The FENE viscoelastic model and thin film flows

Laurent Chupin,

Institut Camille Jordan (CNRS UMR 5208), Villeurbanne, France

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Abstract

This note has as objective to determine, in a rigorous way, a simplified expression of the constitutive law for a visco-elastic fluid of FENE type in thin domains. The proof uses the FENE model behavior for long times and the existence of a stationary solution for this comportemental law. Some possible applications of this study are then briefly described : in the domains of lubrication, of blood flows, of microfluidic, of boundary layers...
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Résumé

Le modèle de fluide visco-élastique FENE et les écoulements minces.
L'objet de cette note est de déterminer, de manière rigoureuse, une expression simplifiée de la loi comportementale d'un fluide visco-élastique de type FENE dans un écoulement en domaine mince. Le principe de la preuve utilise à la fois le comportement en temps long d'un écoulement FENE et l'existence d'une solution stationnaire à ce type de loi. On décrit brièvement quelques applications possibles de cette étude : dans les domaines de la lubrification, des écoulements sanguins, de la microfluidique, des couches limites...
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Version française abrégée

Dans le modèle FENE la contrainte élastique $\boldsymbol{\sigma}$ d'un fluide est donnée en fonction du champ de vitesse $\mathbf{u}$ de ce fluide par la relation $\boldsymbol{\sigma}(t,\mathbf{x}) = \displaystyle\int_B \mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}\psi(t,\mathbf{x},\mathbf{Q})d\mathbf{Q}$ où B est la boule de $\mathbb{R}^3$ de centre $\mathbf{0}$ et de rayon Q_0, $\mathbf{F}$ est la fonction sur B définie par $\mathbf{F}(\mathbf{Q}) = \frac{\mathbf{Q}}{1-\frac{\|\mathbf{Q}\|^2}{Q_0^2}}$ et où $\psi(t,\mathbf{x},\mathbf{Q})$ satisfait l'équation de Fokker-Planck suivante pour tout $(t,\mathbf{x},\mathbf{Q}) \in R_*^+ \times \Omega \times B$:

Email address: `laurent.chupin@insa-lyon.fr` (Laurent Chupin).

$$\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \psi = -\operatorname{div}_{\mathbf{Q}} \left((\nabla_{\mathbf{x}} \mathbf{u})^T \cdot \mathbf{Q} \psi - \frac{1}{2\mathcal{D}e} \mathbf{F}(\mathbf{Q}) \psi - \frac{1}{2\mathcal{D}e} \nabla_{\mathbf{Q}} \psi \right). \tag{1}$$

De nombreux travaux mathématiques récents se sont intéressés à ce type de modèles, citons entre autres [1], [7], [8], [11]. Des résulats plus anciens de Bird et al. [3] donnent les premiers termes du développement de ψ et par conséquent de la contrainte $\boldsymbol{\sigma}$ dans un état stationnaire et pour un écoulement de cisaillement avec une vitesse homogène et supposée petite. Nous montrons ici que ces développements peuvent être vus comme une approximation de (1) dans des domaines minces. Une étude plus détaillée de ce qui est présenté dans cette note se trouve dans [4]. On y montre les résultats suivants.

Etude de l'équation de Fokker-Planck

$\triangleright$ Existence et unicité d'une solution $\psi(\mathbf{x}, \mathbf{Q})$ à l'équation (1) dans le cas sationnaire, sans le terme de transport $\mathbf{u} \cdot \nabla_{\mathbf{x}} \psi$ et satisfaisant $\int_B \psi(\mathbf{x}, \mathbf{Q}) d\mathbf{Q} = \rho(\mathbf{x})$ où $\rho(\mathbf{x}) \in \mathbb{R}$.

$\triangleright$ Existence, unicité et comportement en temps long d'une solution $\psi(t, \mathbf{x}, \mathbf{Q})$ à l'équation (1) dont la condition initiale ψ_0 est connue. Cet étude permet de montrer qu'à chaque vitesse $\mathbf{u}$ correspond une unique contrainte $\boldsymbol{\sigma}$ du modèle FENE et donne (lorsque que $\mathbf{u}$ est assez petit par rapport à la longueur Q_0 et au nombre de Deborah $\mathcal{D}e$) son comportement en temps long.

$\triangleright$ Etude du comportement des solutions $\psi^\varepsilon(t, \mathbf{x}, \mathbf{Q})$ de

$$\varepsilon \left(\frac{\partial \psi^\varepsilon}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \psi^\varepsilon \right) = -\operatorname{div}_{\mathbf{Q}} \left((\nabla_{\mathbf{x}} \mathbf{u} + \varepsilon \boldsymbol{\kappa}(\mathbf{x}))^T \cdot \mathbf{Q} \psi^\varepsilon - \frac{1}{2\mathcal{D}e} \mathbf{F}(\mathbf{Q}) \psi^\varepsilon - \frac{1}{2\mathcal{D}e} \nabla_{\mathbf{Q}} \psi^\varepsilon \right) \tag{2}$$

lorsque le paramètre ε tend vers 0 : la limite de ψ^ε correspond à la valeur de ψ^0 (obtenue pour $\varepsilon = 0$) modulo une couche limite en temps, c'est-à-dire à une fonction de correction près dépendant de t/ε.

Applications aux écoulements en films minces

Considérons un écoulement dans un domaine mince (par exemple avec $\Omega =]0,1[\times]0,\varepsilon[$ et $\varepsilon \ll 1$). Pour un fluide incompressible, il est naturel de supposer que la vitesse $\mathbf{u}^\varepsilon = (u, v)$ est de la forme $(\mathcal{O}(1), \mathcal{O}(\varepsilon))$ de sorte que le gradient de vitesse s'écrit

$$\nabla_{\mathbf{x}} \mathbf{u} = \frac{1}{\varepsilon} \begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ 0 & 0 \end{pmatrix} + \mathcal{O}(1). \tag{3}$$

D'après l'étude de l'équation de Fokker-Planck, lorsque le nombre de Deborah $\mathcal{D}e$ est de l'ordre de ε, on en déduit que la contrainte $\boldsymbol{\sigma}^\varepsilon$ associée à ce type de champ de vitesse s'écrit $\boldsymbol{\sigma}^\varepsilon = \boldsymbol{\sigma}^0 + \mathcal{O}(\varepsilon)$ où $\boldsymbol{\sigma}^0$ est la contrainte obtenue à partir de la solution ψ de l'équation (1) dans le cas sationnaire, sans le terme de transport $\mathbf{u} \cdot \nabla_{\mathbf{x}} \psi$. En utilisant les développements de [3], on peut expliciter $\boldsymbol{\sigma}^0$ en fonction de $\frac{\partial u}{\partial y}$ pour des petits nombres de Deborah. On obtient l'expression suivante :

$$\boldsymbol{\sigma}^0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \frac{\mathcal{D}e}{\varepsilon} \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \frac{\partial u}{\partial y} + \left(\frac{\mathcal{D}e}{\varepsilon} \right)^2 \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \left(\frac{\partial u}{\partial y} \right)^2 + \mathcal{O}(\left(\frac{\mathcal{D}e}{\varepsilon} \right)^3), \tag{4}$$

les coefficients a, b, c et d étant tous non nuls. Notons aussi que $c \neq d$ de sorte qu'il apparait un terme de contrainte normale dans l'expression de la contrainte. L'effet du modèle FENE comparé à un modèle newtonien est double : d'une part la viscosité est modifiée (on lui ajoute le coefficient $2b\mathcal{D}e$), d'autre part des efforts normaux apparaissent. On peut utiliser cette expression de la contrainte dans de nombreux domaines. Par exemple, dans le cadre de la lubrification, on en déduit le modèle de Reynolds généralisé suivant :

$$-\eta \frac{\partial^2 u}{\partial y^2} - 2b\mathcal{D}e \frac{\partial^2 u}{\partial y^2} + \mathcal{D}e^2 (d - c) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\partial p}{\partial x} = 0 , \qquad \frac{\partial p}{\partial y} = 0 \qquad \text{et} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5}$$

1. Introduction

Many natural and synthetic fluids are viscoelastic materials. The classical macroscopic models (Power law model, Oldroyd-B model...) showed their limit and actually numerous efforts are concentrated on micro-macro models such FENE model which is the subject of this note. In thin domains, the equations governing the movement of a fluid can be simplified. We are here interested in a certain asymptotic of the FENE model in order to adapt it to the case of the thin flows.

2. The Fokker-Planck equation and the strain tensor for the FENE model

In the FENE model, the elastic stress $\boldsymbol{\sigma}$ is given with respect to the velocity field $\mathbf{u}$ by the relation $\boldsymbol{\sigma}(t,\mathbf{x}) = \int_B \mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}\psi(t,\mathbf{x},\mathbf{Q})d\mathbf{Q}$ where B is the ball $B(\mathbf{0}, Q_0) \subset \mathbb{R}^3$, $\mathbf{F}$ is the function defined on B by $\mathbf{F}(\mathbf{Q}) = \frac{\mathbf{Q}}{1-\|\mathbf{Q}\|^2/Q_0^2}$ and where ψ satisfies the Fokker-Planck equation (1) for all $(t,\mathbf{x},\mathbf{Q}) \in R_*^+ \times \Omega \times B$. Following [1], it's possible to write the Fokker-Planck equation on a agreeable mathematical form. Introducing the maxwellian $M(\mathbf{Q}) = J\left(1 - \frac{\|\mathbf{Q}\|^2}{Q_0^2}\right)^{Q_0^2/2}$ where J is a coefficient such that $\int_B M = 1$, we can write the Fokker-Planck equation as

$$\frac{\partial \psi}{\partial t} + \mathbf{u}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\psi - \frac{1}{2\mathcal{D}e}\,\mathrm{div}_{\mathbf{Q}}\left(M(\mathbf{Q})\nabla_{\mathbf{Q}}\left(\frac{\psi}{M(\mathbf{Q})}\right)\right) + \mathrm{div}_{\mathbf{Q}}\left((\nabla_{\mathbf{x}}\mathbf{u})^T \cdot \mathbf{Q})\psi\right) = 0. \tag{6}$$

The natural $\mathbf{Q}$-spaces associated to the Fokker-Planck equation (6) are the following

$$\begin{aligned} L_M^2 &= \{\varphi \in L_{loc}^1(B)\ ;\ \int_B M\left|\frac{\varphi}{M}\right|^2 < +\infty\}, \\ H_M^1 &= \{\varphi \in L_{loc}^1(B)\ ;\ \int_B M\left|\frac{\varphi}{M}\right|^2 + M\left|\nabla\left(\frac{\varphi}{M}\right)\right|^2 < +\infty\}. \end{aligned} \tag{7}$$

We denote $H_{M,0}^1$ the subspace $\{\varphi \in H_M^1\ ;\ \int_B \psi = 0\}$ and H_M^{-1} the topological dual of $H_{M,0}^1$.

3. "Stationary" solution

Theorem 3.1 *Assume that $Q_0 > \sqrt{2}$ and $\boldsymbol{\kappa} \in L^\infty(B)$. For all $f \in H_M^{-1}$ there exists an unique weak solution $\psi \in H_{M,0}^1$ to equation*

$$-\frac{1}{2\mathcal{D}e}\,div\left(M\nabla\left(\frac{\psi}{M}\right)\right) + div(\boldsymbol{\kappa}\psi) = f \quad on\ B. \tag{8}$$

Notice that the weak formulation of the equation (8) writes

$$\frac{1}{2\mathcal{D}e}\int_B M\nabla\left(\frac{\psi}{M}\right) \cdot \nabla\left(\frac{\varphi}{M}\right) - \int_B (\boldsymbol{\kappa}\psi) \cdot \nabla\left(\frac{\varphi}{M}\right) = \langle f, \varphi\rangle \qquad \text{for all } \varphi \in H_{M,0}^1. \tag{9}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality brackets between H_M^{-1} and $H_{M,0}^1$. Remark that the assumption $Q_0 > \sqrt{2}$ is not surprising in this context. Indeed, if $Q_0 < \sqrt{2}$, the Fokker-Planck equation is ill-posed since it yields many solutions. More precisely in this case uniqueness of solutions does not hold without the additional requirement to take values in $\overline{B}$ (see [6]). This case will be not studied in this note since, according to H.C. Öttinger [9], Q_0 roughly measures the number of monomer units represented by a bead. It's generally larger than 10.

3.1. *Idea of the existence proof in theorem 3.1*

Because of the non-coercivity of the operator $\varphi \mapsto -\frac{1}{2\mathcal{D}e}\,\text{div}\left(M\nabla\left(\frac{\varphi}{M}\right)\right) + \text{div}\,(\boldsymbol{\kappa}\varphi)$, we study a sequence of approximate problems. The construction of these problems and their solutions ψ_n are described in [4] and proof of theorem 3.1 consists in acquiring estimates on ψ_n in order to be able to pass at the limit when n goes to $+\infty$. More precisely, using the same method as for non-coercive linear elliptic problems (see for instance [5]) and taking care to the singularity due to the cancellation of M on the boundary ∂B, we first estimate $M\ln(1+|\frac{\psi_n}{M}|)$ in H^1_M. We then deduce a control on the measure of $\{\mathbf{Q} \in B\ ;\ |\psi_n(\mathbf{Q})| \geq kM(\mathbf{Q})\}$ which allows us to obtain a bound on ψ_n in H^1_M.

3.2. *Idea of the uniqueness proof in theorem 3.1*

To obtain the uniqueness of the solution, we introduce the following dual problem: for all $g \in H^{-1}_M$, find $\phi \in H^1_{M,0}$ a weak solution to

$$-\frac{1}{2\mathcal{D}e}\,\text{div}\left(M\nabla\left(\frac{\phi}{M}\right)\right) - M\boldsymbol{\kappa}\cdot\nabla\left(\frac{\phi}{M}\right) = g \quad \text{on } B. \tag{10}$$

An existence result for this dual problem can be obtain by using the classical Leray-Schauder topological degree theory. More precisely, we show that the application $G : H^1_{M,0} \to H^1_{M,0}$ where $\phi = G(\widetilde{\phi}) \in H^1_{M,0}$ is the unique weak solution to

$$-\frac{1}{2\mathcal{D}e}\,\text{div}\left(M\nabla\left(\frac{\phi}{M}\right)\right) - M\boldsymbol{\kappa}\cdot\nabla\left(\frac{\widetilde{\phi}}{M}\right) = g \quad \text{on } B$$

is a compact application and we find $R > 0$ such that for all $s \in [0,1]$ there exists no solution of $\phi - sG(\phi) = 0$ satisfying the equality $\|\phi\|_{H^1_M} = R$ (see [4] for more details).
To prove the uniqueness in theorem 3.1, since the equation (8) is linear, it is sufficient to prove that the only solution to (8) with $f = 0$ is the null function. Let ψ be a solution to (8) with $f = 0$ and let ϕ a solution of (10) with $g = \text{sgn}(\psi) \in H^{-1}_M$. By putting $\varphi = \phi$ as test function in the equation satisfied by ψ and $\varphi = \psi$ as test function in the equation satisfied by ϕ, we get $\int_B M|\frac{\psi}{M}| = 0$, that is to say $\psi = 0$.

4. Non stationary solution

Theorem 4.1 *Assume that $Q_0 > \sqrt{2}$ and $\boldsymbol{\kappa} \in \mathcal{C}(0,+\infty; L^\infty(B))$. For all $\psi_0 \in L^2_M$ there exists an unique weak solution $\psi \in \mathcal{C}(0,+\infty; L^2_M) \cap L^2_{loc}(0,+\infty; H^1_M)$ to equation*

$$\frac{\partial\psi}{\partial t} - \frac{1}{2\mathcal{D}e}\,\mathit{div}\left(M(\mathbf{Q})\nabla\left(\frac{\psi}{M(\mathbf{Q})}\right)\right) + \mathit{div}(\boldsymbol{\kappa}(t,\mathbf{Q})\psi) = 0 \quad \text{on } B \tag{11}$$

such that $\psi_0 = \psi(t=0)$. Moreover the mean value $\int_B \psi(t,\mathbf{Q})d\mathbf{Q}$ doesn't depend on time and
$\triangleright$ *if $\psi^0(\mathbf{Q}) \geq 0$ for all $\mathbf{Q} \in B$ then $\psi(t,\mathbf{Q}) \geq 0$ for all $(t,\mathbf{Q}) \in [0,+\infty[\times B$;*
$\triangleright$ *if $\displaystyle\int_B \psi^0 = 0$ and if $\sqrt{2}\mathcal{D}e\|\boldsymbol{\kappa}\|_\infty < 1$ then $\displaystyle\lim_{t\to+\infty}\psi(t,\mathbf{Q}) = 0$ for all $\mathbf{Q} \in B$ (with exponential decreasing).*

This result permits to obtain the same kind of theorem on the Fokker-Planck equation (1). Indeed (see [11]) we can first obtain the estimate for ψ with respect to the lagrangian variables, and then translate them to the eulerian variables considering the flow map $\frac{d\mathbf{X}}{dt}(t,\mathbf{x}) = \mathbf{u}(t,\mathbf{X}(t,\mathbf{x}))$ and $\mathbf{X}(0,\mathbf{x}) = \mathbf{x}$. The variable $\mathbf{X}$ can be consider as a parameter in the lagrangian Fokker-Planck equation. Without take into account this parameter, this corresponds to equation (11).

Idea of the proof of theorem 4.1

To use the properties of the space $H^1_{M,0}$ (for instance a Poincaré lemma which is hold in this space, see [1]), we note ψ (again) the function $\psi - \rho M$ where $\rho = \int_B \psi_0$ and obtain results on this new function ψ. It suffices to add a source term of kind $\operatorname{div}(\mathbf{g})$ into equation (11). Concerning this "new"equation (11), there exists a simple a priori estimate (see estimate (12) below). To prove the existence, it suffices to find an approximate problem (for instance using a Galerkin method) on which we obtain the following results.
$\triangleright$ Average conservation: Taking $\varphi = M \in H^1_M$ as test function in the weak formulation of the equation (11) (see also the weak formulation (9) in the stationary case), we deduce that for all $t \in [0,T]$ we have $\int_B \psi(t,\mathbf{Q})d\mathbf{Q} = \int_B \psi^0(\mathbf{Q})d\mathbf{Q}$.
$\triangleright$ A priori estimate: Taking $\varphi = \psi$ as test function, we get, for all $\varepsilon > 0$ (see [4] for more explications)

$$\frac{d}{dt}\left(\|\psi\|^2_{L^2_M}\right) + \varepsilon\|\psi\|^2_{H^1_M} + \left(\frac{1}{2\mathcal{D}e} - \mathcal{D}e\|\boldsymbol{\kappa}\|^2_{L^\infty} - 3\varepsilon\right)\|\psi\|^2_{L^2_M} \leq \frac{1}{4\varepsilon}\|\mathbf{g}\|^2_{L^2_M}. \tag{12}$$

We then deduce estimates which allow to perform the limit in the approximation and show that $T = +\infty$.
$\triangleright$ Uniqueness: The same kind of estimate holds for the equation satisfied by $\psi_2 - \psi_1$ where ψ_1 and ψ_2 are two solutions. We deduce that $y = \|\psi_2 - \psi_1\|^2_{L^2_M}$ verifies $y'(t) \leq Cy(t)$ on $\mathbb{R}^+$. By a Gronwall type argument, we deduce that $y = 0$ and then $\psi_2 = \psi_1$.
$\triangleright$ Long times behavior: If $\int_B \psi_0 = 0$ then the source term $\mathbf{g} = \mathbf{0}$. Moreover, if $\sqrt{2}\mathcal{D}e\|\boldsymbol{\kappa}\|_\infty < 1$ then the energy estimate (12) writes $z'(t) + Cz(t) \leq 0$ on $\mathbb{R}^+$. The function z being defined by $z = \|\psi\|^2_{L^2_M}$ and C being a positive constant. We deduce $z(t) \leq e^{-Ct}z(0)$ which tends to 0 when t goes to $+\infty$.
$\triangleright$ Positivity of ψ: It's the maximum principle, obtained by taking $\varphi = \psi^-$ as test function in the weak formulation of equation (11).

5. Asymptotic behavior and time boundary layer

Theorem 5.1 *Let ψ^ε be the solution to* (2) *for $\varepsilon > 0$ with ψ_0 as initial condition and ψ^0 be the stationary solution to* (2) *for $\varepsilon = 0$. If $Q_0 > \sqrt{2}$ and $\boldsymbol{\kappa} \in \mathcal{C}(0,+\infty; L^\infty(B))$ then there exists two functions $\widetilde{\psi} \in L^\infty([0,+\infty[; \mathcal{C}(\Omega) \otimes H^1_M)$ and $\Psi \in L^\infty([0,+\infty[; \mathcal{C}(\Omega) \otimes H^1_M)$ such that*

$$\psi^\varepsilon(t,\mathbf{x},\mathbf{Q}) = \psi^0(\mathbf{x},\mathbf{Q}) + \widetilde{\psi}(t/\varepsilon,\mathbf{x},\mathbf{Q}) + \varepsilon\Psi(t,\mathbf{x},\mathbf{Q}) \quad \text{on } [0,+\infty[\times\Omega\times B.$$

$\triangleright$ *Moreover if $\sqrt{2}Q_0\mathcal{D}e\|\mathbf{u}\|_{W^{1,\infty}} < 1$ then the function $\widetilde{\psi}$ is profile of time boundary layer which rapidly decrease to zero.*
$\triangleright$ *If $\psi_0 = \psi^0$ (the so-called well-prepared case) then $\widetilde{\psi} = 0$.*
We easily deduce from this theorem that ψ^ε tends to ψ^0 in $L^2([0,+\infty[; H^1_M)$ and that the convergence takes place in $L^\infty([0,+\infty[; H^1_M)$ in the well-prepared case.

Idea of the proof of theorem 5.1

The proof is organized in three steps. The first consists in building an approximate solution: we carry out a formal asymptotic extension of the solution. In the second step, we solve the profile equations: the first one corresponding to the equations (2) in the case $\varepsilon = 0$ whose the solution ψ^0 comes from theorem 3.1, the second one to an equation in which it is necessary to control the decay in the fast variable. It corresponds to the result given by theorem 4.1. The third step consists in showing that the remainder of the extension is bounded in an adequate space, that is the result given by theorem 4.1 to.

6. Thin film flows applications

Let Ω be a thin domain (for instance $\Omega =]0,1[\times]0,\varepsilon[$ with $\varepsilon \ll 1$). For a flow in such a domain, it's natural to assume that the velocity field $\mathbf{u}^\varepsilon = (u,v)$ is of the form $(\mathcal{O}(1),\mathcal{O}(\varepsilon))$. Hence, the velocity gradient decomposes in power of ε, see for instance the equation (3). From previous results about the Fokker-Planck equation, if the Deborah number $\mathcal{D}e$ is of order of ε, the stress $\boldsymbol{\sigma}^\varepsilon$ associated to this kind of velocity writes $\boldsymbol{\sigma}^\varepsilon = \boldsymbol{\sigma}^0 + \mathcal{O}(\varepsilon)$ where $\boldsymbol{\sigma}^0$ corresponds to the stress which comes from to the solution ψ of the equation (1) in the stationary case without the transport term $\mathbf{u}\cdot\nabla_{\mathbf{x}}\psi$. Using the asymptotic developments introduce in [3], we explicit $\boldsymbol{\sigma}^0$ with respect to $\frac{\partial u}{\partial y}$ for small values of the Deborah number. We obtain the relation (4) where the coefficients a, b, c and d are non zero and $c \neq d$. Hence, the FENE model has two distinct contributions compared with the newtonian model in thin flow: the viscosity is modified (add the value $2b\mathcal{D}e$) and a normal force appear in the equations of motion of an incompressible fluid :

$$\mathcal{R}e\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - \Delta\mathbf{u} = \text{div}\ \boldsymbol{\sigma} \quad \text{and} \quad \text{div}\ \mathbf{u} = 0.$$

Taking into account the thickness (that is the relations (3) and (4)), we obtain a simplified equation for the motion in thin domain. We can use this law in numerous models of thin flows.
$\triangleright$ In the lubrication domain ($p = \mathcal{O}(1/\varepsilon^2)$, see for instance [2]) the pressure term dominates the flows and we have equations (5).
$\triangleright$ In the boundary layer theory ($\mathcal{R}e = \mathcal{O}(1/\varepsilon^2)$, see for instance [10]), the inertial term dominate and we obtain the following generalized Prandtl equation (where U is the outer flow):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \frac{1}{\mathcal{R}e}\left(\frac{\partial^2 u}{\partial y^2} + 2b\mathcal{D}e\frac{\partial^2 u}{\partial y^2} - \mathcal{D}e^2(d-c)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2\right) \quad \text{and} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

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