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Variational inequality solutions and finite stopping time for a class of shear-thinning flows

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Abstract
The aim of this paper is to study the existence of a finite stopping time for solutions in the form of variational inequality to fluid flows following a power law (or Ostwald–DeWaele law) in dimension $N \in \{2, 3\}$. We first establish the existence of solutions for generalized Newtonian flows, valid for viscous stress tensors associated with the usual laws such as Ostwald–DeWaele, Carreau–Yasuda, Herschel–Bulkley and Bingham, but also for cases where the viscosity coefficient satisfies a more atypical (logarithmic) form. To demonstrate the existence of such solutions, we proceed by applying a nonlinear Galerkin method with a double regularization on the viscosity coefficient. We then establish the existence of a finite stopping time for threshold fluids or shear-thinning power-law fluids, i.e. formally such that the viscous stress tensor is represented by a p-Laplacian for the symmetrized gradient for $p \in [1, 2)$.

Keywords Non-Newtonian · Generalized Newtonian · Shear-thinning · Variational inequalities

Mathematics Subject Classification 35K55 · 76D03 · 35Q35 · 76A05

1 Introduction

The aim of this paper is to establish the existence of a finite stopping time for variational inequality solutions of a flow following an Ostwald–DeWaele, Bingham, or Herschel–Bulkley law in a diffusive setting. Such flows can be formally represented by the following system:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla\pi - \Delta u - \operatorname{div}\left(F\left(|D(u)|\right) D(u)\right) = f & \text{in } (0, +\infty) \times \Omega \\ \operatorname{div}(u) = 0 & \text{in } (0, +\infty) \times \Omega \\ u = 0 & \text{on } [0, +\infty) \times \partial\Omega \\ u = u_0 & \text{on } \{0\} \times \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, for $N \in \{2, 3\}$ with a regular enough boundary $\partial\Omega$. Such nonlinear systems describe the flow of incompressible generalized Newtonian fluids

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and give rise to several relevant models. Several types of fluids are described by (1.1). Firstly, if $F(t) = C$, the system (1.1) is the Navier–Stokes equations for a viscous incompressible fluid. Also, by choosing $F(t) = (1+t^2)^{\frac{p-2}{2}}$, system (1.1) describes a Carreau flow. Another relevant example is obtained by choosing, for $p \in (1, 2)$ by $F(t) = t^{p-2}$ for $t > 0$ which leads to an Ostwald–DeWaele (power-law) flow. In the particular case of Bingham arising for $p = 1$ which describes a plastic behavior, we get $F(0) \in [0, 1]$. In the latter case, the function is multivalued at the origin (note that a physical consequence of this phenomenon is the nonexistence of a reference viscosity for threshold fluids, see for example [5]). It is now established that this problem can be circumvented by considering the function outside the origin by a regularization process and by giving a meaning to its limit, in sense of sub-differential. This approach has been successfully carried out in the case of a two-dimensional Bingham flow (see, for instance, [19]).

In the present paper, we focus on the mathematical analysis of shear-thinning flows: a flow is said to be shear-thinning when its viscosity decreases as a function of the stresses applied to it, namely, in the flows we consider, that the function decreases as the shear rate increases. We mainly refer to [8, 12, 25] for the physical motivations of such models. Throughout the present article, we will consider simple fluid flows, that is we will make the assumption that the shear rate is the second invariant of the strain-rate tensor, and moreover it is a scalar quantity given by $|D(u)|$.

In the non-diffusive case (i.e. without Laplacian) the existence and the regularity of distributional solutions for $p > \frac{2N}{N+2}$, which corresponds to the limiting case of the compact Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ (see e.g. [14]) is known for various boundary conditions, both in the stationary and evolutionary cases. We refer for example to [3, 6, 7, 13, 17, 18, 20, 22, 23, 31, 32, 37, 38] and the references therein for more details, as well as to the monograph [11] for a complete and modern presentation of this type of problem. In this non-diffusive case, it is possible to show (see [10]) that the problem can be ill-posed in the sense of distributional solutions in the case $\frac{2N}{N+2} \geq p$. One can avoid such hypotheses on $p \geq 1$ by using dissipative solutions, whose existence has been proved in [1] in the three-dimensional setting. For the above reasons, we consider in the present paper variational inequality solutions in a diffusive setting, which we believe particularly interesting in view of numerical simulations perspective (see for example [26, Chapter 4] or [35]) as for controllability (see for example [24, 27]).

Secondly, we focus on a remarkable property of shear-thinning power-law type fluids: the existence of a finite stopping time. Such a property has been established, for example, in the case of a two-dimensional Bingham flow in [15], in the case of some electrorheological fluids in [2], as well as for the parabolic p-Laplacian operator (see [16]).

Roughly speaking, this property translates into the existence of a time $T_s > 0$ from which the fluid is at a standstill, i.e. such that the velocity field solution to the equation verifies $u(t) = 0$ for almost all $t \geq T_s$. Intuitively, the existence of such a stopping time for the fluid is specific to the shear-thinning character for the Ostwald–DeWaele fluid (as well as for a plastic fluid): the viscosity coefficient given by $F(t) = t^{p-2}$ is decreasing in such a case, which is formally characterized by $1 \leq p < 2$, and amounts to saying that the fluid's viscosity is all the greater the lower the stresses applied to it. It is therefore to be expected that, with no external force adding energy to the system, the time decay of the fluid's energy implies that its viscosity will increase until it stops. Note that in the diffusive case, which we consider here through the system (1.1), we are able to establish the finite stopping time of the kinetic energy associated with the solution of (1.1), i.e. the stopping of its L^2-norm.

Having established the existence of weak solutions in the form of a parabolic variational inequality (see Theorem 3.1 and Definition 2.1) for tensors τ of the form:

$$\tau(D(u)) = F(|D(u)|)D(u),$$

similar to those established, for example, in [19], we will establish the existence of such a stopping time for the kinetic energy of solutions via a differential inequality method (see Theorem 3.2).

Assumptions over the viscosity coefficient F

Throughout this article, we will assume that the viscosity coefficient F satisfies the following assumptions:

(C1) $F : (0, +\infty) \to (0, +\infty)$;
(C2) $F \in W^{1,\infty}_{\text{loc}}((0, +\infty))$;
(C3) $t \mapsto tF(t)$ is non-decreasing on $(0, +\infty)$;
(C4) there exist $p \in [1, 2]$, $t_0 > 0$ and $K > 0$ such that for every $t \geq t_0$, $F(t) \leq Kt^{p-2}$.

Some examples of functions verifying the above assumptions are given in "Appendix A". We emphasize in particular that this takes into account many physical models, such as the Carreau, Bingham, Herschel–Bulkley, Cross, or power law flows.

Remark 1 Assumption (C3) is equivalent to the fact that for all $\varepsilon \geq 0$, the function $t \mapsto tF\left(\sqrt{\varepsilon + t^2}\right)$ is non-decreasing. Indeed, we can write:

$$\forall t \in (0, +\infty),\ tF\left(\sqrt{\varepsilon + t^2}\right) = \left(\frac{t}{\sqrt{\varepsilon + t^2}}\right)\sqrt{\varepsilon + t^2}F\left(\sqrt{\varepsilon + t^2}\right).$$

Hence, $t \mapsto tF\left(\sqrt{\varepsilon + t^2}\right)$ is the product of two non-negative and non-decreasing functions, so it is a non-decreasing function. The opposite implication being obvious by setting $\varepsilon = 0$.

Remark 2 Since the main objective of this article is not to study the existence of solutions, we have established Galerkin method by considering the assumption (C2) in order to make use of Picard–Lindelöf theory, but note that it is possible to weaken this hypothesis by making use of Cauchy-Peano or Carathéodory theory. Then, the assumption (C2) can be replace by $F \in C_{\text{loc}}((0, +\infty))$ without changing the proof of Theorem 3.1.

The existence of a finite stopping time for the kinetic energy associated to variational inequality weak solutions following from Theorem 3.1 is proved in Sect. 5, while considering a viscosity coefficient F verifying (C1)-(C4) and such that it describes a power-type law, namely it satisfies for $1 \leq p < 2$ the additional assumption

$$F(t) \geq Ct^{p-2}. \tag{1.2}$$

Let us conclude with the observation that many fluids are described or approximated by such a law, and are used in a wide range of practical applications. Furthermore, many thixotropic flows (such as blood) also fall into this category, depending on the circumstances of the flow studied (see [33]).

Notations: Throughout the paper, we denote in a generic way the constants by the letter C, and omit their dependence on the parameters in the notations while irrelevant for our study. The functional spaces are defined as follow. We denote by $\mathcal{C}^{\infty}_{0,\sigma}(\Omega)$ the space of divergence-free functions belonging to the space of smooth and compactly supported functions $\mathcal{C}^{\infty}_{0}(\Omega)$, and by $L^2_{\sigma}(\Omega)$ the closure of $\mathcal{C}^{\infty}_{0,\sigma}(\Omega)$ in $L^2(\Omega)$. Then, recalling that $H^1_0(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ into $H^1(\Omega)$ (which is endowed with the norm $u \mapsto \|\nabla u\|_{L^2}$), we consider the Sobolev space $H^1_{0,\sigma}(\Omega)$ defined as

$$H^1_{0,\sigma}(\Omega) := \left\{ w \in L^2_\sigma(\Omega) / \ w = \nabla v, \ v \in H^1_0(\Omega) \right\},$$

which is composed of functions whose trace and divergence are null. We denote $H^{-1}_\sigma(\Omega)$ its dual and $\langle \cdot, \cdot \rangle$ is the duality product between $H^{-1}_\sigma(\Omega)$ and $H^1_{0,\sigma}(\Omega)$. Finally, the space $\mathcal{C}_w(\mathbb{R}_+, L^2_\sigma(\Omega)$ is the functional space whose elements are continuous in the time variable and belonging to $L^2_\sigma(\Omega)$ endowed with its weak topology in the space variable. We should add when necessary the index "loc" to underline that we consider local in time solutions.

2 Weak characterization of solutions by a parabolic variational inequality

In this section we introduce a weak formulation of system (1.1) using a parabolic variational inequality (see Definition 2.1). First, we point out that in the system (1.1), we consider a non-slip boundary condition on $\partial\Omega$.It is thus natural to assume that the initial velocity field u_0 is of null trace on $\partial\Omega$, namely u_0 belongs to $H^1_{0,\sigma}(\Omega)$. Following the ideas employed for showing the existence of solution to Bingham equations in [19, 28], we define a functional j making appear the viscous non-linear term in (1.1) in its derivative.

We fix for the moment $0 \le \varepsilon \le \delta$ and we define a function $G_\varepsilon : (0, +\infty) \to (0, +\infty)$ and a functional $j_\varepsilon : H^1_{0,\sigma}(\Omega) \to \mathbb{R}$ by

$$G_\varepsilon(t) = \int_0^t s F(\sqrt{\varepsilon + s^2}) \, ds \qquad \text{for every } t \in (0, +\infty) \tag{2.1}$$

and

$$j_\varepsilon(v) = \int_\Omega G_\varepsilon(|D(v)|) \, dx, \quad (v \in H^1_{0,\sigma}(\Omega)), \tag{2.2}$$

respectively. We also denote $j = j_0$ and $G = G_0$. One can check that G_ε is a convex functional for ε small enough. Indeed,

$$G'_\varepsilon(t) = t F(\sqrt{\varepsilon + t^2}), \qquad \text{for every } t \in (0, +\infty),$$

and applying the hypothesis (C3) the convexity of G follows immediately. Moreover, we point out that the functional j_ε defined by (2.2) is convex and verifies

$$\langle j'_\varepsilon(v), w \rangle_{-1,1} = \int_\Omega F\left(\sqrt{\varepsilon + |D(v)|^2}\right) (D(v) : D(w)) \ dx \qquad (v, w \in H^1_{0,\sigma}(\Omega)). \tag{2.3}$$

Remark 3 We point out that j' is well defined. Firstly, by our assumptions (C2) and (C3), we can deduce that for all $\beta \in \left(0, \frac{1}{2}\right)$, there exists δ_0 such that:

$$F(t) \le t^{-(1+\beta)} \quad \text{for every } t \in (0, \delta_0).$$

Indeed, assume that this last inequality does not hold, then for every $\delta_0 > 0$, there exists $t_0 \in (0, \delta_0)$ such that:

$$F(t_0) > t_0^{-(1+\beta)}.$$

We can consider without loss of generality that $\delta_0 < \min\left(1, F(1)^{-\frac{1}{\beta}}\right)$, which implies, using our assumption (C3):

$$\delta_0^{-\beta} < t_0^{-\beta} < t_0 F(t_0) \leq F(1).$$

This contradiction shows the result. We recall Korn's L^2 equality for divergence free vector fields:

$$\int_\Omega |D(\varphi)|^2 \, dx = \frac{1}{2}\|\varphi\|^2_{H_0^1}, \quad (\varphi \in H^1_{0,\sigma}(\Omega)).$$

Using these last results and applying Cauchy Schwarz's and Hölder's inequalities, we get:

$$\begin{aligned}
|\langle j'(u), \varphi\rangle_{-1,1}| &= \left|\int_\Omega F(|D(u)|)D(u) : D(\varphi)\, dx\right| \\
&\leq \frac{1}{\sqrt{2}}\left(\int_\Omega F(|D(u)|)^2 |D(u)|^2 \, dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1} \\
&= \frac{1}{\sqrt{2}}\left(\int_{\{|D(u)|\leq\delta_0\}} F(|D(u)|)^2|D(u)|^2\, dx\right. \\
&\quad \left. + \int_{\{|D(u)|>\delta_0\}} F(|D(u)|)^2|D(u)|^2\, dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1} \\
&\leq \frac{1}{\sqrt{2}}\left(\int_{\{|D(u)|\leq\delta_0\}} |D(u)|^{-2\beta}\, dx + \int_{\{|D(u)|>\delta_0\}} F(|D(u)|)^2|D(u)|^2\, dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1} \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{1-2\beta}\int_{\{|D(u)|\leq\delta_0\}} \int_0^{|D(u)|} s^{1-2\beta}\, ds\, dx\right. \\
&\quad \left. + \int_{\{|D(u)|>\delta_0\}} F(|D(u)|)^2|D(u)|^2\, dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1}.
\end{aligned}$$

This implies that j' is well-defined.

We now establish the definition of solutions in the form of variational inequality, which we will consider in the rest of the article.

Definition 2.1 (*Weak solution of* (1.1)) We say that a function $u \in L^2_{\text{loc}}\left(\mathbb{R}_+, H^1_{0,\sigma}(\Omega)\right) \cap \mathcal{C}_{w,\text{loc}}(\mathbb{R}_+, L^2_\sigma(\Omega))$ such that $\partial_t u \in L^{\frac{4}{N}}_{\text{loc}}\left(\mathbb{R}_+, H^{-1}_\sigma(\Omega)\right)$ is a weak solution of (1.1) if and only if u verifies $u_{|t=0} = u_0 \in H^1_{0,\sigma}(\Omega)$, and for every fixed $T > 0$ and all $\varphi \in \mathcal{C}^\infty((0,T)\times\Omega)$ we have:

$$\begin{aligned}
&\int_0^T \langle \partial_t u(t), \varphi(t)\rangle \, dt + \frac{1}{2}\left(\|u_0\|^2_{L^2(\Omega)} - \|u(T)\|^2_{L^2(\Omega)}\right) + \int_0^T\int_\Omega D(u(t)) : D(\varphi(t) - u(t))\, dx \\
&\quad - \int_0^T\int_\Omega (u(t)\cdot\nabla u(t))\cdot\varphi(t)\, dx\, dt + \int_0^T\int_\Omega G(|D(\varphi(t))|) - G(|D(u(t))|)\, dx\, dt \\
&\geq \int_0^T \langle f(t), \varphi(t) - u(t)\rangle\, dt.
\end{aligned} \tag{2.4}$$

Let us quickly motivate this definition with some formal computations. First, we point out that since u belongs to $\mathcal{C}_{w,\text{loc}}(\mathbb{R}_+, L^2_\sigma(\Omega))$, Definition 2.1 makes sense. Then, if we consider for some fixed $T > 0$ that the Lebesgue measure of the set

$$\{(t,x) \in (0,T)\times\Omega \mid |D(u)(t,x)| \leq \delta\}$$

is equal to zero for a small $\delta > 0$, we have that:

$$\int_0^T \langle j'(u), \varphi \rangle \, dt = \int_0^T \int_\Omega F\left(|D(u)|\right)\left(D(u) : D(\varphi)\right) \, dx\, dt.$$

Now, if we replace φ by $u + s\varphi$, with $s > 0$, in the variational inequality (2.4), we obtain after dividing by s:

$$\begin{aligned}&\int_0^T \int_\Omega D(u) : D(\varphi) \, dx\, dt + \int_0^T \int_\Omega \frac{G\left(|D(u+s\varphi)|\right) - G\left(|D(u)|\right)}{s} \, dx\, dt \\ &\quad \geq \int_0^T \int_\Omega \langle f - \partial_t u, \varphi \rangle \, dt - \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi \, dx\, dt.\end{aligned}$$

Since j admits a Fréchet-derivative, it also admits a Gâteaux-derivative and both are the same. Hence, taking the limit as $s \to 0$:

$$\begin{aligned}&\int_0^T \int_\Omega D(u) : D(\varphi) \, dx\, dt + \int_0^T \int_\Omega F\left(|D(u)|\right)\left(D(u) : D(\varphi)\right) \, dx\, dt \\ &\quad \geq \int_0^T \int_\Omega \langle f - \partial_t u, \varphi \rangle \, dt - \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi \, dx\, dt.\end{aligned}$$

Repeating once again the previous reasoning but writing $u - s\varphi$ instead of $u + s\varphi$, we get the following equality:

$$\begin{aligned}&\int_0^T \int_\Omega D(u) : D(\varphi) \, dx\, dt + \int_0^T \int_\Omega F\left(|D(u)|\right)\left(D(u) : D(\varphi)\right) \, dx\, dt \\ &\quad = \int_0^T \int_\Omega \langle f - \partial_t u, \varphi \rangle \, dt - \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi \, dx\, dt.\end{aligned}$$

Therefore, assuming that u is regular enough, we obtain

$$\begin{aligned}&-\frac{1}{2}\int_0^T \int_\Omega \Delta u \cdot \varphi \, dx\, dt - \int_0^T \int_\Omega \operatorname{div}\left(F\left(|D(u)|\right) D(u)\right) \varphi \, dx\, dt \\ &\quad = \int_0^T \int_\Omega \left(f - \partial_t u - u \cdot \nabla u\right) \cdot \varphi \, dx\, dt.\end{aligned}$$

Furthermore De Rham's theorem for a domain with Lipschitz boundary states that there exists a pressure term p such that $f = \nabla p$ into some well chosen space (see [18, Sect. 2] for details). Considering such a function and also the two previous observations, we can write:

$$\begin{aligned}&\int_0^T \int_\Omega \left(\partial_t u + u.\nabla u - \frac{1}{2}\Delta u + \nabla p - \operatorname{div}\left(F\left(|D(u)|\right) D(u)\right) - f\right) \varphi \, dx\, dt = 0, \\ &\left(\varphi \in \mathcal{C}^\infty((0,T) \times \Omega)\right),\end{aligned}$$

which is almost everywhere equivalent to the Eq. (1.1) up to the multiplicative dynamic viscosity constant $\frac{1}{2}$. We have omitted this constant in Definition 2.1 for convenience, and note that it is enough to add the constant 2 in front of the term $\int_0^T \int_\Omega D(u) : D(u - \varphi) \, dx\, dt$ in order to find exactly (1.1).

Finding a solution to the parabolic variational inequality thus amounts to giving meaning to the integral of the nonlinear viscosity coefficient term inherent in the problem, which can be a singular integral in the case of a Bingham fluid.

3 Main results

As announced in the introduction, the main result of this paper is the existence of a finite stopping time for the kinetic energy of solutions of the system (1.1). To this end, we present a proof of the existence of solutions in the form of a variational inequality for this system. The study of the existence of such solutions has initially been developed in [28]. Then, this method was successfully applied for some nonlinear parabolic problems, as the two dimensional Bingham equations in [19], or some power law systems in [30]. Following a similar approach, we get the following existence theorem.

Theorem 3.1 *Assume that the function F satisfies the hypotheses* (C1)–(C4) *and that $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, is a bounded domain with a Lipschitz boundary, and consider an initial datum $u_0 \in H^1_{0,\sigma}(\Omega)$ and a force term $f \in L^2((0,T), H^{-1}_\sigma(\Omega))$. Then, there exists a weak solution u of* (1.1) *having the following regularity*

$$u \in \mathcal{C}_{w,\mathrm{loc}}\left(\mathbb{R}_+, L^2_\sigma(\Omega)\right) \cap L^2_{\mathrm{loc}}\left(\mathbb{R}_+, H^1_{0,\sigma}(\Omega)\right) \quad \textit{and} \quad \partial_t u \in L^{\frac{4}{N}}_{\mathrm{loc}}(\mathbb{R}_+, H^{-1}_\sigma(\Omega)).$$

This result thus ensures the existence of suitable solutions in the two-dimensional and three-dimensional cases. It follows from classical arguments that the solutions are Hölder continuous in time, for a well-chosen Hölder coefficient.

The nonlinear term in the Bingham equations allows us to obtain the rest of the fluid in finite time in the two-dimensional case. This has been demonstrated in [15], using the following approach: it is assumed that the force term will compensate the initial kinetic energy of the fluid, which amounts to establishing a relation between the norm $\|u_0\|_{L^2}$ and an integral of $\|f(t)\|_{L^2}$. This argument is based on the use of the following two-dimensional Nirenberg-Strauss inequality:

$$\exists \gamma > 0,\ \forall u \in H^1_0(\Omega),\ \|u\|_{L^2} \le \gamma \int_\Omega |D(u)|\, dx.$$

We note that such an inequality cannot be true in dimension greater than two, because it would contradict the optimality of Sobolev embedding. We therefore propose to slightly adapt this approach to show the existence of a stopping finite time in both the two and the three-dimensional cases. Firstly, let us formalize the definition.

Definition 3.1 (*Finite stopping time*) Let u be a weak solution in the sense of Definition 2.1 of the system (1.1). We say that $T_0 \in \mathbb{R}_+$ is a finite stopping time for u if:

$$\|u(T_0)\|_{L^2(\Omega)} = 0.$$

In order to prove the existence of a finite stopping time for the solution u provided by Theorem 3.1, we do not make any assumption on the initial velocity field, but we assume that after a certain time the fluid is no longer subjected to any external force. More exactly we make some more assumption on F as stated by the following theorem.

Theorem 3.2 (Existence of a finite stopping time) *Assume that the hypotheses of Theorem 3.1 are verified and that $p \in [1,2)$. Moreover, we assume that there exists two positive constants κ and T_1 such that*

$$F(t) \ge \kappa t^{p-2} \textit{ for every } t \in (0,+\infty) \quad \textit{and} \quad f = 0 \textit{ almost everywhere on } (T_1,+\infty). \tag{3.1}$$

Then, there exists a finite stopping time $T_0 \in \mathbb{R}_+$ *for u in the sense of Definition* 3.1. *Moreover, there exists a constant* $C > 0$ *such that*

$$T_0 \leq T_1 + \frac{C}{1-s(p)} \left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T_1),H_\sigma^{-1}(\Omega))} \right)^{1-s(p)}, \tag{3.2}$$

with

$$s(p) = \frac{5p-4}{4+p}. \tag{3.3}$$

Thus, this result suggests that if no energy is added to the system, then the kinetic energy associated with an Ostwald–DeWaele or Bingham-type flow should become null in a finite time.

Remark 4 Estimate (3.2) in Theorem 3.2 degenerates when $p \to 2$. More exactly, the upper bound on the stopping time goes to $+\infty$ when $p \to 2$. This is related to the loss of the finite stopping time property of the solution in the limit case $p = 2$.

4 Proof of Theorem 3.1

In this section, we establish the proof of Theorem 3.1 in the two-dimensional and three-dimensional settings. In order to prove this result, we begin by establishing an energy estimate for solutions obtained by the Galerkin method in order to obtain uniform bounds with respect to the parameters. We note here that we will have two parameters: a first parameter due to Galerkin's approximation, and a second one due to the regularization proper to the viscosity coefficient F.

First, we briefly establish the Galerkin solutions for the regularized system, with the regularization usually used in numerical methods. Next, we carry out energy estimates to derive, in a third step, weak convergence properties. Finally, we demonstrate the result by making use of properties specific to variational inequalities.

First step: Galerkin scheme
We apply here the usual Galerkin method using the Stokes operator in homogeneous Dirichlet setting, and we use its eigenfunctions $(w_i)_{i\in\mathbb{N}}$ as an orthogonal basis of $H^1_{0,\sigma}(\Omega)$ and orthonormal basis of $L^2_\sigma(\Omega)$ (see [21] for details about this property, and [34, Sect. 2.3] for details concerning the Stokes operator).

For every positive integer m, we denote by P_m the projection of $L^2_\sigma(\Omega)$ onto Span $\big((w_i)_{1\leq i\leq m}\big)$. We would like to formally define our Galerkin system as follows.

$$\begin{cases} \partial_t u_m + P_m(u_m \cdot \nabla u_m) + \nabla P_m(\pi) - \Delta u_m - P_m\left(\operatorname{div}\left(F\left(|D(u_m)|\right) D(u_m)\right)\right) = P_m f \\ \operatorname{div}(u_m) = 0 & \text{on } \mathbb{R}_+ \times \Omega \\ u_m = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ u_m = P_m(u_0) & \text{on } \{0\} \times \Omega. \end{cases} \tag{4.1}$$

In order to avoid the issue posed by the nonlinear term in domains for which the fluid is not deformed we consider the following regularized Galerkin system:

$$\begin{cases} \partial_t u_{m,\varepsilon} - P_m\Big(\mathrm{div}\Big(F\Big(\sqrt{\varepsilon + |D(u_{m,\varepsilon})|^2}\Big)D(u_{m,\varepsilon})\Big)\Big) & \\ \qquad + P_m(u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon}) + \nabla P_m(\pi) - \Delta u_{m,\varepsilon} = P_m f & \text{in } \mathbb{R}_+ \times \Omega \\ \mathrm{div}(u_{m,\varepsilon}) = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u_{m,\varepsilon} = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ u_{m,\varepsilon\,|t=0} = P_m(u_0) & \text{in } \Omega, \end{cases} \tag{4.2}$$

with $0 < \varepsilon < 1$. Applying a Galerkin method, we can see that, writing $u_{m,\varepsilon}(t) = \sum_{i=1}^m d_m^i(t)w_i$, we obtain the ordinary differential system for all $1 \le i \le m$:

$$\begin{aligned} d_m^{i\,\prime}(t) &= \langle f, w_i\rangle - \int_\Omega \frac{1}{2}\|w_i\|^2_{H_0^1} d_m^i(t)\,dx - \int_\Omega D(u_0) : D(w_i)\,dx \\ &- \int_\Omega \frac{1}{2}\|w_i\|^2_{H_0^1} F\left(\sqrt{\varepsilon + \sum_{j=1}^m \frac{1}{2}\|w_j\|^2_{H_0^1}(d_m^j(t))^2 + 2(D(w_j):D(u_0))d_m^j(t) + \frac{1}{2}\|u_0\|^2_{H_0^1}}\right) d_m^i(t)\,dx \\ &- \int_\Omega F\left(\sqrt{\varepsilon + \sum_{j=1}^m \frac{1}{2}\|w_j\|^2_{H_0^1}(d_m^j(t))^2 + 2(D(w_j):D(u_0))d_m^j(t) + \frac{1}{2}\|u_0\|^2_{H_0^1}}\right)(D(u_0):D(w_i))\,dx \\ &- \sum_{j=1}^m \int_\Omega w_j\cdot\nabla w_i d_m^i(t) d_m^j(t)\,dx, \end{aligned} \tag{4.3}$$

completed with initial condition $d_m^i(0) = (u_0, w_i)_{H_0^1}$. This system is described by a locally Lipschitz continuous function with respect to d_m. Indeed, applying the hypothesis (C2), the function $\psi : \mathbb{R}^m \to \mathbb{R}$ defined by

$$\psi(x) = F\left(\sqrt{\varepsilon^2 + \sum_{j=1}^m \frac{1}{2}\|w_j\|^2_{H_0^1} x_j^2 + 2(D(w_j):D(u_0))x_j + \frac{1}{2}\|u_0\|^2_{H_0^1}}\right) \qquad \forall x \in \mathbb{R}^m$$

is locally Lipschitz. The Picard–Lindelöf theorem shows the existence of a solution for system (4.2) defined on $\mathbb{R}_+$.

Second step: Energy estimates and weak convergence
We recall that the solution $u_{m,\varepsilon}$ of (4.2) belongs to Span $\big((w_i)_{1\le i\le m}\big)$, for $(w_i)_{i\in\mathbb{N}}$ the basis of $H^1_{0,\sigma}(\Omega)$ which are the eigenfunctions of the Stokes operator in the homogeneous Dirichlet setting.

In order to clarify our presentation, we specify that we consider the following notion of solution.

Definition 4.1 (*Solution of* (4.2)) We say that $u_{m,\varepsilon} \in L^2((0,T), H^1_{0,\sigma}(\Omega))$, $\partial_t u_{m,\varepsilon} \in L^2((0,T), H^{-1}(\Omega))$ is a weak solution of (4.2) if for every $\varphi \in \mathcal{C}^\infty((0,T)\times\Omega)$ and for a.e. $t \in (0,T)$ it satisfies

$$\begin{aligned} &\langle \partial_t u_{m,\varepsilon}, \varphi\rangle + \int_\Omega D(u_{m,\varepsilon}) : D(\varphi)\,dx \\ &\quad + \langle j'_\varepsilon(u_{m,\varepsilon}), \varphi\rangle - \int_\Omega (u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon})\cdot\varphi\,dx = \langle f, \varphi\rangle. \end{aligned} \tag{4.4}$$

We point out that this definition makes sense since we are studying smooth finite dimensional Galerkin solutions. Then, in order to obtain weak limits into the Galerkin formulation, we establish usual energy estimates, proved in "Appendix B".

Proposition 4.1 *Assume that $u_{m,\varepsilon}$ is a solution of (4.2) in the sense of Definition 4.1. Then, there exists a positive constant C depending on p, Ω, N, $\|u_0\|_{L^2(\Omega)}$ and $\|f\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+,H^{-1}(\Omega))}$ such that the following estimates hold:*

1. $\|u_{m,\varepsilon}\|^2_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+,L^2_\sigma)} + \frac{1}{2}\|u_{m,\varepsilon}\|^2_{L^2_{\mathrm{loc}}\left(\mathbb{R}_+,H^1_{0,\sigma}\right)} \leq C\left(\|f\|^2_{L^2_{\mathrm{loc}}(\mathbb{R}_+,H^{-1})} + \|u_0\|^2_{L^2}\right)$;
2. $\|j'_\varepsilon(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}_{\mathrm{loc}}(\mathbb{R}_+,H^{-1})} \leq C\left(1 + \|f\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+,H^{-1})} + \|u_0\|_{L^2}\right)^{p-1}$;
3. $\|\partial_t u_{m,\varepsilon}\|_{L^{\frac{4}{N}}_{\mathrm{loc}}(\mathbb{R}_+,H^{-1})} \leq C\left(\|f\|^2_{L^2_{\mathrm{loc}}(\mathbb{R}_+,H^{-1})} + \|u_0\|^2_{L^2}\right) + C\left(\|f\|^2_{L^2_{\mathrm{loc}}(\mathbb{R}_+,H^{-1})} + \|u_0\|^2_{L^2}\right)^2$
$$+C\left(1 + \|f\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+,H^{-1})} + \|u_0\|_{L^2}\right)^{p-1}.$$

We then focus in the weak convergence with respect to the above estimates. Here, get suitable convergences by taking the limit with respect to the parameter ε in a first time, then by taking the limit with respect to the Galerkin parameter m. Thus, before proving Theorem 3.1, we show suitable weak convergence properties.

Lemma 4.1 *With the hypotheses of Proposition 4.1 there exists $v_m \in L^2_{\mathrm{loc}}\left(\mathbb{R}_+, H^1_{0,\sigma}(\Omega)\right) \cap L^\infty_{\mathrm{loc}}\left(\mathbb{R}_+, L^2_\sigma(\Omega)\right)$ with $\partial_t v_m \in L^{\frac{4}{N}}_{\mathrm{loc}}\left(\mathbb{R}_+, H^{-1}_\sigma(\Omega)\right)$ such that, up to subsequences:*

1. $\partial_t u_{m,\varepsilon} \rightharpoonup \partial_t v_m$ *in* $L^{\frac{4}{N}}_{\mathrm{loc}}\left(\mathbb{R}_+, H^{-1}_\sigma(\Omega)\right)$;
2. $u_{m,\varepsilon} \rightharpoonup v_m$ *in* $L^2_{\mathrm{loc}}\left(\mathbb{R}_+, H^1_{0,\sigma}(\Omega)\right)$;
3. $u_{m,\varepsilon} \to v_m$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}_+, L^2_\sigma(\Omega))$;
4. $u_{m,\varepsilon} \overset{*}{\rightharpoonup} v_m$ *in* $L^\infty_{\mathrm{loc}}\left(\mathbb{R}_+, L^2_\sigma(\Omega)\right)$.

Moreover, v_m satisfies, for every fixed $T > 0$ and all $\psi \in \mathcal{C}^\infty((0,T)\times\Omega)$:

$$\frac{1}{2}\left(\|v_m(T)\|^2_{L^2} - \frac{1}{2}\|u_0\|^2_{L^2}\right) - \int_0^T \langle \partial_t v_m, \psi\rangle\, dt + \int_0^T\int_\Omega D(v_m) : D(v_m - \psi)\, dx\, dt$$
$$+ \int_0^T j(v_m) - j(\psi)\, dt - \int_0^T\int_\Omega (v_m \cdot \nabla v_m)\cdot\psi\, dx\, dt \leq \int_0^T \langle f, v_m - \psi\rangle\, dt. \tag{4.5}$$

Proof The first and second points follow from the reflexivity of $L^{\frac{4}{N}}_{\mathrm{loc}}\left(\mathbb{R}_+, H^{-1}_\sigma(\Omega)\right)$ and $L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1_{0,\sigma}(\Omega))$ respectively, the third one from Aubin-Lions' Lemma, and the last one by Banach-Alaoglu-Bourbaki's theorem.

Then, since $u_{m,\varepsilon}$ is a solution of (4.2), it satisfies (4.4). Testing against $\varphi = u_{m,\varepsilon} - \psi$ in (4.4) for a test function ψ, we have:

$$\langle \partial_t u_{m,\varepsilon}, u_{m,\varepsilon} - \psi\rangle + \int_\Omega D(u_{m,\varepsilon}) : D(u_{m,\varepsilon} - \psi)\, dx + \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi\rangle$$
$$- \int_\Omega (u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon})\cdot\psi\, dx = \langle f, u_{m,\varepsilon} - \psi\rangle. \tag{4.6}$$

Using (2.3) leads to the well-known convexity inequality:

$$j_\varepsilon(u_{m,\varepsilon}) - j_\varepsilon(\psi) \leq \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi\rangle. \tag{4.7}$$

Using now Lemma B.2 for $u_{m,\varepsilon}$ in (4.7), we get:

$$j(u_{m,\varepsilon}) - C(\varepsilon, u_{m,\varepsilon}) - j_\varepsilon(\psi) \leq \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi\rangle$$

and then, by (C3) applied to $u_{m,\varepsilon}$ for the convergence toward v_m, we get:

$$j(u_{m,\varepsilon}) - C(\varepsilon, u_{m,\varepsilon}) - j_\varepsilon(\psi) \leq \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon} - \psi\rangle.$$

Then, we can write (see [21] part 5.9. for details):

$$\forall \varphi \in H^1_{0,\sigma}(\Omega),\ \int_\Omega u_{m,\varepsilon}(T)\varphi \, dx = \langle u_{m,\varepsilon}(T), \varphi\rangle = \int_0^T \langle \partial_t u_{m,\varepsilon}(t), \varphi\rangle \, dt + \langle u_0, \varphi\rangle \,. \tag{4.8}$$

Now, we also have, using Proposition 4.1:

$$\begin{aligned}\int_0^T \langle \partial_t u_{m,\varepsilon}(t), \varphi\rangle \, dt + \langle u_0, \varphi\rangle &\leq \|\partial_t u_{m,\varepsilon}\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \left(\int_0^T \|\varphi\|_{H^1_0}^{\frac{4}{4-N}} \, dt\right)^{\frac{4-N}{4}} + C\|u_0\|_{L^2}\|\varphi\|_{H^1_0} \\ &\leq C\left(T^{\frac{4-N}{4}} + \|u_0\|_{L^2}\right)\|\varphi\|_{H^1_0}.\end{aligned}$$

In the above inequality we considered φ as a function in $L^\infty((0,T), H^1_0(\Omega))$, so it belongs to $L^{\frac{4}{4-N}}((0,T), H^1_0(\Omega))$ and its left-hand side defines a linear form over $L^{\frac{4}{N}}((0,T), H^{-1}(\Omega))$.

Also, the weak convergence leads to:

$$\int_0^T \langle \partial_t u_{m,\varepsilon}(t), \varphi\rangle \, dt \underset{\varepsilon\to 0}{\longrightarrow} \int_0^T \langle \partial_t v_m(t), \varphi\rangle \, dt. \tag{4.9}$$

Finally, (4.8) and (4.9) imply, up to apply a dominated convergence theorem, to:

$$u_{m,\varepsilon}(T) \underset{\varepsilon\to 0}{\rightharpoonup} v_m(T) \quad \text{in} \quad L^2(\Omega). \tag{4.10}$$

Then, (4.10) implies:

$$\underline{\lim}_{\varepsilon\to 0} \frac{1}{2}\left(\|u_{m,\varepsilon}(T)\|^2_{L^2} - \|P_m(u_0)\|^2_{L^2}\right) \geq \frac{1}{2}\left(\|v_m(T)\|^2_{L^2} - \|P_m(u_0)\|^2_{L^2}\right) \tag{4.11}$$

Also, from usual estimates (see [34, Chapter 4]), since $u_{m,\varepsilon} \underset{\varepsilon\to 0}{\rightharpoonup} v_m$ in $L^2((0,T), H^1_{0,\sigma}(\Omega))$, we have up to extract:

$$\int_0^T \int_\Omega |D(u_{m,\varepsilon})|^2 \, dx \, dt \underset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_\Omega |D(v_m)|^2 \, dx \, dt \tag{4.12}$$

and

$$\int_0^T \int_\Omega (u_{m,\varepsilon} \cdot \nabla u_{m,\varepsilon}) \cdot \psi \, dx \, dt \underset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_\Omega (v_m \cdot \nabla v_m) \cdot \psi \, dx \, dt. \tag{4.13}$$

Integrating in time (4.6), and passing to the limit over ε, combining with (4.12), (4.13), (4.11) and Lemma B.3 leads to (4.5). □

Arguing in the same way, we obtain the following result.

Lemma 4.2 *Under the assumptions of Proposition 4.1, there exists* $u \in L^2_{\mathrm{loc}}\left(\mathbb{R}_+, H^1_{0,\sigma}(\Omega)\right) \cap L^\infty_{\mathrm{loc}}\left(\mathbb{R}_+, L^2_\sigma(\Omega)\right)$ *with* $\partial_t u \in L^{\frac{4}{N}}_{\mathrm{loc}}\left(\mathbb{R}_+, H^{-1}_\sigma(\Omega)\right)$ *such that the function* v_m *given by Lemma 4.1 verifies.*

1. $\partial_t v_m \rightharpoonup \partial_t u$ *in* $L^{\frac{4}{N}}_{\mathrm{loc}}\left(\mathbb{R}_+, H^{-1}_\sigma(\Omega)\right)$;
2. $v_m \to u$ *in* $L^2_{\mathrm{loc}}\left(\mathbb{R}_+, L^2_\sigma(\Omega)\right)$;
3. $v_m \rightharpoonup u$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1_{0,\sigma}(\Omega))$;
4. $v_m \overset{*}{\rightharpoonup} u$ *in* $L^\infty_{\mathrm{loc}}\left(\mathbb{R}_+, L^2_\sigma(\Omega)\right)$.

Moreover, we point out that $u \in C_{w,\mathrm{loc}}(\mathbb{R}_+, L^2_\sigma(\Omega))$ from the above estimates (see [9, Proposition V.1.7. p.363] for details). We can now give the proof of Theorem 3.1.

Proof of Theorem 3.1 We take up again the method previously used, that is we write:

$$\forall \varphi \in H^1_{0,\sigma}(\Omega),\ \int_\Omega v_m(T)\varphi\, dx = \langle v_m(T), \varphi\rangle = \int_0^T \langle \partial_t v_m(t), \varphi\rangle\, dt + \langle P_m(u_0), \varphi\rangle\,. \tag{4.14}$$

Using Proposition 4.1 then leads to:

$$\begin{aligned}\int_0^T \langle \partial_t v_m(t), \varphi\rangle\, dt + \langle u_0, \varphi\rangle &\le \|\partial_t v_m\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \left(\int_0^T \|\varphi\|^{\frac{4}{4-N}}_{H^1_0}\, dt\right)^{\frac{4-N}{4}} + C\|u_0\|_{L^2}\|\varphi\|_{H^1_0}\\ &\le C\left(T^{\frac{4-N}{4}} + \|u_0\|_{L^2}\right)\|\varphi\|_{H^1_0}.\end{aligned} \tag{4.15}$$

Then, the weak convergence leads to:

$$\int_0^T \langle \partial_t v_m(t), \varphi\rangle\, dt \underset{m\to+\infty}{\longrightarrow} \int_0^T \langle \partial_t u(t), \varphi\rangle\, dt. \tag{4.16}$$

Finally, (4.14) and (4.16) imply:

$$v_m(T) \underset{m\to+\infty}{\rightharpoonup} u(T) \quad \text{in} \quad L^2(\Omega). \tag{4.17}$$

Then, (4.10) implies:

$$\underline{\lim}_{m\to+\infty} \frac{1}{2}\left(\|v_m(T)\|^2_{L^2} - \|P_m(u_0)\|^2_{L^2}\right) \ge \frac{1}{2}\left(\|u(T)\|^2_{L^2} - \|u_0\|^2_{L^2}\right) \tag{4.18}$$

Using once again usual estimates for Navier–Stokes equation, since $v_m \underset{m\to+\infty}{\longrightarrow} u$ in $L^2((0,T), H^1_{0,\sigma}(\Omega))$, we have:

$$\int_0^T \int_\Omega |D(v_m)|^2\, dx\, dt \underset{m\to+\infty}{\longrightarrow} \int_0^T \int_\Omega |D(u)|^2\, dx\, dt \tag{4.19}$$

and

$$\int_0^T \int_\Omega (v_m \cdot \nabla v_m)\cdot \psi\, dx\, dt \underset{m\to+\infty}{\longrightarrow} \int_0^T \int_\Omega (u \cdot \nabla u)\cdot \psi\, dx\, dt. \tag{4.20}$$

Applying lemma B.3 with our assumption (C3) and passing to the limit over m, we get:

$$\underline{\lim}_{m\to+\infty} \int_0^T j(v_m)\, dt \ge j(u). \tag{4.21}$$

Passing to the limit over m in (4.15), combining with (4.19), (4.20), (4.21) and (4.18) leads to:

$$\frac{1}{2}\left(\|u(T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) - \int_0^T \langle \partial_t u, \psi \rangle \, dt + \int_0^T \int_\Omega D(u) : D(u-\psi) \, dx \, dt$$
$$+ \int_0^T j(u) - j(\psi) \, dt$$
$$- \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \psi \, dx \, dt \leq \int_0^T \langle f, u - \psi \rangle \, dt \tag{4.22}$$

which is the desired result, that is u is a weak solution of (1.1). □

5 Existence of a finite stopping time for shear-thinning flows

In this part, we assume that hypotheses of the Theorem 3.2 are fulfilled. We are interested to show the existence of a finite stopping time of weak solutions of (1.1) for a viscosity coefficient F which behaves at least as a power-law model.

The finite stopping time profile of a flow is specific to the shear-thinning setting, and is a naturally occurring property in many applications. An illustrative example is paints, whose viscosity is expected to decrease as the applied stresses increase, enabling them to spread well, but which are also expected to avoid dripping once the application is complete. To this end, it is expected that the flow will stop rapidly when the fluid is no longer under stress.

From a mathematical point of view, one can observe that nonlinearities proper to Ostwald–DeWaele or Bingham flows in some special cases imply the existence of such a finite stopping time, as it has already been proved for the two-dimensional Bingham equation under some assumptions in [15]. Moreover, the study of such a profile has been proved in the case of the parabolic p-Laplacian, see [16, Sect. VII.2] for a bounded initial datum or [4, Theorem 4.6] for the case $p = 1$ and with initial datum belonging to $L^2(\Omega)$.

To prove such a result, we proceed to a proof by contradiction. More exactly we show that the regularized Galerkin solutions introduced in the previous section are controlled by a constant $C(\varepsilon)$ tending to zero as the regularization parameter $\varepsilon \to 0$. Then, thanks to the energy estimates used previously, we deduce that the solution necessarily stops in finite time, for viscosity coefficients having a form similar to that of $F(t) = t^{p-2}$, $1 \leq p < 2$.

In this section, we will moreover assume for convenience that the force term belongs to $L^2_{\text{loc}}(\mathbb{R}_+, L^2(\Omega))$ or, if necessary, we will identify the duality bracket $\langle \cdot, \cdot \rangle$ with the L^2 inner product. Note that this assumption is not necessary, the results remain valid for $f \in L^2_{\text{loc}}(\mathbb{R}_+, H^{-1}_\sigma(\Omega))$.

Before proving the Theorem 3.2, we need to prove the following useful interpolation lemma, which is a quantified version of J.-L.-Lions' Lemma (see, for instance [29, Lemme 5.1.]).

Lemma 5.1 *Assume that* $u \in L^6(\Omega)$. *Then, for all* $r \in (0, 3)$, *the following inequality holds:*

$$\|u\|^{2r}_{L^2(\Omega)} \leq \frac{3-r}{3} \|u\|^{\frac{4r}{3-r}}_{L^{\frac{3}{2}}(\Omega)} + \frac{r}{3} \|u\|^2_{L^6(\Omega)}. \tag{5.1}$$

Proof of Lemma 5.1 First, by definition of the L^2-norm, we have, for all $r > 0$:

$$\|u\|^{2r}_{L^2(\Omega)} = \left(\int_\Omega |u|^{\frac{4}{3}} |u|^{\frac{2}{3}} \, dx \right)^r.$$

The Hölder's inequality in the latter relation leads to

$$\|u\|_{L^2(\Omega)}^{2r} \le \|u\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{4r}{3}} \|u\|_{L^6(\Omega)}^{\frac{2r}{3}},$$

and the usual Young's inequality

$$xy \le \frac{3-r}{3} x^{\frac{3}{3-r}} + \frac{r}{3} y^{\frac{3}{r}},$$

holds for all $x, y > 0$ and $r \in (0, 3)$ directly implies the inequality (5.1). □

Moreover, we recall the Nirenberg-Strauss inequality:

Lemma 5.2 [36, Theorem 1] *Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, then there exists a constant $C > 0$ which depends of N and Ω such that for all $u \in W_0^{1,\frac{N}{N-1}}(\Omega)$ the following inequality holds:*

$$\|u\|_{L^{\frac{N}{N-1}}(\Omega)} \le C\|D(u)\|_{L^1(\Omega)}. \tag{5.2}$$

We are now able to prove Theorem 3.2. We point out that the proof being well-known in the two-dimensional case (see [15]) and can be in that last case a direct application of the Korn's inequality and Sobolev's embedding theorem. For this reason, we only give a proof in the three-dimensional setting.

Proof of Theorem 3.2 Let $u_{m,\epsilon}$ be the solution of (4.2). Choosing $\varphi = u_{m,\varepsilon}$ in (4.4) we get:

$$\langle u'_{m,\varepsilon}, u_{m,\varepsilon}\rangle + \int_\Omega |D(u_{m,\varepsilon})|^2 \, dx + \langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon}\rangle - \underbrace{\int_\Omega \left(u_{m,\varepsilon}\nabla u_{m,\varepsilon}\right) u_{m,\varepsilon} \, dx}_{=0} = \langle f, u_{m,\varepsilon}\rangle. \tag{5.3}$$

Combining (2.3) and (3.1), we obtain

$$\langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon}\rangle \ge \kappa^{p-2} \int_\Omega |D(u_{m,\varepsilon})|^2 \left(\varepsilon + |D(u_{m,\varepsilon})|^2\right)^{\frac{p-2}{2}} \, dx.$$

Using $|D(u_{m,\varepsilon})|^2 = (\varepsilon + |D(u_{m,\varepsilon})|^2) - \varepsilon$ we write

$$\langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon}\rangle \ge \kappa^{p-2} \int_\Omega \left(\varepsilon + |D(u_{m,\varepsilon})|^2\right)^{\frac{p}{2}} \, dx - \kappa^{p-2} \int_\Omega \varepsilon \left(\varepsilon + |D(u_{m,\varepsilon})|^2\right)^{\frac{p-2}{2}} \, dx.$$

Since $1 \le p < 2$, we deduce

$$\begin{aligned}\langle j'_\varepsilon(u_{m,\varepsilon}), u_{m,\varepsilon}\rangle &\ge \kappa^{p-2} \int_\Omega |D(u_{m,\varepsilon})|^p \, dx - \kappa^{p-2}\varepsilon \int_\Omega \varepsilon^{\frac{p-2}{2}} \, dx \\ &\ge \kappa^{p-2} \|D(u_{m,\varepsilon})\|_{L^p(\Omega)}^p - \kappa^{p-2} \varepsilon^{\frac{p}{2}} |\Omega|. \end{aligned} \tag{5.4}$$

From (5.3) and (5.4), we get:

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|_{L^2(\Omega)}^2\right) + \|D(u_{m,\varepsilon})\|_{L^2(\Omega)}^2 + \kappa^{p-2}\|D(u_{m,\varepsilon})\|_{L^p(\Omega)}^p \le \langle f, u_{m,\varepsilon}\rangle + \kappa^{p-2}|\Omega|\varepsilon^{\frac{p}{2}}. \tag{5.5}$$

Then, using successively the embedding $L^p(\Omega) \hookrightarrow L^1(\Omega)$, assumption (3.1) and the Lemma 5.2, we get from (5.5), for $t \geq T_1$:

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|^2_{L^2(\Omega)}\right) + \|D(u_{m,\varepsilon})\|^2_{L^2(\Omega)} + C\|u_{m,\varepsilon}\|^p_{L^{\frac{3}{2}}(\Omega)} \leq \kappa^{p-2}|\Omega|\varepsilon^{\frac{p}{2}}. \tag{5.6}$$

Now, from the embedding $\left\{u \in H^1_0(\Omega)/\|D(u)\|_{L^2(\Omega)} < +\infty\right\} \hookrightarrow L^6(\Omega)$ which can be obtained using Korn's L^2 equality and Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$ we get from (5.6):

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|^2_{L^2(\Omega)}\right) + C\|u_{m,\varepsilon}\|^2_{L^6(\Omega)} + C\|u_{m,\varepsilon}\|^p_{L^{\frac{3}{2}}(\Omega)} \leq \kappa^{p-2}|\Omega|\varepsilon^{\frac{p}{2}}. \tag{5.7}$$

Applying Lemma 5.1 with $r = \frac{3p}{4+p}$ (which satisfies $\frac{3}{5} \leq r < 1$ since $1 \leq p < 2$), we have

$$\begin{aligned}\|u_{m,\varepsilon}\|^{1+\frac{5p-4}{4+p}}_{L^2(\Omega)} &\leq \frac{4}{4+p}\|u_{m,\varepsilon}\|^p_{L^{\frac{3}{2}}(\Omega)} + \frac{p}{4+p}\|u_{m,\varepsilon}\|^2_{L^6(\Omega)} \\ &\leq \|u_{m,\varepsilon}\|^p_{L^{\frac{3}{2}}(\Omega)} + \|u_{m,\varepsilon}\|^2_{L^6(\Omega)}.\end{aligned} \tag{5.8}$$

Note that the exponent $s(p) := \frac{5p-4}{4+p}$ in the left-hand side is positive since we have chosen $p \geq 1$. Combining (5.8) with (5.7) we deduce that there exists a constant $C_0 > 0$ (which does not depend on ε) such that, for $t \geq T_1$:

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|^2_{L^2(\Omega)}\right) + C_0\|u_{m,\varepsilon}\|^{1+s(p)}_{L^2(\Omega)} \leq \kappa^{p-2}|\Omega|\varepsilon^{\frac{p}{2}}. \tag{5.9}$$

Assume that for all $t \geq T_1$ we have $C_0\|u_{m,\varepsilon}(t)\|^{1+s(p)}_{L^2(\Omega)} \geq 2\kappa^{p-2}|\Omega|\varepsilon^{\frac{p}{2}}$. Then, we can write from (5.9):

$$\frac{1}{2}\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|^2_{L^2(\Omega)}\right) \leq -\frac{C_0}{2}\|u_{m,\varepsilon}\|^{1+s(p)}_{L^2(\Omega)}. \tag{5.10}$$

Then dividing by $\|u_{m,\varepsilon}(t)\|^{1+s(p)}_{L^2(\Omega)}$ both sides of (5.10), we obtain for all $t \geq T_1$:

$$\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|^{1-s(p)}_{L^2(\Omega)}\right) \leq -\frac{C_0}{2}(1-s(p)). \tag{5.11}$$

Note that $s(p) < 1$ since $p < 2$. Integrating (5.11) with respect to the time leads to $\|u_{m,\varepsilon}(t)\|^{1-s(p)}_{L^2(\Omega)} < 0$ for t large enough. This is a contradiction. Consequently, there exists a time $T_{0,\varepsilon} \geq T_1$ such that $C_0\|u_{m,\varepsilon}(T_{0,\varepsilon})\|^{1+s(p)}_{L^2(\Omega)} \leq 2\kappa^{p-2}|\Omega|\varepsilon^{\frac{p}{2}}$, thus the decay of kinetic energy for smooth Galerkin solutions implies that $C_0\|u_{m,\varepsilon}(t)\|^{1+s(p)}_{L^2(\Omega)} \leq 2\kappa^{p-2}|\Omega|\varepsilon^{\frac{p}{2}}$ for all $t \geq T_{0,\varepsilon}$. Moreover, considering $T_{0,\varepsilon}$ as being the smallest time satisfying such an inequality, we get that for $t \in [T_1, T_{0,\varepsilon}]$, we have that $C_0\|u_{m,\varepsilon}(t)\|^{1+s(p)}_{L^2(\Omega)} \geq 2\kappa^{p-2}|\Omega|\varepsilon^{\frac{p}{2}}$, which means that $\|u_{m,\varepsilon}(t)\|^{-(1+s(p))}_{L^2(\Omega)} \leq \frac{C_0}{2\kappa^{p-2}|\Omega|\varepsilon^{\frac{p}{2}}}$. Dividing (5.9) by $\|u_{m,\varepsilon}(t)\|^{1+s(p)}_{L^2(\Omega)}$ for $t \in [T_1, T_{0,\varepsilon}]$ then leads once again to

$$\frac{d}{dt}\left(\|u_{m,\varepsilon}(t)\|^{1-s(p)}_{L^2}\right) \leq -\frac{C_0}{2}(1-s(p)),$$

which in turn leads, after integrating over $[T_1, T_{0,\varepsilon}]$, to

$$\|u_{m,\varepsilon}(T_{0,\varepsilon})\|^{1-s(p)}_{L^2} - \|u_{m,\varepsilon}(T_1)\|^{1-s(p)}_{L^2} \leq -\frac{C_0}{2}(1-s(p))(T_{0,\varepsilon} - T_1)$$

so that it implies

$$T_{0,\varepsilon} \leq T_1 + \frac{2\|u_{m,\varepsilon}(T_1)\|_{L^2}^{1-s(p)}}{C_0(1-s(p))} \leq T_1 + \frac{2\left(\|u_0\|_{L^2} + \|f\|_{L^2((0,T_1),H_\sigma^{-1})}\right)^{1-s(p)}}{C_0(1-s(p))}. \quad (5.12)$$

Namely, the sequence $(T_{0,\varepsilon})_{\varepsilon>0}$ is uniformly bounded following the parameter $\varepsilon > 0$. Thus, letting $\varepsilon \to 0$ leads to the existence of T_0 (which may depends of m) such that $\|v_m(t)\|_{L^2(\Omega)} = 0$ for almost all $t \in [T_0, +\infty)$ and then $\|v_m\|_{L^2([T_0,+\infty),L^2(\Omega))} = 0$. The same line of arguments shows the existence of a finite stopping time for u in the sense of Definition 3.1. This concludes the proof. □

A Some examples of viscosity coefficients

In this section, we give some examples of functions F satisfying the conditions (C1)–(C4), most of which correspond to models of non-Newtonian coherent flows in the physical sense. This is the case for quasi-Newtonian fluids such as blood, threshold fluids such as mayonnaise, or more generally in the case of polymeric liquids.

1. Firstly, in order to describe power-law fluids (also known as Ostwald–DeWaele flows), we can consider functions $(F_p)_{1<p<2}$ given by:

$$F_p : \left| \begin{array}{l} (0,+\infty) \to (0,+\infty) \\ \\ t \longmapsto t^{p-2}. \end{array} \right.$$

2. Considering functions $(F_{\mu,p})_{\mu>0,p\in[1,2)}$ of the form

$$F_{\mu,p} : \left| \begin{array}{l} (0,+\infty) \to (0,+\infty) \\ \\ t \longmapsto (\mu + t^2)^{\frac{p-2}{2}} \end{array} \right.$$

   leads to Carreau flows.
3. Cross fluids are obtained by choosing function $(F_{\gamma,p})_{\gamma>0,p\in[1,2)}$ given by:

$$F_{\gamma,p} : \left| \begin{array}{l} (0,+\infty) \to (0,+\infty) \\ \\ t \longmapsto (\gamma + t^{2-p})^{-1}. \end{array} \right.$$

4. Another possible choice is to take functions $(F_{p,\beta,\gamma})$ given

$$F_{p,\beta,\gamma} : \left| \begin{array}{l} (0,+\infty) \to (0,+\infty) \\ \\ t \longmapsto \begin{cases} t^{p-2}\log(1+t)^{-\beta} & \text{if } t \in (0,\gamma] \\ \\ \log(1+\gamma)^{-\beta} t^{p-2} & \text{if } t \in (\gamma,+\infty) \end{cases} \end{array} \right.$$

   for $1 < p < 2$ and some $\beta, \gamma > 0$ with γ small enough.

B Useful lemmas and energy estimates

For the sake of clarity, in this "Appendix", we state and prove some useful results employed for the proof of Theorem 3.1. We begin this "Appendix" with some technical lemmas and, in its second part, we give a proof for Proposition 4.1.

B.1 Technical lemmas

Lemma B.1 *Let X be a Banach space, and $\gamma \geq \frac{1}{2}$. Then, the following inequality holds:*

$$\forall (u, v) \in X^2, \ \|u+v\|_X^{\gamma} \leq 2^{\left(\gamma - \frac{1}{2}\right)} \left(\|u\|_X^{\gamma} + \|v\|_X^{\gamma}\right).$$

Proof Using the convexity of $t \mapsto t^{2(2-p)}$ and triangle's inequality of the norm, we get:

$$\|u+v\|_X^{2\gamma} = 2^{2\gamma} \left\| \frac{u+v}{2} \right\|_X^{2\gamma} \leq 2^{2\gamma-1} \left(\|u\|_X^{2\gamma} + \|v\|_X^{2\gamma}\right).$$

Applying now the well-known inequality: $\forall (a, b) \in [0, +\infty)^2, \ \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we get the result. □

Lemma B.2 *Consider that $\varphi \in L^2_{\text{loc}}(\mathbb{R}_+, H_0^1(\Omega))$, then there exists a constant $C(\varepsilon, \varphi) > 0$ which goes to zero as ε does, such that the following inequality holds:*

$$j_\varepsilon(\varphi) + C(\varepsilon, \varphi) \geq j(\varphi), \tag{B.1}$$

where j_ε and j are defined by (2.2).

Proof Recalling that the assumption (C3) states that $t \mapsto tF(t)$ is increasing, we get:

$$\begin{aligned} j(\varphi) &:= \int_\Omega \int_0^{|D(\varphi)|} sF(s)\, ds\, dx \\ &\leq \int_\Omega \int_0^{\sqrt{\varepsilon}} sF(s)\, ds\, dx + \int_\Omega \int_{\sqrt{\varepsilon}}^{\sqrt{\varepsilon}+|D(\varphi)|} sF(s)\, ds\, dx \\ &\leq \varepsilon\sqrt{\varepsilon} F(\varepsilon)|\Omega| + \int_\Omega \int_0^{\sqrt{2|D(\varphi)|\sqrt{\varepsilon}+|D(\varphi)|^2}} sF(\sqrt{\varepsilon + s^2})\, ds\, dx \\ &\leq \underbrace{\varepsilon\sqrt{\varepsilon} F(\varepsilon)|\Omega| + \int_\Omega \int_{|D(\varphi)|}^{2^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}|D(\varphi)|^{\frac{1}{2}}+|D(\varphi)|} sF(\sqrt{\varepsilon + s^2})\, ds\, dx}_{:=C(\varepsilon,\varphi)} + j_\varepsilon(\varphi), \end{aligned}$$

which is the wished result. □

Lemma B.3 *Consider Ω an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, and a sequence $(w_n)_{n\in\mathbb{N}}$ such that there exists a positive constant $C > 0$ satisfying $\|w_n\|_{L^2_{\text{loc}}(\mathbb{R}_+, H^1_{0,\sigma}(\Omega))} \leq C$. Then, for every fixed $T > 0$ and for almost all $(t, x) \in (0, T) \times \Omega$, the following inequality holds:*

$$\varliminf_{n\to+\infty} |D(w_n)(t, x)| \geq |D(w)(t, x)|.$$

Proof Firstly, let us recall that Eberlein-Šmulyan theorem leads up to an extraction to $w_n \rightharpoonup w$ in $L^2_{\text{loc}}(\mathbb{R}_+, H_0^1(\Omega))$ then, for every fixed $T > 0$ and all Lebesgue points $t_0 \in (0, T)$ and $x_0 \in \Omega$, for all $\delta >$ and $R > 0$ small enough, we have $w_n \rightharpoonup w$ in $L^2((t_0-\delta, t_0+\delta), H^1(B(x_0, R))$. Indeed, we have for all test function φ:

$$\int_0^T \int_\Omega \nabla w_n \cdot \nabla\varphi\, dt\, dx \underset{n\to+\infty}{\longrightarrow} \int_0^T \int_\Omega \nabla w \cdot \nabla\varphi\, dt\, dx.$$

Hence, we can take φ, which belongs to $\mathcal{C}_0^\infty((t_0-\delta, t_0+\delta)\times B(x_0,R))$ (up to arguing by density thereafter), satisfying:

$$\nabla\varphi = \begin{cases} \nabla\psi & on (t_0-\delta, t_0+\delta)\times B(x_0,R) \\ 0 & on (0,T)\times\Omega\backslash(t_0-\delta, t_0+\delta)\times B(x_0,R) \end{cases}$$

and so this leads to:

$$\int_{t_0-\delta}^{t_0+\delta}\int_{B(x_0,R)} \nabla w_n\cdot\nabla\psi\; dt\,dx \underset{n\to+\infty}{\longrightarrow} \int_{t_0-\delta}^{t_0+\delta}\int_{B(x_0,R)} \nabla w\cdot\nabla\psi\; dt\,dx.$$

That is $w_n \rightharpoonup w$ in $L^2((t_0-\delta, t_0+\delta), H^1(B(x_0,R)))$. Also, from Korn's L^2 equality and Lebesgue's differentiation theorem over (δ, R) after dividing by $2\delta|B(x_0,R)|$, one gets that for every Lebesgue point $(t_0, x_0)\in(0,T)\times\Omega$:

$$|D(w_n(t_0,x_0))|^2 \le C$$

Following the same line of arguments, we find that:

$$\underline{\lim}_{n\to+\infty}\int_{t_0-\delta}^{t_0+\delta}\int_{B(x_0,R)} |D(w_n)|^2\,dx\,dt \ge \int_{t_0-\delta}^{t_0+\delta}\int_{B(x_0,R)} |D(w)|^2\,dx\,dt.$$

Dividing each side by $2\delta|B(x_0,R)|$, we get:

$$\underline{\lim}_{n\to+\infty}\fint_{t_0-\delta}^{t_0+\delta}\fint_{B(x_0,R)} |D(w_n)|^2\,dx\,dt \ge \fint_{t_0-\delta}^{t_0+\delta}\fint_{B(x_0,R)} |D(w)|^2\,dx\,dt$$

then letting $(\delta, R)\to(0,0)$ leads to the result, after applying a dominated convergence theorem. □

B.2 Proof of Proposition 4.1

We now prove the energy estimates used for the convergence of the nonlinear Galerkin method appearing in the proof of Theorem 3.1.

Proof of Proposition 4.1 1. Setting $\varphi = u_{m,\varepsilon}$ in the weak formulation, we get:

$$\frac{1}{2}\frac{d}{dt}\|u_{m,\varepsilon}\|_{L^2}^2 + \int_\Omega |D(u_{m,\varepsilon})|^2\,dx + \underbrace{\langle j_\varepsilon'(u_{m,\varepsilon}), u_{m,\varepsilon}\rangle}_{\ge 0}$$
$$- \underbrace{\int_\Omega (u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon})\cdot u_{m,\varepsilon}\,dx}_{=0} = \langle f, u_{m,\varepsilon}\rangle.$$

Using the Korn's L^2 equality for divergence free vectors fields, we get

$$\frac{d}{dt}\|u_{m,\varepsilon}(t)\|_{L^2}^2 + \|u_{m,\varepsilon}(t)\|_{H_0^1}^2 \le 2\langle f(t), u_{m,\varepsilon}(t)\rangle \le 2\|f(t)\|_{H^{-1}}^2 + \frac{1}{2}\|u_{m,\varepsilon}(t)\|_{H_0^1}^2.$$

Then, integrating on $(0,t)$ we get

$$\|u_{m,\varepsilon}(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|u_{m,\varepsilon}\|_{H_0^1}^2\,dt \le 2\int_0^t \|f\|_{H^{-1}}^2\,dt + \|u_0\|_{L^2}^2. \tag{B.2}$$

Indeed, we recall that $(P_m(u_0), w_i)_{L^2} = (u_0, P_m w_i)_{L^2} = (u_0, w_i)_{L^2}$, and the conclusion follows. From now on, we will omit to detail this last part which is usual.

2. We have, using Cauchy-Schwarz's inequality and Korn's equality in the divergence free L^2 setting:

$$\begin{aligned}\left\langle j_\varepsilon'(u_{m,\varepsilon}), \varphi\right\rangle &= \int_\Omega F\left(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\right) D(u_{m,\varepsilon}) : D(\varphi)\, dx \\ &\leq \frac{1}{\sqrt{2}}\left(\int_\Omega F\left(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx\right)^{\frac{1}{2}} \|\varphi\|_{H_0^1}. \qquad \text{(B.3)}\end{aligned}$$

From hypothesis (C4), setting $A = \Omega \cap \{|D(u_{m,\varepsilon})| \leq t_0\}$ and B its complement in Ω, we obtain

$$\begin{aligned}\int_\Omega F\left(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx &= \int_A F\left(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx \\ &\quad + \int_B F\left(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx.\end{aligned}$$

Let's estimate these two integrals independently. By assumption (C3), we have that the application $t \mapsto t^2 F\left(\sqrt{\varepsilon+t^2}\right)^2$ is non-decreasing, and we obtain directly:

$$\begin{aligned}\int_A F\left(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx &\leq F\left(\sqrt{\varepsilon+t_0{}^2}\right)^2 t_0{}^2|A| \\ &\leq F\left(\sqrt{\varepsilon+t_0{}^2}\right)^2 t_0{}^2|\Omega| \\ &\leq F\left(\sqrt{1+t_0{}^2}\right)^2 \sqrt{1+t_0{}^2}|\Omega| \\ &\leq C.\end{aligned}$$

Then we have, using again (C4):

$$\begin{aligned}\int_B F\left(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx &\leq K\int_B \frac{|D(u_{m,\varepsilon})|^2}{\left(\varepsilon+|D(u_{m,\varepsilon})|^2\right)^{2-p}}\, dx \\ &\leq K\int_B |D(u_{m,\varepsilon})|^{2(p-1)}\, dx \\ &\leq K\int_B |\nabla u_{m,\varepsilon}|^{2(p-1)}\, dx \\ &\leq C\|u_{m,\varepsilon}\|_{H_0^1}^{2(p-1)},\end{aligned}$$

where we used Jensen's inequality in the concave setting with $t \mapsto t^{p-1}$ in the last line. So, we obtain:

$$\left(\int_\Omega F\left(\sqrt{\varepsilon+|D(u_{m,\varepsilon})|^2}\right)^2 |D(u_{m,\varepsilon})|^2\, dx\right)^{\frac{1}{2}} \leq \left(C + C\|u_{m,\varepsilon}\|_{H_0^1}^{2(p-1)}\right)^{\frac{1}{2}}. \qquad \text{(B.4)}$$

Thus, combining the inequality (B.3)–(B.4), using Lemma B.1 with $\gamma = \frac{2}{N}$, and integrating in time leads to:

$$\|j_\varepsilon'(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})}^{\frac{4}{N}} \leq C + C\|u_{m,\varepsilon}\|_{L^{\frac{4(p-1)}{N}}((0,T),H_0^1)}^{\frac{4(p-1)}{N}}.$$

Then, since $0 < \frac{4(p-1)}{N} \leq 2$, we get, using the embedding $L^2 \hookrightarrow L^{\frac{4(p-1)}{N}}$ and Lemma B.1 with $X := H_0^1$, $q = \frac{4(p-1)}{N}$ and $p = 2$ on $\|u_{m,\varepsilon}\|_{L^{\frac{4(p-1)}{N}}((0,T),H_0^1)}$:

$$\|j_\varepsilon'(u_{m,\varepsilon})\|^{\frac{4}{N}}_{L^{\frac{4}{N}}((0,T),H^{-1})} \leq C + C\|u_{m,\varepsilon}\|^{\frac{4(p-1)}{N}}_{L^2((0,T),H_0^1)}.$$

Using the first point of the proposition for $t = T$, and since $\frac{4(p-1)}{N} \geq 0$, we get:

$$\|j_\varepsilon'(u_{m,\varepsilon})\|^{\frac{4}{N}}_{L^{\frac{4}{N}}((0,T),H^{-1})} \leq C + C(\|f\|_{L^2((0,T),H^{-1})} + \|u_0\|_{L^2})^{\frac{4(p-1)}{N}}.$$

Then, using the exponent $\frac{N}{4}$ on both sides and applying once again Lemma B.1 with $\gamma = \frac{N}{4}$ on the right-hand side in the inequality above leads us to:

$$\|j_\varepsilon'(u_{m,\varepsilon})\|_{L^{\frac{4}{N}}((0,T),H^{-1})} \leq C + C(\|f\|_{L^2((0,T),H^{-1})} + \|u_0\|_{L^2})^{p-1}.$$

This is the wished result.

3. From the weak formulation (4.4) we get

$$\begin{aligned}\langle \partial_t u_{m,\varepsilon}, \varphi\rangle \ &= -\int_\Omega D(u_{m,\varepsilon}) : D(\varphi)\,dx - \langle j_\varepsilon'(u_{m,\varepsilon}), \varphi\rangle \\ &\quad + \int_\Omega (u_{m,\varepsilon}\cdot\nabla u_{m,\varepsilon})\cdot\varphi\,dx + \langle f, \varphi\rangle. \end{aligned} \tag{B.5}$$

Let us point out that

$$\int_\Omega D(u_{m,\varepsilon}) : D(\varphi)\,dx = \frac{1}{2}\int_\Omega \nabla u_{m,\varepsilon}\cdot\nabla\varphi\,dx \leq \frac{1}{2}\|u_{m,\varepsilon}\|_{H_0^1}\|\varphi\|_{H_0^1}. \tag{B.6}$$

Also, from Gagliardo-Nirenberg's inequality, we get the existence of a positive constant C which only depends on N and Ω such that:

$$\|u\|^2_{L^4} \leq C\|\nabla u\|^{\frac{N}{2}}_{L^2}\|u\|^{\frac{4-N}{2}}_{L^2}. \tag{B.7}$$

The latter leads, as for the Navier–Stokes equations:

$$\left|\int_\Omega (u_{m,\varepsilon}.\nabla u_{m,\varepsilon}).\varphi\,dx\right| \leq C\|u_{m,\varepsilon}\|^{\frac{4-N}{2}}_{L^2}\|u_{m,\varepsilon}\|^{\frac{N}{2}}_{H_0^1}\|\varphi\|_{H_0^1}.$$

So, putting (B.6)–(B.8) and the second estimate of the Proposition 4.1 in (B.5), we obtain

$$\begin{aligned}\langle \partial_t u_{m,\varepsilon}, \varphi\rangle \leq \frac{1}{2}\|u_{m,\varepsilon}\|_{H_0^1}\|\varphi\|_{H_0^1} + \|j_\varepsilon'(u_{m,\varepsilon})\|_{H^{-1}}\|\varphi\|_{H_0^1} + C\|u_{m,\varepsilon}\|^{\frac{4-N}{2}}_{L^2}\|u_{m,\varepsilon}\|^{\frac{N}{2}}_{H_0^1}\|\varphi\|_{H_0^1} \\ + \|f\|_{H^{-1}}\|\varphi\|_{H_0^1},\end{aligned}$$

and therefore

$$\|\partial_t u_{m,\varepsilon}(t)\|_{H^{-1}} \leq \frac{1}{2}\|u_{m,\varepsilon}\|_{H_0^1} + \|j_\varepsilon'(u_{m,\varepsilon})\|_{H^{-1}} + C\|u_{m,\varepsilon}\|^{\frac{4-N}{2}}_{L^2}\|u_{m,\varepsilon}\|^{\frac{N}{2}}_{H_0^1} + \|f\|_{H^{-1}}.$$

Now, using the following convexity inequality

$$\forall k \in \mathbb{N},\ \forall (x_i)_{1\leq i\leq k} \in (0,+\infty)^k,\ \exists C > 0,\ \left(\sum_{i=1}^k x_i\right)^{\frac{4}{N}} \leq C\sum_{i=1}^k x_i^{\frac{4}{N}}$$

we get, after integrating in time an using the the embedding $L^2(\Omega) \hookrightarrow L^{\frac{4}{N}}(\Omega)$ (which is valid since $N \in \{2, 3\}$, so that we have $\frac{4}{N} \leq 2$):

$$\|\partial_t u_{m,\varepsilon}\|^{\frac{4}{N}}_{L^{\frac{4}{N}}((0,T),H^{-1})} \leq C\left(\|u_{m,\varepsilon}\|^{\frac{4}{N}}_{L^2((0,T),H_0^1)} + \|j'_\varepsilon(u_{m,\varepsilon})\|^{\frac{4}{N}}_{L^{\frac{4}{N}}((0,T),H^{-1})}\right) + C\|u_{m,\varepsilon}\|^{\frac{8-2N}{N}}_{L^\infty((0,T),L^2)}\|u_{m,\varepsilon}\|^{\frac{4}{N}}_{L^2((0,T),H_0^1)} + C\|f\|^{\frac{4}{N}}_{L^2((0,T),H^{-1})}.$$

Using the previously given convexity inequality and the first and second points of the proposition we obtain the desired result.

□

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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