Steady state solutions for a lubrication multi-fluid flow

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Abstract - In this paper, we describe possible solutions for a stationary flow of two superposed fluids between two close surfaces in relative motion. Physically, this study is within the lubrication framework, in which it is of interest to predict the relative positions of the lubricant and the air in the device. Mathematically, we observe that this problem corresponds to finding the interface between the two fluids, and we prove that it is equivalent to solve some polynomial equation.

We solve this equation using an original method of polynomial resolution. First, we check that our results are consistent with previous work. Next, we use this solution to answer some physically relevant questions related to the lubrication setting. For instance, we obtain theoretical and numerical results enabling to predict the apparition of a full film with respect to physical parameters (fluxes, shear velocity, . . .). In particular, we present a figure giving the number of stationary solutions depending on the physical parameters.

Moreover, in the last part, we give some indications for a better understanding of the multi-fluid case.

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Key words: Cavitation, Lubrication, Multi-fluid, Saturation, Steady state, Stokes equations, Reynolds equation.

1 Introduction

Many works are devoted to the study of free boundary problems for hydrodynamic flows in the thin film setting, and in particular to the cavitation phenomenon. This phenomenon corresponds to the occurrence of air in a lubricant flow, and is of particular interest for industrial applications, since it is essential to have a full film of lubricant in order to avoid damage to the lubricated surfaces. The understanding and possible prediction of cavitation is very challenging. One of the main questions is to foresee the experimental conditions preventing the cavitation to happen (geometry of the lubricated device, injection velocity, shear velocity...).

Several mathematical models have been developed to take into account the cavitation [5, 6]. One approach is to consider the mixture air-lubricant as a two-fluid flow. Although early studies of multi-fluid problems in the lubrication context assumed that the boundary between the two immiscible fluids was known [2, 10], the real problem is a free boundary one. More generally, it consists in solving the Navier-Stokes or Stokes

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equations for two fluids of different viscosities. The corresponding model has been studied from a mathematical point of view in general domains [7]. Further, this model has been investigated in the thin film setting under the assumption that the two fluids are separated by the graph of a function. In [8], Paoli justified the asymptotic model in thin domains and obtained a limit problem coupling a Buckley-Leverett equation and the Reynolds law. More recently, in [3], Bayada, Martin, Vázquez studied this system with a non-zero shear velocity, which is realistic for most lubrication problems due to the difference of velocities between the surrounding surfaces. Their numerical results indicated the existence of stationary solutions; in fact, the time scale of lubrication applications is large enough to neglect the transient state and consider stationary solutions.

However, the solutions in [3] seem to be in contradiction with the non-miscibility property proved in [7, Lemma 2.3]. In fact, these stationary solutions have (vertical) interfaces between the two fluids along which the flow velocity is not tangent. Note that the non-miscibility property, satisfied in the general setting, can be proved in the thin film approximation in the case of regular enough interfaces. For less regular interfaces, although we can not prove rigorously this property, it seems physically relevant to keep this property as an additional assumption. In this paper, we are thus concerned with solutions satisfying this non-miscibility property. We propose a new formulation for this model, which consists in a polynomial equation on the function describing the interface separating the two fluids. On the one hand, we recover the numerical results obtained in [3], which treated the case of a small viscosity ratio between the two fluids. On the other hand, we obtain theoretical and numerical results for the existence and uniqueness of solutions for more general data. Moreover, these results allow us to predict the saturation phenomenon, namely to predict for which parameters a full film of lubricant is formed, which avoids cavitation.

This paper is organized as follows: in section 2, we present the two-fluid model of [7] and recall the non-miscibility property. Then we derive formally the limit model in a thin domain. Assuming that the two fluids are separated by the graph of a function, the model is written as a polynomial equation (see equation (2.8)). Section 3 is devoted to existence and uniqueness results, in the no-shear case (subsection 3.1) and in the lubrication context (subsection 3.2). More precisely, in subsection 3.2.1, we show how our model allows us to recover the previous results of [3]. Subsection 3.2.2 is concerned with the theoretical study of the small viscosity ratio case, which is a case of physical interest (e.g. lubricant/air case). In subsection 3.2.3, we obtain by theoretical and numerical methods saturation criteria, which allow to prevent cavitation. Finally, in section 4 we tackle the multi-fluid case. In particular we show, using the example of a three-layers flow, how to generalize the approach developed in section 3. In Appendix, we recall some useful results on Sturm's theorem, which is a key step for the resolution of our polynomial equation.

2 Governing equations

2.1 Two-fluid model in a thin domain

We consider the flow of two different fluids of viscosities η_1, η_2 in a Lipschitz domain $\Omega \subset \mathbb{R}^2$. The flow is described by the Stokes equations, coupling the velocity $\boldsymbol{v} = (u, w)$ and the pressure p. The global viscosity $\eta : \Omega \to \{\eta_1, \eta_2\}$ satisfies a transport equation. The existence of a solution to this system is proved in [7]. In the stationary setting, the model is written:

$$\begin{cases} \operatorname{div}(2\eta\, D\boldsymbol{v}) = \nabla p, \\ \operatorname{div} \boldsymbol{v} = 0, \\ \boldsymbol{v} \cdot \nabla \eta = 0, \\ \boldsymbol{v}\big|_{\partial\Omega} \text{ given}, \\ \eta\big|_{\partial\Omega_{\text{in}}} \text{ given}, \end{cases} \tag{2.1}$$

where $\partial\Omega_{\text{in}}$ denotes the part of the boundary $\partial\Omega$ where the velocity $\boldsymbol{v}$ is incoming. It is underlined in [7, Lemma 2.1] that system (2.1) in its unsteady form implies that the velocity field $\boldsymbol{v}$ and the stress tensor $2\eta\, D\boldsymbol{v} - p\mathrm{Id}$ are continuous. Furthermore, the results in [7] are obtained under the following assumption:

$$\Omega_1 = \eta^{-1}(\eta_1) \text{ and } \Omega_2 = \eta^{-1}(\eta_2) \text{ are Lipschitz domains.} \tag{2.2}$$

To our knowledge, no existence or uniqueness results has been proved for the stationary equations. However, some results of [7] can easily be adapted for system (2.1). Thus, Lemma 2.3 of [7] becomes:

Property 2.1 *For any solution $(\boldsymbol{v}, p, \eta)$ of (2.1)-(2.2), the velocity field $\boldsymbol{v}$ is continuous and tangent to the discontinuity lines of η.*

Since we are interested here in lubrication applications, we consider two fluids flowing between two close surfaces in relative motion. The domain is assumed to be thin by depeding on a small parameter ε and is written

$$\Omega_\varepsilon = \left\{(x, z) \in \mathbb{R}^2, \quad 0 < x < 1, \quad 0 < z < \varepsilon h(x)\right\},$$

where the function h describes the gap between the two surfaces. Moreover, the relative motion of the two surfaces is modeled by a shear velocity s at the bottom of the domain.

Formally, system (2.1) tends to the following system when ε tends to zero:

$$\begin{cases} \partial_z(\eta\partial_z u) = \partial_x p, \quad \partial_z p = 0, \\ \partial_x\left(\displaystyle\int_0^h u\,dz\right) = 0, \\ \partial_x\left(\displaystyle\int_0^h \eta u\,dz\right) = 0, \\ u(x,0) = s, \quad u(x,h(x)) = 0, \quad \displaystyle\int_0^{h(0)} u\,dz \text{ given}, \\ \displaystyle\int_0^{h(0)} \eta u\,dz \text{ given}. \end{cases} \tag{2.3}$$

The rigorous asymptotic study when ε tends to zero has been done in [8]. Clearly, solutions of (2.3) are not solutions of (2.1) for two reasons:

- When passing to the limit $\varepsilon \to 0$, we loose the x-derivatives in the $\operatorname{div}(2\eta\, D\cdot)$-operator.
- Nothing proves that solutions of (2.3) satisfy property 2.1, except in some special cases (see remark 2.7).

However, we can prove the following proposition.

Proposition 2.2 *For any solution (u,p,η) of (2.3) satisfying property 2.1, there exists $\boldsymbol{v} = (u,w)$ satisfying in a weak sense the following equations of (2.1):*

$$\begin{cases} div\,\boldsymbol{v} = 0, \\ \boldsymbol{v}\cdot\nabla\eta = 0, \\ \boldsymbol{v}(x,0) = (s,0), \quad \boldsymbol{v}(x,h(x)) = 0. \end{cases}$$

Proof For any solution (u,p,η) of (2.3), let us introduce $w = -\partial_x\left(\int_0^z u\right)$, and define $\boldsymbol{v} = (u,w)$.

- We have $\operatorname{div}\boldsymbol{v} = \partial_x u + \partial_z w = 0$.
- It is also clear that $w(x,0) = 0$, and using that $\partial_x\left(\int_0^h u\,dz\right) = 0$, we compute $w(x,h(x)) = 0$.
- Moreover, let $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, and $\boldsymbol{n}$ the normal vector to Γ. For any $\varphi \in \mathcal{C}_0^\infty(\Omega)$, since $\operatorname{div}\boldsymbol{v} = 0$, we have
$$-\int_\Omega \boldsymbol{v}\cdot\nabla\eta\,\varphi = \int_\Omega \eta u\partial_x\varphi + \eta w\partial_z\varphi.$$
Using that $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ and the divergence theorem, we get
$$\begin{aligned} -\int_\Omega \boldsymbol{v}\cdot\nabla\eta\,\varphi &= \eta_1\int_{\Omega_1}\Big(u\partial_x\varphi + w\partial_z\varphi\Big) + \eta_2\int_{\Omega_2}\Big(u\partial_x\varphi + w\partial_z\varphi\Big), \\ &= (\eta_1-\eta_2)\int_\Gamma \boldsymbol{v}\cdot\boldsymbol{n} = 0. \end{aligned}$$

$\square$

Therefore, if we find a solution of (2.3) satisfying property 2.1, then it is almost a solution of (2.1) (except for the Stokes equation, due to the thin film approximation).

2.2 Fluid interface described as a graph

As in previous works in the lubrication context [8, 3], we are looking for solutions of (2.3) so that the two fluids are separated by the graph of a function h_1 (see Figure 1 (a)).

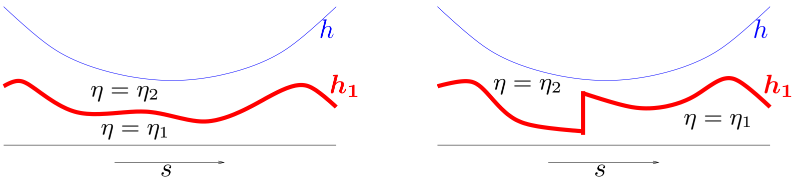

FIGURE 1. Flow domain and fluid interface: function h_1 (a) continuous and (b) discontinuous

Remark 2.3 *If the function h_1 is not continuous, we consider the curve separating the two fluids which is the graph of h_1 almost everywhere, see Figure 1 (b).*

As a convention, we denote by η_1 the viscosity of the bottom fluid and η_2 the viscosity of the top fluid. In system (2.3), we can decompose

$$\partial_x\left(\int_0^h u\,dz\right) = \partial_x\left(\int_0^{h_1} u\,dz\right) + \partial_x\left(\int_{h_1}^h u\,dz\right),$$
$$\partial_x\left(\int_0^h \eta u\,dz\right) = \eta_1\,\partial_x\left(\int_0^{h_1} u\,dz\right) + \eta_2\,\partial_x\left(\int_{h_1}^h u\,dz\right).$$

Consequently, since $\eta_1 \neq \eta_2$, system (2.3) is equivalent to the following system:

$$\begin{cases} \partial_z(\eta\partial_z u) = \partial_x p, \quad \partial_z p = 0, \\ \partial_x\left(\displaystyle\int_0^{h_1} u\,dz\right) = 0, \\ \partial_x\left(\displaystyle\int_{h_1}^{h} u\,dz\right) = 0, \\ u(x,0) = s, \quad u(x,h(x)) = 0, \\ \displaystyle\int_0^{h_1} u\,dz = Q_1, \quad \int_{h_1}^{h} u\,dz = Q_2, \end{cases} \tag{2.4}$$

where Q_1 and Q_2 correspond respectively to the flux of each fluid. They can be determined from the given data of system (2.3). Integrating the first equation of (2.4) with respect to the vertical variable on the interval $[0,z]$, and using the continuity of the stress $\eta\partial_z u$, we obtain

$$\eta\partial_z u = z\partial_x p - A,$$

where the constant $A = -\eta\partial_z u|_{z=0}$ corresponds physically to the stress on the bottom of the domain. For each subdomain, we can deduce the value of $\partial_z u$ with respect to the pressure p:

$$\partial_z u = \begin{cases} \dfrac{z}{\eta_1}\partial_x p - \dfrac{A}{\eta_1} & z \in [0, h_1), \\ \dfrac{z}{\eta_2}\partial_x p - \dfrac{A}{\eta_2} & z \in (h_1, h]. \end{cases}$$

Integrating again with respect to the vertical variable z (and using the velocity continuity and the boundary condition $u(x, 0) = s$), we obtain

$$u = \begin{cases} \dfrac{z^2}{2\eta_1}\partial_x p - \dfrac{A}{\eta_1} z + s & z \in [0, h_1), \\ \dfrac{h_1^2}{2\eta_1}\partial_x p - \dfrac{A}{\eta_1} h_1 + s + \dfrac{z^2 - h_1^2}{2\eta_2}\partial_x p - \dfrac{A}{\eta_2}(z - h_1) & z \in (h_1, h]. \end{cases} \tag{2.5}$$

The constant A can be determined from the boundary condition $u(x, h(x)) = 0$:

$$A = \frac{q}{2\ell}\partial_x p + \frac{s}{\ell},$$

with

$$\ell = \frac{h_1}{\eta_1} + \frac{h - h_1}{\eta_2}, \qquad q = \frac{h_1^2}{\eta_1} + \frac{h^2 - h_1^2}{\eta_2}.$$

Recall that the two fluxes $Q_1 = \int_0^{h_1} u$ and $Q_2 = \int_{h_1}^{h} u$ are given data in system (2.4). From the previous expression of the velocity u we deduce a relation between these fluxes and the pressure p:

$$\begin{cases} 12\eta_1 \ell Q_1 = h_1^2\Big(2\ell h_1 - 3q\Big)\partial_x p + 6h_1(2\eta_1\ell - h_1)s, \\ 12\eta_2 \ell Q_2 = (h - h_1)^2\Big(3q - 2\ell h_1 - 4\ell h\Big)\partial_x p + 6(h - h_1)^2 s. \end{cases} \tag{2.6}$$

Remark 2.4 *When $h_1 = h$ we recover the case of a single fluid of viscosity η_1 and a flux Q_1:*

$$Q_1 = -\frac{h^3}{12\eta_1}\partial_x p + \frac{h}{2}s.$$

The conservation of this flux corresponds to the classical Reynolds equation. In the same way, when $h_1 = 0$, we obtain the case of a single fluid of viscosity η_2 and a flux Q_2.

Remark 2.5 *Note that in the case of one single fluid, the choice of the boundary conditions is arbitrary. Indeed, it follows immediately from the Reynolds equation (see remark 2.4) that the boundary conditions we chose (imposing the flux, and one condition on the pressure)*

$$p(L) = 0, \quad Q_1 \text{ given},$$

are equivalent to pressure boundary conditions

$$p(L) = 0, \quad p(0) \text{ given},$$

since the flux Q_1 can be computed from $p(0)$ and $p(L)$. However, it is not clear that this equivalence still holds in the two-fluid case, since relation (2.7) between $\partial_x p$ and Q_1

depends on the unknown h_1. We chose to impose

$$p(L) = 0, \quad Q_1 \text{ given} \quad \text{and} \quad Q_2 \text{ given},$$

which allows us to write equation (2.8) on h_1 as a function of the given data. If we had chosen pressure boundary conditions, we would not have known how to write this relation.

Using the Sturm sequences (see Appendix, page 27 and example 5.1), we prove that there exists no height $h_1 \in (0, h)$ such that $2\ell h_1 - 3q = 0$. In other words, we can express $\partial_x p$ from the first equation of system (2.6):

$$\partial_x p = \frac{12\eta_1 \ell Q_1 - 6h_1(2\eta_1\ell - h_1)s}{h_1^2(2\ell h_1 - 3q)}. \tag{2.7}$$

Using this expression in the second equation of system (2.6) we obtain a polynomial equation for the unknown h_1:

$$(h-h_1)^2\Big(2\eta_1\ell Q_1 - h_1(2\eta_1\ell - h_1)s\Big)\Big(3q - 2\ell h_1 - 4\ell h\Big) = h_1^2\Big(2\eta_2\ell Q_2 - (h-h_1)^2 s\Big)\Big(2\ell h_1 - 3q\Big). \tag{2.8}$$

Recall that in this equation, the quantity h models the height of the domain (it depends on the coordinate x so that the solution h_1 depends on x too), the data η_1, Q_1, η_2 and Q_2 respectively correspond to the viscosity and the flux of each fluid, and the values ℓ and q are respectively linear and quadratic with respect to the unknown h_1:

$$\ell = \frac{h_1}{\eta_1} + \frac{h - h_1}{\eta_2}, \qquad q = \frac{h_1^2}{\eta_1} + \frac{h^2 - h_1^2}{\eta_2}.$$

Solving system (2.4) is then equivalent to finding a function h_1 defined on $[0, L]$ and satisfying equation (2.8). Next, the pressure is explicitly given, up to an additive constant, from equation (2.7). The additive constant is computed using the boundary condition $p(L) = 0$. Finally, the velocity is computed from equation (2.5).

Remark 2.6 *For all $x \in (0, 1)$, the velocity profiles $u(x, \cdot)$ are piecewise parabolic. More precisely $u(x, \cdot)$ is parabolic on $[0, h_1(x))$ and on $(h_1(x), h(x)]$.*

From a mathematical point of view, system (2.4) can admit solutions without physical sense. In fact, we will see that there exists solutions of equation (2.8) (and consequently solutions of system (2.4)) for which property 2.1 is not satisfied.
For instance, if the function h_1, solution of (2.8), is discontinuous with respect to the variable x, then, by virtue of the previous remark, the corresponding horizontal velocity u is discontinuous too. Such discontinuous functions h_1 can correspond to the case where there exists, for all $x \in [0, L]$, more than one solution to equation (2.8): we can choose for each x a different solution, and "jump" from one solution to the other.
In the following, we only consider solutions of system (2.4) which satisfy property 2.1. In that case, the function h_1 is necessarily continuous (and it is not a restrictive assumption in definition 2.8).

Remark 2.7 *It is important to note that if the function h_1 is not only continuous but more regular, for instance if $h_1 \in \mathcal{C}^1(0, L)$, then property 2.1 is automatically satisfied* *for all solution* (u, p, η) *of system* (2.4)*. In fact, let* $w = -\int_0^z \partial_x u$*. For* $x \in (0, L)$ *we compute*

$$w(x, h_1(x)) = -\int_0^{h_1(x)} \partial_x u(x, z)\, dz = -\partial_x \left(\int_0^{h_1(x)} u(x, z)\, dz \right) + h_1'(x)\, u(x, h_1(x)).$$

Since any solution of (2.4) *satisfies* $\partial_x \left(\int_0^{h_1(x)} u(x, z)\, dz \right) = 0$*, we obtain the equality*

$$w(x, h_1(x)) - h_1'(x)\, u(x, h_1(x)) = 0, \qquad i.e. \qquad \boldsymbol{v} \cdot \boldsymbol{n} = (u, w) \cdot {}^t(-h_1'(x), 1) = 0.$$

Definition 2.8 *We call* interface *a function* $h_1 \in \mathcal{C}(0, L)$ *such that* $0 \le h_1 \le h$ *and such that there exists a solution* (u, p, η) *of system* (2.4) *satisfying*

$$u \in H^1(\Omega), \quad p \in L^2(\Omega) \quad \text{and} \quad \eta(x, z) = \begin{cases} \eta_1 & \text{for } z < h_1(x), \\ \eta_2 & \text{for } z > h_1(x). \end{cases}$$

In these terms, the goal of this paper is to study the existence and uniqueness of such interfaces.

Remark 2.9 *It seems that the method proposed in this paper cannot be extended to the three-dimensional case, namely when the domain* Ω *is a bounded open subset of* $\mathbb{R}^3$*. In fact, in the three-dimensional case, the quantity* $\int_0^{h(x)} u(x, z)\, dz$ *is not constant but only satisfies* $div_x \left(\int_0^{h(x)} u(x, z)\, dz \right) = 0$.

3 Existence and uniqueness problems

3.1 No-shear case

In this subsection, we are interested in the no-shear case, which models for example a two-fluid flow in a channel. Moreover, the results in this simplified case give us a first understanding of equation (2.8).
Let $s = 0$. Equality (2.8) is then written (since $\ell > 0$ for $0 < h_1 < h$)

$$\eta_1 Q_1 (h - h_1)^2 (3q - 2\ell h_1 - 4\ell h) = \eta_2 Q_2 h_1^2 (2\ell h_1 - 3q).$$

Consequently an interface h_1 is a solution of the problem if and only if it satisfies $0 < h_1 < h$ and $\mathcal{P}_0(h_1/h) = 0$, where the polynomial $\mathcal{P}_0$ has degree 4 and is defined by

$$\mathcal{P}_0(X) = (1 - X)^2 (3\widetilde{q} - 2\widetilde{\ell} X - 4\widetilde{\ell}) - M\widetilde{Q} X^2 (2\widetilde{\ell} X - 3\widetilde{q}),$$

with $\widetilde{Q} = Q_2/Q_1$, $M = \eta_2/\eta_1$, $\widetilde{\ell} = 1 + (M - 1)X$ and $\widetilde{q} = 1 + (M - 1)X^2$.
This formulation highlights the dimensionless quantities governing the behavior of the flow. Note that in this particular case without shear, it is determined by the ratio Q_2/Q_1, whereas we will see that in the shear case each of the two fluxes Q_1 and Q_2 can be chosen independently.

Remark 3.1 *In the limit case* $M = 0$*, the polynomial* $\mathcal{P}_0$ *is very simple:* $\mathcal{P}_0 = -(1-X)^4$*. The only solution* $(X = 1)$ *corresponds to a full film of the fluid of viscosity* $\eta_1 \gg \eta_2$*.*

In the same way, when $M = +\infty$, the only solution is again a full film of the fluid of greatest viscosity.

3.1.1 *Theoretical results*

In fact, the polynomial $\mathcal{P}_0$ is simple enough to prove existence and uniqueness of a solution for $\widetilde{Q} > 0$, that is when the two fluxes have the same sign.
The existence of a solution is clear since $\mathcal{P}_0(0) = -1 < 0$ and $\mathcal{P}_0(1) = M^2\widetilde{Q} > 0$. The polynomial $\mathcal{P}_0$ having degree 4, there can be one or three real solutions. Using the Sturm sequences (see Appendix), we compute:

$$\mathcal{P}_0^{(1)}(0) = 4(1-M), \qquad \mathcal{P}_0^{(1)}(1) = 4M\widetilde{Q}(M-1),$$
$$\mathcal{P}_0^{(2)}(0) = \frac{3M(1+\widetilde{Q})}{2(M\widetilde{Q}+1)}, \qquad \mathcal{P}_0^{(2)}(1) = -\frac{3M^2\widetilde{Q}(1+\widetilde{Q})}{2(M\widetilde{Q}+1)}.$$

We deduce the following sign table:

	$\mathcal{P}_0^{(0)}$	$\mathcal{P}_0^{(1)}$	$\mathcal{P}_0^{(2)}$	$\mathcal{P}_0^{(3)}$	$\mathcal{P}_0^{(4)}$
$X = 0$	$-$	$+$	$+$	?	?
$X = 1$	$+$	$-$	$-$	?	?

which is enough to exclude the three solutions case (in fact, we must have $4-1$ or $3-0$ sign changes to obtain three solutions). We have proved the following result:

Proposition 3.2 *Let $s = 0$. If Q_1 and Q_2 have the same sign, there exists a unique interface h_1 solution of equation* (2.8).

In other words, for any positive fluxes Q_1 and Q_2 and for any viscosities η_1 and η_2 there is always a unique two-fluid stationary solution with one fluid above the other with no shear velocity.

3.1.2 *Numerical validation*

In order to validate these theoretical results, we developed an algorithm with Scilab and performed numerical simulations. This algorithm plots the (possibly several) interface solutions h_1 with respect to the geometry h and the following physical parameters: shear velocity s, fluxes of each fluids Q_1, Q_2 and viscosities of each fluids η_1, η_2. Moreover, for each interface solution h_1, we can plot the pressure p and a horizontal velocity profile $u(x, \cdot)$ for some $x \in [0, L]$.

We present the no-shear case results (with $s = 0$) in the case of an oil/water flow. The two viscosities approximatively correspond to the oil viscosity $\eta_1 = 10^{-2}$ Pa · s and water viscosity $\eta_2 = 10^{-3}$ Pa · s. The ratio M is then given by $M = 0.1$. The two fluxes take the same values: $Q_1 = Q_2 = 1\ \mathrm{m}^2 \cdot \mathrm{s}^{-1}$. The domain geometry is described by its length $L = 1$ and by its height $h(x) = 4\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}$. It corresponds to a so-called convergent-divergent domain.

Figure 2 (a) represents the form of the domain (that is the function h, thin line) and the position of the interface (function h_1, thick line). On Figure 2 (b), we plot the pressure profile. Figure 3 corresponds to two different horizontal velocity profiles $u(x, \cdot)$ for $x = 0$ and $x = L/2$. The velocity is greater in the less viscous fluid.

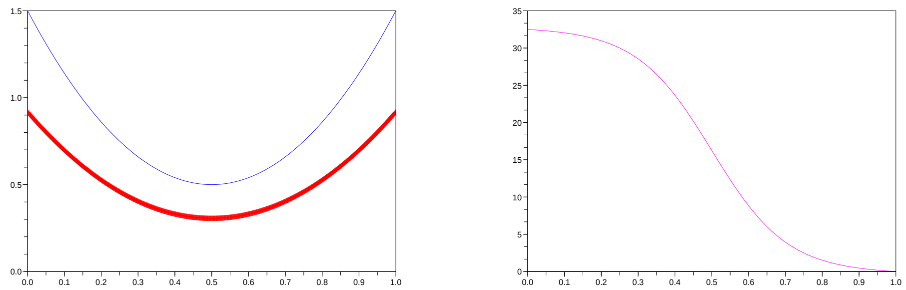

FIGURE 2. (a) Interface position and (b) pressure profile

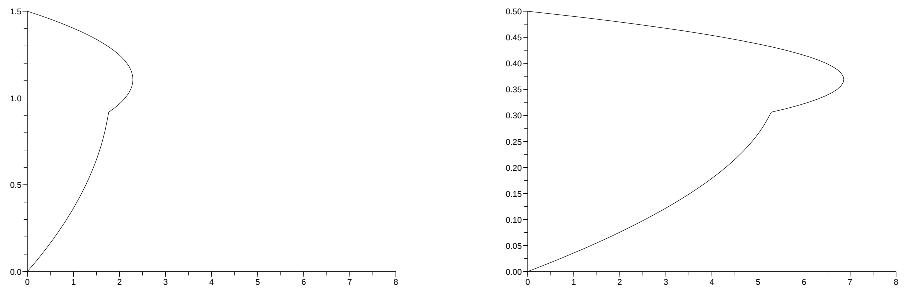

FIGURE 3. Horizontal velocity profile at (a) $x = 0$ and (b) $x = L/2$.

3.2 Lubrication context

In this part we are interested in lubrication applications. We thus take into account the relative motion of the two surrounding surfaces, *via* the non zero velocity $s \neq 0$. In this case, the reference flux is given by sh and it is natural to introduce the following non dimensional quantities:

$$\widetilde{Q}_1 = \frac{Q_1}{sh}, \quad \widetilde{Q}_2 = \frac{Q_2}{sh}, \quad X = \frac{h_1}{h}.$$

Note that the non dimensional fluxes $\widetilde{Q}_1$ and $\widetilde{Q}_2$ depend on the variable x through the function h.

With these new notations, there exists an interface $h_1 = Xh$ solution to (2.8) if and only if there exists $X \in (0,1)$ root of the following polynomial of degree 6:

$$\begin{aligned}\mathcal{P}(X) =& \Big(1-X\Big)^2 \Big(2\widetilde{\ell}\widetilde{Q}_1 - X(2\widetilde{\ell} - MX)\Big)\Big(3\widetilde{q} - 2\widetilde{\ell}X - 4\widetilde{\ell}\Big) \\ & - MX^2\Big(2\widetilde{\ell}\widetilde{Q}_2 - (1-X)^2\Big)\Big(2\widetilde{\ell}X - 3\widetilde{q}\Big),\end{aligned} \tag{3.1}$$

where

$$M = \frac{\eta_2}{\eta_1}, \quad \widetilde{\ell} = 1 + (M-1)X \quad \text{and} \quad \widetilde{q} = 1 + (M-1)X^2.$$

The idea is to apply Sturm's theorem (see Appendix) to count the number of roots of $\mathcal{P}$ in $(0,1)$. Since $\mathcal{P}(0) = -2\widetilde{Q}_1$ and $\mathcal{P}(1) = 2M^3\widetilde{Q}_2$, as in the no-shear case, there exists at least one solution (provided $\widetilde{Q}_1$ and $\widetilde{Q}_2$ are both positive). However, there are too many parameters ($\widetilde{Q}_1$, $\widetilde{Q}_2$ and M) to compute a general sign table manually. Therefore we consider some special cases of physical interest, corresponding to some values of the parameters.

- First, in subsection 3.2.1, we set the parameters as in [3] and show how to recover their numerical results, which do not satisfy the non-miscibility property (see property 2.1).
- From the point of view of lubrication applications, there are two relevant particular cases. In the first, see subsection 3.2.2, we consider a small viscosity ratio (for example, for an air/water mixture, the viscosity ratio is about 10^{-3}), and obtain theoretical existence results.
- The second important application is the understanding of the cavitation phenomenon (subsection 3.2.3). To prevent the cavitation to happen, we prove that the parameter $\widetilde{Q}_2$ has to be zero, and we obtain theoretical and numerical results enabling to predict the apparition of a full film with respect to the two other parameters $\widetilde{Q}_1$ and M.

3.2.1 *Comparison with previous works*

Since we use an original approach, it is first interesting to compare our results to other well-known results in the same physical setting.
For instance, in [3], the authors analyze the asymptotic system corresponding to a thin film flow with two different fluids. As stated in the introduction of this article, their approach differs from ours in the sense that they obtain solutions which do not respect property (2.1). Nevertheless, we will see that our results can be compared to their results. Also note that in [3] the authors compare their model to the Elrod-Adams one, which is the reference model in tribology, as cavitation phenomena occur [5, 1]. Roughly speaking, the Elrod-Adams model is obtained when the ratio M (which is denoted ε in the reference [3]) tends to 0.
More precisely, one of their results is given by Figure 4. On this figure, they plotted different interfaces[3] h_1 depending on the value of the ratio M (curves $\varepsilon = 10^{-1}$, 10^{-2} and 10^{-3}). Moreover, they plotted the interface corresponding to the Elrod-Adams model.
All the other parameters (the height h, the total flux $Q_1 + Q_2$, the saturation at the

[3] Notice that they plot the ratio $h_1(x)/h(x)$, which is thus in the interval $(0,1)$. In order to compare our results with theirs, we plot in this part the same rescaled interfaces h_1/h.

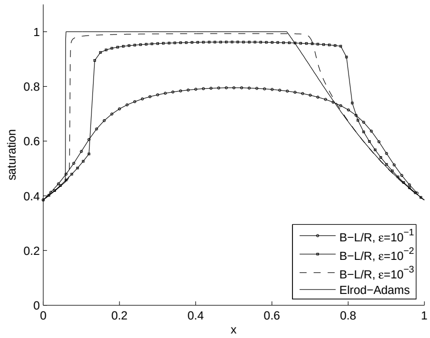

FIGURE 4. Results obtained in [3]: rescaled interfaces for different viscosity ratios.

entrance $h_1(0)$ and the shear velocity s) are given in [3]. We simply note that the fluxes Q_1 and Q_2 are not directly given but can be deduced from the ratio M, the total flux Q_1+Q_2 and the value of the saturation at the entrance of the domain $h_1(0)$. We use the same parameters with our model:

$$h(x) = 4\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}, \qquad s = 1$$

and Q_1, Q_2 such that $Q_1 + Q_2 = 0.58$ and $h_1(0) = 0.38 \times h(0)$. Note that the values of the two fluxes may thus depend on M.

As far as the pressure is concerned, let us emphasize that they impose in [3] two boundary conditions on the pressure : $p(0) = 0$ and $p(L) = 0$, and therefore the system they solve is somehow overdetermined (see remark 2.5). In our case, only the right-hand side condition is imposed : $p(L) = 0$. Therefore, it is not straightforward to compare the values of the pressure.

- In the case $M = 10^{-1}$, we compute the respective values of the fluxes $Q_1 = 0.5$ and $Q_2 = 0.08$. We plot the function h_1/h on Figure 5 (a). The profile of the interface obtained is similar to the one obtained in [3] (see Fig. 4 in the case $\varepsilon = 10^{-1}$).

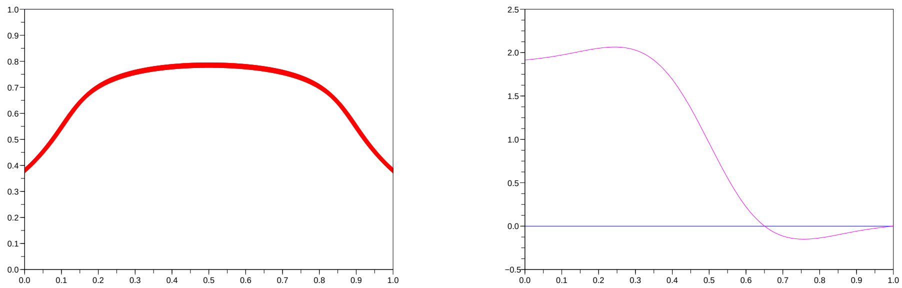

FIGURE 5. (a) Interface and (b) pressure obtained with our method, with the same parameters as in Figure 4 for $M = 10^{-1}$.

- In the case $M = 10^{-3}$ we have $Q_1 = 0.575$ and $Q_2 = 0.005$. In this case, depending on the height $h(x)$, and thus depending on x, there may be one or several interface solutions $h_1(x)$. All solutions are plotted for each x with a thick light line on Figure 6. It is important to note that there is no (continuous) interface on $(0, L)$. However, if we allow discontinuous solutions (that is if we violate property 2.1) then a possible solution is drawn with a thin dark line in figure 6. In this case, we recover the solution proposed by [3] with $M = 10^{-3}$ (see Fig. 4 in the case $\varepsilon = 10^{-3}$).

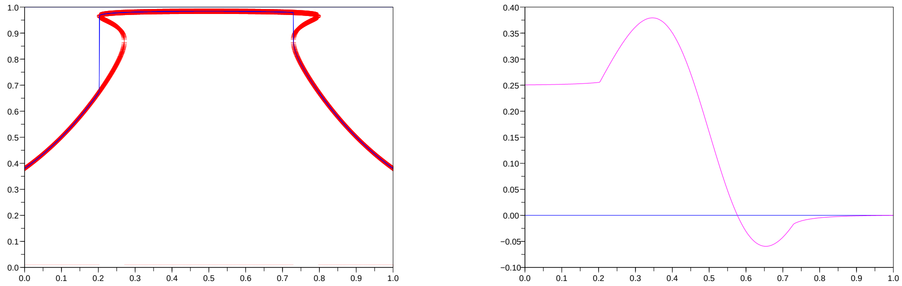

FIGURE 6. (a) Interface and (b) pressure obtained with our method, with the same parameters as in Figure 4 for $M = 10^{-3}$.

In order to compare the pressures in a better way with the model of Bayada, Martin, Vàzquez [3], the authors provided us new results when relaxing the Dirichlet boundary condition on p at $x = 0$. If they impose Neumann condition on p, they obtained for the pressure the curves showed on Figure 7. The red curve (with $p(0) \neq 0$) corresponds to the Buckley-Leverett/Reynolds system, and the blue one (with $p(0) = 0$) to the Elrod-Adams model. As far as the interface is concerned, this new boundary condition does not influence the results of Figure 4. We can see that our results presented on Figure 6 (b) are quite similar to this curve, both qualitatively and quantitatively.

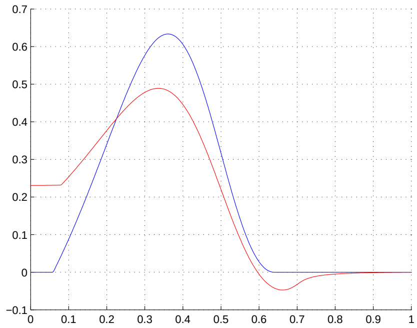

FIGURE 7. Case $M = 10^{-3}$: pressure for the model of [3] with $p(0) \neq 0$ and for the Elrod-Adams model (with $p(0) = 0$).

3.2.2 *Small viscosity ratio*

Note that for $M = 0$ the polynomial $\mathcal{P}$ defined by equation (3.1), is very simple:

$$\mathcal{P}(X) = 2(1-X)^5(X - \widetilde{Q}_1).$$

This polynomial admits exactly two real solutions. In terms of dimensional quantities, we recall that this equation is solved for any $x \in [0, L]$, and that we look for $0 \leq h_1 \leq h$. Two cases can occur:

- either $x \in [0, L]$ is such that $h(x) \leq Q_1/s$: then there exists only one solution $h_1(x)$ such that $h_1(x) \in [0, h(x)]$, it is the full film solution $h_1(x) = h(x)$;
- either we are interested in $x \in [0, L]$ such that $h(x) > Q_1/s$ and there is in addition to the full film solution a constant solution $h_1(x) = Q_1/s$.

Figure 8 corresponds to such a case. To obtain this figure, we imposed the shear velocity $s = 1$, the two fluxes $Q_1 = 0.7$ and $Q_2 = 0.1$ and the height $h(x) = 4(x - \frac{1}{2})^2 + \frac{1}{2}$ so to have both cases: one solution (for small height $h(x)$) and the case with two solutions (for large height $h(x)$).

Physically the viscosity ratio M is not equal to zero, but there are many interesting cases

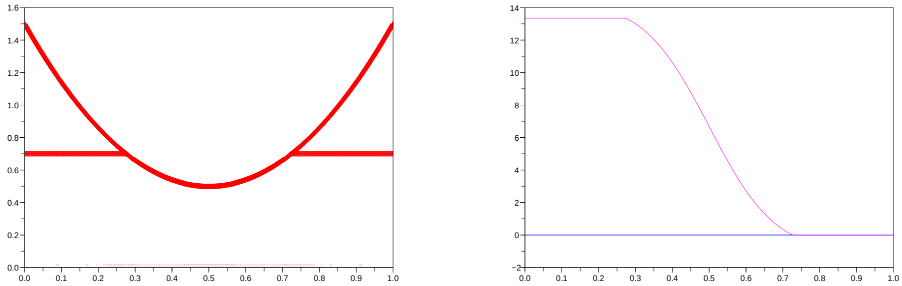

FIGURE 8. Shear case ($s = 1$) with zero viscosity ratio $M = 0$: (a) possible fluid interfaces in the domain and (b) pressure.

for which M is very small (e.g. lubricant/air). We show that such cases are a perturbation of the case $M = 0$. More precisely, the implicit function theorem gives the behavior of the roots $X = 1$ and $X = \widetilde{Q}_1$ for small values of the viscosity ratio M. For instance, near the point $X = \widetilde{Q}_1$, $M = 0$ we have (in the case $\widetilde{Q}_1 \neq 1$):

$$\mathcal{P}(X) = 0 \quad \Longleftrightarrow \quad X = \widetilde{Q}_1 - \frac{\widetilde{Q}_1^2\Big((3+\widetilde{Q}_1)(2\widetilde{Q}_2 - 1 - \widetilde{Q}_1) - (1-\widetilde{Q}_1)^2\Big)}{2(1-\widetilde{Q}_1)^3} M + O(M^2).$$

This means that the root $X = \widetilde{Q}_1$ corresponding to the case $M = 0$ is little changed when M is small.

Near the point $X = 1$, $M = 0$ the implicit function theorem cannot be directly applied since $\partial_X \mathcal{P}(1)|_{M=0} = 0$. However, we can introduce the variable $Y = (1-X)^5$, and apply the implicit function theorem to the function $g(Y,M) = \mathcal{P}(X)$. We obtain

$$\mathcal{P}(X) = 0 \quad \Longleftrightarrow \quad X = 1 + \sqrt[5]{\frac{\widetilde{Q}_2}{1-\widetilde{Q}_1}} M^{3/5} + O(M^{4/5}).$$

Note that for $\widetilde{Q}_1 < 1$ (and $M > 0$), this root becomes greater than 1 and is not considered in our situation. Conversely, for $\widetilde{Q}_1 > 1$ this root has to be taken into account.

We have numerically observed this situation. Taking the same parameters as previously ($s = 1$, $Q_1 = 0.7$, $Q_2 = 0.1$) and $M = 10^{-3}$ (whereas $M = 0$ in Figure 8), we obtain Figure 9. We clearly observe the first solution $h_1 \approx Q_1/s$ on the left and on the right (i.e. for big heights). The second solution $h_1 = h$ disappeared for these heights, since it became greater than h, whereas for smaller heights, we observe this solution $h_1 \approx h$, with $h_1 < h$.

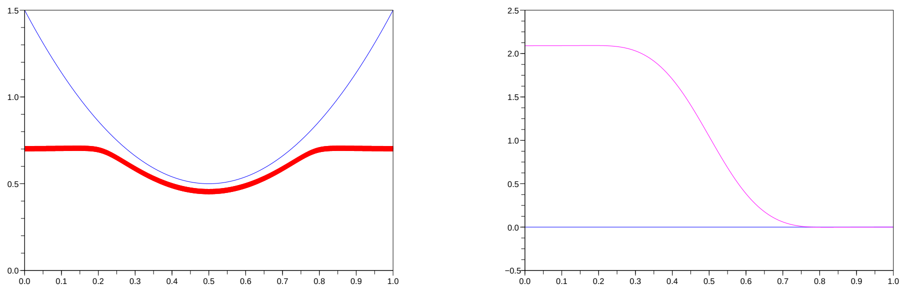

FIGURE 9. Shear case ($s = 1$) with small viscosity ratio $M = 10^{-3}$: (a) possible fluid interface in the domain and (b) pressure.

3.2.3 *Can the two-fluid model describe saturation phenomenon?*

Roughly speaking, we say that there is saturation in the flow when there exists a zone in which there is only one fluid, typically when a lubricant fills the gap between two surfaces in relative motion. More precisely, we introduce the following mathematical definition of the saturation phenomenon:

Definition 3.3 *We say that an interface* $h_1 \in \mathcal{C}(0,L)$ saturates *if*

$$\exists (x_1,x_2) \in [0,L]^2, \quad x_1 < x_2, \quad h_1(x_1) < h(x_1) \quad \textit{and} \quad h_1(x_2) = h(x_2).$$

From this definition, we easily prove that if the interface is more regular[1] then the saturation can be locally characterized:

[1] Let us denote by $\mathcal{C}^1_{pm}$ the set of piecewise $\mathcal{C}^1$-functions. For such a function f, the quantity f'_g denotes its left derivative.

Proposition 3.4 *An interface $h_1 \in \mathcal{C}^1_{pm}(0,L)$ saturates if and only if*

$$\exists x_2 \in [0,L], \quad h_1(x_2) = h(x_2) \quad \textit{and} \quad {h_1}'_g(x_2) > h'(x_2).$$

Such a point x_2 is called a saturation point *for the interface h_1.*

Figure 10 shows an example of a saturation point x_2 associated to an interface h_1. From a

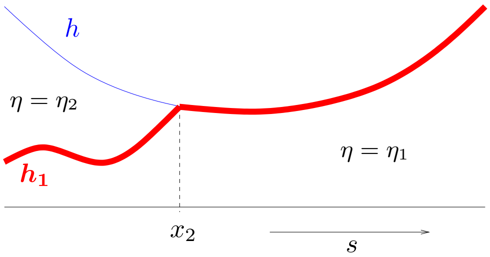

FIGURE 10. Saturating interface

physical point of view, it seems natural that in such a situation the flux Q_2, corresponding to the flux of the top fluid, should be equal to 0. This fact can be mathematically proved in a simple way using the conservation of the flux Q_2: if the interface h_1 saturates then there exists $x_2 \in [0,L]$ such that $h_1(x_2) = h(x_2)$. Consequently $Q_2 = \displaystyle\int_{h_1(x_2)}^{h(x_2)} u\,dz = 0$.

Property 3.5 *If the interface h_1 saturates then the flux of the top fluid vanishes: $Q_2 = 0$.*

Hence, in a saturation case, the polynomial $\mathcal{P}$ is written

$$\mathcal{P}|_{Q_2=0}(X) = 2\widetilde{\ell}(1-X)^2\left((\widetilde{Q}_1 - X)(3\widetilde{q} - 2\widetilde{\ell}X - 4\widetilde{\ell}) - 2MX^2\right).$$

Theoretical results

Among the roots of the polynomial $\mathcal{P}|_{Q_2=0}$ there is the obvious solution $X = 1$ (*i.e.* $h_1 = h$). Since ℓ does not vanish for $X \in [0,1]$, we look for the other roots introducing

$$\mathcal{R}(X, \widetilde{Q}_1) = (\widetilde{Q}_1 - X)(3\widetilde{q} - 2\widetilde{\ell}X - 4\widetilde{\ell}) - 2MX^2.$$

Let us come back to the dimensional variables in order to use proposition 3.4. Assuming that there exists an interface $h_1 \in \mathcal{C}^1_{pm}(0,L)$ (not equal to h), the function h_1 satisfies, for all $x \in [0,L]$, the relation

$$\mathcal{R}\left(\frac{h_1(x)}{h(x)}, \frac{Q_1}{sh(x)}\right) = 0. \tag{3.2}$$

Taking the left-derivative with respect to x, we obtain

$$s\Big({h_1}'_g(x)h(x) - h'_g(x)h_1(x)\Big)\partial_1\mathcal{R}\left(\frac{h_1(x)}{h(x)}, \frac{Q_1}{sh(x)}\right) - Q_1 h'_g(x)\partial_2\mathcal{R}\left(\frac{h_1(x)}{h(x)}, \frac{Q_1}{sh(x)}\right) = 0.$$

From proposition 3.4 and due to the fact that Q_1, s, $h(x)$ are positive, if the interface h_1 saturates at a point $x_2 \in [0, L]$ then

$$\mathcal{R}(1, \widetilde{Q}_1) = 0 \quad \text{and} \quad h'(x_2)\frac{\partial_2 \mathcal{R}(1, \widetilde{Q}_1)}{\partial_1 \mathcal{R}(1, \widetilde{Q}_1)} > 0, \quad \text{with} \quad \widetilde{Q}_1 = \frac{Q_1}{sh(x_2)}. \tag{3.3}$$

In fact $\mathcal{R}(1, \widetilde{Q}_1) = 2M^2(1 - 3\widetilde{Q}_1)$ vanishes for $\widetilde{Q}_1 = 1/3$. Moreover, we get

$$\frac{\partial_2 \mathcal{R}(1, 1/3)}{\partial_1 \mathcal{R}(1, 1/3)} = -9.$$

We deduce the following necessary condition for saturation:

Proposition 3.6 *If an interface $h_1 \in \mathcal{C}^1_{pm}(0, L)$ saturates at $x_2 \in [0, L]$ then*

$$h(x_2) = \frac{3Q_1}{s} \quad \text{and} \quad h'(x_2) < 0. \tag{3.4}$$

This proposition 3.6 indicates that in order to have a saturation phenomenon the domain must be convergent, through the condition $h'(x_2) < 0$.

Remark 3.7 *In the same way it is possible to define the notion of "desaturation", when an interface appears in a full film. Using the same methods as those used to prove proposition 3.6, we can prove that an interface can desaturate only if there exists a point $x_2 \in [0, L]$ for which the height $h(x_2)$ is imposed, namely $h(x_2) = \frac{3Q_1}{s}$, and for which the domain is locally divergent: $h'(x_2) > 0$.*

Proposition 3.6 contains an important assumption: the function h_1 must be an interface, that is the function h_1 must be defined on $[0, L]$. In particular, condition (3.4) does not imply that h_1 is an interface in the sense of definition 2.8. Nevertheless, following the proof of proposition 3.6, condition (3.4) implies that $\partial_1 \mathcal{R}(1, \widetilde{Q}_1) \neq 0$ (see (3.3)). Consequently, the implicit function theorem proves that there exists a function f defined in a neighborhood of x_2 such that

$$\mathcal{R}\left(\frac{h_1}{h(x)}, \frac{Q_1}{sh(x)}\right) = 0 \quad \Longleftrightarrow \quad h_1 = f(x).$$

In other words, condition (3.4) implies that there exists locally (with respect to the spatial variable x) a saturating interface.

Proposition 3.8 *If there exists $x_2 \in [0, L]$ such that (3.4) holds then there exists $\varepsilon > 0$ and an interface $h_1 \in \mathcal{C}^1_{pm}(x_2 - \varepsilon, x_2 + \varepsilon)$ which saturates at x_2.*

Propositions 3.6 and 3.8 give a characterization of (local) saturation. It is important to notice that such characterization does not depend on the value of the viscosity ratio M. It depends only on the geometry, that is on the height h.

Moreover, the implicit function theorem is valid as long as $\partial_1 \mathcal{R}(X, \widetilde{Q}_1) \neq 0$. To see how "global" the result of proposition 3.8 is, we must find the common zeros of the two polynomials $\partial_1\mathcal{R}$ and $\mathcal{R}$. On Figure 11, we plot the set of values $(M, \widetilde{Q}_1)$ for which the discriminant of $\partial_1\mathcal{R}$ and $\mathcal{R}$ vanishes.

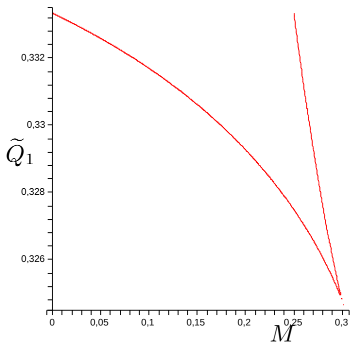

FIGURE 11. The set of values $(M, \widetilde{Q}_1)$ for which the discriminant of $\partial_1\mathcal{R}$ and $\mathcal{R}$ vanishes.

From this Figure 11 we can emphasize two fundamental results. First, for M large enough (that is $M \gtrsim 0.3$), the discriminant of the two polynomials $\partial_1\mathcal{R}$ and $\mathcal{R}$ does not vanish. Consequently, for $M \gtrsim 0.3$ the result of proposition 3.8 holds globally:

Proposition 3.9 *Let* $0.3 \lesssim M \leq 1$. *There exists an interface* $h_1 \in \mathcal{C}^1_{pm}(0, L)$ *which saturates at* $x_2 \in [0, L]$ *if and only if the condition* (3.4) *holds.*

The second remark we can deduce from Figure 11 is that for $M \lesssim 0.3$, a saturated solution can exist only locally, that is only on a small interval around the saturation point x_2. Such a situation will be highlighted by numerical results in the sequel.

Numerical validation

The results of the previous propositions (3.6, 3.8 and 3.9) can be observed numerically. We conducted three numerical tests in which there exists a point $x_2 \in [0, L]$ satisfying condition (3.4). With the following data

$$L = 1, \quad h(x) = 4\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}, \quad s = 1 \quad \text{and} \quad Q_1 = \frac{1}{3},$$

the probable saturation point x_2 is given by $x_2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) \approx 0.146$.

- On Figure 12, we use a viscosity ratio $M = 0.6 \gtrsim 0.3$ so that proposition 3.9 ensures the existence of an interface on $[0, L]$ which saturates at x_2. We plot on Figure 12 the full film solution $h_1 = h$ and the saturated solution.
- Next, we give an example with $M = 0.2 \lesssim 0.3$. We observe that the saturated solution can be defined as a function only near the saturation point x_2. Consequently, in this configuration, there exists only one interface on $[0, L]$: it is the full film solution $h_1 = h$.

  In this case, there is no solution existing on the whole interval $[0, L]$, except for the full film solution, thus it is not of interest to plot the pressure.
- For lower values of M, we can observe another phenomenon. On Figure 14, we use

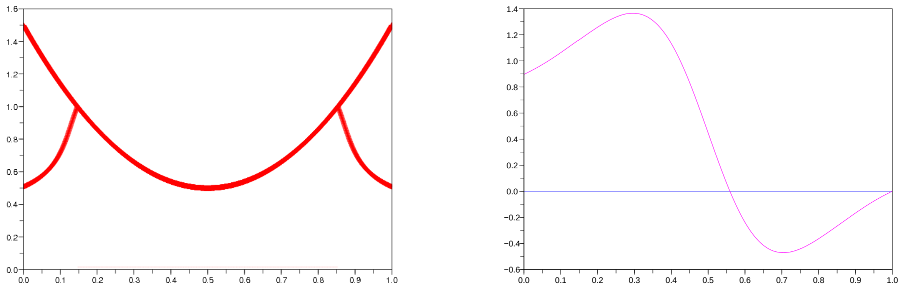

FIGURE 12. (a) the full film solution and the saturated solution obtained for $M = 0.6$ and (b) the pressure corresponding to the saturated solution

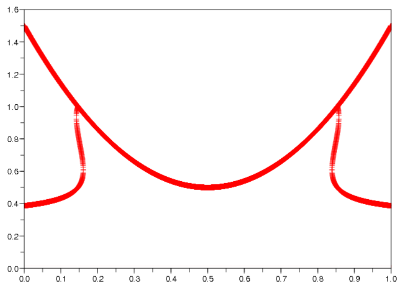

FIGURE 13. The full film solution and the other solutions $h_1(x)$ of the polynomial equation (3.2) obtained for $M = 0.2$.

$M = 0.005$. In such a case, there always exists the full film solution. There exists also a locally saturated solution (but as for the case $M = 0.2$ this solution does not allow to define an interface on $[0, L]$). Moreover, there exists a third solution, which is global (that is corresponds to an interface on $[0, L]$) but which does not saturate.

As in the previous case, apart from the full film solution, there is no saturated solution existing globally, and we do not plot any pressure.

We will see in the next paragraph that it is possible to predict the configuration (corresponding to Figure 13 or corresponding to Figure 14) according to the value of the flux Q_1 and the viscosity ratio M.

Prediction

In this part, we assume that $\widetilde{Q}_2 = 0$. There may therefore be a saturation phenomenon for $\widetilde{Q}_1 = 1/3$. If we want to understand the configurations of the number of solution(s) near a saturation point, we must predict the solution(s) for $\widetilde{Q}_1 \neq 1/3$.

In this situation, we have two independent parameters: M and $\widetilde{Q}_1$. We will plot a map of the number of solutions depending on M and $\widetilde{Q}_1$. To obtain such a map, we use the Sturm sequences (see Appendix 5): from the polynomial $\mathcal{R}$, we define a sequence $\mathcal{R}^{(i)}$,

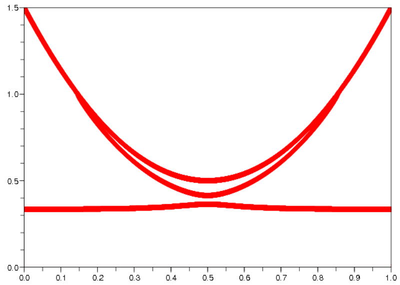

FIGURE 14. The full film solution and the other solutions $h_1(x)$ of the polynomial equation (3.2) obtained for $M = 0.005$.

$i = 0..3$. In practice, the number of roots $n(M, \widetilde{Q}_1)$ of the polynomial $\mathcal{R}$ between 0 and 1 is given by

$$n(M, \widetilde{Q}_1) = \frac{\left| -a_0a_1 - a_1a_2 - a_2a_3 + b_0b_1 + b_1b_2 + b_2b_3 \right|}{2} ,$$

$$\text{where} \quad a_i = \frac{\mathcal{R}^{(i)}(0)}{|\mathcal{R}^{(i)}(0)|} \quad \text{and} \quad b_i = \frac{\mathcal{R}^{(i)}(1)}{|\mathcal{R}^{(i)}(1)|}.$$

Using the software Maple, we obtain the map given by Figure 15. We find that the value $\widetilde{Q}_1 = 1/3$ plays a particular role. Thus, when the height $h(x)$ varies so that $\widetilde{Q}_1$ crosses the value 1/3, the behavior is given by the previous figures: Figure 12 for large values of M, and Figures 13 and 14 for small values of M. Note that the small white zone (corresponding to the 3 roots case) can be observe on Figures 13 and 14 near the saturation point $x \approx 0.146$: for such x, there exists three possible positions of the interface.

4 More layers cases

The previous study only concerns the two layers configuration, that is the case where two fluids are superposed one above the other. In fact, the same theoretical study can be applied for more than two fluids, the only additional difficulty coming from the size of the polynomial systems to solve. We briefly present in this section the case with three layers and its main application in lubrication context: the existence or not of so called "fingers" in a stationary configuration (see Figure 16).

4.1 Three-fluid model in a thin domain

For such a situation, we respectively denote by η_1, η_2 and η_3 the viscosities of the three fluids (upwards), and we denote by h_1 the interface between the two bottom fluids, by h_2 the interface between the two top fluids. Starting from the Stokes equations (2.1), we deduce as for the two layers case (see equation 2.5) the following horizontal velocity

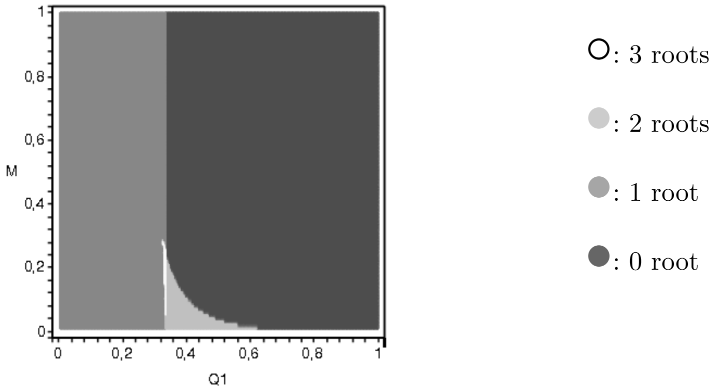

FIGURE 15. Number of roots in $[0,1[$ of the polynomial $\mathcal{R}$ for different values of M and $\widetilde{Q}_1$.

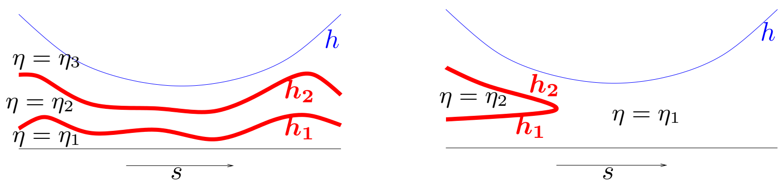

FIGURE 16. Three layers: (a) usual superposed fluids and (b) possible finger

profile:

$$u = \begin{cases} \dfrac{z^2}{2\eta_1}\partial_x p - \dfrac{A}{\eta_1} z + s & z \in [0, h_1), \\ \dfrac{h_1^2}{2\eta_1}\partial_x p - \dfrac{A}{\eta_1} h_1 + s + \dfrac{z^2 - h_1^2}{2\eta_2}\partial_x p - \dfrac{A}{\eta_2}(z - h_1) & z \in [h_1, h_2], \\ \dfrac{h_1^2}{2\eta_1}\partial_x p - \dfrac{A}{\eta_1} h_1 + s + \dfrac{h_2^2 - h_1^2}{2\eta_2}\partial_x p - \dfrac{A}{\eta_2}(h_2 - h_1) & \\ \qquad + \dfrac{z^2 - h_2^2}{2\eta_3}\partial_x p - \dfrac{A}{\eta_3}(z - h_2) & z \in (h_2, h]. \end{cases} \tag{4.1}$$

The constant A can be determined from the boundary condition $u(x, h(x)) = 0$:

$$A = \frac{q}{2\ell}\partial_x p + \frac{s}{\ell},$$

with

$$\ell = \frac{h_1}{\eta_1} + \frac{h_2 - h_1}{\eta_2} + \frac{h - h_2}{\eta_3}, \qquad q = \frac{h_1^2}{\eta_1} + \frac{h_2^2 - h_1^2}{\eta_2} + \frac{h^2 - h_2^2}{\eta_3}.$$

The fluxes are naturally denoted by $Q_1 = \int_0^{h_1} u$, $Q_2 = \int_{h_1}^{h_2} u$ and $Q_3 = \int_{h_2}^{h} u$, and we deduce a system of three equations for the three unknowns $\partial_x p$, h_1 and h_2. Clearly, when $h_1 = h_2$ (or when $h_1 = 0$ or $h_2 = h$) we must recover the system corresponding to the two layers case, that is system (2.6). We can write the system under the following form (which simply corresponds to the integration with respect to the variable z of the velocity u):

$$\begin{cases} 12\eta_1 \ell Q_1 = h_1\, f_1(\partial_x p, h_1, h_2), \\ 12\eta_2 \ell Q_2 = (h_2 - h_1)\, f_2(\partial_x p, h_1, h_2), \\ 12\eta_3 \ell Q_3 = (h - h_2)\, f_3(\partial_x p, h_1, h_2), \end{cases} \tag{4.2}$$

with

$$\begin{aligned} f_1(\partial_x p, h_1, h_2) &= h_1\Big(2\ell h_1 - 3q\Big)\partial_x p + 6\Big(2\eta_1\ell - h_1\Big)s, \\ f_2(\partial_x p, h_1, h_2) &= \Big\{(h_2 - h_1)\Big(2\ell(h_2 + 2h_1) - 3q + 6\frac{\eta_2}{\eta_1}h_1 h_2\big(\frac{1}{\eta_3} - \frac{1}{\eta_2}\big)\Big) \\ &\qquad + 6\frac{\eta_2}{\eta_1\eta_3}h_1 h(h_1 - h)\Big\}\,\partial_x p + 6\Big((h_2 - h_1) + 2\frac{\eta_2}{\eta_3}(h - h_2)\Big)s, \\ f_3(\partial_x p, h_1, h_2) &= \Big\{(h_2 - h_1)\Big(6\frac{\eta_3}{\eta_2}\Big(\ell(h_2 + h_1) - q\Big) + 6\frac{\eta_3}{\eta_1}h_1 h_2\big(\frac{1}{\eta_3} - \frac{1}{\eta_2}\big)\Big) \\ &\qquad + (h - h_2)\Big(2\ell(h + 2h_2) - 3q\Big) + \frac{6}{\eta_1}h_1 h(h_1 - h)\Big\}\,\partial_x p + 6\Big(h - h_2\Big)s. \end{aligned}$$

In fact, each function f_i is affine with respect to its first variable (that is $\partial_x p$) and it is not difficult to express $\partial_x p$ as a function of the two heights h_1 and h_2. Moreover, we observe that the first equation of this system (4.2) is exactly the same as the first equation of system (2.6) corresponding to the two-layers study. The expression of the pressure derivative $\partial_x p$ can thus be given by equation (2.7). We then obtain a polynomial system of two equations with two unknowns h_1 and h_2. Theoretically, it suffices to determine the roots (h_1, h_2) of such a system and select those which satisfy $0 < h_1 \le h_2 < h$. We can then explicitly find the pressure (using system (4.2) and an expression of $\partial_x p$ with respect to h_1 and h_2), and next the velocity using equation (4.1).
For system (4.2), there are many physical parameters: the three fluxes Q_1, Q_2 and Q_3, the shear velocity s, the three viscosities η_1, η_2 and η_3, and the function defining the height h. Using non-dimensionalization process, we can see that only some ratios are important. Nevertheless the study of the general case seems rather tedious.

4.2 Finger configuration ?

The purpose of this subsection is to investigate whether there is a stationary solution where the relative position of the fluids is given as in Figure 16 (b). Such a configuration is called "finger configuration" (see for instance [4, 9] in which they define such configurations in a non stationary context). Observe that in this case, we clearly have $\eta_3 = \eta_1$ and $Q_2 = 0$.

Necessary condition to have a finger configuration
To have a finger configuration, there must be a point $x_2 \in [0, L]$ (which corresponds to the "tip of the finger") for which $h_1(x_2) = h_2(x_2)$, and such that $h_1(x_1) < h_2(x_1)$ for $x_1 < x_2$. This implies in particular that $h'_{1g}(x_2) < h'_{2g}(x_2)$. In other words, the solution $h_1 = h_2$ to the second equation of system (4.2) must be at least of order 2. In mathematical terms, we deduce from the fact that $Q_2 = 0$ that

$$\forall x \in (0,1), \quad (h_2(x) - h_1(x)) f_2(\partial_x p(x), h_1(x), h_2(x)) = 0.$$

Taking the left-derivative of this relation with respect to x, evaluating it at x_2 and using that $h'_{1g}(x_2) \neq h'_{2g}(x_2)$ whereas $h_1(x_2) = h_2(x_2)$, we deduce that

$$f_2(\partial_x p(x_2), h_1(x_2), h_2(x_2)) = 0.$$

A simple calculation indicates that

$$f_2(\partial_x p, h_1, h_1) = \frac{h_1\, h}{\eta_1} \partial_x p - 2s.$$

We deduce that if there exists a finger then there exists $x_2 \in [0, L]$ and a solution $h_1 \in (0, h)$ to the equation

$$h_1(x_2)\, h(x_2)\, \partial_x p(x_2) = 2\eta_1\, s.$$

We easily deduce this first result about the existence of finger configuration (we will note that $s = 0$ and $\partial_x p = 0$ are incompatible with the other equations of system (4.2)):

Proposition 4.1 *In the no-shear case (i.e. $s = 0$), there is no finger configuration.*

More generally, we will see later (see proposition 4.2) that there is no finger configuration if s is not large enough.

Assume that there exists a finger configuration in the shear case ($s \neq 0$). Consequently, the second equation of system (4.2) is written $h_1\, h\, \partial_x p = 2\eta_1\, s$. At point $x_2 \in [0, L]$ where $h_1(x_2) = h_2(x_2)$ the two other equations of system (4.2) are written

$$\begin{cases} 12\eta_1 \ell Q_1 = h_1\, f_1\left(\dfrac{2\eta_1\, s}{h_1\, h}, h_1, h_1\right), \\[2ex] 12\eta_1 \ell Q_3 = (h - h_1)\, f_3\left(\dfrac{2\eta_1\, s}{h_1\, h}, h_1, h_1\right). \end{cases} \tag{4.3}$$

With the expressions of f_1 and f_3, this system is written

$$\begin{cases} 12\eta_1 \ell Q_1 = 2h_1 s(3h - h_1), \\[1ex] 12\eta_1 \ell Q_3 = 2s(h - h_1)^2 \left(1 - \dfrac{h}{h_1}\right). \end{cases}$$

As in the previous section, we introduce the dimensionless quantities $\widetilde{Q}_1 = \frac{Q_1}{sh}$, $\widetilde{Q}_3 = \frac{Q_3}{sh}$ and $X = \frac{h_1}{h}$. The previous system is equivalent to the following one:

$$\begin{cases} 12\widetilde{Q}_1 = 2(3 - X)X, \\ 12X\widetilde{Q}_3 = -2(1 - X)^3. \end{cases} \tag{4.4}$$

It is not difficult to observe that the first polynomial equation (of degree 2) admits a solution $X \in (0;1)$ if and only if $0 < \widetilde{Q}_1 < \frac{1}{3}$. The solution is then given by

$$X = \frac{3 - \sqrt{9 - 24\widetilde{Q}_1}}{2}.$$

Next, the second equation of (4.4) implies

$$\widetilde{Q}_3 = -\frac{(1-X)^3}{6X}.$$

Moreover, summing the two equations of system (4.4), we deduce that

$$6X(\widetilde{Q}_1 + \widetilde{Q}_3) = 3X - 1.$$

Using the physical variables, we obtain the following result

Proposition 4.2 *If there exists a finger configuration then $Q_1 > 0$, $Q_2 = 0$, $Q_3 < 0$ and there exists $x_2 \in [0, L]$ such that*

$$h(x_2) > \frac{3Q_1}{s} \quad \textit{and} \quad Q_3 = -\frac{\left(\sqrt{9s\,h(x_2) - 24Q_1} - \sqrt{s\,h(x_2)}\right)^3}{24\left(3\sqrt{s\,h(x_2)} - \sqrt{9s\,h(x_2) - 24Q_1}\right)}.$$

Moreover the height of the finger tip is explicitly given by

$$h_1(x_2) = \frac{s\,h(x_2)^2}{3(s\,h(x_2) - 2(Q_1 + Q_3))}.$$

Remark 4.3 (1) *Note that the total flux can be positive or negative:*

$$Q_1 + Q_3 \in]-\infty, \frac{h(x_2)s}{3}[.$$

It is positive if and only if $h(x_2) < \frac{27Q_1}{4s}$.

(2) *In a sense, this proposition completes proposition* (4.1)*: if the shear velocity s is too small, then there is no finger configuration.*

(3) *As for the existence of a saturation point in the two layers case (see proposition 3.6), the necessary condition to have a finger configuration does not depend on the viscosities of the fluids.*

Sufficient condition to have a finger configuration

Proposition 4.2 gives some necessary conditions to have a finger configuration, more precisely to have a point x_2 such that $h_1(x_2) = h_2(x_2)$. In order to obtain sufficient conditions, we have to determine if there is some set of parameters for which there actually exists a solution (h_1, h_2) such that for any $0 \le x < x_2$ (or at least in a neighborhood of x_2) we have $h_1(x) < h_2(x)$. We assume that $Q_2 = 0$, $\eta_3 = \eta_1$, $0 < \widetilde{Q}_1 < \frac{1}{3}$ and that $\widetilde{Q}_3$ is given with respect to $\widetilde{Q}_1$ by proposition 4.2. We then have only two independent parameters: the viscosity ratio $M = \frac{\eta_2}{\eta_1}$ and the non-dimensional flux of the bottom fluid $\widetilde{Q}_1$.

From a theoretical point of view, we could apply the implicit function theorem to determine if there is a solution x_2 such that locally, the right condition on h_1 and h_2 is satisfied. However, this study is quite complicated.

From a numerical point of view, we only observe a finger configuration for a "divergent" domain. Such a case is represented on Figure 17, for $Q_1 = 0.2$, $Q_3 = 0.05$, $h(x) = 0.9 + 0.2x$, and the viscosity ratio $M = \eta_2/\eta_1 = 0.4$.

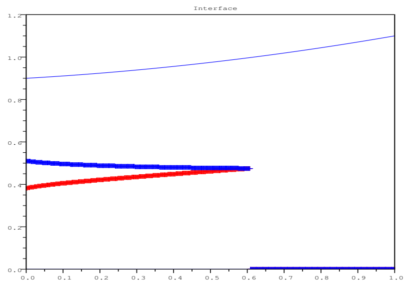

FIGURE 17. Possible finger for a divergent domain: the bottom solution h_1 (light line) and the top solution h_2 (dark line)

For a "convergent" domain, the configuration observed in the previous case is reverted, and corresponds to a "finger coming from the inside". This is shown on Figure 18. This figure is obtained for the same parameters $Q_1 = 0.2$, $Q_3 = 0.05$, $M = 0.4$ and the following geometry: $h(x) = 1.1 - 0.2x$.

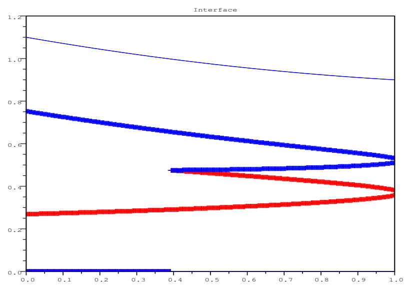

FIGURE 18. Possible solutions for a convergent domain: finger coming from the convergent part, and a solution corresponding to an "open finger"

On Figure 18, we observe that there is a second couple (h_1, h_2) of solutions, but which do not cross (there is no x_2 such that $h_1(x_2) = h_2(x_2)$, thus the finger is not "closed"). In fact, for smaller h as those represented on this figure, this couple of solutions does not exist anymore.

Remark 4.4 *If we allow the jump between different solutions (of course, as in the case* *of two layers, this violates the non-miscibility condition given by property 2.1), then it is possible to obtain an infinite number of different fingers.*

Let us explain more precisely what happens. System (4.2) can be rewritten, extracting $\partial_x p$ from the second equation and plugging it into the two others:

$$\begin{cases} F_1\left(h_1, h_2\right) = 0, \\ F_3\left(h_1, h_2\right) = 0, \end{cases}$$

where the functions F_i are rational fractions depending on the ratios Q_1/sh, Q_3/sh, η_2/η_1. It is of course equivalent to find roots of the numerators of the F_i, thus the system to be solved is polynomial.

If we fix a value of the viscosity ratio $M = \eta_2/\eta_1$ and plot the resolvant of the two polynomials for several values of Q_i/sh, $i = 1, 3$, we obtain again necessary conditions on Q_i/sh for a finger configuration (i.e. for the existence of a finger tip at some $x_2 \in [0, L]$). Then, in order to know how the two solutions h_1 and h_2 behave locally around x_2, and thus to know whether there exists a finger for $x < x_2$, we plot the values of $X = h_1/h$ and $Y = h_2/h$ for which the two polynomials cancel. Figure 19 (a) corresponds to a value of the $\widetilde{Q}_i = Q_i/sh$ for which there exists a mutual root $X = Y$: $M = 0.2$, $\widetilde{Q}_1 = 0.2$, $\widetilde{Q}_3 = -0.05$. In Figure 19 (b), we plot a small perturbation of this case, namely a slightly modified value of Q_i/sh. This perturbation consists in decreasing slightly the values of Q_i/sh by multiplying them by 0.85, that is increasing the value of h (and thus shifting to the left for a convergent geometry).

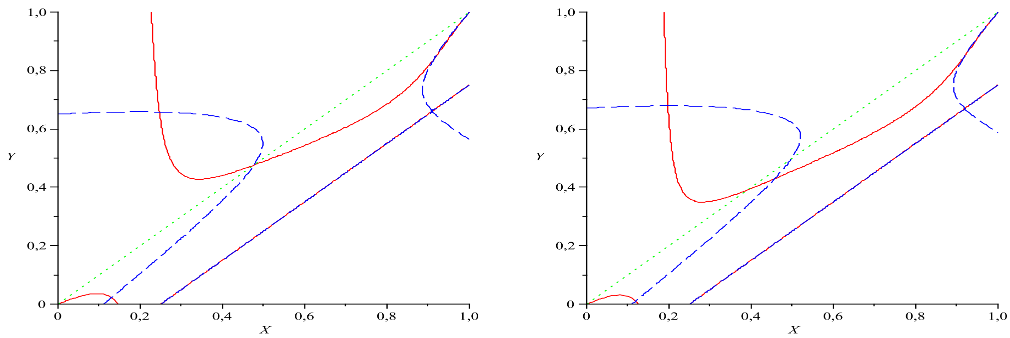

FIGURE 19. Values of $X = h_1/h$ (solid line) and $Y = h_2/h$ (dashed line) solutions for (a) some Q_i and (b) perturbed values of these Q_i ($\times 0.85$)

Of course, we recover on Figure 19 (a) that there is a mutual root of the two polynomials such that $h_1(x_2) = h_2(x_2)$. A finger configuration exists if the mutual root of the two polynomials moves above the diagonal (i.e. $h_1(x_1) < h_2(x_1)$) when decreasing h. We see that this is not the case in Figure 19 (b).

Moreover, we observe that there exists another mutual root to the two polynomials, but this root never crosses the diagonal, so there is no "tip" ($h_1 = h_2$) for this possible finger. This second root is observed on Figure 18, and we already stated that the finger could not be "closed".

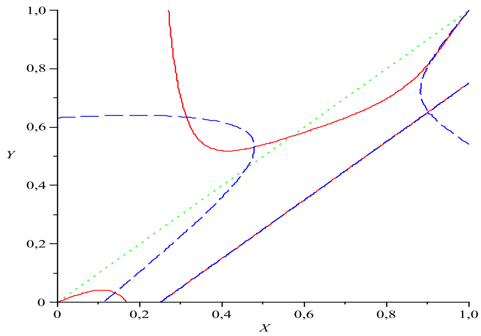

FIGURE 20. Values of $X = h_1/h$ (solid line) and $Y = h_2/h$ (dashed line) solutions for other perturbed values of the Q_i of Figure 19 (a) ($\times 1.15$)

Conversely, if we increase $\widetilde{Q}_i$, and thus decrease h and shift to the right for a convergent geometry, we observe a solution satisfying $X < Y$, see Figure 20.

This last result indicates that it is possible to obtain "inclusions" as a stationary solution. Such a situation is given on Figure 21 where the height is a classical convergent-divergent domain $h(x) = \frac{1}{2}(x - \frac{1}{2})^2 + 0.9$, $L = 1$ and where the physical parameters are given by $Q_1 = 0.2$, $Q_3 = 0.05$ and $M = 0.4$.

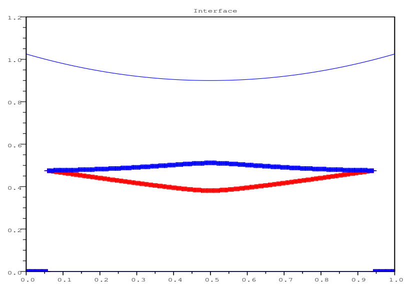

FIGURE 21. Possible stationary solutions for a convergent-divergent domain : "inclusion"

5 Appendix : Sturm sequences

Let $\mathcal{P} \in \mathbb{R}[X]$ be a polynomial. Sturm's theorem is a symbolic procedure to determine the number of distinct real roots of P in a given interval.
First, we build the Sturm sequence as follows:

$$\mathcal{P}^{(0)} = \mathcal{P}, \qquad \mathcal{P}^{(1)} = \mathcal{P}' \quad \text{and for all } k \in \mathbb{N},$$
$$\mathcal{P}^{(k)} = \mathcal{Q}^{(k+1)}\mathcal{P}^{(k+1)} - \mathcal{P}^{(k+2)} \qquad \text{with} \quad \deg(\mathcal{P}^{(k+2)}) < \deg(\mathcal{P}^{(k+1)}).$$

In other terms, it consists in taking the opposite of the remainders of the successive polynomial divisions. Since $\deg(\mathcal{P}^{(k+1)}) < \deg(\mathcal{P}^{(k)})$, the algorithm terminates with a constant polynomial $\mathcal{P}^{(n)}$.
Let $V(x)$ be the number of sign changes (zeros are not counted) in the sequence $\mathcal{P}^{(0)}(x)$,

$\mathcal{P}^{(1)}(x)$, ..., $\mathcal{P}^{(n)}(x)$.
Sturm's theorem states that for two real numbers $a < b$, the number of distinct roots in the half-open interval $(a, b]$ is $|V(a) - V(b)|$.

Example 5.1 *Let $h > 0$ and $0 < M < 1$ be given data. We are looking for the number of roots in the interval $[0, h]$ for the polynomial $\mathcal{P}$ of degree 2 defined by*

$$\mathcal{P}(h_1) = 2\ell h_1 - 3q \quad \textit{with} \quad \ell = h + (M-1)h_1,\ q = h^2 + (M-1)h_1^2.$$

Using the Sturm procedure, we define the three following polynomials $\mathcal{P}^{(0)} = \mathcal{P}$, $\mathcal{P}^{(1)} = \mathcal{P}'$ and $\mathcal{P}^{(2)} = C$ (C is a constant and we see that its value does not matter here). We obtain the following signs:

$$\mathcal{P}^{(0)}(0) = -3h^2 < 0, \quad \mathcal{P}^{(0)}(h) = -Mh^2 < 0,$$
$$\mathcal{P}^{(1)}(0) = h > 0 \quad \textit{and} \quad \mathcal{P}^{(1)}(h) = 2(2-M)h > 0.$$

Next, we deduce the following sign table:

	$\mathcal{P}^{(0)}(h_1)$	$\mathcal{P}^{(1)}(h_1)$	$\mathcal{P}^{(2)}(h_1)$
$h_1 = 0$	$-$	$+$	$sgn(C)$
$h_1 = h$	$-$	$+$	$sgn(C)$

Thus, whatever the sign of the constant C is, there is exactly the same number of signs in the second row of this table that in the third. In conclusion, the polynomial $\mathcal{P}$ has no root between 0 and h.

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