Weak solution for a dense granular materials model using Drucker-Prager flow constraint and Bagnold scaling for viscosity

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Abstract

This article addresses the existence of solutions for partial differential equation (PDE) models describing granular flows. We emphasize the essential role of flow dilatation, coupled with complex rheology, in ensuring both stability and the existence of dissipative energy. A central focus of the paper is to understand how this energy, arising from strongly nonlinear and singular terms, contributes to the existence of weak solutions. We first establish an existence result for a model that reflects some mathematical difficulties of the complete system. In particular, the dilatancy law describes local volume changes in terms of the velocity divergence, which depends on the shear rate and the square root of the pressure, reflecting a balance between these two quantities. Although the model studied rigorously in this article does not contain all the difficulties of a complete physical model - the case of a variable volume fraction being treated only by using regularisation, this work represents a significant advance in the mathematical analysis of models for such complex flows.

keywords: granular model, weak existence, rheology, dilatance.

1 Introduction

From grain to continuous model Granular flow modeling is a major challenge in many contexts. Indeed, this type of flow is involved in a wide variety of fields, such as the earth sciences, with pyroclastic flows or dune movements; the food industry in the broadest sense of the term, from cereal silos to food design; medicine, and in particular the pharmaceutical industry, which makes extensive use of powders and capsules corresponding to granular media with diverse behaviors. The tricky aspect in this type of medium lies in its intermediate position between a fluid-like medium (such as gases, where the particles are too small to be individually described) and a solid medium, which requires understanding the interactions between a limited number of macroscopic particles.

With the advent of computational tools, we could be led to use these models even when the media comprise a very large number of particles (generally several billion). Currently this approach is hampered by our limits knowledge of the local interactions: at the grain level, the solid contact laws between multiple particles involve highly nonlinear relations such as frictional or inelastic shocks, that remain difficult to fully understand. The well-known Discrete Element Method (see for instance [25]) attempts to take into account as much information as possible, but the lack of physical understanding means that not all the expected results can be obtained.

An alternative strategy is to define spatially averaged quantities to obtain continuous models, similar to what is done for fluid flows with the Navier-Stokes equations. One of the foundational works of this approach for granular flows is arguably that of the GdR MIDI, which highlighted the $\mu(I)$-rheology model as a basis for granular flow models (see, for instance, the collective work [22]).

Issues and open questions Since the $\mu(I)$-model is primarily derived from experimental considerations, numerous questions remain regarding its applicability in more general contexts those in which it was established (for instance, at larger scales). Furthermore, its implementation raises additional challenges to be

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used numerically. Indeed, to be able to numerically discretise a model and perform a computational implementation, it is highly desirable that it has a solution and that this solution is stable, in particular not too sensitive to perturbations in the data. Unfortunately, from this perspective, these continuous models often exhibit significant limitations. Their implementation in a numerical framework is feasible, but their reliability cannot be guaranteed: instabilities have been frequently observed with classical formulations (see [16] for instance). At first glance, these drawbacks appear to be closely linked to the nonlinear and singular nature of the model. Several authors have presented mathematical results based on these models, but they have introduced assumptions that are not always physically relevant. This is the case, for example, in [1, 2, 17] where the authors examine the existence of solutions in a model with a pressure-dependent stress threshold. However, the pressure considered in these works differs from the total pressure, which results in a simpler coupling. Hence, from both a theoretical and a numerical point of view, these continuous models often exhibit significant limitations. To address these limitations, one would need to incorporate more realistic models commonly used to describe granular flows–models capable of capturing phenomena such as normal stress differences. Such models have been developed and discussed in the review article [19] and it is this type of model that we aim to explore in the present work.
The major challenge is therefore to develop a model for dense granular flows that is both physically consistent and mathematically well-posed. A first step in this direction was taken in [15], where a stable and physically consistent model was proposed. A particularly noteworthy feature of this model is that its stability results from the coupling between rheology and dilatancy. Accounting for dilatancy, which reflects local volume changes, is essential: stability issues arise in granular flow models based on classical rheologies, such as the $\mu(I)$ or Drucker–Prager laws, which generate unstable solutions at high wave-numbers when incompressibility is assumed, as shown in the work of Barker and co-authors (see [5] and the references therein). Moreover, they demonstrated that allowing for flow compressibility can yield a stable formulation of the model. In these studies, both the divergence of the velocity field and the yield threshold in the constitutive relation are functions of the inertial number, the pressure, and the volume fraction. Stability of the model is ensured if these functions satisfy a system of variational inequalities. In [15], we went further by proposing explicit functions, in the context of the aforementioned rheologies, that satisfy these stability conditions and are consistent with the known physics of granular flows. Furthermore, the resulting model fulfills additional properties—namely, the existence of a system energy that is dissipated over time and the preservation of uniform bounds on the volume fraction.
It is worth noting that in [9, 10, 11], similar dilatancy laws were used in depth-averaged models for granular flows. Also, in [23], a granular model with dilatancy effects was used to study through numerical simulations immersed granular avalanches. However, in these works, the question of the existence of solutions is not addressed.
To the best of our knowledge, the existence of solutions for such models has not yet been investigated, and a comprehensive theoretical framework is still lacking. The strong nonlinearities of the model proposed in [15], along with the presence of yield thresholds in the stress expression, raise important questions regarding the existence of solutions. The work presented in this article represents a first attempt to adress this issue.

Results and article outline In this article, we propose a functional framework in which it is possible to establish the existence of a solution. Once again, it is the combination of several physical ingredients that guarantees these results. In the first part (Section 2), we present the complete model and then focus on a reduced version that retains the two main physical features which are also essential for ensuring the existence of a solution:

- the incorporation of threshold rheology, with a pressure-dependent yield criterion, as in the $\mu(I)$-model;
- the consideration of dilatancy, as proposed in [28], specifically through a velocity field whose divergence, representing volume changes, is governed by a balanced between shear and pressure effects.

The second part forms the core of the article and is devoted to the question of the existence of solutions for the proposed model. It is structured into several subsections. After introducing the mathematical notations in Subsection 3.1, the following subsection presents a key component of the main result: it provides the weak formulation of the problem studied. In particular, we observed that this formulation appears to suppress certain threshold-related nonlinearities. Nevertheless, Proposition 1 demonstrates that the weak formulation faithfully represents the full physical model, allowing these nonlinear effects to be recovered. The final subsections (Subsections 3.3 and 3.4) are devoted to the proof of the existence result, with a particular focus on handling the remaining nonlinear terms. These results represent a real innovation in the mathematical analysis of this type of model for complex flows: they demonstrate that a granular model, incorporating threshold rheology and dilatation, is mathematically well-posed thanks to these relevant physical laws.

In the final Section 4, we return to the full problem and show how the preceding analysis can be adapted. A specific study of the $\mu(I)$-model highlights how it relates to the reduced version initially considered. This section concludes by noting that many questions remain open regarding this class of models.

2 Modeling dense granular flow

Guided by the following principles–maximizing the use of physical laws (conservation principles) and phenomenological analyses (constitutive relations, equations of state); ensuring linear stability of the model, based on Barker's stability criteria as presented in [29]; and obtaining an energy that dissipates over time–a complete model for dense granular flows was proposed in [15]. This model couples four unknown fields: the scalar volume fraction ϕ, the velocity vector field $\mathbf{u}$, the symmetric extra-stress tensor field $\boldsymbol{\sigma}$ and the scalar pressure field p. Depending on the chosen rheology for the granular medium, the expression of the different terms differs; however, in the case of the Druker-Prager rheology, the model is written as follows

$$\begin{cases} \phi\rho_0(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p - \operatorname{div}(2\nu(\phi,|\mathbf{Du}|)\mathbf{Du}) = \phi\rho_0\mathbf{g} + \operatorname{div}\boldsymbol{\sigma}, & (1)\\ \boldsymbol{\sigma} : \mathbf{Su} = 2\alpha(\phi)p|\mathbf{Su}|, \quad |\boldsymbol{\sigma}| \le \alpha(\phi)p, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}} \quad \text{and} \quad \operatorname{tr}\boldsymbol{\sigma} = 0, & (2)\\ \operatorname{div}\mathbf{u} = 2\alpha(\phi)|\mathbf{Su}| - \beta(\phi)\sqrt{p}, & (3)\\ \partial_t\phi + \operatorname{div}(\phi\mathbf{u}) = 0. & (4) \end{cases}$$

In this version $\mathbf{g}$ denotes the gravity vector, ρ_0 is the grain density, and $\phi_{\max}$ corresponds to the maximum volume fraction (on the order of 0.6 for laboratory-scale experimental flows, see [27]). The functions α and β are positive and depends on ϕ as well as on several physical parameters such as the mean grain size, friction coefficient, and so on.

The function ν depends on the chosen model: it reflects the possibly nonlinear effect of viscous forces and depends on the strain-rate tensor which is the symmetrical part of the velocity gradient, namely $\mathbf{Du} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}})$. By experiments described in his 1954 paper, Bagnold [4] showed that when a shear flow is applied to the suspension, then the shear and normal stresses in the suspension may vary quadratically with the shear rate. In other words, if one wishes to adopt this type of approach, it is consistent to impose a viscosity contribution of the form $\nu(\phi,|\mathbf{Du}|)\mathbf{Du} = \nu_0|\mathbf{Du}|\mathbf{Du}$. As we will demonstrate, this choice is not only physically consistent but crucial for ensuring sufficient regularity of the velocity field, thereby allowing us to rigorously define the product $p|\mathbf{Su}|$.

Remark 1. *In [15], the viscosity ν is not included, although the authors mention this possibility in their remarks. In contrast, in the present study, viscosity is essential to ensure a minimum of regularity.*

The rheology also introduces the deviatoric strain-rate tensor $\mathbf{Su} = \mathbf{Du} - \frac{1}{3}(\operatorname{div}\mathbf{u})\mathbf{I_3}$. For all these tensors, the norm is defined by $|\mathbf{A}|^2 = \frac{1}{2}\mathbf{A} : \mathbf{A}$ where $\mathbf{A} : \mathbf{B} = \operatorname{tr}(\mathbf{A}^{\mathsf{T}}\mathbf{B})$. The rheology introduced here corresponds to the Drucker-Prager model in the sense that the relations (2), which describe the connection between stress and deformation, can be rewritten as follows:

$$\begin{aligned} &\boldsymbol{\sigma} = \alpha(\phi)p\frac{\mathbf{Su}}{|\mathbf{Su}|} && \text{if } |\mathbf{Su}| \neq 0,\\ &|\boldsymbol{\sigma}| \le \alpha(\phi)p, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}} \quad \text{and} \quad \operatorname{tr}\boldsymbol{\sigma} = 0 && \text{if } |\mathbf{Su}| = 0. \end{aligned} \qquad (5)$$

In this formulation, the quantity $\alpha(\phi)p$ represents the yield threshold, while $\alpha(\phi)$ itself is associated with the internal friction angle of the granular material. Another fundamental feature of the model (1)–(4) is the expression for the divergence of the velocity field, as given by equation (3). This expression provides an interpretation of the dilatancy law derived from the work of F. Radjaï and S. Roux in [28]. According to this law, the local volume variation (*i.e.*, $\operatorname{div}\mathbf{u}$) is related to changes in the volume fraction ϕ. More precisely, $\operatorname{div}\mathbf{u}$ is proportional to the deviation of ϕ from an equilibrium state ϕ_{eq}, which depends on the shear rate magnitude $|\mathbf{Su}|$ and on the square root of the pressure $\sqrt{p}$; see Subsection 4.1 for further details.

Remark 2. *As explained in [15], more complex rheologies, such as the $\mu(I)$-rheology, can be considered. However, in such cases, the dilatation law* (3) *must be adapted accordingly. We will revisit the important case of the $\mu(I)$-rheology in Section 4.*

Remark 3. *This model does not account for the interstitial gas in which the grain particles move. In [15], this additional contribution is incorporated by including the pressure forces ∇p_f exerted by the gas in* (1)*, together with an evolution equation for the pore gas pressure p_f. This evolution equation is essentially of the heat-equation type, see [15] for further details. Notably, for this more complete model, as well as for the model presented here, energy is dissipated over time.*

In the following section, to better emphasize the main analytical challenges and the originality of the approach, we focus on a simplified model that captures the core difficulties of the problem–specifically the coupling between rheology and dilatancy–while assuming a constant volume fraction. Moreover, from a mathematical point of view, the requirement that pressure remains positive (a condition necessary to write $\sqrt{p}$) is generally not reasonable. For instance, if we consider flows without internal friction (i.e., with $\alpha = 0$), condition (3) reduces to $\operatorname{div} \mathbf{u} = -\beta(\phi)\sqrt{p} \leq 0$. Given the boundary condition $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$, this implies $\operatorname{div} \mathbf{u} = 0$ and, consequently, $p = 0$, which is incompatible with (1). It is therefore essential to have a model that accounts for both the positive and negative components of pressure, respectively defined by $p_+ = \max(0, p)$ and $p_- = -\min(0, p)$.
The model under study is formulated as follows:

$$\begin{cases} \phi_0\rho_0\partial_t \mathbf{u} + \nabla p - \operatorname{div}(2\nu_0|\mathbf{Du}|\mathbf{Du}) = \mathbf{f} + \operatorname{div} \boldsymbol{\sigma}, & (6) \\ \boldsymbol{\sigma} : \mathbf{Su} = 2\alpha_0 p_+|\mathbf{Su}|, \quad |\boldsymbol{\sigma}| \leq \alpha_0 p_+, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^\intercal \quad \text{and} \quad \operatorname{tr} \boldsymbol{\sigma} = 0, & (7) \\ \operatorname{div} \mathbf{u} = 2\alpha_0|\mathbf{Su}| - \beta_0\sqrt{p_+} + \gamma_0\sqrt{p_-}, & (8) \end{cases}$$

where $\mathbf{f}$ are the external forces, namely $\mathbf{f} = \phi_0\rho_0\mathbf{g}$. It is important to note that this model is not a special case of the full model obtained by simply replacing ϕ with a constant ϕ_0, since the velocity field $\mathbf{u}$ is not divergence-free. In Subsection 4.2, we show that this reduced system (6)–(8) corresponds to the leading-order behavior of the full model (1)–(4) in a particular asymptotic regime, namely when ϕ remains close to a constant value near $\phi_{\max}$. In Section 4, we discuss how these results can be extended to more complete frameworks, particularly by incorporating the evolution of the volume fraction ϕ. As previously noted, it is necessary to take into account both the positive and negative components of pressure. Physically, when pressure becomes negative, it should not contribute to the yield threshold. However, the negative pressure part $p_- = -\min(0, p)$ must still influence constraint (3). This effect is incorporated in (8) through the additional term $\gamma_0\sqrt{p_-}$.

3 Mathematical results

3.1 Mathematical framework and notations

For the purposes of mathematical analysis, we assume that all physical constants have been normalized. Naturally, these assumptions do not qualitatively affect the results that follow. The model considered in this section can thus be written as follows:

$$\begin{cases} \partial_t \mathbf{u} + \nabla p - \operatorname{div}(2|\mathbf{Du}|\mathbf{Du}) = \mathbf{f} + \operatorname{div} \boldsymbol{\sigma}, & (9) \\ \boldsymbol{\sigma} : \mathbf{Su} = 2p_+|\mathbf{Su}|, \quad |\boldsymbol{\sigma}| \leq p_+, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^\intercal \quad \text{and} \quad \operatorname{tr} \boldsymbol{\sigma} = 0, & (10) \\ \operatorname{div} \mathbf{u} = 2|\mathbf{Su}| - \sqrt{p_+} + \sqrt{p_-}. & (11) \end{cases}$$

Given a time $T > 0$ and a bounded open domain $\Omega \subset \mathbb{R}^3$ with a boundary of class $\mathcal{C}^2$, we consider the problem (9)–(11) on the space-time domain $(0, T) \times \Omega$. It is completed by the following initial and boundary conditions:

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_{\text{init}} \qquad \text{and} \qquad \mathbf{u}\big|_{\partial\Omega} = \mathbf{0}. \tag{12}$$

Notations - In order to define the notion of weak solution and to formulate the results, we need to fix the notations. The symbol $\|\cdot\|_q$ for $1 \leq q \leq +\infty$ stands for the L^q-norm in the usual Lebesgue space $L^q(\Omega)$ while $\|\cdot\|_{s,q}$ refers to the norm of the Sobolev space $W^{s,q}(\Omega)$, $s \in \mathbb{R}$ and $1 \leq q \leq +\infty$.

To indicate that a function is assumed to be non-negative, we use a "+" superscript, as in $L^p_+(\Omega)$.
We will make repeated use of the Sobolev space $W^{1,3}_0(\Omega)$, consisting of functions of $W^{1,3}(\Omega)$ that vanish at the boundary, together with its dual space $W^{-1,\frac{3}{2}}(\Omega)$. The associated duality bracket will be denoted $\langle\cdot,\cdot\rangle$. Finally, for a Banach space X, we denote the corresponding Bochner space by $L^q(0,T;X)$, with its associated norm explicitly written as $\|\cdot\|_{L^q(0,T;X)}$.

Remark 4. — *As explained in the previous section, the relation* (11) *arises from physical observations that capture the competing effects of shear and pressure on volume changes. However, we can also identify a fundamental mathematical element that clarifies the connection between relation* (11) *and the rheology given by* (10). *Indeed, much like the role of pressure in the Stokes equations for Newtonian fluid flow, the pressure p appearing in equations* (9)–(10) *acts as a Lagrange multiplier associated with constraint* (11).
As a simpler example, consider the minimization problem $\mathbf{u} = \operatorname{argmin}\{f(\mathbf{v})\,;\, g(\mathbf{v}) = \mathbf{0}\}$ withe

$$f : \mathbf{v} \in W^{1,2}_0(\Omega) \longmapsto \frac{1}{2}\int_\Omega |\nabla \mathbf{v}|^2 \in \mathbb{R}$$
$$\text{and} \qquad g : \mathbf{v} \in W^{1,2}_0(\Omega) \longmapsto \operatorname{div}\mathbf{v} - 2|\mathbf{S}\mathbf{v}| - q \in L^2(\Omega),$$

where the source term q is given in $L^2(\Omega)$. Note that the applications f and g are differentiable in any non-vanishing function $\mathbf{v} \in W^{1,2}_0(\Omega)$ (if $\mathbf{v}$ vanishes, we should probably consider the sub-differential of g, as proposed by Beck [8]):

$$df(\mathbf{v}) : \boldsymbol{\varphi} \in W^{1,2}_0(\Omega) \longmapsto \int_\Omega \nabla \mathbf{v} : \nabla \boldsymbol{\varphi} \in \mathbb{R}$$
$$\text{and} \qquad dg(\mathbf{v}) : \boldsymbol{\varphi} \in W^{1,2}_0(\Omega) \longmapsto \operatorname{div}\boldsymbol{\varphi} - \frac{\mathbf{S}\mathbf{v}}{|\mathbf{S}\mathbf{v}|} : \mathbf{S}\boldsymbol{\varphi} \in L^2(\Omega).$$

If a solution $\mathbf{u}$, minimizing f subject to the constraint $g(\mathbf{u}) = 0$, exists and is nonzero, then according to the Lagrange multiplier Theorem, there exists a linear functional $\widetilde{p} : L^2(\Omega) \longrightarrow \mathbb{R}$ such that $df(\mathbf{u}) = \widetilde{p} \circ dg(\mathbf{u})$. According to the Riesz representation Theorem, the linear functional $\widetilde{p}$ can be represented via the scalar product in $L^2(\Omega)$: there exists $p \in L^2(\Omega)$ such that for all $\boldsymbol{\psi} \in L^2(\Omega)$ we have $\widetilde{p}(\boldsymbol{\psi}) = \int_\Omega p\boldsymbol{\psi}$. Thus, using the expressions for the differentials of f and g, we deduce that

$$\forall \boldsymbol{\varphi} \in W^{1,2}_0(\Omega) \qquad \int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} = \int_\Omega p\big(\operatorname{div}\boldsymbol{\varphi} - \frac{\mathbf{S}\mathbf{u}}{|\mathbf{S}\mathbf{u}|} : \mathbf{S}\boldsymbol{\varphi}\big)$$

which implies that, in the sense of distributions, $-\Delta\mathbf{u} + \nabla p = \operatorname{div}\boldsymbol{\sigma}$ where $\boldsymbol{\sigma} : \mathbf{S}\mathbf{u} = 2p|\mathbf{S}\mathbf{u}|$.

The approach described in Remark 4, which interprets the pressure as a Lagrange multiplier, will not be pursued further in this article, although it may yield interesting results. Instead, we present here an original velocity-pressure formulation, with a particular emphasis on the energy structure of the system.

3.2 Weak formulation and statement of the main result

A key aspect is the definition of the weak solution for the system (9)–(12). A difficulty arises from the presence of the positive and negative parts of the pressure in equality (11). We propose here an equivalent formulation, motivated by the following considerations. The positive and negative parts of the pressure, defined respectively as $p_+ = \max(0,p)$ and $p_- = -\min(0,p)$, admit the following characterization

$$\big(a = p_+ \quad \text{and} \quad b = p_-\big) \quad \Longleftrightarrow \quad \big(p = a - b, \quad a \geq 0, \quad b \geq 0, \quad ab = 0\big).$$

Moreover, we obtain the following result, whose proof is relatively straightforward.

Lemma 1. *Let $f \in \mathbb{R}$, $a \geq 0$ and $b \geq 0$. We have*

$$\Big(f = \sqrt{b} - \sqrt{a} \quad \textit{and} \quad ab = 0\Big) \quad \Longleftrightarrow \quad \begin{pmatrix} f + \sqrt{a} = 0 & \textit{if } a > 0 & & f - \sqrt{b} = 0 & \textit{if } b > 0 \\ & & \textit{and} & & \\ f + \sqrt{a} \geq 0 & \textit{if } a = 0 & & f - \sqrt{b} \leq 0 & \textit{if } b = 0 \end{pmatrix}$$

We therefore consider the following weak formulation.

Definition 1 (weak solution)**.** *Let* $\mathbf{u}_{\text{init}} \in L^2(\Omega)$ *and* $\mathbf{f} \in L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega))$.
We say that $(\mathbf{u}, p, \boldsymbol{\sigma})$ *is a weak solution of* (9)–(12) *if*

$$\begin{aligned} &\mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^3(0,T;W_0^{1,3}(\Omega)), \quad \partial_t \mathbf{u} \in L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega)), \\ &p = a - b \quad \textit{with} \quad a \in L_+^{\frac{3}{2}}((0,T)\times\Omega) \quad \textit{and} \quad b \in L_+^{\frac{3}{2}}((0,T)\times\Omega), \\ &\boldsymbol{\sigma} \in L^{\frac{3}{2}}((0,T)\times\Omega) \quad \textit{with } |\boldsymbol{\sigma}| \le a, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^\intercal \quad \textit{and} \quad \operatorname{tr}\boldsymbol{\sigma} = 0 \quad \textit{a.e.} \end{aligned}$$

and for all $\boldsymbol{\varphi} \in L^3(0,T;W_0^{1,3}(\Omega))$*, for all* $\psi \in L_+^{\frac{3}{2}}((0,T)\times\Omega)$*, for all* $\xi \in L^\infty((0,T)\times\Omega)$ *we have*

$$\langle \partial_t \mathbf{u}, \boldsymbol{\varphi}\rangle - \int_\Omega p \operatorname{div}\boldsymbol{\varphi} + \int_\Omega 2|\mathbf{D}\mathbf{u}|\mathbf{D}\mathbf{u} : \mathbf{D}\boldsymbol{\varphi} + \int_\Omega \boldsymbol{\sigma} : \mathbf{S}\boldsymbol{\varphi} = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle, \tag{13}$$

$$\int_\Omega \xi a\big(\operatorname{div}\mathbf{u} - 2|\mathbf{S}\mathbf{u}| + \sqrt{a}\big) = 0, \qquad \int_\Omega \xi b\big(\operatorname{div}\mathbf{u} - 2|\mathbf{S}\mathbf{u}| - \sqrt{b}\big) = 0, \tag{14}$$

$$\int_\Omega \psi\big(\operatorname{div}\mathbf{u} - 2|\mathbf{S}\mathbf{u}| + \sqrt{a}\big) \ge 0, \qquad \int_\Omega \psi\big(\operatorname{div}\mathbf{u} - 2|\mathbf{S}\mathbf{u}| - \sqrt{b}\big) \le 0, \tag{15}$$

$$\frac{1}{2}\int_\Omega |\mathbf{u}(T)|^2 + 4\int_0^T\int_\Omega |\mathbf{D}\mathbf{u}|^3 + \int_0^T\int_\Omega a^{\frac{3}{2}} + \int_0^T\int_\Omega b^{\frac{3}{2}} + \int_0^T\int_\Omega 2b|\mathbf{S}\mathbf{u}| \le \frac{1}{2}\int_\Omega |\mathbf{u}_{\text{init}}|^2 + \int_0^T \langle \mathbf{f}, \mathbf{u}\rangle, \tag{16}$$

and the initial condition $\mathbf{u}\big|_{t=0} = \mathbf{u}_{\text{init}}$ *holds in* $L^2(\Omega)$.

Relation (13) provides the weak form of equation (9), while relations (14) and (15) correspond to equation (11), following the equivalent formulation established in Lemma 1. In particular, note that the quantities a and b respectively represent the positive and negative parts of the pressure p.
Although not immediately evident, this weak formulation fully captures the threshold rheology given by relation (10), as demonstrated in the proof below.

Proposition 1. *If* $(\mathbf{u}, p, \boldsymbol{\sigma})$ *is a weak solution of* (9)–(12)*, then the relations* (10) *are satisfied almost everywhere.*

Proof - Let $(\mathbf{u}, p, \boldsymbol{\sigma})$ be a weak solution of (9)–(12). Taking $\boldsymbol{\varphi} = \mathbf{u}$ as test function in (13), $\xi = 1$ in (14) and combining the results, we immediately deduce

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2 + \int_\Omega 4|\mathbf{D}\mathbf{u}|^3 + \int_\Omega a^{\frac{3}{2}} + \int_\Omega b^{\frac{3}{2}} + \int_\Omega \Big(\boldsymbol{\sigma} : \mathbf{S}\mathbf{u} - 2a|\mathbf{S}\mathbf{u}|\Big) + \int_\Omega 2b|\mathbf{S}\mathbf{u}| = \langle \mathbf{f}, \mathbf{u}\rangle.$$

Integrating with respect to time and comparing with (16), we get

$$\int_0^T\int_\Omega \Big(\boldsymbol{\sigma} : \mathbf{S}\mathbf{u} - 2a|\mathbf{S}\mathbf{u}|\Big) \ge 0. \tag{17}$$

But, from the Cauchy-Schwarz inequality $\boldsymbol{\sigma} : \mathbf{S}\mathbf{u} \le 2|\boldsymbol{\sigma}||\mathbf{S}\mathbf{u}|$ and using the fact that $|\boldsymbol{\sigma}| \le a$, we obtain $\boldsymbol{\sigma} : \mathbf{S}\mathbf{u} - 2a|\mathbf{S}\mathbf{u}| \le 0$. In view of (17), this implies that $\boldsymbol{\sigma} : \mathbf{S}\mathbf{u} = 2a|\mathbf{S}\mathbf{u}|$ almost everywhere. ■

The main result, which will be established in the remainder of the paper, states:

Theorem 1. *If* $\mathbf{u}_{\text{init}} \in L^2(\Omega)$ *and* $\mathbf{f} \in L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega))$*, then there exists a weak solution* $(\mathbf{u}, p, \boldsymbol{\sigma})$ *to problem* (9)–(12).

3.3 Approximate system

To obtain a weak solution, we first construct a solution to an approximate problem. To this end, consider two sequences $(\mathbf{f}_\varepsilon)_{\varepsilon>0}$ and $(\mathbf{u}_{\text{init},\varepsilon})_{\varepsilon>0}$ of regular functions which converge respectively to $\mathbf{f}$ in $L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega))$ and to $\mathbf{u}_{\text{init}}$ in $L^2(\Omega)$. For $\varepsilon > 0$, we consider the following problem

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon + \nabla(a_\varepsilon - b_\varepsilon) - \operatorname{div}(2|\mathbf{D}\mathbf{u}_\varepsilon|\mathbf{D}\mathbf{u}_\varepsilon) = \mathbf{f}_\varepsilon + \operatorname{div}\boldsymbol{\sigma}_\varepsilon, & (18) \\ \operatorname{div}\mathbf{u}_\varepsilon = 2\dfrac{|\mathbf{S}\mathbf{u}_\varepsilon|^2}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon} - V_\varepsilon(a_\varepsilon) - \varepsilon(\partial_t a_\varepsilon - \Delta a_\varepsilon), & (19) \\ \operatorname{div}\mathbf{u}_\varepsilon = 2\dfrac{|\mathbf{S}\mathbf{u}_\varepsilon|^2}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon} + V_\varepsilon(b_\varepsilon) + \varepsilon(\partial_t b_\varepsilon - \Delta b_\varepsilon), & (20) \\ \boldsymbol{\sigma}_\varepsilon = a_\varepsilon \dfrac{\mathbf{S}\mathbf{u}_\varepsilon}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon}, & (21) \end{cases}$$

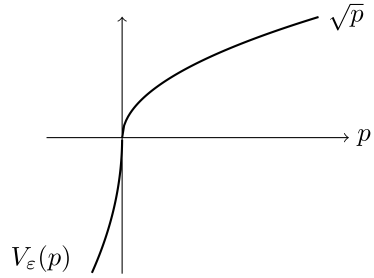
with the following initial and boundary conditions:

$$\mathbf{u}_\varepsilon\big|_{t=0} = \mathbf{u}_{\mathrm{init},\varepsilon}, \qquad \mathbf{u}_\varepsilon\big|_{\partial\Omega} = \mathbf{0}, \qquad a_\varepsilon\big|_{t=0} = b_\varepsilon\big|_{t=0} = 0 \qquad \text{and} \qquad a_\varepsilon\big|_{\partial\Omega} = b_\varepsilon\big|_{\partial\Omega} = 0. \tag{22}$$

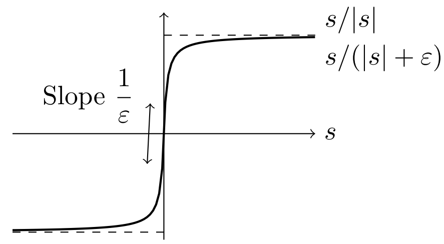
The function V_ε is an extension of the square root that facilitates the proof that the limits a and b, of the respective sequences $(a_\varepsilon)_{\varepsilon>0}$ and $(b_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$, are non-negative. More precisely, the function V_ε is defined on $\mathbb{R}$ by

$$V_\varepsilon(x) = \begin{cases} \sqrt{x} & \text{if } x > 0, \\ -\dfrac{1}{\varepsilon}\sqrt{-x} & \text{if } x \leq 0. \end{cases} \tag{23}$$

The following two figures illustrate the approximation of the square root and Heaviside functions.



Extension of $\sqrt{p}$



Approximation of $s/|s|$

Proposition 2. *For $\varepsilon > 0$ (ε small enough), there exists a weak solution $(\mathbf{u}_\varepsilon, a_\varepsilon, b_\varepsilon, \boldsymbol{\sigma}_\varepsilon)$ to equations* (18)–(20) *satisfying*

$$\begin{aligned} &\mathbf{u}_\varepsilon \in L^\infty(0,T;L^2(\Omega)) \cap L^3(0,T;W_0^{1,3}(\Omega)), && \partial_t \mathbf{u}_\varepsilon \in L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega)), \\ &a_\varepsilon,\ b_\varepsilon \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega)), && \partial_t a_\varepsilon,\ \partial_t b_\varepsilon \in L^2(0,T;W^{-1,2}(\Omega)), \\ &\boldsymbol{\sigma}_\varepsilon \in L^{\frac{3}{2}}((0,T)\times\Omega), \end{aligned}$$

and such that for all $\boldsymbol{\varphi} \in L^3(0,T;W_0^{1,3}(\Omega))$

$$\langle \partial_t \mathbf{u}_\varepsilon, \boldsymbol{\varphi}\rangle - \int_\Omega (a_\varepsilon - b_\varepsilon)\,\mathrm{div}\,\boldsymbol{\varphi} + \int_\Omega 2|\mathbf{D}\mathbf{u}_\varepsilon|\mathbf{D}\mathbf{u}_\varepsilon : \mathbf{D}\boldsymbol{\varphi} + \int_\Omega \boldsymbol{\sigma}_\varepsilon : \mathbf{S}\boldsymbol{\varphi} = \langle \mathbf{f}_\varepsilon, \boldsymbol{\varphi}\rangle, \tag{24}$$

for all $\psi \in L^2(0,T;W_0^{1,2}(\Omega))$

$$\langle \varepsilon\partial_t a_\varepsilon, \psi\rangle + \int_\Omega \varepsilon\nabla a_\varepsilon \cdot \nabla\psi + \int_\Omega \psi\,\mathrm{div}\,\mathbf{u}_\varepsilon - \int_\Omega 2\psi\frac{|\mathbf{S}\mathbf{u}_\varepsilon|^2}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon} + \int_\Omega \psi V_\varepsilon(a_\varepsilon) = 0, \tag{25}$$

for all $\phi \in L^2(0,T;W_0^{1,2}(\Omega))$

$$\langle \varepsilon\partial_t b_\varepsilon, \psi\rangle + \int_\Omega \varepsilon\nabla b_\varepsilon \cdot \nabla\psi - \int_\Omega \phi\,\mathrm{div}\,\mathbf{u}_\varepsilon + \int_\Omega 2\phi\frac{|\mathbf{S}\mathbf{u}_\varepsilon|^2}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon} + \int_\Omega \phi V_\varepsilon(b_\varepsilon) = 0, \tag{26}$$

and for all $\boldsymbol{\Sigma} \in L^3((0,T)\times\Omega)$

$$\int_\Omega \boldsymbol{\sigma}_\varepsilon : \boldsymbol{\Sigma} = \int_\Omega a_\varepsilon \frac{\mathbf{S}\mathbf{u}_\varepsilon : \boldsymbol{\Sigma}}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon}. \tag{27}$$

The initial condition is satisfied in $L^2(\Omega)$: $\mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_{\mathrm{init},\varepsilon}$ and $a_\varepsilon|_{t=0} = b_\varepsilon|_{t=0} = 0$.

Proof - Adding $\partial_t a_\varepsilon - \Delta a_\varepsilon$ into equation (19) allows us to define, for each $\varepsilon > 0$, the pressure a_ε as the solution to a heat-type equation. The same observation applies to the existence of the pressure b_ε. The explicit expression of $\boldsymbol{\sigma}_\varepsilon$ given by (21) corresponds to a standard regularized version of condition (10). This type of regularization is very common in problems of this kind (see, for example, [2]). The existence of a solution to the complete system (18)–(19) then follows by classical arguments, for instance *via* a Galerkin method (see [2]). In practice, the Galerkin method applies as soon as an energy estimate for the solution is available. We therefore show below how such an estimate can be derived.

Energy-type estimate - We choose $\boldsymbol{\varphi} = \mathbf{u}_\varepsilon$ as test function in (24), $\psi = a_\varepsilon$ in (25), $\phi = b_\varepsilon$ in (26) and $\mathbf{\Sigma} = \mathbf{S}\mathbf{u}_\varepsilon$ in (27). We note that, for $\mathbf{u}_\varepsilon \in L^3(0,T;W_0^{1,3}(\Omega))$ and $\partial_t \mathbf{u}_\varepsilon \in L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega))$, we have (see [12, p.99])

$$\langle \partial_t \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon \rangle = \frac{1}{2}\frac{d}{dt}\|\mathbf{u}_\varepsilon\|_2^2.$$

Applying the same reasoning to the time derivative of the pressure terms, we finally obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{u}_\varepsilon\|_2^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|a_\varepsilon\|_2^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|b_\varepsilon\|_2^2 + 4\|\mathbf{D}\mathbf{u}_\varepsilon\|_3^3 + \varepsilon\|\nabla a_\varepsilon\|_2^2 + \varepsilon\|\nabla b_\varepsilon\|_2^2 \\ + \int_\Omega a_\varepsilon V_\varepsilon(a_\varepsilon) + \int_\Omega b_\varepsilon V_\varepsilon(b_\varepsilon) + \int_\Omega 2b_\varepsilon \frac{|\mathbf{S}\mathbf{u}_\varepsilon|^2}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon} = \langle \mathbf{f}_\varepsilon, \mathbf{u}_\varepsilon \rangle. \end{aligned} \tag{28}$$

We begin by estimating the term on the right-hand side of equation (28). Thus, by successively applying the duality between $W^{-1,\frac{3}{2}}(\Omega)$ and $W_0^{1,3}(\Omega)$, Korn's inequality $\|\mathbf{u}\|_{1,3} \le c_K\|\mathbf{D}\mathbf{u}\|_3$ (see [21, Chapter V, Theorem 1.10]), and Young's inequality $\alpha\beta \le \frac{2}{9}\alpha^{\frac{3}{2}} + 3\beta^3$, we obtain

$$\begin{aligned} \langle \mathbf{f}_\varepsilon, \mathbf{u}_\varepsilon \rangle &\le \|\mathbf{f}_\varepsilon\|_{-1,\frac{3}{2}}\|\mathbf{u}_\varepsilon\|_{1,3} \\ &\le c_K\|\mathbf{f}_\varepsilon\|_{-1,\frac{3}{2}}\|\mathbf{D}\mathbf{u}_\varepsilon\|_3 \\ &\le \frac{2c_K^{3/2}}{9}\|\mathbf{f}_\varepsilon\|_{-1,\frac{3}{2}}^{\frac{3}{2}} + 3\|\mathbf{D}\mathbf{u}_\varepsilon\|_3^3. \end{aligned}$$

Due to the definition (23) of the function V_ε, we have (as soon as $\varepsilon \le \frac{1}{2}$)

$$\int_\Omega a_\varepsilon V_\varepsilon(a_\varepsilon) = \int_{a_\varepsilon \le 0} \frac{|a_\varepsilon|^{\frac{3}{2}}}{\varepsilon} + \int_{a_\varepsilon > 0} |a_\varepsilon|^{\frac{3}{2}} \ge \frac{1}{2\varepsilon}\|(a_\varepsilon)_-\|_{\frac{3}{2}}^{\frac{3}{2}} + \|a_\varepsilon\|_{\frac{3}{2}}^{\frac{3}{2}}.$$

and similarly,

$$\int_\Omega b_\varepsilon V_\varepsilon(b_\varepsilon) \ge \frac{1}{2\varepsilon}\|(b_\varepsilon)_-\|_{\frac{3}{2}}^{\frac{3}{2}} + \|b_\varepsilon\|_{\frac{3}{2}}^{\frac{3}{2}}.$$

Using Hölder's inequality and Young's inequality, $\alpha\beta \le \frac{1}{4\varepsilon}\alpha^{\frac{3}{2}} + \frac{2^6\varepsilon^2}{3^3}\beta^3$, we also obtain the following estimate

$$\int_\Omega 2(b_\varepsilon)_- \frac{|\mathbf{S}\mathbf{u}_\varepsilon|^2}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon} \le 2\|(b_\varepsilon)_-\|_{\frac{3}{2}}\|\mathbf{S}\mathbf{u}_\varepsilon\|_3 \le \frac{2^9\varepsilon^2}{3^3}\|\mathbf{D}\mathbf{u}_\varepsilon\|_3^3 + \frac{1}{4\varepsilon}\|(b_\varepsilon)_-\|_{\frac{3}{2}}^{\frac{3}{2}}, \tag{29}$$

so that, for ε small enough (that is $\frac{2^9\varepsilon^2}{3^3} \le \frac{1}{2}$), the inequality (28) becomes

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{u}_\varepsilon\|_2^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|a_\varepsilon\|_2^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|b_\varepsilon\|_2^2 + \frac{1}{2}\|\mathbf{D}\mathbf{u}_\varepsilon\|_3^3 + \varepsilon\|\nabla a_\varepsilon\|_2^2 + \varepsilon\|\nabla b_\varepsilon\|_2^2 \\ + \|a_\varepsilon\|_{\frac{3}{2}}^{\frac{3}{2}} + \|b_\varepsilon\|_{\frac{3}{2}}^{\frac{3}{2}} + \frac{1}{2\varepsilon}\|(a_\varepsilon)_-\|_{\frac{3}{2}}^{\frac{3}{2}} + \frac{1}{4\varepsilon}\|(b_\varepsilon)_-\|_{\frac{3}{2}}^{\frac{3}{2}} \le \frac{2c_K^{3/2}}{9}\|\mathbf{f}_\varepsilon\|_{-1,\frac{3}{2}}^{\frac{3}{2}}. \end{aligned} \tag{30}$$

Integrating in time and using the convergences of $\mathbf{u}_\varepsilon$ to $\mathbf{u}_{\text{init}}$ in $L^2(\Omega)$ and $\mathbf{f}_\varepsilon$ to $\mathbf{f}$ in $L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega))$, we deduce

$$\begin{aligned} \sup_{(0,T)}\|\mathbf{u}_\varepsilon\|_2^2 + \varepsilon\sup_{(0,T)}\|a_\varepsilon\|_2^2 + \varepsilon\sup_{(0,T)}\|b_\varepsilon\|_2^2 + \int_0^T \|\mathbf{D}\mathbf{u}_\varepsilon\|_3^3 + 2\varepsilon\int_0^T \|\nabla a_\varepsilon\|_2^2 + 2\varepsilon\int_0^T \|\nabla b_\varepsilon\|_2^2 \\ + 2\int_0^T \|a_\varepsilon\|_{\frac{3}{2}}^{\frac{3}{2}} + 2\int_0^T \|b_\varepsilon\|_{\frac{3}{2}}^{\frac{3}{2}} + \frac{1}{\varepsilon}\int_0^T \|(a_\varepsilon)_-\|_{\frac{3}{2}}^{\frac{3}{2}} + \frac{1}{2\varepsilon}\int_0^T \|(b_\varepsilon)_-\|_{\frac{3}{2}}^{\frac{3}{2}} \le c_1, \end{aligned} \tag{31}$$

where the constant c_1 only depends on the Korn's constant c_K, on the initial value $\|\mathbf{u}_{\text{init}}\|_2$ and on the contribution of the source term $\|\mathbf{f}\|_{-1,\frac{3}{2}}$.

Stress estimate - The estimate on the stress $\boldsymbol{\sigma}_\varepsilon$ follows directly by duality from (27): for all $\boldsymbol{\Sigma} \in L^3((0,T)\times\Omega)$ we have

$$\Big|\int_0^T \int_\Omega \boldsymbol{\sigma}_\varepsilon : \boldsymbol{\Sigma}\Big| \le 2\|a_\varepsilon\|_{L^{\frac{3}{2}}((0,T)\times\Omega)} \|\boldsymbol{\Sigma}\|_{L^3((0,T)\times\Omega)},$$

which corresponds to the bound

$$\|\boldsymbol{\sigma}_\varepsilon\|_{L^{\frac{3}{2}}((0,T)\times\Omega)} \le \|a_\varepsilon\|_{L^{\frac{3}{2}}((0,T)\times\Omega)} \le \frac{c_1}{2}. \tag{32}$$

Time-derivative estimates - Let us return to the weak formulation (24). More precisely, using Hölder's inequality and the duality $W^{-1,\frac{3}{2}}(\Omega) \leftrightarrow W_0^{1,3}(\Omega)$, we know that for all $\boldsymbol{\varphi} \in L^3(0,T;W_0^{1,3}(\Omega))$ we have

$$\langle \partial_t \mathbf{u}_\varepsilon, \boldsymbol{\varphi}\rangle \le \|a_\varepsilon - b_\varepsilon\|_{\frac{3}{2}} \|\operatorname{div}\boldsymbol{\varphi}\|_3 + 2\|\mathbf{D}\mathbf{u}_\varepsilon\|_3^2 \|\mathbf{D}\boldsymbol{\varphi}\|_3 + \|\boldsymbol{\sigma}_\varepsilon\|_{\frac{3}{2}} \|\mathbf{S}\boldsymbol{\varphi}\|_3 + \|\mathbf{f}_\varepsilon\|_{-1,\frac{3}{2}} \|\boldsymbol{\varphi}\|_{1,3}.$$

Integrating in time and applying Hölder's inequalities (with respect to integration in time), we derive the following estimate

$$\begin{aligned}\Big|\int_0^T \langle \partial_t \mathbf{u}_\varepsilon, \boldsymbol{\varphi}\rangle\Big| \le& \Big(\int_0^T \|a_\varepsilon - b_\varepsilon\|_{\frac{3}{2}}^{\frac{3}{2}}\Big)^{\frac{2}{3}} \Big(\int_0^T \|\operatorname{div}\boldsymbol{\varphi}\|_3^3\Big)^{\frac{1}{3}} + 2\Big(\int_0^T \|\mathbf{D}\mathbf{u}_\varepsilon\|_3^3\Big)^{\frac{2}{3}} \Big(\int_0^T \|\mathbf{D}\boldsymbol{\varphi}\|_3^3\Big)^{\frac{1}{3}} \\ &+ \Big(\int_0^T \|\boldsymbol{\sigma}_\varepsilon\|_{\frac{3}{2}}^{\frac{3}{2}}\Big)^{\frac{2}{3}} \Big(\int_0^T \|\mathbf{S}\boldsymbol{\varphi}\|_3^3\Big)^{\frac{1}{3}} + \Big(\int_0^T \|\mathbf{f}_\varepsilon\|_{-1,\frac{3}{2}}^{\frac{3}{2}}\Big)^{\frac{2}{3}} \Big(\int_0^T \|\boldsymbol{\varphi}\|_{1,3}^3\Big)^{\frac{1}{3}}.\end{aligned}$$

By virtue of the estimates (31), we deduce that there exists a constant c_2, independent of ε (in practice, c_2 only depends on c_1), such that

$$\Big|\int_0^T \langle \partial_t \mathbf{u}_\varepsilon, \boldsymbol{\varphi}\rangle\Big| \le c_2 \Big(\int_0^T \|\boldsymbol{\varphi}\|_{1,3}^3\Big)^{\frac{1}{3}} = c_2 \|\boldsymbol{\varphi}\|_{L^3(0,T;W_0^{1,3}(\Omega))}.$$

Thus, by the definition of the dual norm, we obtain

$$\|\partial_t \mathbf{u}_\varepsilon\|_{L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega))} \le c_2. \tag{33}$$

We use (25) to estimate $\partial_t a_\varepsilon$ (and similarly $\partial_t b_\varepsilon$). For all $\psi \in L^2(0,T;W_0^{1,2}(\Omega))$, we have

$$\langle \varepsilon\partial_t a_\varepsilon, \psi\rangle \le \varepsilon\|\nabla a_\varepsilon\|_2 \|\nabla\psi\|_2 + \|\psi\|_{\frac{3}{2}} \|\operatorname{div}\mathbf{u}_\varepsilon\|_3 + 2\|\psi\|_{\frac{3}{2}} \|\mathbf{S}\mathbf{u}_\varepsilon\|_3 + \|\psi\|_{\frac{3}{2}} \|V_\varepsilon(a_\varepsilon)\|_3. \tag{34}$$

We integrate in time and use the previously established bounds (31) on $\varepsilon^{\frac{1}{2}}\nabla a_\varepsilon$, $\operatorname{div}\mathbf{u}_\varepsilon$ and $\mathbf{S}\mathbf{u}_\varepsilon$ (the latter two relying on the bound for $\mathbf{D}\mathbf{u}_\varepsilon$), so that it remains to estimate $\|V_\varepsilon(a_\varepsilon)\|_3$.
To this end, we set $\psi = |V_\varepsilon(a_\varepsilon)|V_\varepsilon(a_\varepsilon)$ in (25) which yields

$$\frac{d}{dt}\int_\Omega \varepsilon W_\varepsilon(a_\varepsilon) + \int_\Omega \varepsilon W_\varepsilon''(a_\varepsilon)|\nabla a_\varepsilon|^2 + \|V_\varepsilon(a_\varepsilon)\|_3^3 \le \int_\Omega 2V_\varepsilon(a_\varepsilon)^2|\mathbf{S}\mathbf{u}_\varepsilon| + \int_\Omega V_\varepsilon(a_\varepsilon)^2\,|\operatorname{div}\mathbf{u}_\varepsilon|, \tag{35}$$

where W_ε is such that $W_\varepsilon' = |V_\varepsilon|V_\varepsilon$ and $W_\varepsilon(0) = 0$. The right-hand side is bounded using the previous estimates : for instance, applying successively Hölder's inequality followed by Young's inequality, we derive

$$\int_\Omega V_\varepsilon(a_\varepsilon)^2\,|\operatorname{div}\mathbf{u}_\varepsilon| \le \|V_\varepsilon(a_\varepsilon)\|_3^2 \|\operatorname{div}\mathbf{u}_\varepsilon\|_3 \le \frac{1}{3}\|V_\varepsilon(a_\varepsilon)\|_3^3 + \frac{4}{3}\|\operatorname{div}\mathbf{u}_\varepsilon\|_3^3.$$

Since W_ε and W_ε'' are non-negative, and $W_\varepsilon(a_\varepsilon|_{t=0}) = W_\varepsilon(0) = 0$, integration of (35) with respect to time over the interval $(0,T)$ yields the desired bound on $V_\varepsilon(a_\varepsilon)$. The relation (34) provides a bound for $\|\varepsilon\partial_t a_\varepsilon\|_{L^2(0,T;W^{-1,2}(\Omega))}$. Similarly, we deduce a bound for $\|\varepsilon\partial_t b_\varepsilon\|_{L^2(0,T;W^{-1,2}(\Omega))}$. There exists a constant c_3, independent of ε, such that

$$\|\varepsilon\partial_t a_\varepsilon\|_{L^2(0,T;W^{-1,2}(\Omega))} \le c_3 \quad \text{and} \quad \|\varepsilon\partial_t b_\varepsilon\|_{L^2(0,T;W^{-1,2}(\Omega))} \le c_3. \tag{36}$$

Conclusion of the proof - The estimates (31), (32), (33) and (36) allow us to demonstrate the existence of a solution to equations (18)–(20) in the classical way. ■

Note that the estimates performed in the previous proof are independent of ε as long as $0 < \varepsilon < \frac{3^{\frac{3}{2}}}{2^5} \approx 0.16$. We therefore obtain the following result.

Corollary 1. *The quantities*

$$\begin{array}{lll}
\|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}, & \|\nabla\mathbf{u}_\varepsilon\|_{L^3((0,T)\times\Omega)}, & \|\partial_t\mathbf{u}_\varepsilon\|_{L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega))}, \\
\|\boldsymbol{\sigma}_\varepsilon\|_{L^{\frac{3}{2}}((0,T)\times\Omega)}, & \|a_\varepsilon\|_{L^{\frac{3}{2}}((0,T)\times\Omega)}, & \|b_\varepsilon\|_{L^{\frac{3}{2}}((0,T)\times\Omega)}, \\
\|\varepsilon^{\frac{1}{2}}a_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}, & \|\varepsilon^{\frac{1}{2}}\nabla a_\varepsilon\|_{L^2((0,T)\times\Omega)}, & \|\varepsilon\partial_t a_\varepsilon\|_{L^2(0,T;W^{-1,2}(\Omega))}, \\
\|\varepsilon^{\frac{1}{2}}b_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}, & \|\varepsilon^{\frac{1}{2}}\nabla b_\varepsilon\|_{L^2((0,T)\times\Omega)}, & \|\varepsilon\partial_t b_\varepsilon\|_{L^2(0,T;W^{-1,2}(\Omega))}, \\
\|\varepsilon^{-\frac{2}{3}}(a_\varepsilon)_-\|_{L^{\frac{3}{2}}((0,T)\times\Omega)}, & \|\varepsilon^{-\frac{2}{3}}(b_\varepsilon)_-\|_{L^{\frac{3}{2}}((0,T)\times\Omega)}. &
\end{array}$$

are bounded independently of $\varepsilon > 0$ (ε small enough).

3.4 Limit process

By virtue of the bounds obtained in Corollary 1, we deduce the following convergences (up to extraction of sub-sequences):

Proposition 3. *There exists a function $\mathbf{u}$, such that*

$$\begin{array}{ll}
\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} & \textit{weakly-}\star \textit{ in } L^\infty(0,T;L^2(\Omega)), \\
\nabla\mathbf{u}_\varepsilon \rightharpoonup \nabla\mathbf{u} & \textit{weakly in } L^3(0,T;L^3(\Omega)), \\
\partial_t\mathbf{u}_\varepsilon \rightharpoonup \partial_t\mathbf{u} & \textit{weakly in } L^{\frac{3}{2}}(0,T;W^{-1,\frac{3}{2}}(\Omega)), \\
\mathbf{u}_\varepsilon \to \mathbf{u} & \textit{in } L^3(0,T;L^2(\Omega)).
\end{array}$$

There exists two non-negative functions a and b, such that

$$\begin{array}{ll}
a_\varepsilon \rightharpoonup a & \textit{weakly in } L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(\Omega)), \\
b_\varepsilon \rightharpoonup b & \textit{weakly in } L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(\Omega)), \\
\varepsilon\nabla a_\varepsilon\ ,\ \varepsilon\nabla b_\varepsilon \rightharpoonup \mathbf{0} & \textit{weakly in } L^2(0,T;L^2(\Omega)), \\
\varepsilon\partial_t a_\varepsilon\ ,\ \varepsilon\partial_t b_\varepsilon \rightharpoonup 0 & \textit{weakly in } L^2(0,T;W^{-1,2}(\Omega)).
\end{array}$$

There exists a symmetric and traceless tensor function $\boldsymbol{\sigma}$, such that

$$\boldsymbol{\sigma}_\varepsilon \rightharpoonup \boldsymbol{\sigma} \qquad \textit{weakly in } L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(\Omega)).$$

Proof - Weak convergence, up to a sub-sequence, for $\mathbf{u}_\varepsilon$, $\nabla\mathbf{u}_\varepsilon$, $\partial_t\mathbf{u}_\varepsilon$, a_ε, b_ε and $\boldsymbol{\sigma}_\varepsilon$ follows from the bounds derived in Corollary 1. Strong convergence on velocity $\mathbf{u}_\varepsilon$ can be deduced from the Aubin-Lions-Simon Lemma (see [12, p.102]). It should be noted that this strong convergence is not used in the proof of Theorem 1. However, it will be useful for handling nonlinear terms of the form $\mathbf{u}\cdot\nabla\mathbf{u}$ that will be present in the complete model, see Subsection 4.1.

Since a_ε weakly converges to a, we know that $\partial_t a_\varepsilon$ and ∇a_ε converge respectively to $\partial_t a$ and ∇a in the sense of distributions. Thus, we deduce that $\varepsilon\partial_t a_\varepsilon$ and $\varepsilon^{\frac{1}{2}}\nabla a_\varepsilon$ converge to zero in the sense of distributions. Since these two sequences are bounded in $L^2(0,T;W^{-1,2}(\Omega))$ and $L^2((0,T)\times\Omega)$, respectively, they converge weakly to zero in these spaces. The same arguments apply to the convergence of b_ε to b.

The last point that remains to be proven is the non-negativity of the limits a and b. This follows directly from the energy estimate (31) and the bounds $\|(a_\varepsilon)_-\|_{\frac{3}{2}}^{\frac{3}{2}} \leq c_1\varepsilon$ and $\|(b_\varepsilon)_-\|_{\frac{3}{2}}^{\frac{3}{2}} \leq 2c_1\varepsilon$. Passing to the limit $\varepsilon \to 0$, these imply that $a_- = b_- = 0$. ∎

The aim at the end of this section is to demonstrate that we can pass to the limit in relations (24), (25), (26) and (28), in particular by carefully handling nonlinear terms.

Step 1: Limit in (24) Given $\boldsymbol{\varphi} \in L^3(0,T;W_0^{1,3}(\Omega))$, the convergences established in Proposition 3 imply that

$$\langle\partial_t\mathbf{u},\boldsymbol{\varphi}\rangle - \int_\Omega (a-b)\,\mathrm{div}\,\boldsymbol{\varphi} + \int_\Omega 2\overline{|\mathbf{Du}|\mathbf{Du}} : \mathbf{D}\boldsymbol{\varphi} + \int_\Omega \boldsymbol{\sigma} : \mathbf{S}\boldsymbol{\varphi} = \langle\mathbf{f},\boldsymbol{\varphi}\rangle, \tag{37}$$

where the notation $\overline{|\mathbf{Du}|\mathbf{Du}}$ denotes the weak limit of $|\mathbf{Du}_\varepsilon|\mathbf{Du}_\varepsilon$ in $L^{\frac{3}{2}}((0,T)\times\Omega)$.

Step 2: Limit in (25) **and** (26) To perform the limit $\varepsilon \to 0$ in (25), we first note that

$$\Big|\frac{|\mathbf{S}\mathbf{u}_\varepsilon|^2}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon} - |\mathbf{S}\mathbf{u}_\varepsilon|\Big| = \varepsilon \frac{|\mathbf{S}\mathbf{u}_\varepsilon|}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon} \leq \varepsilon \longrightarrow 0.$$

Thus, for $\psi \in L^{\frac{3}{2}}((0,T)\times\Omega)$ (in practice, this is done for more regular functions, typically in $L^2(0,T;W_0^{1,2}(\Omega))$, and which converge to ψ), when ε goes to 0 in (25), we get

$$\int_\Omega \psi \operatorname{div} \mathbf{u} - \int_\Omega 2\psi\overline{|\mathbf{S}\mathbf{u}|} + \int_\Omega \psi \overline{V(a)} = 0, \tag{38}$$

where $\overline{|\mathbf{S}\mathbf{u}|}$ and $\overline{V(a)}$ respectively correspond to the weak limits of $|\mathbf{S}\mathbf{u}_\varepsilon|$ and $V_\varepsilon(a_\varepsilon)$ in $L^3((0,T)\times\Omega)$. Similarly, for $\phi \in L^{\frac{3}{2}}((0,T)\times\Omega)$, when ε goes to 0 in (26), we get

$$\int_\Omega \phi \operatorname{div} \mathbf{u} - \int_\Omega 2\phi\overline{|\mathbf{S}\mathbf{u}|} - \int_\Omega \phi \overline{V(b)} = 0, \tag{39}$$

where $\overline{V(b)}$ corresponds to the weak limit of $V_\varepsilon(b_\varepsilon)$ in $L^3((0,T)\times\Omega)$.

Step 3: Limit in (28) After integrating equation (28) with respect to time, we obtain

$$\frac{1}{2}\|\mathbf{u}_\varepsilon(T)\|_2^2 + \frac{\varepsilon}{2}\|a_\varepsilon\|_2^2 + \frac{\varepsilon}{2}\|b_\varepsilon\|_2^2 + 4\int_0^T \|\mathbf{D}\mathbf{u}_\varepsilon\|_3^3 + \varepsilon\int_0^T \|\nabla a_\varepsilon\|_2^2 + \varepsilon\int_0^T \|\nabla b_\varepsilon\|_2^2$$
$$+ \int_0^T\int_\Omega a_\varepsilon V_\varepsilon(a_\varepsilon) + \int_0^T\int_\Omega b_\varepsilon V_\varepsilon(b_\varepsilon) + \int_0^T\int_\Omega 2b_\varepsilon \frac{|\mathbf{S}\mathbf{u}_\varepsilon|^2}{|\mathbf{S}\mathbf{u}_\varepsilon| + \varepsilon} = \frac{1}{2}\|\mathbf{u}_{\mathrm{init},\varepsilon}\|_2^2 + \int_0^T \langle \mathbf{f}_\varepsilon, \mathbf{u}_\varepsilon\rangle.$$

Since $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ weakly converges to $\mathbf{u}$ in $L^\infty(0,T;L^2(\Omega))$, we know that $\liminf_{\varepsilon\to 0}\|\mathbf{u}_\varepsilon\| \geq \|\mathbf{u}\|$. We also use the strong convergence of the sequences $(\mathbf{u}_{\mathrm{init},\varepsilon})_{\varepsilon>0}$ and $(\mathbf{f}_\varepsilon)_{\varepsilon>0}$ to pass tom the limit $\varepsilon \to 0$ and obtain

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}(T)\|_2^2 + 4\int_0^T\int_\Omega \overline{|\mathbf{D}\mathbf{u}|^3} + \int_0^T\int_\Omega \overline{aV(a)} + \int_0^T\int_\Omega \overline{bV(b)}&\\ + \int_0^T\int_\Omega 2\overline{b|\mathbf{S}\mathbf{u}|} \leq \frac{1}{2}\|\mathbf{u}_{\mathrm{init}}\|_2^2 + \int_0^T\langle \mathbf{f},\mathbf{u}\rangle,&\end{aligned} \tag{40}$$

where $\overline{|\mathbf{D}\mathbf{u}|^3}$, $\overline{aV(a)}$, $\overline{bV(b)}$ and $\overline{b|\mathbf{S}\mathbf{u}|}$ respectively denote the weak limits of $(|\mathbf{D}\mathbf{u}_\varepsilon|^3)_{\varepsilon>0}$, $(a_\varepsilon V_\varepsilon(a_\varepsilon))_{\varepsilon>0}$, $(b_\varepsilon V_\varepsilon(b_\varepsilon))_{\varepsilon>0}$ and $(b_\varepsilon|\mathbf{S}\mathbf{u}_\varepsilon|)_{\varepsilon>0}$ in $L^1((0,T)\times\Omega)$.

To conclude the proof of existence, it remains to identify the weak limits of the nonlinear terms that appear in the relationships (37), (38), (39) and (40).

3.5 Convergence of the nonlinear terms

By choosing $\boldsymbol{\varphi} = \mathbf{u}$ in (37), $\psi = a$ in (38) and $\phi = -b$ in (39), integrating both equations over time, and summing the resulting expressions, we derive

$$\frac{1}{2}\|\mathbf{u}(T)\|_2^2 + \int_0^T\int_\Omega 2\overline{|\mathbf{D}\mathbf{u}|\mathbf{D}\mathbf{u}} : \mathbf{D}\mathbf{u} + \int_0^T\int_\Omega a\overline{V(a)} + \int_0^T\int_\Omega b\overline{V(b)}$$
$$+ \int_0^T\int_\Omega \Big(\boldsymbol{\sigma} : \mathbf{S}\mathbf{u} - 2a\overline{|\mathbf{S}\mathbf{u}|}\Big) + \int_0^T\int_\Omega 2b\overline{|\mathbf{S}\mathbf{u}|} = \frac{1}{2}\|\mathbf{u}_{\mathrm{init}}\|_2^2 + \int_0^T\langle\mathbf{f},\mathbf{u}\rangle.$$

By subtracting Equation (40), we obtain

$$\begin{aligned}&\underbrace{\int_0^T\int_\Omega 2\Big(\overline{|\mathbf{D}\mathbf{u}|\mathbf{D}\mathbf{u}} : \mathbf{D}\mathbf{u} - 2\overline{|\mathbf{D}\mathbf{u}|^3}\Big)}_{I_1} + \underbrace{\int_0^T\int_\Omega \Big(a\overline{V(a)} - \overline{aV(a)}\Big)}_{I_2} + \underbrace{\int_0^T\int_\Omega \Big(b\overline{V(b)} - \overline{bV(b)}\Big)}_{I_3}\\ &+ \underbrace{\int_0^T\int_\Omega \Big(\boldsymbol{\sigma} : \mathbf{S}\mathbf{u} - 2a\overline{|\mathbf{S}\mathbf{u}|}\Big)}_{I_4} + \underbrace{\int_0^T\int_\Omega 2\Big(b\overline{|\mathbf{S}\mathbf{u}|} - \overline{b|\mathbf{S}\mathbf{u}|}\Big)}_{I_5} \geq 0.\end{aligned} \tag{41}$$

The results below will show that each of the terms I_k, $k \in \{1,...,5\}$, identified above are non-positive and, by virtue of inequality (41), is therefore zero.

3.5.1 Non-positivity of I_1

The following lemma directly implies that $I_1 \leq 0$. It can be found in [30, Lemma 1], see also [18, p.1715 and p.1730]:

Lemma 2. *If* $\mathbf{Du}_\varepsilon \rightharpoonup \mathbf{Du}$ *weakly in* $L^3((0,T)\times\Omega)$ *and* $|\mathbf{Du}_\varepsilon|\mathbf{Du}_\varepsilon \rightharpoonup \overline{|\mathbf{Du}|\mathbf{Du}}$ *weakly in* $L^{\frac{3}{2}}((0,T)\times\Omega)$ *then*

$$\int_0^T \int_\Omega \left(\overline{|\mathbf{Du}|\mathbf{Du}} : \mathbf{Du} - 2\overline{|\mathbf{Du}|^3} \right) \leq 0.$$

Furthermore, if there is equality, then $\overline{|\mathbf{Du}|\mathbf{Du}} = |\mathbf{Du}|\mathbf{Du}$.

3.5.2 Non-positivity of I_2 and I_3

These two integrals are treated in a similar way. The results $I_2 \leq 0$ and $I_3 \leq 0$ are direct consequences of the following lemma.

Lemma 3. *We have the following inequalities for the function a (and similarly for the function b)*

$$\overline{aV(a)} \geq a\sqrt{a} \qquad \text{and} \qquad \overline{V(a)} \leq \sqrt{a}.$$

Proof - In order to prove the first inequality, we consider $\varphi \in L^\infty_+((0,T)\times\Omega)$. For all $\varepsilon > 0$, we have

$$\int_0^T \int_\Omega a_\varepsilon V_\varepsilon(a_\varepsilon)\,\varphi = \iint_{a_\varepsilon \leq 0} a_\varepsilon V_\varepsilon(a_\varepsilon)\,\varphi + \iint_{a_\varepsilon > 0} a_\varepsilon \sqrt{a_\varepsilon}\,\varphi.$$

✓ Since for all $x \leq 0$, we have $xV_\varepsilon(x) \geq 0$, the first integral on the right-hand side is non-negative.
✓ Next, since $x \in \mathbb{R}_+ \longmapsto x\sqrt{x}$ is convex and continuous, the function

$$\Phi : h \in L^{\frac{3}{2}}_+((0,T)\times\Omega) \longmapsto \int_0^T \int_\Omega h\sqrt{h}\,\varphi$$

is convex and lower semi-continuous. Following [14, Corollaire III.8] (the statement is reproduced below, see Lemma 4) we deduce that the weak convergence of a_ε to a in $L^{\frac{3}{2}}((0,T)\times\Omega)$ implies $\liminf_{\varepsilon\to 0} \Phi(a_\varepsilon) \geq \Phi(a)$, that is

$$\liminf_{\varepsilon\to 0} \iint_{a_\varepsilon > 0} a_\varepsilon \sqrt{a_\varepsilon}\,\varphi \geq \int_0^T \int_\Omega a\sqrt{a}\,\varphi.$$

We then conclude that

$$\int_0^T \int_\Omega \overline{aV(a)}\,\varphi \geq \int_0^T \int_\Omega a\sqrt{a}\,\varphi.$$

The second inequality can be proven in a similar way. Consider $\varphi \in L^{\frac{3}{2}}_+((0,T)\times\Omega)$. For all $\varepsilon > 0$ we have

$$\int_0^T \int_\Omega V_\varepsilon(a_\varepsilon)\,\varphi = \iint_{a_\varepsilon \leq 0} V_\varepsilon(a_\varepsilon)\,\varphi + \iint_{a_\varepsilon > 0} \sqrt{a_\varepsilon}\,\varphi.$$

✓ The first integral on the right-hand side is non-positive since for all $x \leq 0$, we have $V_\varepsilon(x) \leq 0$.
✓ The function $x \in \mathbb{R}_+ \longmapsto \sqrt{x}$ is concave and continuous. By applying the same convexity result as previously, see Lemma 4, we deduce

$$\liminf_{\varepsilon\to 0} \iint_{a_\varepsilon > 0} \sqrt{\varepsilon(a_\varepsilon)}\,\varphi \leq \int_0^T \int_\Omega \sqrt{a}\,\varphi.$$

We then conclude that

$$\int_0^T \int_\Omega \overline{V(a)}\,\varphi \leq \int_0^T \int_\Omega \sqrt{a}\,\varphi,$$

which completes the proof of Lemma 3. ∎

It should be noted that the previous lemma is closely related to the following classical result [14, Corollaire III.8], that we will reuse several times:

Lemma 4. *Given a topological space E, if $\Phi : E \longrightarrow \mathbb{R}$ is lower semicontinuous and convex, then for any sequence $(h_\varepsilon)_{\varepsilon>0}$ in E weakly converging to h as $\varepsilon \to 0$, we have*

$$\Phi(h) \leq \liminf_{\varepsilon\to 0} \Phi(h_\varepsilon).$$

3.5.3 Non-positivity of I_4

By the convexity of $x \mapsto |x|$, Lemma 4 ensures that $|\mathbf{Su}| \leq \overline{|\mathbf{Su}|}$. Furthermore, since $|\boldsymbol{\sigma}| \leq a$, applying the Cauchy-Schwarz inequality gives

$$\boldsymbol{\sigma} : \mathbf{Su} \leq 2|\boldsymbol{\sigma}||\mathbf{Su}| \leq 2a\overline{|\mathbf{Su}|}. \tag{42}$$

which implies that $I_4 \leq 0$.

3.5.4 Non-positivity of I_5

First, we decompose b_ε into its positive and negative parts. For the latter, the energy estimate (31) provides the inequality $\|(b_\varepsilon)_-\|_{\frac{3}{2}} \leq (2c_1\varepsilon)^{\frac{2}{3}}$ This shows that $(b_\varepsilon)_-$ converges strongly to 0 in $L^{\frac{3}{2}}$. Moreover, since b is non-negative, we can therefore rewrite

$$I_5 = \int_0^T \int_\Omega 2\Big(b_+\overline{|\mathbf{Su}|} - \overline{b_+|\mathbf{Su}|}\Big).$$

For $\varphi \in L^\infty_+((0,T)\times\Omega)$, the function

$$\Phi : (h,k) \in L^{\frac{3}{2}}_+((0,T)\times\Omega) \times L^3_+((0,T)\times\Omega) \longmapsto \int_0^T \int_\Omega hk\varphi$$

is convex and lower semicontinuous. Lemma 4 can therefore be applied to the sequence $(h_\varepsilon, k_\varepsilon) = ((b_\varepsilon)_+, |\mathbf{Su}_\varepsilon|)$, yielding

$$b_+\overline{|\mathbf{Su}|} \leq \overline{b_+|\mathbf{Su}|} \tag{43}$$

which implies that $I_5 \leq 0$.

3.5.5 Preliminary conclusions on nonlinear convergences

The estimate (41) reads $I_1 + I_2 + I_3 + I_4 + I_5 \geq 0$. We have just shown that each of the integrals I_k, $k \in \{1, ..., 5\}$, is non-positive, and therefore they must all be equal to zero. From this (using the conclusion of Lemma 2 for I_1, the inequalities obtained in Lemma 3 for I_2 and I_3, and the inequality (43) for I_5 with $b_+ = b$), we deduce that

$$\overline{|\mathbf{Du}|\mathbf{Du}} = |\mathbf{Du}|\mathbf{Du}, \quad \overline{aV(a)} = a\sqrt{a}, \quad \overline{bV(b)} = b\sqrt{b}, \quad \boldsymbol{\sigma} : \mathbf{Su} = 2a\overline{|\mathbf{Su}|}, \quad \overline{b|\mathbf{Su}|} = b\overline{|\mathbf{Su}|}.$$

It remains to show that $|\mathbf{Su}| = \overline{|\mathbf{Su}|}$.

3.5.6 Proof that $|\mathbf{Su}| = \overline{|\mathbf{Su}|}$

We proceed in the following four steps.

✓ As a consequence of Lemma 2 (in the equality case), we have $\overline{|\mathbf{Du}|\mathbf{Du}} = |\mathbf{Du}|\mathbf{Du}$, so that

$$\int_0^T \int_\Omega \Big(|\mathbf{Du}|^3 - \overline{|\mathbf{Du}|^3}\Big) = 0.$$

The function $x \mapsto |x|^3$ being convex, Lemma 4 also yields $|\mathbf{Du}|^3 - \overline{|\mathbf{Du}|^3} \leq 0$. We therefore conclude that $\overline{|\mathbf{Du}|^3} = |\mathbf{Du}|^3$.

✓ Next, using the convexity of $x \mapsto x^2$ and the concavity of $x \mapsto x^{\frac{2}{3}}$, from Lemma 4 (noting that if Φ is concave, then $-\Phi$ is convex) we deduce

$$|\mathbf{Du}|^2 \leq \overline{|\mathbf{Du}|^2} = \overline{(|\mathbf{Du}|^3)^{\frac{2}{3}}} \leq \overline{|\mathbf{Du}|^3}^{\frac{2}{3}} = |\mathbf{Du}|^2.$$

All these terms are thus equal, and in particular $\overline{|\mathbf{Du}|^2} = |\mathbf{Du}|^2$.

✓ Now, we use the identity

$$|\mathbf{Du}_\varepsilon|^2 = |\mathbf{Su}_\varepsilon|^2 + \frac{1}{6}|\mathrm{div}\,\mathbf{u}_\varepsilon|^2. \tag{44}$$

Let $\xi \geq 0$ and $\zeta \geq 0$ be the elements of $L^{\frac{3}{2}}_{+}((0,T)\times\Omega)$ such that $\overline{|\mathbf{Su}|^2} = |\mathbf{Su}|^2 + \xi$ and $\overline{|\operatorname{div}\mathbf{u}|^2} = |\operatorname{div}\mathbf{u}|^2 + \zeta$ then we pass to the limit in (44) using the fact that $\overline{|\mathbf{Du}|^2} = |\mathbf{Du}|^2$:

$$|\mathbf{Du}|^2 = |\mathbf{Su}|^2 + \xi + \frac{1}{6}|\operatorname{div}\mathbf{u}|^2 + \frac{1}{6}\zeta.$$

Since we also have $|\mathbf{Du}|^2 = |\mathbf{Su}|^2 + \frac{1}{6}|\operatorname{div}\mathbf{u}|^2$, it follows that $\xi + \frac{1}{6}\zeta = 0$. Given the non-negativity of ξ and ζ, we conclude that $\xi = \zeta = 0$, and, in particular, $\overline{|\mathbf{Su}|^2} = |\mathbf{Su}|^2$.

✓ Finally, using the convexity of $x \mapsto |x|$, the concavity of $x \mapsto x^{\frac{1}{2}}$ and Lemma 4 again, we deduce that

$$|\mathbf{Su}| \leq \overline{|\mathbf{Su}|} = \overline{(|\mathbf{Su}|^2)^{\frac{1}{2}}} \leq \overline{|\mathbf{Su}|^2}^{\frac{1}{2}} = |\mathbf{Su}|.$$

All these terms are thus equal, and in particular $\overline{|\mathbf{Su}|} = |\mathbf{Su}|$.

3.5.7 Conclusion of the existence proof (Theorem 1)

✓ The equality $\overline{|\mathbf{Du}|\mathbf{Du}} = |\mathbf{Du}|\mathbf{Du}$ implies that (13) is satisfied (see the limit of (24) obtain in (37)).

✓ Using the equality $\overline{|\mathbf{Su}|} = |\mathbf{Su}|$, the limit equation (38) shows that for all $\psi \in L^{\frac{3}{2}}((0,T)\times\Omega)$ we have

$$\int_\Omega \psi \operatorname{div}\mathbf{u} - \int_\Omega 2\psi|\mathbf{Su}| + \int_\Omega \psi\overline{V(a)} = 0. \tag{45}$$

For $\xi \in L^\infty((0,T)\times\Omega)$, taking $\psi = a\xi$ and using the equality $a\overline{V(a)} = a\sqrt{a}$, we also deduce the equality (14). Since $\overline{V(a)} \leq \sqrt{a}$, inequality (15) follows whenever $\psi \geq 0$ (the same results hold replacing a by b).

✓ Given that $\overline{|\mathbf{Du}|^3} = |\mathbf{Du}|^3$, $\overline{aV(a)} = a\sqrt{a}$, $\overline{bV(b)} = b\sqrt{b}$ and $\overline{b|\mathbf{Su}|} = b|\mathbf{Su}|$, inequality (40) coincides exactly with inequality (16).

Finally, the equations (13), (14), (15) and (16) are satisfied by $(\mathbf{u}, a, b, \boldsymbol{\sigma})$. Consequently, $(\mathbf{u}, p = a - b, \boldsymbol{\sigma})$ is a weak solution of (9)–(12). ■

4 Adaptation to more general models

4.1 Taking into account the variation of the volume fraction

As explained in the introduction, the complete granular flow model (1)–(4) also accounts for the evolution of the grain volume fraction, denoted ϕ. While the previous results address several challenges related to complex flows, such as nonlinear rheology and dilatancy phenomena, the full model described by (1)–(4) presents additional difficulties. A key issue is the specification of the constitutive functions α and β, which depend on the volume fraction ϕ. Of particular importance, is the characterization of the dilatancy law for $\operatorname{div}\mathbf{u}$. Following the work of Roux and Radjaï [28], this law can be expressed as

$$\operatorname{div}\mathbf{u} = 2K|\mathbf{Su}|(I - I_{\mathrm{eq}}(\phi)) \qquad \text{where} \quad I = \frac{2d|\mathbf{Su}|}{\sqrt{p_+/\rho_0}}.$$

The non-dimensional number I denotes the inertial number, which depends on the diameter d of the grains and on their density ρ_0, and $I_{\mathrm{eq}}(\phi)$ denotes its equilibrium value. Several expressions for this equilibrium value can be found in the literature (see [13, 20, 22, 26, 29]). In this work, we adopt the formulation given in [29, p.929]:

$$I_{\mathrm{eq}}(\phi) = \frac{\phi_{\max} - \phi}{\phi - \phi_{\min}},$$

where the constants $0 < \phi_{\min} < \phi_{\max} < 1$ correspond to the extreme values of the volume fraction ϕ. This choice ensures, from a mathematical point of view, that ϕ remains effectively bounded between $\phi_{\min}$

and $\phi_{\max}$; see Proposition 5 below. In order to ensure stability of the model (see [5, 6, 7, 15]), we choose $K = \dfrac{\phi - \phi_{\min}}{\delta\phi\, I}$ with $\delta\phi = \phi_{\max} - \phi_{\min}$ so that the dilatation law takes the form

$$\operatorname{div} \mathbf{u} = 2\frac{\phi - \phi_{\min}}{\delta\phi}|\mathbf{Su}| - \frac{\phi_{\max} - \phi}{\delta\phi}\frac{\sqrt{p_+}}{d\sqrt{\rho_0}}.$$

In order to take into account possible negative pressure values (see Section 3.1), the dilatation law must be modified. In terms of mathematical analysis, we present a preliminary theoretical result on the existence of solution for a model that is relatively close to the full model. The model considered is as follows:

$$\begin{cases} \mathcal{D}_t(\phi, \mathbf{u}) + \nabla p - \operatorname{div}(2|\mathbf{Du}|\mathbf{Du}) = \mathbf{f} + \operatorname{div}\boldsymbol{\sigma}, & (46)\\ \boldsymbol{\sigma} : \mathbf{Su} = 2(\phi - \phi_{\min})p_+|\mathbf{Su}|, \quad |\boldsymbol{\sigma}| \le (\phi - \phi_{\min})p_+, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}} \quad \text{and} \quad \operatorname{tr}\boldsymbol{\sigma} = 0, & (47)\\ \operatorname{div}\mathbf{u} = 2(\phi - \phi_{\min})|\mathbf{Su}| - (\phi_{\max} - \phi)\sqrt{p_+} + (\phi - \phi_{\min})\sqrt{p_-}, & (48)\\ \partial_t\phi + \operatorname{div}(\phi\mathbf{u}) = \xi\mathcal{H}, & (49) \end{cases}$$

where

$$\mathcal{D}_t(\phi, \mathbf{u}) = \frac{1}{2}\Big(\big(\partial_t(\phi\mathbf{u}) + \operatorname{div}(\phi\mathbf{u}\otimes\mathbf{u})\big) + \big(\phi(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u})\big)\Big).$$

In practice, the only difference between the full model (1)–(4) and (46)–(49) is the additional term $\xi\mathcal{H}$, where ξ is a fixed and positive real parameter (which may be very small) satisfying $0 < \xi \le \phi_{\max} - \phi_{\min}$. The additional term $\mathcal{H}$ is chosen of the following form:

$$\mathcal{H} = \Delta\phi - \phi\sqrt{p_+}.$$

The contribution $\Delta\phi$ gives meaning to nonlinear terms such as $(\phi_{\max} - \phi)\sqrt{p_+}$. Indeed, if we follow the same proof of existence as in the simplified case (model (1)–(4) in Section 3), we use a Galerkin method by approximating the solution ϕ by a sequence of regular solutions $(\phi_\varepsilon)_{\varepsilon>0}$. The presence of $\Delta\phi_\varepsilon$ ensures the compactness of the sequence $(\phi_\varepsilon)_{\varepsilon>0}$ and therefore a strong convergence of $(\phi_\varepsilon)_{\varepsilon>0}$ to ϕ.
Furthermore, as we will see below, the contribution $\phi\sqrt{p_+}$ guarantes that $\phi \le \phi_{\max} - \xi$, thus allowing the energy estimate to provide a bound on the pressure.

Remark 5. *Taking the average of the convective and conservative derivatives for the temporal derivative guarantees an estimate of the energy independent of the evolution equation* (49) *on* ϕ. *Note that when* $\xi = 0$, *the two forms are equivalent. Currently, we do not know how to obtain sufficient regularity of the solutions to assert their existence when* $\xi = 0$. *In particular, we do not know whether the solutions constructed here converge to a solution when* ξ *tends toward* 0.

Theorem 2. *Let* $\xi > 0$, $\mathbf{u}_{\text{init}} \in L^2(\Omega)$, $\phi_{\text{init}} \in L^2(\Omega)$ *and* $\mathbf{f} \in L^{\frac{3}{2}}(0, T; W^{-1,\frac{3}{2}}(\Omega))$.
If $\phi_{\min} \le \phi_{\text{init}} \le \phi_{\max} - \xi$, *there exists a weak solution of* (46)–(49) *satisfying the initial conditions* $\mathbf{u}|_{t=0} = \mathbf{u}_{\text{init}}$ *and* $\phi|_{t=0} = \phi_{\text{init}}$.

The proof follows the same arguments as in Theorem 1. The only difference lies in controlling ϕ. Therefore, we must ensure that there is always an energy estimate that allows us to control $\mathbf{u}$ and p, as well as ϕ. To avoid repeating the complete proof of the previous theorem, we simply derive the energy estimate formally.

Proposition 4. *Any regular solution* $(\mathbf{u}, p, \boldsymbol{\sigma}, \phi)$ *to* (46)–(49) *satisfies*

$$\frac{d}{dt}\int_\Omega \phi\frac{|\mathbf{u}|^2}{2} + 4\int_\Omega |\mathbf{Du}|^3 + \int_\Omega (\phi_{\max} - \phi)p_+^{\frac{3}{2}} + \int_\Omega (\phi - \phi_{\min})p_-^{\frac{3}{2}} + 2\int_\Omega (\phi - \phi_{\min})p_-|\mathbf{Su}| = \langle \mathbf{f}, \mathbf{u}\rangle. \qquad (50)$$

Proof - The result essentially follows by taking the scalar product of equation (46) with $\mathbf{u}$. We note that, independently of equation (49), we have

$$\mathcal{D}_t(\phi, \mathbf{u})\cdot\mathbf{u} = \partial_t\Big(\phi\frac{|\mathbf{u}|^2}{2}\Big) + \operatorname{div}\Big(\phi\frac{|\mathbf{u}|^2}{2}\mathbf{u}\Big).$$

Equation (48) allows us to control the pressure term

$$\nabla p\cdot\mathbf{u} = \operatorname{div}(p\mathbf{u}) - p\operatorname{div}\mathbf{u} = \operatorname{div}(p\mathbf{u}) - 2(\phi - \phi_{\min})p|\mathbf{Su}| + (\phi_{\max} - \phi)p_+^{\frac{3}{2}} + (\phi - \phi_{\min})p_-^{\frac{3}{2}},$$

while equation (47) is useful for cancelling out the stress part:

$$-\mathrm{div}\,\boldsymbol{\sigma}\cdot\mathbf{u} = -\mathrm{div}(\boldsymbol{\sigma}\cdot\mathbf{u}) + \boldsymbol{\sigma} : \mathbf{S}\mathbf{u} = -\mathrm{div}(\boldsymbol{\sigma}\cdot\mathbf{u}) + 2(\phi-\phi_{\min})p_+|\mathbf{S}\mathbf{u}|.$$

After integration over Ω, the sum of these contributions provides the result stated in Proposition 4. ■

In order to use equation (50), it is necessary to prove that the terms containing $\phi|\mathbf{u}|^2$, $(\phi_{\max}-\phi)p_+^{\frac{3}{2}}$, $(\phi-\phi_{\min})p_-^{\frac{3}{2}}$ and $(\phi-\phi_{\min})p_-|\mathbf{S}\mathbf{u}|$ are positive. In fact, it is sufficient to prove that $0<\phi_{\min}\le\phi\le\phi_{\max}$. Moreover, to derive an estimate of the pressure, we must ensure that $\phi_{\max}-\phi$ does not equal zero.

Proposition 5. *Any regular solution* $(\mathbf{u}, p, \boldsymbol{\sigma}, \phi)$ *to* (46)–(49)*, such that* $\phi_{\min}\le\phi|_{t=0}\le\phi_{\max}-\xi$*, satisfies*

$$\phi_{\min}\le\phi\le\phi_{\max}-\xi, \tag{51}$$

and

$$\frac{d}{dt}\|\phi\|_2^2 + 2\xi\|\nabla\phi\|_2^2 \le \phi_{\max}^2\int_\Omega(|\mathrm{div}\,\mathbf{u}| + 2\sqrt{p_+}). \tag{52}$$

Proof – Step 1: Upper bound. First, note that equation (49) implies that, for $\beta:\mathbb{R}\longrightarrow\mathbb{R}$, we have

$$\partial_t\beta(\phi) + \mathrm{div}(\beta(\phi)\mathbf{u}) + (\phi\beta'(\phi)-\beta(\phi))\mathrm{div}\,\mathbf{u} = \xi\beta'(\phi)\Delta\phi - \xi\phi\beta'(\phi)\sqrt{p_+}, \tag{53}$$

To derive the upper bound for ϕ, we choose the function β defined as follows

$$\beta(\phi) = \begin{cases} 0 & \text{if } \phi<\phi_{\max}-\xi, \\ \phi-(\phi_{\max}-\xi) & \text{if } \phi\ge\phi_{\max}-\xi. \end{cases}$$

Integrating (53) over Ω, and thanks to this choice for the function β, we obtain

$$\frac{d}{dt}\int_\Omega\beta(\phi) + \int_{\mathcal{E}_+}(\phi_{\max}-\xi)\mathrm{div}\,\mathbf{u} = \xi\int_{\mathcal{E}_+}\Delta\phi - \int_{\mathcal{E}_+}\xi\phi\sqrt{p_+}, \tag{54}$$

where $\mathcal{E}_+ = \{x\in\Omega\ ;\ \phi(t,x)\ge\phi_{\max}-\xi\}$. The unit outgoing normal vector at $\mathcal{E}_+$ is given by $-\nabla\phi/|\nabla\phi|$ so that Stokes' formula allows us to write

$$\xi\int_{\mathcal{E}_+}\Delta\phi = -\xi\int_{\partial\mathcal{E}_+}|\nabla\phi|\le 0.$$

Moreover, from the divergence of the velocity given by equation (48), we deduce that

$$\phi\ge\phi_{\max}-\xi \quad\Longrightarrow\quad \mathrm{div}\,\mathbf{u} + \xi\sqrt{p_+}\ge 0.$$

We finally obtain

$$\int_{\mathcal{E}_+}(\phi_{\max}-\xi)\mathrm{div}\,\mathbf{u} + \int_{\mathcal{E}_+}\xi\phi\sqrt{p_+} \ge \int_{\mathcal{E}_+}(\phi_{\max}-\xi)(\mathrm{div}\,\mathbf{u}+\xi\sqrt{p_+})\ge 0.$$

Consequently, equation (54) implies $\frac{d}{dt}\int\beta(\phi)\le 0$. Thus, if $\phi|_{t=0}\le\phi_{\max}-\xi$, i.e. $\beta(\phi|_{t=0})=0$, then $\beta(\phi)=0$ so that the result follows, namely $\phi\le\phi_{\max}-\xi$.

Step 2: Lower bound. Similarly, to derive the lower bound, we use equation (53) together with the following choice for the function β:

$$\beta(\phi) = \begin{cases} 0 & \text{if } \phi>\phi_{\min}, \\ \phi_{\min}-\phi & \text{if } \phi\le\phi_{\min}. \end{cases}$$

By integrating (53) over Ω, we obtain

$$\frac{d}{dt}\int_\Omega\beta(\phi) - \int_{\mathcal{E}_-}\phi_{\min}\mathrm{div}\,\mathbf{u} = -\xi\int_{\mathcal{E}_-}\Delta\phi + \int_{\mathcal{E}_-}\xi\phi\sqrt{p_+}, \tag{55}$$

where $\mathcal{E}_- = \{x \in \Omega \; ; \; \phi(t,x) \leq \phi_{\min}\}$. This time, the unit outgoing normal vector at $\mathcal{E}_-$ is given by $\nabla\phi/|\nabla\phi|$ so that Stokes' formula allows us to write

$$-\xi \int_{\mathcal{E}_-} \Delta\phi = -\xi \int_{\partial\mathcal{E}_-} |\nabla\phi| \leq 0.$$

From the divergence of the velocity field (48), we derive

$$\phi \leq \phi_{\min} \quad \Longrightarrow \quad \operatorname{div} \mathbf{u} \leq -(\phi_{\max} - \phi_{\min})\sqrt{p_+}.$$

Since $\xi \leq \phi_{\max} - \phi_{\min}$, we obtain the following inequality

$$\int_{\mathcal{E}_-} \xi\phi\sqrt{p_+} + \int_{\mathcal{E}_-} \phi_{\min} \operatorname{div} \mathbf{u} \leq \int_{\mathcal{E}_-} \phi_{\min}(\xi - (\phi_{\max} - \phi_{\min}))\sqrt{p_+} \leq 0.$$

Consequently, equation (55) implies that $\frac{d}{dt}\int \beta(\phi) \leq 0$. Thus, if we have $\phi|_{t=0} \geq \phi_{\min}$, i.e. $\beta(\phi|_{t=0}) = 0$, then $\beta(\phi) = 0$ so that the result follows, namely $\phi \geq \phi_{\min}$.

Step 3: H^1 estimate. By multiplying equation (49) by 2ϕ and integrating over Ω, we obtain

$$\frac{d}{dt}\int_\Omega |\phi|^2 + 2\xi \int_\Omega |\nabla\phi|^2 = -\int_\Omega 2\phi \operatorname{div}(\phi\mathbf{u}) - \int_\Omega 2\phi^2\sqrt{p_+}.$$

Using integration by parts, we can write the right-hand side in the form

$$\frac{d}{dt}\|\phi\|_2^2 + 2\xi\|\nabla\phi\|_2^2 = -\int_\Omega \phi^2(\operatorname{div}\mathbf{u} + 2\sqrt{p_+}).$$

The bound $|\phi| \leq \phi_{\max}$ allows us to conclude. ■

Ideas for the proof of Theorem 2. The method follows the same strategy as in the proof of Theorem 1. We construct a sequence of solutions $(\mathbf{u}_\varepsilon, p_\varepsilon, \boldsymbol{\sigma}_\varepsilon, \phi_\varepsilon)_{\varepsilon>0}$ to an approximate problem, and show that this sequence converges to the solution of the problem (46)–(49) when ε tends to 0.

We focus in particular on estimating the volume fraction ϕ_ε, since the other quantities can be bounded as in the proof of Theorem 1. The crucial original step is to ensure that the sequence $(\phi_\varepsilon)_{\varepsilon>0}$ converges strongly to a limit ϕ, which will enable us to pass to the limit in all numerous nonlinear terms.
Note here that one of the other differences not detailed here concerns the treatment of nonlinear terms of the form $\mathbf{u}\cdot\nabla\mathbf{u}$ that did not appear in model (9)–(11). For such terms, as in the classic case of the Navier-Stokes equations, it suffices to use strong convergence of the velocity sequence $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$.

Given that $\phi_{\max} - \phi_\varepsilon \geq \xi > 0$, estimate (50) ensures that the sequences $(\operatorname{div}\mathbf{u}_\varepsilon)_{\varepsilon>0}$ and $(\sqrt{p_\varepsilon})_{\varepsilon>0}$ are uniformly bounded in $L^3((0,T)\times\Omega)$ independently of ε. Moreover, from estimate (52), we deduce that $(\phi_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$.

To obtain compactness, we consider the sequence $(\partial_t\phi_\varepsilon)_{\varepsilon>0}$. By rewriting the equation (49) as follows

$$\partial_t\phi = \xi\Delta\phi - \operatorname{div}(\phi\mathbf{u}) - \phi\sqrt{p_+},$$

we deduce from the previous estimates that $(\partial_t\phi_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$. Finally, the Aubin-Lions-Simon Lemma (see [12, p.102]) implies that the sequence $(\phi_\varepsilon)_{\varepsilon>0}$ converges strongly to ϕ in $L^2((0,T)\times\Omega)$. ■

Remark 6. *Propositions 4 and 5 are true even if $\xi = 0$. However, the proof of Theorem 2 is no longer correct. Indeed, when $\xi = 0$, we do not have an H^1 estimate for ϕ (see relation (52)), nor an estimate for p (see relation (50), particularly when $\phi_{\max} - \phi$ vanishes). Therefore, proving such a theorem in the case $\xi = 0$ requires different arguments.*

4.2 The $\mu(I)$-rheology

The $\mu(I)$-rheology relates stress, pressure and shear in a manner analogous to equation (5). The main difference lies in the expression of the stress threshold: instead of $\alpha(\phi)p$ as in (5), it takes the form $\mu(I)p$, where μ is an experimentally determined function of the inertial number I (see, for example, [3]). In this framework, to satisfy Barker's stability conditions [29], the dilatancy relation must also be prescribed, and involves functions that depend explicitly on I. These principles lead to the following model, proposed in [15]:

$$\begin{cases} \phi\rho_0\big(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\big) + \nabla p = \phi\rho_0\mathbf{g} + \operatorname{div}\boldsymbol{\sigma}, & (56)\\ \boldsymbol{\sigma} : \mathbf{S}\mathbf{u} = 2\mu(I)p_+|\mathbf{S}\mathbf{u}|, \quad |\boldsymbol{\sigma}| \le \mu(I)p_+, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^\intercal \quad \text{and} \quad \operatorname{tr}\boldsymbol{\sigma} = 0, & (57)\\ \operatorname{div}\mathbf{u} = 2F(I)|\mathbf{S}\mathbf{u}| - \gamma I_{\mathrm{eq}}(\phi)F(I_{\mathrm{eq}}(\phi))\sqrt{p_+}, & (58)\\ \partial_t\phi + \operatorname{div}(\phi\mathbf{u}) = 0, & (59) \end{cases}$$

where $\gamma = 2/d\sqrt{\rho_0}$ and where the dimensionless functions μ and F are explicitly given (see [15] for more details). In particular, these functions are smooth, continuous, and coincide at zero: we denote $\alpha_0 = \mu(0) = F(0)$.

Dimensionless procedure. To rewrite the system in dimensionless form, we apply the following change of variables and unknowns:

$$t = T\widetilde{t}, \quad x = L\widetilde{x}, \quad \mathbf{u} = U\widetilde{\mathbf{u}}, \quad p = gL\rho_0\widetilde{p}, \quad \boldsymbol{\sigma} = gL\rho_0\widetilde{\boldsymbol{\sigma}},$$

where T, L and U denote the characteristic time, length and velocity scales, respectively. We then introduce the dimensionless numbers ε, $\mathfrak{Fr}$ and $\mathfrak{Di}$ defined by

$$\varepsilon = \frac{TU}{L}, \quad \mathfrak{Fr}^2 = \frac{U^2}{gL}, \quad \mathfrak{Di} = \frac{d}{L}\mathfrak{Fr}.$$

This yields the following system where the tilde notation has been omitted for conciseness:

$$\begin{cases} \phi\big(\partial_t \mathbf{u} + \varepsilon\mathbf{u}\cdot\nabla\mathbf{u}\big) + \dfrac{\varepsilon}{\mathfrak{Fr}^2}\nabla p = -\dfrac{\varepsilon}{\mathfrak{Fr}^2}\phi\mathbf{e} + \dfrac{\varepsilon}{\mathfrak{Fr}^2}\operatorname{div}\boldsymbol{\sigma}, & (60)\\ \boldsymbol{\sigma} : \mathbf{S}\mathbf{u} = 2\mu(\mathfrak{Di}\,I)p_+|\mathbf{S}\mathbf{u}|, \quad |\boldsymbol{\sigma}| \le \mu(\mathfrak{Di}\,I)p_+, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^\intercal \quad \text{and} \quad \operatorname{tr}\boldsymbol{\sigma} = 0, & (61)\\ \operatorname{div}\mathbf{u} = 2F(\mathfrak{Di}\,I)|\mathbf{S}\mathbf{u}| - \dfrac{2}{\mathfrak{Di}}I_{\mathrm{eq}}(\phi)F(I_{\mathrm{eq}}(\phi))\sqrt{p_+}, & (62)\\ \partial_t\phi + \varepsilon\operatorname{div}(\phi\mathbf{u}) = 0. & (63) \end{cases}$$

Note that in this dimensionless form, the inertial number I is written as: $I = \dfrac{2|\mathbf{S}\mathbf{u}|}{\sqrt{p_+}}$.

Asymptotic problem. Our aim is to study this system when the volume fraction ϕ is nearly constant and close to the maximum packing fraction $\phi_{\max}$. By choosing ε as a small parameter, we can introduce

$$\phi = \phi_0 + \varepsilon\psi,$$

where ϕ_0 is a constant close to $\phi_{\max}$ in the sense that: $\phi_{\max} - \phi_0 = \mathcal{O}(\varepsilon)$. More generally, we write

$$\mathbf{u} = \mathbf{v} + \mathcal{O}(\varepsilon), \quad p = q + \mathcal{O}(\varepsilon) \quad \text{and} \quad \boldsymbol{\sigma} = \boldsymbol{\tau} + \mathcal{O}(\varepsilon).$$

To retain both rheological and expansion-related contributions, we also assume that

$$\mathfrak{Fr}^2 = \mathcal{O}(\varepsilon) \quad \text{and} \quad \mathfrak{Di} = \mathcal{O}(\varepsilon).$$

A typical illustration of this situation is shown below:

$$L = 10^{-1}\,\mathrm{m}, \quad U = 10^{-1}\,\mathrm{m.s}^{-1}, \quad T = 10^{-2}\,\mathrm{s} \quad d = 10^{-2}\,\mathrm{m} \quad \text{and} \quad g = 10\,\mathrm{m.s}^{-2}.$$

In this case, we have $\varepsilon = 10^{-2}$, $\mathfrak{Fr}^2 = 10^{-2}$ and $\mathfrak{Di} = 10^{-2}$. By keeping only the terms of leading order in ε in the system (60)–(63), we obtain

$$\begin{cases} \phi_0 \partial_t \mathbf{v} + \lambda_0 \nabla q = -\phi_0 \lambda_0 \mathbf{e} + \lambda_0 \mathrm{div}\, \boldsymbol{\tau}, & (64) \\ \boldsymbol{\tau} : \mathbf{Sv} = 2\alpha_0 q_+ |\mathbf{Sv}|, \quad |\boldsymbol{\tau}| \le \alpha_0 q_+, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^\intercal \quad \text{and} \quad \mathrm{tr}\, \boldsymbol{\tau} = 0, & (65) \\ \mathrm{div}\, \mathbf{v} = 2\alpha_0 |\mathbf{Sv}| - \gamma_0 \sqrt{q_+}, & (66) \\ \partial_t \psi + \phi_0 \mathrm{div}\, \mathbf{v} = 0, & (67) \end{cases}$$

where $\lambda_0 = \dfrac{\varepsilon}{\mathfrak{Fr}^2} = \mathcal{O}(1)$ and $\gamma_0 = \dfrac{2\alpha_0(\phi_{\max} - \phi_0)}{\mathfrak{Di}(\phi_0 - \min)} = \mathcal{O}(1)$.
In this system, the evolution of the volume fraction ψ is decoupled from the velocity-stress system, which is described by equations (64)–(66). This independent subsystem for $(\mathbf{v}, q, \boldsymbol{\tau})$ closely resembles the system (1)–(3), with the addition of viscosity. In other words, the system studied in the earlier sections can be regarded as an approximation of the full model incorporating the $\mu(I)$-rheology.

4.3 Alternative models and potential developments.

Granular materials immersed in a fluid. Note that in a recent paper by Barker et al [5], the authors use the $\mu(J)$-rheology, rather than $\mu(I)$, to describe fluidised granular flow, *i.e.* granular material immersed in water. The dimensionless number J, defined by $J = \eta_f |\mathbf{S}|/p$ is used for granular flows with a low Stokes number ($St = \rho_0 d^2 |\mathbf{S}|/\eta_f$) and is particularly appropriate for granular flows in liquids. As in the case of the models using the $\mu(I)$-rheology discussed in the introduction, instability issues frequently arise, and regularization terms are often introduced without rigorous justification of the resulting system's stability. For instance, the models presented in [23, 24] are relatively close to the one proposed here. However, the closure they adopt—specifically, the inclusion of time derivatives in the pressure terms—does not appear to guarantee either stability or the existence of solutions. The analytical framework developed in the present article may offer a pathway for adapting the analysis to such fluid–granular configurations.

Shallow water model. In gravity-driven flow applications, the domain geometry is often highly anisotropic. For example, pyroclastic flows typically extend over several kilometers in length, while their vertical thickness rarely exceeds a few meters. In such settings, many authors adopt depth-averaged (or vertically averaged) models, which reduce the number of unknowns and simplify the computational complexity–see, for instance, [9, 10, 11]. A natural question that arises is how the model introduced in the present work behaves under such vertical averaging, and whether it can be formulated in a simplified but consistent reduced form.

Numerical simulations. The aim of this model is to take the physics of granular flows into account as much as possible in order to carry out numerical simulations and compare the results with those obtained from laboratory experiments. In fact, ongoing work is focused on developing a numerical scheme that remains consistent with the theoretical results presented here—namely, one that preserves as many of the proven properties as possible, such as bounds on the volume fraction, energy dissipation, and structural stability.

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