{x}} \mathcal{G}(\cdot, \mu) \, d\mu \right\} \, d\zeta \, d\eta \, dy \right) \\ &- \operatorname{div}_{\mathbf{x}} \left(\frac{h^+}{2} \int_0^{h^+} \int_0^\eta \left\{ \mathcal{F}(\cdot, \zeta) - \int_0^\zeta \nabla_{\mathbf{x}} \mathcal{G}(\cdot, \mu) \, d\mu \right\} \, d\zeta \, d\eta \right). \end{aligned} \quad (13)$$

Lemma 3.8 *Under the compatibility condition (9), problem (R) admits a unique solution $(\mathbf{u}, v, p) \in C^\infty(\omega_R)^{d+2}$.*

Proof. Obviously, Eq. (13) with assumption (9) can be written as

$$\operatorname{div}_{\mathbf{x}}(\mathcal{A}\nabla_{\mathbf{x}}\bar{p}) = \operatorname{div}_{\mathbf{x}}\mathcal{B} - \mathcal{C} \quad \text{with} \quad \int_{\mathbb{T}^d} \mathcal{C} = 0,$$

where the left-hand side and right-hand side obviously depend on all the data, i. e.

$$\mathcal{A} := \frac{h^{+3}}{12}, \quad \mathcal{B} := \mathcal{B}(h^+, \mathcal{F}, \mathcal{G}, \mathbf{U}^0), \quad \mathcal{C} := \mathcal{V}^0,$$

with

$$\mathcal{A} \in C^\infty(\mathbb{T}^d), \quad \mathcal{B} \in C^\infty(\mathbb{T}^d)^d, \quad \mathcal{C} \in C^\infty(\mathbb{T}^d).$$

Thus, existence and uniqueness of a solution $\bar{p} \in H^1(\mathbb{T})$ (defined up to an additive constant) immediately follows from the Lax-Milgram theorem. Then \mathbf{u} , v and p are defined and uniquely determined by means of integration, see Eq. (10), (11) and (12), and the regularity of (\mathbf{u}, v, p) follows from the regularity of \bar{p} (with respect to the \mathbf{x} -variable) and the data. @blacksquare

Since the compatibility condition (9) is satisfied for the systems $(\mathbf{R}^{(j)})$ (see the Remark 3.3), we deduce the following result:

Corollary 3.9 *For $j \in \mathbb{N}$ the problem $(\mathbf{R}^{(j)})$ admits a unique solution $(\mathbf{u}_j, v_j, p_j) \in C^\infty(\omega_R)^{d+2}$.*

As we noted in the subsection 2.3 for the first order term (\mathbf{u}_0, v_0, p_0) , see for instance the Remark 2.1, we can easily show by induction that the solution (\mathbf{u}_j, v_j, p_j) of the problem $(\mathbf{R}^{(j)})$ is polynomial with respect to the variable Z . Moreover the degree of these polynomials are given by, for any $n \in \mathbb{N}$,

$$\begin{aligned} \deg p_{2n} &= \deg p_{2n+1} = 2n, \\ \deg \mathbf{u}_{2n} &= \deg \mathbf{u}_{2n+1} = 2n + 2, \\ \deg v_{2n} &= \deg v_{2n+1} = 2n + 3. \end{aligned}$$

It is therefore natural to extend the velocity field (\mathbf{u}_j, v_j) for $Z < 0$, putting

$$\mathbf{u}_j(\mathbf{x}, Z) = \sum_{k=0}^{j+2} \partial_Z^k \mathbf{u}_j(\mathbf{x}, 0) \frac{Z^k}{k!} \quad \text{and} \quad v_j(\mathbf{x}, Z) = \sum_{k=0}^{j+3} \partial_Z^k v_j(\mathbf{x}, 0) \frac{Z^k}{k!}.$$

Due to the boundary dirichlet condition imposed on (\mathbf{u}_j, v_j) for $Z = 0$, and due to the divergence equation on this velocity (see the divergence equation for the problem $(\mathbf{R}^{(j)})$), we obtain for all $j \in \mathbb{N}^*$

$$\begin{aligned} \mathbf{u}_j(\mathbf{x}, Z) &= \sum_{k=1}^{j+2} \partial_Z^k \mathbf{u}_j(\mathbf{x}, 0) \frac{Z^k}{k!}, \\ v_j(\mathbf{x}, Z) &= -C_j(\mathbf{x}) - \sum_{k=2}^{j+3} \operatorname{div}_{\mathbf{x}} \partial_Z^{k-1} \mathbf{u}_j(\mathbf{x}, 0) \frac{Z^k}{k!}. \end{aligned} \tag{14}$$

These are the terms which, measured in $Z = -\varepsilon h^- \left(\frac{\mathbf{x} - \mathbf{st}}{\varepsilon^2} \right)$, must be compensated the boundary layer corrector.

By Lemmas 3.3, 3.5 and 3.8 (and related corollaries), we have proved that each term of the asymptotic expansion satisfies a well-posed problem. Moreover, we have characterized the behaviour of each solution.

3.3 Algorithm

In the two previous subsections, we have proved that the intermediate problems – Stokes problems ($S^{(j)}$) and Reynolds-type problems ($R^{(j)}$) – were all well posed, independently of each other. Clearly, to solve the Stokes problem, you must know some solution of the problem of Reynolds and vice versa. Here, we describe the procedure to really solve all the problems thoroughly.

To evaluate the development up to order N (see the *ansatz*, see Eq. (1) on page 11), we theoretically just add all intermediate profiles: (\mathbf{u}_0, v_0, p_0) , $(\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1)$, (\mathbf{u}_1, v_1, p_1) , $(\tilde{\mathbf{u}}_2, \tilde{v}_2, \tilde{p}_2)$ etc. In practice the first terms are obtained as described in the introduction, see subsection 1.3. More generally, assuming known the terms (\mathbf{u}_k, v_k, p_k) and $(\tilde{\mathbf{u}}_k, \tilde{v}_k, \tilde{p}_k)$ for any $k < j$, we compute the terms (\mathbf{u}_j, v_j, p_j) and $(\tilde{\mathbf{u}}_j, \tilde{v}_j, \tilde{p}_j)$ as follows.

INITIALIZATION:

0. **Main flow:** (\mathbf{u}_0, v_0, p_0) solves ($R^{(0)}$) with

$$\mathbf{A}_0 = \mathbf{0}, \quad B_0 = 0, \quad C_0 = 0, \quad \beta_1 = 0$$

1.A **Corrective Stokes flow:** $(\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1)$ solves ($S^{(1)}$) with

$$\tilde{\mathbf{U}}_1(\mathbf{X}) = h^-(\mathbf{X}) \partial_Z \mathbf{u}_0(\mathbf{x}, 0), \quad \tilde{\mathcal{V}}_1 \equiv 0$$

1.B. **Corrective Reynolds flow:** (\mathbf{u}_1, v_1, p_1) solves ($R^{(1)}$) with

$$\alpha_1 = \lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{\mathbf{u}}_1(\cdot, \mathbf{X}, Y) d\mathbf{X}, \quad \beta_2 = 0, \\ \mathbf{A}_1 = \mathbf{0}, \quad B_1 = 0, \quad C_1 = \operatorname{div}_{\mathbf{x}} \alpha_1.$$

ITERATIVE PROCEDURE: Assume that, for $1 \leq k \leq j-1$,

- problem ($S^{(k)}$) is defined, i. e. in particular, the source terms $(\tilde{\mathcal{F}}_k, \tilde{\mathcal{G}}_k, \tilde{\mathcal{H}}_k)$ and the boundary terms $(\tilde{\mathcal{U}}_k, \tilde{\mathcal{V}}_k)$ have been defined. Let $(\tilde{\mathbf{u}}_k, \tilde{v}_k, \tilde{p}_k)$ be its solution.
- problem ($R^{(k)}$) is defined, i. e. in particular, the source terms $(\mathcal{F}_k, \mathcal{G}_k, \mathcal{H}_k)$ and the boundary terms (α_k, β_{k+1}) have been defined, meaning that coefficients \mathbf{A}_k, B_k and C_k have been also defined. Let (\mathbf{u}_k, v_k, p_k) be its solution.

j.A **Corrective Stokes flow:** $(\tilde{\mathbf{u}}_j, \tilde{v}_j, \tilde{p}_j)$ solves $(S^{(j)})$ with

- the following source terms

$$\begin{aligned}\tilde{\mathcal{F}}_j(\mathbf{X}, Y) &= \mathbf{A}_{j-2} + (2\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \tilde{\mathbf{u}}_{j-2} + \Delta_{\mathbf{x}} \tilde{\mathbf{u}}_{j-4} - \nabla_{\mathbf{x}} \tilde{p}_{j-2})(\cdot, \mathbf{X}, Y), \\ \tilde{\mathcal{G}}_j(\mathbf{X}, Y) &= B_{j-1} + (2\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \tilde{v}_{j-2} + \Delta_{\mathbf{x}} \tilde{v}_{j-4})(\cdot, \mathbf{X}, Y), \\ \tilde{\mathcal{H}}_j(\mathbf{X}, Y) &= C_{j-2} - \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_{j-2}(\cdot, \mathbf{X}, Y).\end{aligned}$$

- the following boundary conditions

$$\begin{aligned}\tilde{\mathbf{u}}_j(\mathbf{X}) &= - \sum_{k=1}^{[\frac{j+1}{2}]+1} \frac{(-1)^k h^-(\mathbf{X})^k}{k!} \partial_Z^k \mathbf{u}_{j-k}(\mathbf{x}, 0), \\ \tilde{v}_j(\mathbf{X}) &= -h^-(\mathbf{X}) C_{j-2}(\mathbf{x}) + \sum_{k=2}^{[\frac{j}{2}]+2} \frac{(-1)^k h^-(\mathbf{X})^k}{k!} \operatorname{div}_{\mathbf{x}} \partial_Z^{k-1} \mathbf{u}_{j-k-1}(\mathbf{x}, 0).\end{aligned}$$

j.B **Corrective Reynolds flow:** (\mathbf{u}_j, v_j, p_j) solves $(R^{(j)})$ with

- the following boundary values

$$\begin{aligned}\alpha_j &= \lim_{Y \rightarrow +\infty} \int_{]0,1[^d} \tilde{\mathbf{u}}_j(\cdot, \mathbf{X}, Y) \, d\mathbf{X} \\ \beta_{j+1} &= \operatorname{div}_{\mathbf{x}} \left(\int_{\{Y < 0\}} \tilde{\mathbf{u}}_{j-1}(\cdot, \mathbf{X}, Y) \, d\mathbf{X} dY \right) \\ &\quad - \operatorname{div}_{\mathbf{x}} \left(\sum_{k=2}^{[\frac{j+1}{2}]+2} \frac{(-1)^k}{k!} \left(\int_{]0,1[^d} h^-(\mathbf{X})^k \, d\mathbf{X} \right) \partial_Z^{k-1} \mathbf{u}_{j-k}(\cdot, 0) \right).\end{aligned}$$

- the following constants

$$\mathbf{A}_j = -\Delta_{\mathbf{x}} \alpha_{j-2}, \quad B_j = -\Delta_{\mathbf{x}} \beta_{j-3}, \quad C_j = \operatorname{div}_{\mathbf{x}} \alpha_j.$$

- the following source terms

$$\begin{aligned}\mathcal{F}_j(\mathbf{x}, Z) &= -\mathbf{A}_j(\mathbf{x}) + \Delta_{\mathbf{x}} \mathbf{u}_{j-2}(\mathbf{x}, Z), \\ \mathcal{G}_j(\mathbf{x}, Z) &= -B_j(\mathbf{x}) + \partial_Z^2 v_{j-2}(\mathbf{x}, Z) + \Delta_{\mathbf{x}} v_{j-4}(\mathbf{x}, Z), \\ \mathcal{H}_j(\mathbf{x}) &= -C_j(\mathbf{x}).\end{aligned}$$

4 Error analysis

The error analysis is based on a three-step procedure: *i*) first we recall classical estimates related to the Stokes system satisfied by the remainder. At this stage, the estimates do depend on the small parameter ε through the expression of the source terms and also through the domain Ω_ε whose measure tends to zero

as ε tends to zero ; *ii*) then we establish estimates which allow us to control the source terms ; *iii*) finally we translate the previous estimates (expressed in a norm which depends on the small parameter) into estimates which are relevant with respect to a convergence procedure: the chosen norm preserves the constant states defined in thin domains.

The remainder is defined by the *ansatz* proposed on Eq. (1). Using the linearity of the Stokes system, we easily deduce, after some formal computations, that the remainder $(\mathcal{R}^{(N)}, \mathcal{S}^{(N)}, \mathcal{Q}^{(N)})$ satisfies a Stokes-type system:

$$(A_\varepsilon) \left\{ \begin{array}{ll} -\Delta_{\mathbf{x}} \mathcal{R} - \partial_y^2 \mathcal{R} + \nabla_{\mathbf{x}} \mathcal{Q} & = \mathcal{F}_\varepsilon^{(N)}, \quad \text{on } \Omega_\varepsilon(t), \\ -\Delta_{\mathbf{x}} \mathcal{S} - \partial_y^2 \mathcal{S} + \partial_y \mathcal{Q} & = \mathcal{G}_\varepsilon^{(N)}, \quad \text{on } \Omega_\varepsilon(t), \\ \operatorname{div}_{\mathbf{x}} \mathcal{R} + \partial_y \mathcal{S} & = \mathcal{H}_\varepsilon^{(N)}, \quad \text{on } \Omega_\varepsilon(t), \\ \mathcal{R} & = \mathcal{U}_\varepsilon^{(N)}, \quad \text{on } \Gamma_\varepsilon^+, \\ \mathcal{S} & = \mathcal{V}_\varepsilon^{(N)}, \quad \text{on } \Gamma_\varepsilon^+, \\ \mathcal{R} & = \mathcal{U}_\varepsilon^{-(N)}, \quad \text{on } \Gamma_\varepsilon^-, \\ \mathcal{S} & = \mathcal{V}_\varepsilon^{-(N)}, \quad \text{on } \Gamma_\varepsilon^-. \end{array} \right.$$

where the source terms take the following forms:

$$\begin{aligned} \mathcal{F}_\varepsilon^{(N)}(\mathbf{x}, y, t) &= \mathcal{F}_\varepsilon^R\left(\mathbf{x}, \frac{y}{\varepsilon}\right) + \mathcal{F}_\varepsilon^{bl}\left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2}\right), \\ \mathcal{G}_\varepsilon^{(N)}(\mathbf{x}, y, t) &= \mathcal{G}_\varepsilon^R\left(\mathbf{x}, \frac{y}{\varepsilon}\right) + \mathcal{G}_\varepsilon^{bl}\left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2}\right), \\ \mathcal{H}_\varepsilon^{(N)}(\mathbf{x}, y, t) &= \mathcal{H}_\varepsilon^R\left(\mathbf{x}, \frac{y}{\varepsilon}\right) + \mathcal{H}_\varepsilon^{bl}\left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2}\right), \end{aligned}$$

with the following precise definitions:

$$\begin{aligned} \mathcal{F}_\varepsilon^R &:= \varepsilon^{N-1}(\varepsilon \Delta_{\mathbf{x}} \mathbf{u}_N + \Delta_{\mathbf{x}} \mathbf{u}_{N-1}), \\ \mathcal{F}_\varepsilon^{bl} &:= \varepsilon^{N-2}(\varepsilon^3 \Delta_{\mathbf{x}} \tilde{\mathbf{u}}_{N+1} + \varepsilon^2 \Delta_{\mathbf{x}} \tilde{\mathbf{u}}_N + \varepsilon \Delta_{\mathbf{x}} \tilde{\mathbf{u}}_{N-1} + \Delta_{\mathbf{x}} \tilde{\mathbf{u}}_{N-2} \\ &\quad + 2\varepsilon \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \tilde{\mathbf{u}}_{N+1} + 2\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \tilde{\mathbf{u}}_N - \varepsilon \nabla_{\mathbf{x}} \tilde{p}_{N+1} - \nabla_{\mathbf{x}} \tilde{p}_N), \\ \mathcal{G}_\varepsilon^R &:= \varepsilon^{N-2}(\varepsilon^3 \Delta_{\mathbf{x}} v_N + \varepsilon^2 \Delta_{\mathbf{x}} v_{N-1} + \varepsilon \Delta_{\mathbf{x}} v_{N-2} + \Delta_{\mathbf{x}} v_{N-3} \\ &\quad + \varepsilon \partial_Z^2 v_N + \partial_Z^2 v_{N-1}), \\ \mathcal{G}_\varepsilon^{bl} &:= \varepsilon^{N-2}(\varepsilon^3 \Delta_{\mathbf{x}} \tilde{v}_{N+1} + \varepsilon^2 \Delta_{\mathbf{x}} \tilde{v}_N + \varepsilon \Delta_{\mathbf{x}} \tilde{v}_{N-1} + \Delta_{\mathbf{x}} \tilde{v}_{N-2} \\ &\quad + 2\varepsilon \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \tilde{v}_{N+1} + 2\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \tilde{v}_N), \\ \mathcal{H}_\varepsilon^R &:= 0, \\ \mathcal{H}_\varepsilon^{bl} &:= -\varepsilon^N(\varepsilon \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_{N+1} + \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_N). \end{aligned}$$

About the boundary condition, using the *ansatz* at the boundary Γ_ε^+ and Γ_ε^-

we get

$$\begin{aligned}\mathbf{u}_\varepsilon^{(N)}(\mathbf{x}) &= \sum_{j=1}^{N+1} \varepsilon^j \left(\boldsymbol{\alpha}_j(\mathbf{x}) - \tilde{\mathbf{u}}_j\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon^2}, \frac{h^+(\mathbf{x})}{\varepsilon}\right) \right) - \varepsilon^{N+1} \boldsymbol{\alpha}_{N+1}(\mathbf{x}), \\ \mathcal{V}_\varepsilon^{(N)}(\mathbf{x}) &= \sum_{j=1}^{N+1} \varepsilon^j \left(\beta_j(\mathbf{x}) - \tilde{v}_j\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon^2}, \frac{h^+(\mathbf{x})}{\varepsilon}\right) \right), \\ \mathbf{u}_\varepsilon^{-(N)}(\mathbf{x}) &= \varepsilon^{N+2} \times \text{function}\left(h^-\left(\frac{\mathbf{x}-\mathbf{st}}{\varepsilon}\right), \mathbf{u}_N(\mathbf{x}, 0), \dots, \mathbf{u}_0(\mathbf{x}, 0)\right), \\ \mathcal{V}_\varepsilon^{-(N)}(\mathbf{x}) &= \varepsilon^{N+2} \times \text{function}\left(h^-\left(\frac{\mathbf{x}-\mathbf{st}}{\varepsilon}\right), v_N(\mathbf{x}, 0), \dots, v_0(\mathbf{x}, 0)\right).\end{aligned}$$

For sake of simplicity we do not explicitly give the functions appearing in the boundary terms $\mathbf{u}_\varepsilon^{-(N)}$ and $\mathcal{V}_\varepsilon^{-(N)}$. They write like the boundary term $\tilde{\mathbf{u}}_j$ and \tilde{v}_j in the Stokes problem $(S^{(j)})$.

The existence and uniqueness results of such a problem are well-known (see for example [8]). We will endeavour to obtain estimates of the solution according to the sources terms and to the dependence into ε . By means of construction, as the initial Stokes problem is well-posed and all intermediate problems are also well-posed, we have necessarily

$$\int_{\Omega_\varepsilon} \mathcal{H}_\varepsilon^{(N)} = \int_{\Gamma_\varepsilon^+} \begin{pmatrix} \mathcal{R} \\ \mathcal{S} \end{pmatrix} \cdot \mathbf{n} = \int_{\mathbb{T}^d} \mathcal{V}_\varepsilon^{(N)} - \int_{\mathbb{T}^d} \mathbf{u}_\varepsilon^{(N)} \cdot \nabla_{\mathbf{x}} h^+. \quad (15)$$

In the sequel, we will drop the overscripts $(\cdot)^{(N)}$ for the sake of clarity.

4.1 Lift procedure

4.1.1 Lift velocity at the boundary

To obtain estimates on the remainder $(\mathcal{R}, \mathcal{S}, \mathcal{Q})$ with respect to ε we first introduce a explicit velocity field which has the same boundary conditions. We introduce

$$f(\mathbf{x}, y) = \frac{y + \varepsilon^2 h^-(\mathbf{x}/\varepsilon^2)}{\varepsilon h^+(\mathbf{x}) + \varepsilon^2 h^-(\mathbf{x}/\varepsilon^2)}$$

and we consider the following velocity field $(\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}})$ defined on Ω_ε by

$$\begin{aligned}\tilde{\mathcal{R}}_{\text{bound}}(\mathbf{x}, y) &= f(\mathbf{x}, y) \mathbf{u}_\varepsilon(\mathbf{x}) + (1 - f(\mathbf{x}, y)) \mathbf{u}_\varepsilon^-(\mathbf{x}), \\ \tilde{\mathcal{S}}_{\text{bound}}(\mathbf{x}, y) &= f(\mathbf{x}, y) \mathcal{V}_\varepsilon(\mathbf{x}) + (1 - f(\mathbf{x}, y)) \mathcal{V}_\varepsilon^-(\mathbf{x}).\end{aligned}$$

Due to the definition of the function f , this vector field satisfies

$$\begin{aligned}(\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}}) &= (\mathbf{u}_\varepsilon, \mathcal{V}_\varepsilon) \quad \text{on } \Gamma_\varepsilon^+, \\ (\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}}) &= (\mathbf{u}_\varepsilon^-, \mathcal{V}_\varepsilon^-) \quad \text{on } \Gamma_\varepsilon^-.\end{aligned}$$

4.1.2 Lift velocity using the Bogovskii formulae

One of the features of the previous Stokes system is that the divergence of $(\mathcal{R}, \mathcal{S})$ is not equal to zero. A classic method consists in making an lifting of the velocity field $(\mathcal{R}, \mathcal{S})$ by introducing a solution $(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})$ of the following problem:

$$(A'_\varepsilon) \left\{ \begin{array}{l} \operatorname{div}_{\mathbf{x}} \tilde{\mathcal{R}}_{\text{div}} + \partial_y \tilde{\mathcal{S}}_{\text{div}} = H, \quad \text{on } \Omega_\varepsilon, \\ \tilde{\mathcal{R}}_{\text{div}} = \mathbf{0}, \quad \text{on } \Gamma_\varepsilon^-, \\ \tilde{\mathcal{S}}_{\text{div}} = 0, \quad \text{on } \Gamma_\varepsilon^-, \\ \tilde{\mathcal{R}}_{\text{div}} = \mathbf{0}, \quad \text{on } \Gamma_\varepsilon^+, \\ \tilde{\mathcal{S}}_{\text{div}} = 0, \quad \text{on } \Gamma_\varepsilon^+. \end{array} \right.$$

where $H = \mathcal{H}_\varepsilon - (\operatorname{div}_{\mathbf{x}} \tilde{\mathcal{R}}_{\text{bound}} + \partial_y \tilde{\mathcal{S}}_{\text{bound}})$. An explicit solution of this system exists, it corresponds to the Bogovskii formulae (see [6]). The advantage of this formula is to allow to have precise estimations of the solution. In particular, we have (see for instance [13, p.121]):

Proposition 4.1 (Bogovskii [6]) *If $H \in H^m(\Omega_\varepsilon)$, $m \geq 0$ satisfies*

$$\int_{H^m(\Omega_\varepsilon)} H = 0, \quad (16)$$

then there exists a solution $(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}}) \in H^{m+1}(\Omega_\varepsilon)$ of problem (A'_ε) such that

$$\|\nabla_{\mathbf{x},y}(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{H^m(\Omega_\varepsilon)} \leq \frac{c}{\varepsilon} \|H\|_{H^m(\Omega_\varepsilon)},$$

where the constant c does not depend on ε . Besides, one has also

$$\|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{L^2(\Omega_\varepsilon)} \leq c \|H\|_{L^2(\Omega_\varepsilon)}.$$

Remark 4.1 *In fact, the constant c/ε that appears in the right hand side member is explicitly given in [13]. It depends on the geometry of the domain Ω_ε and, more precisely, it depends on the number of star-shaped subdomains with respect to some open ball to cover Ω_ε . For the rugous domain Ω_ε , let us focus on the boundary layer $\Omega_\varepsilon^-(t)$: the average slope of the roughness patterns is 1 whereas the thickness of the domain is ε so that the bottom of a roughness can be “seen” from a ball of radius $\mathcal{O}(\varepsilon)$. Thus, covering up the domain, whose length is of order 1, with such balls, we need $\mathcal{O}(1/\varepsilon)$ balls. Besides, a straightforward use of the Poincaré inequality (note that the domain thickness is of order ε) provides the L^2 -bound.*

Remark 4.2 *Note that the condition (16) exactly corresponds to the condition (15) satisfied for the Stokes system (A_ε) .*

4.2 Classical Stokes estimates

In order to cancel the boundary condition and the divergence of the vector field considered, we define $\overline{\mathcal{R}} = \mathcal{R} - (\widetilde{\mathcal{R}}_{\text{bound}} + \widetilde{\mathcal{R}}_{\text{div}})$ and $\overline{\mathcal{S}} = \mathcal{S} - (\widetilde{\mathcal{S}}_{\text{bound}} + \widetilde{\mathcal{S}}_{\text{div}})$. We have

$$\left\{ \begin{array}{ll} -\Delta_{\mathbf{x}} \overline{\mathcal{R}} - \partial_y^2 \overline{\mathcal{R}} + \nabla_{\mathbf{x}} Q &= \overline{\mathcal{F}}_{\varepsilon}, & \text{on } \Omega_{\varepsilon}, \\ -\Delta_{\mathbf{x}} \overline{\mathcal{S}} - \partial_y^2 \overline{\mathcal{S}} + \partial_y Q &= \overline{\mathcal{G}}_{\varepsilon}, & \text{on } \Omega_{\varepsilon}, \\ \operatorname{div}_{\mathbf{x}} \overline{\mathcal{R}} + \partial_y \overline{\mathcal{S}} &= 0, & \text{on } \Omega_{\varepsilon}, \\ \overline{\mathcal{R}} &= \mathbf{0}, & \text{on } \Gamma_{\varepsilon}^-, \\ \overline{\mathcal{S}} &= 0, & \text{on } \Gamma_{\varepsilon}^-, \\ \overline{\mathcal{R}} &= \mathbf{0}, & \text{on } \Gamma_{\varepsilon}^+, \\ \overline{\mathcal{S}} &= 0, & \text{on } \Gamma_{\varepsilon}^+. \end{array} \right.$$

where $\overline{\mathcal{F}}_{\varepsilon} = \mathcal{F}_{\varepsilon} - \Delta_{\mathbf{x},y}(\widetilde{\mathcal{R}}_{\text{bound}} + \widetilde{\mathcal{R}}_{\text{div}})$ and $\overline{\mathcal{G}}_{\varepsilon} = \mathcal{G}_{\varepsilon} - \Delta_{\mathbf{x},y}(\widetilde{\mathcal{S}}_{\text{bound}} + \widetilde{\mathcal{S}}_{\text{div}})$. We are now able to derive classical estimates:

Proposition 4.2 *One has:*

i) Estimates in the L^2 -norm:

$$\|(\overline{\mathcal{R}}, \overline{\mathcal{S}})\|_{L^2(\Omega_{\varepsilon})} \lesssim \varepsilon^2 \|(\overline{\mathcal{F}}_{\varepsilon}, \overline{\mathcal{G}}_{\varepsilon})\|_{L^2(\Omega_{\varepsilon})},$$

$$\|Q\|_{L^2(\Omega_{\varepsilon})} \lesssim \varepsilon \|(\overline{\mathcal{F}}_{\varepsilon}, \overline{\mathcal{G}}_{\varepsilon})\|_{L^2(\Omega_{\varepsilon})}.$$

ii) Estimates in the H^1 -norm:

$$\|(\overline{\mathcal{R}}, \overline{\mathcal{S}})\|_{H^1(\Omega_{\varepsilon})} \lesssim \varepsilon \|(\overline{\mathcal{F}}_{\varepsilon}, \overline{\mathcal{G}}_{\varepsilon})\|_{L^2(\Omega_{\varepsilon})},$$

$$\|Q\|_{H^1(\Omega_{\varepsilon})} \lesssim \|(\overline{\mathcal{F}}_{\varepsilon}, \overline{\mathcal{G}}_{\varepsilon})\|_{L^2(\Omega_{\varepsilon})}.$$

Proof. Choosing $\overline{\mathcal{R}}$ as a test function in the first equation, $\overline{\mathcal{S}}$ as test function in the second one and using the free divergence relation to cancel the pressure term, we obtain the following estimate

$$\begin{aligned} & \|\nabla_{\mathbf{x}} \overline{\mathcal{R}}\|_{L^2(\Omega_{\varepsilon})}^2 + \|\partial_y \overline{\mathcal{R}}\|_{L^2(\Omega_{\varepsilon})}^2 + \|\nabla_{\mathbf{x}} \overline{\mathcal{S}}\|_{L^2(\Omega_{\varepsilon})}^2 + \|\partial_y \overline{\mathcal{S}}\|_{L^2(\Omega_{\varepsilon})}^2 \\ & \leq \|\overline{\mathcal{F}}_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \|\overline{\mathcal{R}}\|_{L^2(\Omega_{\varepsilon})} + \|\overline{\mathcal{G}}_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \|\overline{\mathcal{S}}\|_{L^2(\Omega_{\varepsilon})}. \end{aligned}$$

Successively using the Poincaré inequality and the Young inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ in the right-hand side of the previous inequality, we obtain

$$\begin{aligned} \|\overline{\mathcal{F}}_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \|\overline{\mathcal{R}}\|_{L^2(\Omega_{\varepsilon})} & \leq c\varepsilon \|\overline{\mathcal{F}}_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})} \|\partial_y \overline{\mathcal{R}}\|_{L^2(\Omega_{\varepsilon})} \\ & \leq \frac{1}{2} \|\partial_y \overline{\mathcal{R}}\|_{L^2(\Omega_{\varepsilon})}^2 + \frac{1}{2} c^2 \varepsilon^2 \|\overline{\mathcal{F}}_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})}^2, \end{aligned}$$

where the constant c does not depend on ε . A similar estimate holds for the other source terms $\|\overline{\mathcal{G}}_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})}$ and $\|\overline{\mathcal{S}}\|_{L^2(\Omega_{\varepsilon})}$. Hence, we successively get (omitting the constants for the sake of simplicity)

$$\begin{aligned} & \|\nabla_{\mathbf{x}} \overline{\mathcal{R}}\|_{L^2(\Omega_{\varepsilon})}^2 + \|\partial_y \overline{\mathcal{R}}\|_{L^2(\Omega_{\varepsilon})}^2 + \|\nabla_{\mathbf{x}} \overline{\mathcal{S}}\|_{L^2(\Omega_{\varepsilon})}^2 + \|\partial_y \overline{\mathcal{S}}\|_{L^2(\Omega_{\varepsilon})}^2 \\ & \lesssim \varepsilon^2 \|\overline{\mathcal{F}}_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})}^2 + \varepsilon^2 \|\overline{\mathcal{G}}_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})}^2, \end{aligned} \quad (17)$$

Then, using the Poincaré inequality again we obtain

$$\|\overline{\mathcal{R}}\|_{L^2(\Omega_\varepsilon)}^2 + \|\overline{\mathcal{S}}\|_{L^2(\Omega_\varepsilon)}^2 \lesssim \varepsilon^4 \|\overline{\mathcal{F}}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon^4 \|\overline{\mathcal{G}}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2. \quad (18)$$

In the same way, taking respectively $-\Delta_x \overline{\mathcal{R}} - \partial_y^2 \overline{\mathcal{R}}$ and $-\Delta_x \overline{\mathcal{S}} - \partial_y^2 \overline{\mathcal{S}}$ as test functions in the two first equations of the Stokes problem, we get

$$\|\Delta_x \overline{\mathcal{R}}\|_{L^2(\Omega_\varepsilon)}^2 + \|\partial_y^2 \overline{\mathcal{R}}\|_{L^2(\Omega_\varepsilon)}^2 + \|\Delta_x \overline{\mathcal{S}}\|_{L^2(\Omega_\varepsilon)}^2 + \|\partial_y^2 \overline{\mathcal{S}}\|_{L^2(\Omega_\varepsilon)}^2 \lesssim \|\overline{\mathcal{F}}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\overline{\mathcal{G}}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2.$$

It is then easy to estimate the pressure:

$$\|\nabla_x \mathcal{Q}\|_{L^2(\Omega_\varepsilon)}^2 + \|\partial_y \mathcal{Q}\|_{L^2(\Omega_\varepsilon)}^2 \lesssim \|\overline{\mathcal{F}}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\overline{\mathcal{G}}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2. \quad (19)$$

Using the Poincaré-Wirtinger inequality we obtain

$$\|\mathcal{Q}\|_{L^2(\Omega_\varepsilon)}^2 \lesssim \varepsilon^2 \|\overline{\mathcal{F}}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon^2 \|\overline{\mathcal{G}}_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2. \quad (20)$$

All these estimate imply the result announced by the proposition. @blacksquare

Corollary 4.3 *In terms of velocities $(\mathcal{R}, \mathcal{S})$, one has:*

i) Estimates in the L^2 -norm:

$$\begin{aligned} \|(\mathcal{R}, \mathcal{S})\|_{L^2(\Omega_\varepsilon)} &\lesssim \varepsilon^2 \|(\mathcal{F}_\varepsilon, \mathcal{G}_\varepsilon)\|_{L^2(\Omega_\varepsilon)} + \varepsilon^2 \|(\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}})\|_{H^2(\Omega_\varepsilon)} \\ &\quad + \varepsilon^2 \|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{H^2(\Omega_\varepsilon)} + \|(\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}})\|_{L^2(\Omega_\varepsilon)} \\ &\quad + \|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{L^2(\Omega_\varepsilon)}, \end{aligned}$$

$$\begin{aligned} \|\mathcal{Q}\|_{L^2(\Omega_\varepsilon)} &\lesssim \varepsilon \|(\mathcal{F}_\varepsilon, \mathcal{G}_\varepsilon)\|_{L^2(\Omega_\varepsilon)} + \varepsilon \|(\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}})\|_{H^2(\Omega_\varepsilon)} \\ &\quad + \varepsilon \|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{H^2(\Omega_\varepsilon)}. \end{aligned}$$

ii) Estimates in the H^1 -norm:

$$\begin{aligned} \|(\mathcal{R}, \mathcal{S})\|_{H^1(\Omega_\varepsilon)} &\lesssim \varepsilon \|(\mathcal{F}_\varepsilon, \mathcal{G}_\varepsilon)\|_{L^2(\Omega_\varepsilon)} + \varepsilon \|(\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}})\|_{H^2(\Omega_\varepsilon)} \\ &\quad + \varepsilon \|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{H^2(\Omega_\varepsilon)} + \|(\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}})\|_{H^1(\Omega_\varepsilon)} \\ &\quad + \|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{H^1(\Omega_\varepsilon)}, \end{aligned}$$

$$\begin{aligned} \|\mathcal{Q}\|_{H^1(\Omega_\varepsilon)} &\lesssim \|(\mathcal{F}_\varepsilon, \mathcal{G}_\varepsilon)\|_{L^2(\Omega_\varepsilon)} + \|(\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}})\|_{H^2(\Omega_\varepsilon)} \\ &\quad + \|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{H^2(\Omega_\varepsilon)}. \end{aligned}$$

4.3 Explicit estimates with respect to ε

4.3.1 Control of the source terms

As the norms $\|\cdot\|_{L^2(\Omega_\varepsilon)}$ and $\|\cdot\|_{H^1(\Omega_\varepsilon)}$ highly depend on ε , the control of the source terms $\|\mathcal{F}_\varepsilon\|_{H^s(\Omega_\varepsilon)}^2$, $\|\mathcal{G}_\varepsilon\|_{H^s(\Omega_\varepsilon)}^2$, ... stated in Proposition 4.6, is not sufficient. Thus, the dependancy with respect to ε has to be given in an explicit

way: for this we must analyse each term of $\mathcal{F}_\varepsilon, \mathcal{G}_\varepsilon, \dots$. These terms are twofold: some of them are related to a thin film flow (in which case they depend on \mathbf{x} and y/ε) and some others are related to the boundary layer (in which case they depend on $\mathbf{x}, \mathbf{x}/\varepsilon^2$ and y/ε^2). We prove, in the Appendix A and Appendix B, page 46, the following propositions whose the goal is to control the source terms with respect to ε .

Proposition 4.4 *Let $f \in \mathcal{C}^1(\mathbb{T}^d \times \omega_{bl})$, such that $\mathbf{X} \mapsto f(\cdot, \mathbf{X}, \cdot)$ is periodic, $f(\cdot, \cdot, Y) = \mathcal{O}(e^{-Y})$ for $Y \rightarrow +\infty$ (uniformly w.r.t. the other variables). Let us consider the function f^ε defined by*

$$\forall (\mathbf{x}, y) \in \Omega_\varepsilon(t), \quad f^\varepsilon(\mathbf{x}, y) = f\left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2}\right).$$

Then we have

$$\|f^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2 \lesssim \varepsilon^2, \quad \|f^\varepsilon\|_{H^1(\Omega_\varepsilon(t))}^2 \lesssim 1/\varepsilon^2. \quad (21)$$

Proposition 4.5 *Let $g \in \mathcal{C}^0(\omega_R)$ defined on $\{(\mathbf{x}, Z), Z < 0\}$ by a regular extension and let us consider a function g^ε defined by*

$$\forall (\mathbf{x}, y) \in \Omega_\varepsilon(t), \quad g^\varepsilon(\mathbf{x}, y) = g\left(\mathbf{x}, \frac{y}{\varepsilon}\right).$$

Then we have

$$\|g^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2 \lesssim \varepsilon. \quad (22)$$

Recalling the definition of the source terms $\mathcal{F}_\varepsilon, \mathcal{G}_\varepsilon$ and \mathcal{H}_ε (see the beginning of Subsection 4.1), Equations (21) and (22) allow us to derive L^2 -estimates with respect to ε :

$$\begin{aligned} \|\mathcal{F}_\varepsilon\|_{L^2(\Omega_\varepsilon)} &\lesssim \varepsilon^{N-1}, \\ \|\mathcal{G}_\varepsilon\|_{L^2(\Omega_\varepsilon)} &\lesssim \varepsilon^{N-3/2}, \\ \|\mathcal{H}_\varepsilon\|_{L^2(\Omega_\varepsilon)} &\lesssim \varepsilon^{N+1}, \end{aligned} \quad (23)$$

and also H^1 -estimates:

$$\|\mathcal{H}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \lesssim \varepsilon^{N-1}.$$

4.3.2 Boundary lift

We first use the following estimates about the function f introduced in the subsection 4.1.1:

$$\begin{aligned} \|f\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{1/2}, & \|\nabla_{\mathbf{x}} f\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{-1/2}, & \|\partial_y f\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{-1/2}, \\ \|\nabla_{\mathbf{x}} \partial_y f\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{-3/2}, & \|\nabla_{\mathbf{x}}^2 f\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{-5/2}, & \partial_y^2 f &\equiv 0. \end{aligned}$$

We easily deduce the following bounds for the lift velocity at the boundary:

$$\begin{aligned}
\|\tilde{\mathcal{R}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N+3/2}, & \|\tilde{\mathcal{S}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N+5/2}, \\
\|\nabla_{\mathbf{x}}\tilde{\mathcal{R}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N+1/2}, & \|\nabla_{\mathbf{x}}\tilde{\mathcal{S}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N+1/2}, \\
\|\Delta_{\mathbf{x}}\tilde{\mathcal{R}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N-3/2}, & \|\Delta_{\mathbf{x}}\tilde{\mathcal{S}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N-3/2}, \\
\|\partial_y\tilde{\mathcal{R}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N+1/2}, & \|\partial_y\tilde{\mathcal{S}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N+3/2}, \\
\partial_y^2\tilde{\mathcal{R}}_{\text{bound}} &\equiv \mathbf{0}, & \partial_y^2\tilde{\mathcal{S}}_{\text{bound}} &\equiv 0, \\
\|\partial_y\nabla_{\mathbf{x}}\tilde{\mathcal{R}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N-1/2}, & \|\partial_y\nabla_{\mathbf{x}}\tilde{\mathcal{S}}_{\text{bound}}\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N-1/2}.
\end{aligned}$$

4.3.3 Bogovskii lift

From the estimates for the boundary lift $(\tilde{\mathcal{R}}_{\text{bound}}, \tilde{\mathcal{S}}_{\text{bound}})$, we can use the proposition 4.1 with a control on the source term H with respect to ε . We obtain

$$\begin{aligned}
\|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{L^2(\Omega_\varepsilon)} &\leq \varepsilon^{N+1/2} \\
\|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{H^1(\Omega_\varepsilon)} &\leq \varepsilon^{N-1/2} \\
\|(\tilde{\mathcal{R}}_{\text{div}}, \tilde{\mathcal{S}}_{\text{div}})\|_{H^2(\Omega_\varepsilon)} &\leq \varepsilon^{N-3/2}.
\end{aligned} \tag{24}$$

4.3.4 Estimates

Coupling the estimates on the source terms (23) and the estimates on the lift (24) we can rewrite the corollary 4.3 as follows:

Corollary 4.6 (Estimates on the remainder) *One has:*

$$\begin{aligned}
\|(\mathcal{R}, \mathcal{S})\|_{L^2(\Omega_\varepsilon)} &\lesssim \varepsilon^{N+1/2} \\
\|\mathcal{Q}\|_{L^2(\Omega_\varepsilon)} &\lesssim \varepsilon^{N-1/2} \\
\|(\mathcal{R}, \mathcal{S})\|_{H^1(\Omega_\varepsilon)} &\lesssim \varepsilon^{N-1/2} \\
\|\mathcal{Q}\|_{H^1(\Omega_\varepsilon)} &\lesssim \varepsilon^{N-3/2}
\end{aligned}$$

4.4 Error analysis on adapted spaces

In this subsection, we translate the previous estimates (which highly depend on the thin domain Ω_ε) into similar estimates in which the chosen norm does not depend on the thickness ε . This is motivated by the fact that the Ω_ε -norm of any constant function vanishes as ε tends to 0, as the measure of the domain tends to 0. Thus, estimates have to be expressed in suitable norms that do not depend on ε and allow us to capture the scale effects in both the rescaled “thin film domain” (i. e. the Reynolds domain) and the rescaled boundary layer (i. e. the Stokes domain).

Definition 4.7 (Rescaling operator and unfolding operator) *Let δ be a positive integer, and let $(\mathbf{x}, y) \in \mathbb{T}^d \times]a, b[$.*

i) The “rescaling operator”

$$\begin{aligned} \mathfrak{R}_\delta & : L^2(\mathbb{T}^d \times]a, b[) \rightarrow L^2(\mathbb{T}^d \times]a, b[) \\ f & \quad \mapsto \quad \mathfrak{R}_\delta(f), \end{aligned}$$

is defined by

$$\forall Z \in]a, b[, \quad \mathfrak{R}_\delta(f)(\cdot, Z) := f(\cdot, \delta Z).$$

ii) The “unfolding operator”

$$\begin{aligned} \mathfrak{U}_\delta & : L^2(\mathbb{T}^d \times]a, b[) \rightarrow L^2(\mathbb{T}^d \times]0, 1[^d \times]a, b[) \\ f & \quad \mapsto \quad \mathfrak{U}_\delta(f), \end{aligned}$$

is defined by

$$\forall \mathbf{x} \in \mathbb{T}^d, \quad \forall \mathbf{X} \in]0, 1[^d \quad \mathfrak{U}_\delta(f)(\mathbf{x}, \mathbf{X}, \cdot) := f\left(\delta \left\lfloor \frac{\mathbf{x}}{\delta} \right\rfloor + \delta \mathbf{X}, \cdot\right),$$

where $\lfloor \cdot \rfloor$ denotes the integer part in \mathbb{Z}^d .

Notice that the so-called “rescaling operator” only rescales the (vertical) last coordinate ; the “unfolding operator” only acts on the (horizontal) first variable. The main properties of these operators, from [12], are recalled in the appendix C. The formal development we have introduced requires the separation of the domain Ω_ε into two “sub-domains”: ω_R and ω_{bl} . To take into account the anisotropy of each of these domains, we express the usual $L^2(\Omega_\varepsilon)$ -norm as follows (the proof of this lemma is given in Appendix C).

Lemma 4.8 *Let $f \in H^1(\Omega_\varepsilon)$. The following estimates hold:*

i) *Zeroth order derivative:*

$$\|f\|_{L^2(\Omega_\varepsilon)}^2 = \varepsilon \|\mathfrak{R}_\varepsilon(f)\|_{L^2(\omega_R)}^2 + \varepsilon^2 \|\mathfrak{R}_{\varepsilon^2} \circ \mathfrak{U}_{\varepsilon^2}(f)\|_{L^2(\omega_{bl})}^2.$$

ii) *First order derivatives:*

$$\begin{aligned} \|\nabla_{\mathbf{x}} f\|_{L^2(\Omega_\varepsilon)}^2 &= \varepsilon^2 \|\nabla_{\mathbf{x}} (\mathfrak{R}_{\varepsilon^2} \circ \mathfrak{U}_{\varepsilon^2}(f))\|_{L^2(\omega_{bl})}^2 \\ &\quad + \frac{1}{\varepsilon^2} \|\nabla_{\mathbf{x}} (\mathfrak{R}_{\varepsilon^2} \circ \mathfrak{U}_{\varepsilon^2}(f))\|_{L^2(\omega_{bl})}^2 + \varepsilon \|\nabla_{\mathbf{x}} (\mathfrak{R}_\varepsilon(f))\|_{L^2(\omega_R)}^2, \\ \|\partial_y f\|_{L^2(\Omega_\varepsilon)}^2 &= \frac{1}{\varepsilon} \|\partial_Z (\mathfrak{R}_\varepsilon(f))\|_{L^2(\omega_R)}^2 + \frac{1}{\varepsilon^2} \|\partial_Y (\mathfrak{R}_{\varepsilon^2} \circ \mathfrak{U}_{\varepsilon^2}(f))\|_{L^2(\omega_{bl})}^2. \end{aligned}$$

Now let us define a norm that is adapted to the measure of a function for both the “thin film” approximation and the “roughness boundary layer” aspect:

$$\|f\|_s^2 = \|\mathfrak{R}_\varepsilon(f)\|_{H^s(\omega_R)}^2 + \|\mathfrak{R}_{\varepsilon^2} \circ \mathfrak{U}_{\varepsilon^2}(f)\|_{H^s(\omega_{bl})}^2.$$

Unlike the Ω_ε -norms whose drawback is to fail at capturing concentration effects, this norm preserves the constant states independantly from the value of ε . Thus, it is a correct way to characterize convergence results in both the thin film region and the boundary layer. We can re-write Proposition 4.6 using the norm $\|\cdot\|_s$:

Theorem 4.9 (Third estimates on the remainder) *One has:*

i) L^2 -estimates:

$$\|(\mathbf{u}, v) - (\mathbf{u}^{(N)}, v^{(N)})\|_0 \lesssim \varepsilon^{N-1/2}, \quad \|p - p^{(N)}\|_0 \lesssim \varepsilon^{N-3/2},$$

ii) H^1 -estimates:

$$\|(\mathbf{u}, v) - (\mathbf{u}^{(N)}, v^{(N)})\|_1 \lesssim \varepsilon^{N-3/2}, \quad \|p - p^{(N)}\|_1 \lesssim \varepsilon^{N-5/2}.$$

5 Discussion on the different scale effects

In this section, we discuss two qualitative aspects of the asymptotic expansion. Firstly, we focus on the contribution of the roughness patterns on the thin film approximation, considering the additive terms that tend to alter the approximation of the corresponding thin film flow. Then we focus on a typical case that is considered in most of the boundary layer studies and that is much simpler than in our study: the constant cross-section channel, meaning that function h^+ is constant. Not only this situation would not be relevant in lubrication, but also we show that such an assumption would fail in capturing essential coupling effects that we have described for the statement of the equivalent flow.

5.1 Contribution of the rugosities in the thin film approximation

Let us consider the thin film approximation with a flat bottom, i. e.

$$\forall \mathbf{X} \in]0, 1[^d, \quad h^-(\mathbf{X}) = 0.$$

Then the asymptotic expansion reduces to

$$\begin{aligned} \mathbf{u}_\Delta^{(N)}(\mathbf{x}, y, t) &= \mathbf{u}_0\left(\mathbf{x}, \frac{y}{\varepsilon}\right) + \sum_{j=1}^N \varepsilon^{2j} \mathbf{u}_{2j}\left(\mathbf{x}, \frac{y}{\varepsilon}\right), \\ v_\Delta^{(N)}(\mathbf{x}, y, t) &= \varepsilon v_0\left(x, \frac{y}{\varepsilon}\right) + \sum_{j=1}^N \varepsilon^{2j+1} v_{2j}\left(x, \frac{y}{\varepsilon}\right), \\ p_\Delta^{(N)}(\mathbf{x}, y, t) &= \varepsilon^{-2} p_0\left(\mathbf{x}, \frac{y}{\varepsilon}\right) + \sum_{j=1}^N \varepsilon^{2j-2} p_{2j-2}\left(\mathbf{x}, \frac{y}{\varepsilon}\right), \end{aligned}$$

meaning that only half of the sequence of Reynolds problems ($R^{(2j)}$) is considered. Moreover, each Reynolds problem is easy to treat: the pressure obeys an elliptic (Reynolds) equation and the corresponding velocity field can be deduced from the pressure gradient by means of a simple integration, see for instance [24]. When considering the roughness patterns, we can check that each Reynolds-type solution of problem ($R^{(2j)}$) has to be modified by a two-shot correction: *i)* a

Stokes boundary layer correction (in order to treat the boundary value default due to the extension of the Reynolds solution in the boundary layer) and *ii*) a Reynolds correction (in order to compensate the behaviour of the Stokes correction at infinity).

Note also that the introduction of the roughness patterns needs two levels of correction in the description of the thin film approximation, unlike most of the related boundary layer analysis: in particular, a Stokes flow in a constant cross-section channel with rugosities needs only one level of correction: the sequence of Stokes-type solutions is composed of a main classical Stokes solution, corrected by one-single Stokes boundary layer solution.

Besides, it is possible to draw a quantitative study of the convergence of the asymptotic expansion, with or without roughness correction, towards the solution of the full problem. More precisely, suppose that, aiming at evaluating the exact solution defined on $\Omega_\varepsilon(t)$, one uses the asymptotic expansion related to the thin film approximation only (i. e. voluntarily omitting the boundary layer corrections) ; then the error is not controlled by the remainder anymore. If the boundary layer corrections are neglected, the error may be controlled by the lack of precision due to the neglect of the first order boundary layer correction and the related remainder, namely

$$\begin{aligned} & \varepsilon \tilde{\mathbf{u}}_1 \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2} \right) + \varepsilon \mathbf{u}_1 \left(\mathbf{x}, \frac{y}{\varepsilon} \right) + \mathcal{O}(\varepsilon^2), \\ & \varepsilon \tilde{v}_1 \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2} \right) + \varepsilon^2 v_1 \left(\mathbf{x}, \frac{y}{\varepsilon} \right) + \mathcal{O}(\varepsilon^2), \\ & \varepsilon^{-1} \tilde{p}_1 \left(\mathbf{x}, \frac{\mathbf{x} - \mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2} \right) + \varepsilon^{-1} p_1 \left(\mathbf{x}, \frac{y}{\varepsilon} \right) + \mathcal{O}(1). \end{aligned}$$

Thus the error is controlled by the following estimates (using the results of propositions 4.4 and 4.5)

$$\|(\mathbf{u}, v) - (\mathbf{u}_\Delta^{(N)}, v_\Delta^{(N)})\|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon^{3/2}, \quad \|\varepsilon^2(p - p_\Delta^{(N)})\|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon^{1/2},$$

whereas the full asymptotic expansion satisfies (see Corollary 4.6):

$$\|(\mathbf{u}, v) - (\mathbf{u}^{(N)}, v^{(N)})\|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon^{N+1/2}, \quad \|\varepsilon^2(p - p^{(N)})\|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon^{N+1/2}.$$

Note that, at main order (i. e. $N = 1$), the truncated velocity field is of order $\varepsilon^{3/2}$ with or without roughness correction. In contrast, the approximation of the pressure distribution is severely altered by the neglect of the roughness correction ; this is a key-point in the framework of lubrication as one may be interested in the control of the load, defined as the L^1 -norm of the pressure in the whole domain.

5.2 Multiscale coupling effects due to the curvature of the film thickness

In most of the boundary layer problems dealing with Stokes or Navier-Stokes equations, the typical situation is concerned with a simple geometrical case, as the considered domain is a constant cross-section channel, perturbed by the roughness patterns. In the context of lubrication, this situation is not relevant as a converging-diverging profile has to be considered for the modelling of lubricated devices such as journal bearings, plain bearings, roller bearings etc. As we will state further, neglecting the curvature of the macroscopic thin film gap h^+ leads to a much simpler boundary layer analysis, in the sense that the related Stokes or Reynolds subproblems lead to straightforward computations of the corresponding solutions. More precisely, assuming that

$$\forall \mathbf{x} \in \mathbb{T}^d, \quad h^+(\mathbf{x}) := H > 0.$$

The computation of the solutions becomes much easier than in the full lubrication problem, as simplifications lead to a straightforward determination of solution which only depend on the second space variable:

- As a straightforward consequence of the main assumption in this subsection, any Stokes boundary layer problems is such that coefficients do not depend on \mathbf{x} as a parameter, i. e.

$$\forall j \in \mathbb{N}, \quad \mathbf{A}_j = \mathbf{0}, \quad B_j = 0, \quad C_j = 0.$$

- The main Reynolds problem ($R^{(0)}$) reduces to a simple Couette flow:

$$\mathbf{u}_0(\mathbf{x}, Z) = (1 - Z/H) \mathbf{s}, \quad v_0 \equiv 0, \quad p_0 \equiv 0.$$

The corrective Stokes problem ($S^{(1)}$) now reads

$$(S^{(1)}) \left\{ \begin{array}{ll} -\Delta_{\mathbf{x}} \tilde{\mathbf{u}}_1 - \partial_Y^2 \tilde{\mathbf{u}}_1 + \nabla_{\mathbf{x}} \tilde{p}_1 = \mathbf{0}, & \text{on } \omega_{bl}(0), \\ -\Delta_{\mathbf{x}} \tilde{v}_1 - \partial_Y^2 \tilde{v}_1 + \partial_Y \tilde{p}_1 = 0, & \text{on } \omega_{bl}(0), \\ \operatorname{div}_{\mathbf{x}} \tilde{\mathbf{u}}_1 + \partial_Y \tilde{v}_1 = 0, & \text{on } \omega_{bl}(0), \\ \tilde{\mathbf{u}}_1 = \tilde{\mathbf{u}}_1, & \text{on } \gamma_{bl}(0), \\ \tilde{v}_1 = 0, & \text{on } \gamma_{bl}(0), \\ (\tilde{\mathbf{u}}_1, \tilde{v}_1, \tilde{p}_1) \text{ is } \mathbf{X}\text{-periodic,} & \end{array} \right.$$

where the boundary condition should be read as $\tilde{\mathbf{u}}_1(\mathbf{X}) = -\frac{h^-(\mathbf{X})}{H} \mathbf{s}$. In the sequel, we will introduce the linear operators \mathcal{M}_u , \mathcal{M}_v and \mathcal{M}_p , which only depends on the data h^- and H , such that the solution of the Stokes problem ($S^{(1)}$) reads

$$\tilde{\mathbf{u}}_1 = \mathcal{M}_u \mathbf{s}, \quad \tilde{v}_1 = \mathcal{M}_v \mathbf{s}, \quad \tilde{p}_1 = \mathcal{M}_p \mathbf{s}.$$

Remark 5.1 *In practice, these operators \mathcal{M}_u , \mathcal{M}_v and \mathcal{M}_p are represented by matrices which are easily obtained by resolving d Stokes problem like ($S^{(1)}$) with the vectors of a base of \mathbb{R}^d instead of \mathbf{s} .*

Using the linearity of this Stokes problem and the proposition 3.2 we deduce that the limits α_1 and β_1 satisfy

$$\alpha_1 = \mathbf{L} \mathbf{s} \quad \text{and} \quad \beta_1 = 0,$$

where $\mathbf{L} \in \mathbb{M}_d(\mathbb{R})$ is the matrix of the application $\mathbf{s} \mapsto \frac{-1}{H} \mathbf{L}_u(0, h^-(\mathbf{X}) \mathbf{s}, 0)$.

Remark 5.2 *In practice, using the definition of \mathbf{L}_u , we have*

$$\mathbf{L} \mathbf{s} = -\frac{1}{H} \int_{[0,1]^d} \mathcal{M}_u \mathbf{s}(\mathbf{X}, 0) \, d\mathbf{X}.$$

• In that case, the corrective Reynolds problem ($\mathbf{R}^{(1)}$) can be written with respect to the unknowns $\mathbf{u}_1 + \alpha_1$, v_1 and p_1 . It is similar to the Reynolds problem ($\mathbf{R}^{(0)}$), replacing \mathbf{s} with $\alpha_1 = \mathbf{L} \mathbf{s}$. We obtain

$$\mathbf{u}_1(\mathbf{x}, Z) = -(Z/H) \mathbf{L} \mathbf{s}, \quad v_1 \equiv 0, \quad p_1 \equiv 0.$$

• In the same way, the corrective Stokes problem ($\mathbf{S}_C^{(2)}$) is similar to the Reynolds problem ($\mathbf{S}^{(1)}$), replacing \mathbf{s} with $\mathbf{L} \mathbf{s}$. By linearity, we deduce

$$\tilde{\mathbf{u}}_2 = \mathcal{M}_u \mathbf{L} \mathbf{s}, \quad \tilde{v}_2 = \mathcal{M}_v \mathbf{L} \mathbf{s}, \quad \tilde{p}_2 = \mathcal{M}_p \mathbf{L} \mathbf{s}.$$

• More generally, we obtain, for all $j \in \mathbb{N}^*$,

$$\begin{aligned} \mathbf{u}_j(\mathbf{x}, Z) &= -(Z/H) \mathbf{L}^j \mathbf{s}, & v_j &\equiv 0, & p_j &\equiv 0. \\ \tilde{\mathbf{u}}_j &= \mathcal{M}_u \mathbf{L}^{j-1} \mathbf{s}, & \tilde{v}_j &= \mathcal{M}_v \mathbf{L}^{j-1} \mathbf{s}, & \tilde{p}_j &= \mathcal{M}_p \mathbf{L}^{j-1} \mathbf{s}. \end{aligned}$$

Such analysis implies that, for $N \in \mathbb{N}$ we obtain, for instance for the horizontal velocity $\mathbf{u}^{(N)}$:

$$\mathbf{u}^{(N)} = \mathbf{s} - \left(\frac{\mathcal{Z}}{H} - \varepsilon \mathcal{M}_u \right) \left((\mathbf{Id} + \varepsilon \mathbf{L} + \dots + (\varepsilon \mathbf{L})^N) \mathbf{s} \right),$$

where the symbol \mathcal{Z} denotes the application $(\mathbf{x}, y, t) \mapsto y/\varepsilon$. Passing to the limit $N \rightarrow +\infty$ we introduce

$$\begin{aligned} \mathbf{u}^{(\infty)} &= \mathbf{s} - \left(\frac{\mathcal{Z}}{H} - \varepsilon \mathcal{M}_u \right) \left((\mathbf{Id} - \varepsilon \mathbf{L})^{-1} \mathbf{s} \right), \\ v^{(\infty)} &= \varepsilon \mathcal{M}_v \left((\mathbf{Id} - \varepsilon \mathbf{L})^{-1} \mathbf{s} \right), \\ p^{(\infty)} &= \frac{1}{\varepsilon} \mathcal{M}_p \left((\mathbf{Id} - \varepsilon \mathbf{L})^{-1} \mathbf{s} \right). \end{aligned}$$

The functions $(\mathbf{u}^{(\infty)}, v^{(\infty)}, p^{(\infty)})$ exactly satisfies the Stokes equations in Ω_ε , and the bottom boundary condition corresponding to the initial Stokes problem. Using an Fourier analysis (exactly as in the part 3.1.1), we show that the boundary condition on the top boundary is satisfied with an error of order $e^{-H/\varepsilon}$. Following the same method that in the part 4, we deduce that, for any sobolev norms,

$$\|\mathbf{u} - \mathbf{u}^{(\infty)}\| \lesssim e^{-H/\varepsilon}, \quad \|v - v^{(\infty)}\| \lesssim e^{-H/\varepsilon}, \quad \|p - p^{(\infty)}\| \lesssim e^{-H/\varepsilon}.$$

Proposition 5.1 *If the height h^+ is constant then it suffices to solve d Stokes problems to deduce an approximation with an exponential decreasing error.*

A Proof of Proposition 4.4

Expressing the L^2 -norm of function f^ε , we have

$$\|f^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2 = \int_{\mathbb{T}^d} \left(\int_{-\varepsilon^2 h^-((\mathbf{x}-\mathbf{st})/\varepsilon^2)}^{\varepsilon h^+(\mathbf{x})} \left| f\left(\mathbf{x}, \frac{\mathbf{x}-\mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2}\right) \right|^2 dy \right) d\mathbf{x}.$$

Using the change of variable $Y = y/\varepsilon^2$, we get

$$\begin{aligned} \|f^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2 &= \varepsilon^2 \int_{\mathbb{T}^d} \left(\int_{-h^-((\mathbf{x}-\mathbf{st})/\varepsilon^2)}^{h^+(\mathbf{x})/\varepsilon} \left| f\left(\mathbf{x}, \frac{\mathbf{x}-\mathbf{st}}{\varepsilon^2}, Y\right) \right|^2 dY \right) d\mathbf{x} \\ &\leq \varepsilon^2 \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}} \left| f\left(\mathbf{x}, \frac{\mathbf{x}-\mathbf{st}}{\varepsilon^2}, Y\right) \right|^2 \mathbf{1}_{[Y > -h^-((\mathbf{x}-\mathbf{st})/\varepsilon^2)]}(Y) dY \right) d\mathbf{x} \end{aligned}$$

Now considering the function

$$\begin{aligned} \mathcal{F} : \mathbb{T}^d \times]0, 1[^d &\rightarrow \mathbb{R} \\ (\mathbf{x}, \mathbf{X}) &\mapsto \int_{\mathbb{R}} \left| f\left(\mathbf{x}, \mathbf{X}, Y\right) \right|^2 \mathbf{1}_{[Y > -h^-(\mathbf{x})]}(Y) dY, \end{aligned}$$

we use a straightforward adaptation of Theorem 2 in [19] to obtain:

$$\int_{\mathbb{T}^d} |\mathcal{F}(\mathbf{x}, \mathbf{x}/\varepsilon)| d\mathbf{x} \leq \int_{\mathbb{T}^d} \sup_{\mathbf{X} \in]0, 1[^d} |\mathcal{F}(\mathbf{x}, \mathbf{X})| d\mathbf{x}.$$

By periodicity with respect to the second variable, the same argument applies to function $(\mathbf{x}, \mathbf{X}) \mapsto \mathcal{F}(\mathbf{x}, \mathbf{X} - \mathbf{st}/\varepsilon^2)$ so that, defining the constant

$$C(f) := \int_{\mathbb{T}^d} \left(\sup_{\mathbf{X} \in]0, 1[^d} \left| \int_{\mathbb{R}} \left| f\left(\mathbf{x}, \mathbf{X}, Y\right) \right|^2 \mathbf{1}_{[Y > -h^-(\mathbf{x})]}(Y) dY \right| \right) d\mathbf{x},$$

we obtain:

$$\|f^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2 \leq \varepsilon^2 C(f).$$

In order to state the H^1 -estimates, we proceed as follows: first we have

$$\begin{aligned} \nabla_{\mathbf{x}} f^\varepsilon(\mathbf{x}, y, t) &= \nabla_{\mathbf{x}} f\left(\mathbf{x}, \frac{\mathbf{x}-\mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2}\right) + \frac{1}{\varepsilon^2} \nabla_{\mathbf{x}} f\left(\mathbf{x}, \frac{\mathbf{x}-\mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2}\right). \\ \partial_y f^\varepsilon(\mathbf{x}, y, t) &= \frac{1}{\varepsilon^2} \partial_Y f\left(\mathbf{x}, \frac{\mathbf{x}-\mathbf{st}}{\varepsilon^2}, \frac{y}{\varepsilon^2}\right). \end{aligned}$$

Then we apply the previous computation related to the L^2 -estimate in order to get the result:

$$\|\nabla_{\mathbf{x}, y} f^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2 = \|\nabla_{\mathbf{x}} f^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2 + \|\partial_y f^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2.$$

Using the formula $(a + b)^2 \leq 2(a^2 + b^2)$ for the derivative with respect to \mathbf{x} , and using the previous L^2 -estimate, we get:

$$\|\nabla_{\mathbf{x},y} f^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2 \leq 2\varepsilon^2 C(\nabla_{\mathbf{x}} f) + 2\varepsilon^{-2} C(\nabla_{\mathbf{x}} f) + \varepsilon^{-2} C(\partial_Y f).$$

Finally, the following H_1 -estimate is obtained:

$$\|f^\varepsilon\|_{H^1(\Omega_\varepsilon(t))}^2 \leq \varepsilon^2 C(f) + 2\varepsilon^2 C(\nabla_{\mathbf{x}} f) + 2\varepsilon^{-2} C(\nabla_{\mathbf{x}} f) + \varepsilon^{-2} C(\partial_Y f).$$

which concludes the proof.

B Proof of Proposition 4.5

By means of a simple calculation, we get

$$\begin{aligned} \|g^\varepsilon\|_{L^2(\Omega_\varepsilon(t))}^2 &= \int_{\mathbb{T}^d} \left(\int_{-\varepsilon^2 h - ((\mathbf{x}-\mathbf{s}t)/\varepsilon^2)}^{\varepsilon h^+(\mathbf{x})} \left| g\left(\mathbf{x}, \frac{y}{\varepsilon}\right) \right|^2 dy \right) d\mathbf{x} \\ &= \varepsilon \int_{\mathbb{T}^d} \left(\int_{-\varepsilon h - ((\mathbf{x}-\mathbf{s}t)/\varepsilon^2)}^{h^+(\mathbf{x})} |g(\mathbf{x}, Z)|^2 dZ \right) d\mathbf{x} \\ &= \varepsilon \int_{\mathbb{T}^d} \left(\int_0^{h^+(\mathbf{x})} |g(\mathbf{x}, Z)|^2 dZ \right) d\mathbf{x} + \mathcal{O}(\varepsilon^2), \end{aligned}$$

which states the result.

C Proof of Lemma 4.8

First, we infer from [12] the following property of the unfolding operator:

Proposition C.1 *One has:*

i) For any $f, g \in L^2(\mathbb{T}^d \times]a, b[)$,

$$\mathfrak{U}_\delta(fg) = \mathfrak{U}_\delta(f) \mathfrak{U}_\delta(g).$$

ii) For any $f \in L^1(\mathbb{T}^d \times]a, b[)$,

$$\int_{\mathbb{T}^d} \int_a^b f(\mathbf{x}, y) dy d\mathbf{x} = \int_{\mathbb{T}^d} \int_{]0,1[^d} \int_a^b \mathfrak{U}_\delta(f)(\mathbf{x}, \mathbf{X}, y) dy d\mathbf{X} d\mathbf{x}.$$

Using this proposition, we now prove the Lemma 4.8. More precisely, we prove item *i*) (other items may be proven with a straightforward computation). Thus we have:

$$\|f\|_{L^2(\Omega_\varepsilon)}^2 = \|f\|_{L^2(\Omega_\varepsilon^+)}^2 + \|f\|_{L^2(\Omega_\varepsilon^-)}^2.$$

Using the properties of the rescaling operator,

$$\begin{aligned}
 \|f\|_{L^2(\Omega_\varepsilon^+)}^2 &:= \int_{\mathbb{T}^d} \int_0^{\varepsilon h^+(\mathbf{x})} |f(\mathbf{x}, y)|^2 dy d\mathbf{x} \\
 &= \varepsilon \int_{\mathbb{T}^d} \int_0^{h^+(\mathbf{x})} |f(\mathbf{x}, \varepsilon Z)|^2 dZ d\mathbf{x} \quad (\text{a}) \\
 &= \varepsilon \int_{\mathbb{T}^d} \int_0^{h^+(\mathbf{x})} |\mathfrak{R}_\varepsilon(f)(\mathbf{x}, Z)|^2 dZ d\mathbf{x} \quad (\text{b})
 \end{aligned}$$

where we have used (a) the change of variables $y = \varepsilon Z$, (b) the definition of the rescaling operator. Using the properties of the rescaling and unfolding operators,

$$\begin{aligned}
 \|f\|_{L^2(\Omega_\varepsilon^-)}^2 &:= \int_{\mathbb{T}^d} \int_{-\varepsilon^2 h^-(\frac{\mathbf{x}}{\varepsilon^2})}^0 |f(\mathbf{x}, y)|^2 dy d\mathbf{x} \\
 &= \int_{\mathbb{T}^d} \int_{\mathbb{R}} |f(\mathbf{x}, y)|^2 \mathbf{1}_{]-\varepsilon^2 h^-(\frac{\mathbf{x}}{\varepsilon^2}), 0[}(y) dy d\mathbf{x} \\
 &= \int_{\mathbb{T}^d} \int_{]0, 1[^d} \mathcal{U}_{\varepsilon^2} \left(\int_{\mathbb{R}} |f(\cdot, y)|^2 \mathbf{1}_{]-\varepsilon^2 h^-(\frac{\mathbf{x}}{\varepsilon^2}), 0[}(y) dy \right) (\mathbf{x}, \mathbf{X}) d\mathbf{X} d\mathbf{x} \quad (\text{a}) \\
 &= \int_{\mathbb{T}^d} \int_{]0, 1[^d} \left(\int_{\mathbb{R}} |\mathfrak{U}_{\varepsilon^2}(f)|^2(\mathbf{x}, \mathbf{X}, y) \mathbf{1}_{]-\varepsilon^2 h^-(\mathbf{x}), 0[}(y) dy \right) d\mathbf{X} d\mathbf{x} \quad (\text{b}) \\
 &= \int_{\mathbb{T}^d} \int_{]0, 1[^d} \left(\int_{-\varepsilon^2 h^-(\mathbf{x})}^0 |\mathfrak{U}_{\varepsilon^2}(f)|^2(\mathbf{x}, \mathbf{X}, y) dy \right) d\mathbf{X} d\mathbf{x} \\
 &= \varepsilon^2 \int_{\mathbb{T}^d} \int_{]0, 1[^d} \left(\int_{-h^-(\mathbf{x})}^0 |\mathfrak{U}_{\varepsilon^2}(f)|^2(\mathbf{x}, \mathbf{X}, \varepsilon^2 Y) dY \right) d\mathbf{X} d\mathbf{x} \quad (\text{c}) \\
 &= \varepsilon^2 \int_{\mathbb{T}^d} \int_{]0, 1[^d} \left(\int_{-h^-(\mathbf{x})}^0 |\mathfrak{R}_{\varepsilon^2} \circ \mathfrak{U}_{\varepsilon^2}(f)|^2(\mathbf{x}, \mathbf{X}, Y) dY \right) d\mathbf{X} d\mathbf{x} \quad (\text{d})
 \end{aligned}$$

where we have used (a) Proposition C.1-ii), (b) Proposition C.1-i), (c) the change of variables $y = \varepsilon^2 Y$, (d) the definition of the rescaling operator.

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