

ROUGHNESS-INDUCED EFFECT AT MAIN ORDER ON THE REYNOLDS APPROXIMATION*

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Abstract. Usually the Stokes equations that govern a flow in a smooth thin domain (with thickness of order ε) are related to the Reynolds equation for the pressure p_{smooth} . In this paper, we show that for a rough thin domain (with rugosities of order ε^2) the flow is governed by a modified Reynolds equation for a pressure p_{rough} . Moreover, we find the relation $p_{\text{rough}} = K p_{\text{smooth}}$, where K is an explicit coefficient depending only on the form of the rugosities and on the viscosity of the fluid. In some sense, we see that the flow may be accelerated using adequate rugosity profiles on the bottom. The limit system is mathematically justified through a variant of the notion of two-scale convergence, the originality and difficulty being the anisotropy in the height profile.

Key words. thin films, Stokes and Reynolds equations, rough boundaries, homogenization, two-scale convergence, microfluidic

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Introduction. We study in this paper the effect of very small domain irregularities on a thin film flow governed by the Stokes equations. In some papers, the height of flow, denoted h and depending on the horizontal component x , is fixed and the effect of small scales of the boundary is studied; see [1, 10]. Other papers (see [5, 8, 9]) are related to lubrication from a mechanical point of view, that is, when the height of flow, denoted h^ε , is assumed to be small. Various limit models, in special regimes, are obtained depending on the ratio between the size of the rugosities and the mean height of the domain. In [3, 4, 8, 9], the ratio is assumed to be of order 1, namely

$$h^\varepsilon(x) = \varepsilon h\left(x, \frac{x}{\varepsilon}\right),$$

and an asymptotic analysis is performed using an homogenization process. More recently, in [5], the authors study the case where the narrow gap is smaller than the roughness, namely

$$h^\varepsilon(x) = \varepsilon h\left(x, \frac{x}{\varepsilon^\alpha}\right) \quad \text{with } \alpha \leq 1.$$

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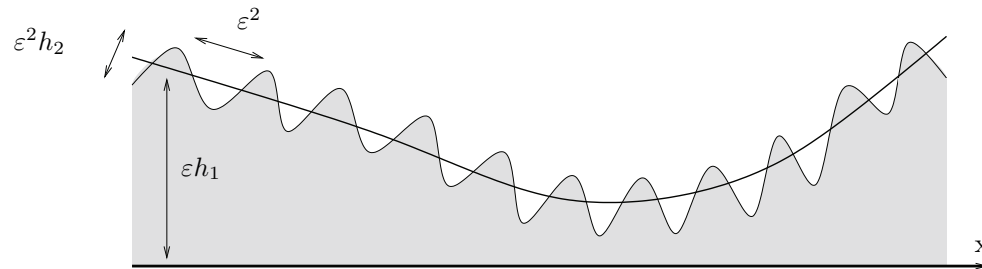


FIG. 1. Domain Ω^ϵ .

Here we consider the particular case which does not enter in the previous framework,

$$h^\epsilon(x) = \epsilon h_1(x) + \epsilon^2 h_2\left(\frac{x}{\epsilon^2}\right),$$

and we mathematically justify (through a variant of the notion of two-scale convergence) that an extra term modifies the standard Reynolds equation. More precisely the roughness is defined by a periodical function with period of order ϵ^2 ($\epsilon > 0$), so that the three-dimensional domain occupied by the fluid is (see Figure 1)

$$\Omega^\epsilon = \left\{ (x, z) \in \mathbb{R}^2 \times \mathbb{R} ; \quad x \in \omega \quad \text{and} \quad 0 < z < h^\epsilon(x) \right\},$$

where ω is a domain of \mathbb{R}^2 . More general rugosity profiles will be considered in a forthcoming paper; see [6]. In the following, the applications h_1 and h_2 are assumed to be regular and satisfy $h_1 \geq \delta > 0$ and h_2 periodic with 0 as average. For the sake of simplicity, we assume the flow to be governed by the stationary Stokes system

$$(1) \quad \begin{cases} -\eta \Delta \mathbf{u}^\epsilon + \nabla p^\epsilon = 0 & \text{on } \Omega^\epsilon, \\ \operatorname{div}(\mathbf{u}^\epsilon) = 0 & \text{on } \Omega^\epsilon. \end{cases}$$

The vector field $\mathbf{u}^\epsilon = (u^\epsilon, w^\epsilon) \in \mathbb{R}^2 \times \mathbb{R}$ describes the fluid velocity, whereas the pressure is given by the scalar function p^ϵ . The positive real number $\eta > 0$ corresponds to the viscosity of the fluid. Finally, we complete the Stokes system with the following boundary conditions: adherence condition at the top, namely

$$w^\epsilon|_{z=h^\epsilon(x)} = 0 \quad \text{and} \quad u^\epsilon|_{z=h^\epsilon(x)} = 0;$$

imposed velocity at the bottom (physically, the plane $z = 0$ is moving horizontally with a constant velocity $u_b \in \mathbb{R}^2$)

$$w^\epsilon|_{z=0} = 0 \quad \text{and} \quad u^\epsilon|_{z=0} = u_b;$$

and along the lateral boundary (that is, for $x \in \partial\omega$) two kinds of boundary conditions will be considered. The choice of the conditions highly depends on the devices to be considered. In most of the physical problems, two types of boundary conditions are simultaneously used: Neumann-type conditions and Dirichlet-type conditions. Thus, in the general case, let

$$\mathbf{u}^\epsilon|_{\partial\omega^q} = \mathbf{u}_\ell \quad \text{and} \quad (-p^\epsilon \operatorname{Id} + 2\eta D(\mathbf{u}^\epsilon)) \cdot n|_{\partial\omega^p} = \mathbf{0},$$

where $\partial\omega^p$ and $\partial\omega^q$ define a partition of the boundary $\partial\omega$. Since the velocity field \mathbf{u}^ϵ is a free-divergence vector field, let us notice that a compatibility condition on the

total flux is needed if $\partial\omega^p = \emptyset$: $\int_{\partial\omega} \mathbf{u}_\ell \cdot \mathbf{n} = 0$. For the sake of simplicity, since it does not change the analysis, we will consider the more simple boundary condition

$$\mathbf{u}^\varepsilon|_{\partial\omega^q} = \mathbf{u}_\ell \quad \text{and} \quad \partial\mathbf{u}^\varepsilon/\partial n|_{\partial\omega^p} = \mathbf{0}$$

instead of the previous one.

This paper is organized as follows. In the first section, we give the definition and properties of a convenient two-scale convergence and give the main result that justifies the asymptotic model. The second section is dedicated to formal analysis. At first, an equivalent formulation is given after rescaling the real domain and defining different scales. Then, for the reader’s convenience, the case without rugosity is recalled, where we recover the simplified Stokes equations in a thin domain and the well-known Reynolds equation. Namely, we find the equations

$$\begin{cases} -\eta\partial_Z^2 u_{\text{smooth}} + h_1^2 \nabla_x p_{\text{smooth}} = 0, \\ \partial_Z p_{\text{smooth}} = 0, \end{cases}$$

with

$$\operatorname{div}_x \left(\frac{h_1^3}{12\eta} \nabla_x p_{\text{smooth}} \right) = \operatorname{div}_x \left(\frac{h_1}{2} u_b \right).$$

In the rugosity case, we show that the roughness may be responsible for some acceleration of the flow, which is the main contribution of this paper. Namely, we find that the flow is governed by the equations

$$\begin{cases} -\eta\partial_Z^2 u_{\text{rough}} + h_1^2 \nabla_x p_{\text{rough}} + \eta M Z \partial_Z u_{\text{rough}} = 0, \\ \partial_Z p_{\text{rough}} = 0, \end{cases}$$

with

$$\operatorname{div}_x \left(\frac{h_1^3}{12\eta} \nabla_x p_{\text{rough}} \right) = \operatorname{div}_x \left(\frac{h_1}{2} K u_b \right),$$

where M and $K \geq 1$ are coefficients depending on the viscosity of the fluid η and on the rugosities. In the third section, we illustrate numerically the effect of rugosities depending on the lateral boundary condition we consider. These figures show the acceleration of the flow due to effects of the considered rugosities. Finally, in the last section, we rigorously derive the limit system through a variant of the usual two-scale convergence. Namely, we prove Theorem 1 given in section 1.2. The main difficulty is the anisotropy in the height profile. We need to define “anisotropic” two-scale limit; see, for instance, Lemma 5.4 or section 5.2 in order to study the divergence equation.

1. Definition and properties of a convenient two-scale convergence and the main result.

1.1. Homogenization: Definition and basic properties. The proof of the homogenization process will be carried out by using a variant of the two-scale convergence introduced by Nguetseng in [12] and developed by Allaire in [2]. Let us give the basic definition and properties of this concept. We set $\Omega = \omega \times (0, 1) \subset \mathbb{R}^{d-1} \times \mathbb{R}$ and denote by \mathbb{T}^{d-1} the torus of dimension $d - 1$.

PROPOSITION 1.1. We say that a sequence $(v^\varepsilon(x, Z))_{\varepsilon>0}$ of functions in $L^2(\Omega)$ two-scale converges to a limit $v^0(x, Z, X)$ belonging to $L^2(\Omega \times \mathbb{T}^{d-1})$, and we denote it by $v^\varepsilon \xrightarrow{2} v^0$, if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v^\varepsilon(x, Z) \Psi(x, Z, x/\varepsilon^2) dx dZ = \int_{\Omega} \int_{\mathbb{T}^{d-1}} v^0(x, Z, X) \Psi(x, Z, X) dX dx dZ$$

for any test function $\Psi(x, Z, X)$, X -periodic in the third variable, satisfying

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\Psi(x, Z, x/\varepsilon^2)|^2 dx dZ = \int_{\Omega} \int_{\mathbb{T}^{d-1}} |\Psi(x, Z, X)|^2 dX dx dZ.$$

Note that Z is only a parameter for this definition.

- (i) From each bounded sequence (v^ε) in $L^2(\Omega)$ one can extract a subsequence which two-scale converges to some limit $v^0 \in L^2(\Omega \times \mathbb{T}^{d-1})$. The weak L^2 -limit of v^ε is $v(x, Z) = \int_{\mathbb{T}^{d-1}} v^0(x, Z, X) dX$.
- (ii) Let (v^ε) be a bounded sequence in $L^2(0, 1; H^1(\omega))$ which converges weakly to v in $L^2(0, 1; H^1(\omega))$. Then $v^\varepsilon \xrightarrow{2} v$ and there exists a function $v^1 \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$ such that, up to a subsequence, $\nabla_x v^\varepsilon \xrightarrow{2} \nabla_x v(x, Z) + \nabla_X v^1(x, Z, X)$.
- (iii) Let (v^ε) be a bounded sequence in $L^2(\Omega)$ such that $(\varepsilon^2 \nabla_x v^\varepsilon)$ is bounded in $L^2(\Omega)$. Then there exists a function $v^0 \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$ such that, up to a subsequence, $v^\varepsilon \xrightarrow{2} v^0$ and $\varepsilon^2 \nabla_x v^\varepsilon \xrightarrow{2} \nabla_X v^0(x, Z, X)$.

An immediate consequence of the above definition is the following.

LEMMA 1.1. Let (v^ε) be a bounded sequence in $L^2(\Omega)$ such that $(\varepsilon \nabla_x v^\varepsilon)$ is bounded in $L^2(\Omega)$. Then the two-scale limit $v^0 \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$ of v^ε is such that

$$\nabla_X v^0 = 0.$$

Proof. By (iii) of Proposition 1.1, we have $\varepsilon^2 \nabla_x v^\varepsilon \xrightarrow{2} \nabla_X v^0$. Since $(\varepsilon \nabla_x v^\varepsilon)$ is bounded in $L^2(\Omega)$, it two-scale converges to some limit $\eta^0 \in L^2(\Omega; L^2(\mathbb{T}^{d-1}))$, and $(\varepsilon^2 \nabla_x v^\varepsilon)$ two-scale converges to 0. Thus $\nabla_X v^0 = 0$. \square

1.2. Application to rugosities in a thin film: Statement of the main result. Recall that the problem studied in this paper relates to Stokes equation (1) in this domain Ω^ε . The boundary of Ω^ε is defined by its rugous border

$$h^\varepsilon(x) = \varepsilon h_1(x) + \varepsilon^2 h_2\left(\frac{x}{\varepsilon^2}\right).$$

We define h_2^ε by $h_2^\varepsilon(x) = h_2(x/\varepsilon^2)$. To apply the homogenization process presented in the previous section, we have to assume only that the functions h_2^ε and $\nabla_X h_2^\varepsilon$ are strongly two-scale converging to $h_2(X)$ and $\nabla_X h_2(X)$, that is,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |h_2(x/\varepsilon^2)|^2 dx &= \int_{\Omega} \int_{\mathbb{T}^{d-1}} |h_2(X)|^2 dx dX, \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla_X h_2(x/\varepsilon^2)|^2 dx &= \int_{\Omega} \int_{\mathbb{T}^{d-1}} |\nabla_X h_2(X)|^2 dx dX. \end{aligned}$$

Such a function is also called an ‘‘admissible test function’’ for the two-scale convergence. The latter relations are fulfilled because we have assumed that $h_2 \in C^1(\mathbb{T}^{d-1})$. By the way, notice that the periodicity of h_2 is not a necessary assumption. It would

be sufficient for our needs to assume that $h_2(x/\varepsilon^2)$ and $\nabla h_2(x/\varepsilon^2)$ strongly two-scale converge to $h_2(X)$ and $\nabla_X h_2(X)$, respectively (see Remark 2 at the end of this paper).

The existence issues are not addressed in this paper since they correspond to standard linear PDEs of Stokes type. We state now the main result.

THEOREM 1. *Let $(u^\varepsilon, w^\varepsilon, p^\varepsilon)$ be a sequence of weak solutions of (1).*

The rescaled quantity $(u^\varepsilon, \frac{1}{\varepsilon}w^\varepsilon, \varepsilon^2 p^\varepsilon) \circ (x, h^\varepsilon(x)Z)$ two-scale converges to the weak solution (u_0, w_1, p_0) of

$$\begin{cases} -\eta\partial_Z^2 u_0 + h_1^2 \nabla_x p_0 + \eta M Z \partial_Z u_0 = 0, \\ \partial_Z p_0 = 0, \\ \operatorname{div}_x(h_1 u_0) + \partial_Z(w_1 - Z \nabla_x h_1 \cdot u_0) = 0 \end{cases}$$

on $\omega \times (0, 1)$, where $M = \int_{\mathbb{T}^{d-1}} |\nabla_X h_2|^2$.

REMARK 1. *Thanks to the boundary conditions on u_0 and w_1 , we can deduce (see section 2.3.2 for details) that the limit model is equivalent to the following Reynolds modified equation on the pressure p_0 :*

$$\operatorname{div}_x \left(\frac{h_1^3}{12\eta} \nabla_x p_0 \right) = \operatorname{div}_x \left(\frac{h_1}{2} K u_b \right) \quad \text{on } \omega,$$

where K depends only on M (its explicit expression is given in section 2.3.2).

2. Formal derivation.

2.1. New variables. In view of taking into account the oscillations of the domain, we introduce a new variable, namely $X = x/\varepsilon^2$. Then we write the height h^ε as

$$h(x, X) = \varepsilon h_1(x) + \varepsilon^2 h_2(X),$$

where $h_1 \in H^2(\Omega)$ and $h_2 \in H^2(\mathbb{T}^{d-1})$.

In what follows, to use an ε -independent domain, let us introduce the change of variables $Z = z/h(x, X)$. According to this change of variables, we seek the unknowns (velocity and pressure) as $\mathbf{u}^\varepsilon(x, z) = \mathbf{u}(x, X, Z)$ and $p^\varepsilon(x, z) = p(x, X, Z)$. More generally, if $f^\varepsilon(x, z) = f(x, X, Z)$, then

$$\nabla_x f^\varepsilon(x, z) = \nabla_x f(x, X, Z) + \frac{1}{\varepsilon^2} \nabla_X f(x, X, Z) - \frac{1}{h(x, X)} \nabla h(x, X) Z \partial_Z f(x, X, Z),$$

$$\partial_z f^\varepsilon(x, z) = \frac{1}{h(x, X)} \partial_Z f(x, X, Z),$$

where we use the notation

$$\nabla h(x, X) = \varepsilon \nabla_x h_1(x) + \nabla_X h_2(X).$$

In the same way, the second derivative of f^ε can be expressed with respect to the second derivative of f , introducing

$$\Delta h(x, X) = \varepsilon \Delta_x h_1(x) + \frac{1}{\varepsilon^2} \Delta_X h_2(X).$$

Then, the two first velocity components in (1) read as

$$\begin{aligned} & -\eta h^2 \Delta_x u - \frac{2\eta}{\varepsilon^2} h^2 \nabla_x \cdot \nabla_X u - \frac{\eta}{\varepsilon^4} h^2 \Delta_X u + \frac{2\eta}{\varepsilon^2} h \nabla h \cdot Z \nabla_X \partial_Z u \\ (2) \quad & + \eta h \Delta h Z \partial_Z u - \eta |\nabla h|^2 Z \partial_Z u + 2\eta h \nabla h \cdot Z \nabla_x \partial_Z u - \eta |\nabla h|^2 Z^2 \partial_Z^2 u \\ & - \eta \partial_Z^2 u + h^2 \nabla_x p + \frac{1}{\varepsilon^2} h^2 \nabla_X p - h \nabla h Z \partial_Z p = 0. \end{aligned}$$

The process is similar for the third velocity component in (1). We have

$$\begin{aligned}
 (3) \quad & -\eta h^2 \Delta_x w - \frac{2\eta}{\varepsilon^2} h^2 \nabla_x \cdot \nabla_X w - \frac{\eta}{\varepsilon^4} h^2 \Delta_X w + \frac{2\eta}{\varepsilon^2} h \nabla h \cdot Z \nabla_X \partial_Z w \\
 & + \eta h \Delta h Z \partial_Z w - \eta |\nabla h|^2 Z \partial_Z w + 2\eta h \nabla h \cdot Z \nabla_x \partial_Z w \\
 & - \eta |\nabla h|^2 Z^2 \partial_Z^2 w - \eta \partial_Z^2 w + h \partial_Z p = 0.
 \end{aligned}$$

Finally, the free-divergence condition (corresponding to the last equation in system (1)) in the new variables reads as

$$(4) \quad h \operatorname{div}_x u + \frac{1}{\varepsilon^2} h \operatorname{div}_X u - \nabla h \cdot Z \partial_Z u + \partial_Z w = 0.$$

In the system described by (2), (3), and (4), the domain does not depend on ε : it can be written as

$$\Omega \times \mathbb{R}^2 = \{(x, X, Z) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}, \quad x \in \omega, \quad 0 < Z < 1\}.$$

2.2. Case without rugosity. Here we consider the classical case where the oscillations are not taken into account. Although the asymptotic analysis in this case is well known, we briefly present the arguments and the result. We hope that this will help the reader to compare the results with and without rugosity. In this case the unknowns do not depend on the fast variable X . The height h of the domain satisfies

$$h(x) = \varepsilon h_1(x), \quad \nabla h(x) = \varepsilon \nabla_x h_1(x), \quad \text{and} \quad \Delta h(x) = \varepsilon \Delta_x h_1(x).$$

2.2.1. Equations related to a thin domain. Equation (2) can be rewritten as

$$\begin{aligned}
 (5) \quad & -\eta \varepsilon^2 h_1^2 \Delta_x u + \eta \varepsilon^2 h_1 \Delta_x h_1 Z \partial_Z u - \eta \varepsilon^2 |\nabla_x h_1|^2 Z \partial_Z u \\
 & + 2\eta \varepsilon^2 h_1 \nabla_x h_1 \cdot Z \nabla_x \partial_Z u - \eta \varepsilon^2 |\nabla_x h_1|^2 Z^2 \partial_Z^2 u - \eta \partial_Z^2 u \\
 & + \varepsilon^2 h_1^2 \nabla_x p - \varepsilon^2 h_1 \nabla_x h_1 Z \partial_Z p = 0.
 \end{aligned}$$

In the same way, (3) on the third component of the velocity reads as

$$\begin{aligned}
 (6) \quad & -\eta \varepsilon^2 h_1^2 \Delta_x w + \eta \varepsilon^2 h_1 \Delta_x h_1 Z \partial_Z w - \eta \varepsilon^2 |\nabla_x h_1|^2 Z \partial_Z w \\
 & + 2\eta \varepsilon^2 h_1 \nabla_x h_1 \cdot Z \nabla_x \partial_Z w - \eta \varepsilon^2 |\nabla_x h_1|^2 Z^2 \partial_Z^2 w - \eta \partial_Z^2 w + \varepsilon h_1 \partial_Z p = 0.
 \end{aligned}$$

Finally, the free-divergence condition (4) reads in the new variables as

$$(7) \quad \varepsilon h_1 \operatorname{div}_x u - \varepsilon \nabla_x h_1 \cdot Z \partial_Z u + \partial_Z w = 0.$$

Formally, when ε goes to zero, assuming that the pressure is of order $1/\varepsilon^2$ (see [3]), one constructs an expansion of the unknowns u , w , and p in the following form:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \quad w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots, \quad p = \frac{1}{\varepsilon^2} p_0 + \frac{1}{\varepsilon} p_1 + p_2 + \dots.$$

The main order terms in (5) and (6) are

$$(8) \quad \begin{cases} -\eta \partial_Z^2 u_0 + h_1^2 \nabla_x p_0 = 0, \\ \partial_Z p_0 = 0. \end{cases}$$

2.2.2. The standard Reynolds equations. System (8), coupled with the boundary conditions $u_0|_{Z=0} = u_b$ and $u_0|_{Z=1} = 0$, allows us to express u_0 with respect to p_0 :

$$(9) \quad u_0 = \frac{h_1^2}{2\eta} Z(Z-1)\nabla_x p_0 + (1-Z)u_b.$$

Next, we use (7) on a conservative form, namely

$$\varepsilon (\operatorname{div}_x(h_1 u) - \partial_Z(\nabla_x h_1 \cdot Zu)) + \partial_Z w = 0.$$

Using the boundary condition for u and w , we deduce that $\int_0^1 \operatorname{div}_x(h_1 u) dZ = 0$. With expression (9), we obtain the classical Reynolds equation:

$$(10) \quad \operatorname{div}_x \left(\frac{h_1^3}{12\eta} \nabla_x p_0 \right) = \operatorname{div}_x \left(\frac{h_1}{2} u_b \right).$$

Concerning the lateral boundary conditions for the velocity (Neumann-type conditions and Dirichlet-type conditions; see the introduction), they become conditions (through (9)) on the pressure (Dirichlet-type conditions and Neumann-type conditions, respectively)

$$\left. \frac{\partial p_0}{\partial \mathbf{n}} \right|_{\partial \omega^q} = q_\ell \quad \text{and} \quad p_0|_{\partial \omega^p} = 0,$$

the flux q_ℓ being given with respect to the Z -average of u_b . Note that the compatibility condition on the total flux in the case $\partial \omega^p = \emptyset$ reads as $\int_{\partial \omega} q_\ell = 0$.

2.3. Rugosity case.

2.3.1. Effect of rugosities on the flow at main order. As in the previous case (without rugosity), we consider a characteristic pressure of order $1/\varepsilon^2$ and velocity of order 1:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \quad w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots, \quad p = \frac{1}{\varepsilon^2} p_0 + \frac{1}{\varepsilon} p_1 + p_2 + \dots.$$

This choice comes naturally from the estimates derived for $(u^\varepsilon, w^\varepsilon, p^\varepsilon)$; see Lemmas 4.1 and 4.2 below.

We use the following approach. We formally replace (u, w, p) by its asymptotic expansion in (2), (3), and (4). We determine the profiles (u_i, w_i, p_i) by identifying all terms of the same order with respect to ε . We show that (8) is modified by the roughness profile, which adds an extra term in the horizontal part of the momentum equation.

• *Fast variable independence for the main velocity and pressures $u_0, w_0,$ and p_0 .* We obtained the main terms of (2), (3), and (4) (that is, of order $1/\varepsilon^2$ in (2), order $1/\varepsilon^2$ in (3), and order $1/\varepsilon$ in (4)). Notice that $h = \mathcal{O}(\varepsilon)$, $\nabla h = \mathcal{O}(1)$, and $\Delta h = \mathcal{O}(1/\varepsilon^2)$, which gives

$$\begin{cases} -\eta h_1^2 \Delta_X u_0 + h_1^2 \nabla_X p_0 = 0, \\ -\eta h_1^2 \Delta_X w_0 = 0, \\ h_1 \operatorname{div}_X u_0 = 0. \end{cases}$$

We then obtain a Stokes system with respect to the variable X for (u_0, p_0) and a Laplace equation in X for the unknown w_0 . We immediately find (see the boundary conditions with respect to X) that

$$\nabla_X u_0 = 0, \quad \nabla_X p_0 = 0, \quad \text{and} \quad \nabla_X w_0 = 0.$$

• *Relation between the main order and first order horizontal velocity components u_0 and u_1 .* Writing the next expansion order for (2), (3), and (4) (order $1/\varepsilon$ for (2), order $1/\varepsilon$ for (3), and order 1 for (4)), we get

$$(11) \quad \begin{cases} -\eta h_1^2 \Delta_X u_1 + h_1^2 \nabla_X p_1 + \eta h_1 \Delta_X h_2 Z \partial_Z u_0 - h_1 \nabla_X h_2 \cdot Z \partial_Z p_0 = 0, \\ -\eta h_1^2 \Delta_X w_1 + \eta h_1 \Delta_X h_2 Z \partial_Z w_0 + h_1 \partial_Z p_0 = 0, \\ h_1 \operatorname{div}_X u_1 - \nabla_X h_2 \cdot Z \partial_Z u_0 + \partial_Z w_0 = 0. \end{cases}$$

We compute the mean value in X of the last equation. Since h_1 , u_0 , and w_0 do not depend on X , we find $\partial_Z w_0 = 0$. Using the boundary conditions for the horizontal velocity at the top and the bottom of the domain, we deduce that

$$w_0 = 0.$$

Then the free-divergence condition written at order ε gives

$$(12) \quad h_1 \operatorname{div}_X u_1 = \nabla_X h_2 \cdot Z \partial_Z u_0.$$

Taking the mean value in X of the second equation of (11), we conclude that

$$\partial_Z p_0 = 0 \quad \text{and next} \quad \nabla_X w_1 = 0.$$

Taking the curl_X operator of the horizontal component in the previous system (11), we get

$$h_1 \operatorname{curl}_X \Delta_X u_1 = \nabla_X^\perp \Delta_X h_2 \cdot Z \partial_Z u_0;$$

that is, since h_1 and u_0 do not depend on X ,

$$(13) \quad h_1 \operatorname{curl}_X u_1 = \nabla_X^\perp h_2 \cdot Z \partial_Z u_0.$$

We have $\Delta_X u_1 = \nabla_X \operatorname{div}_X u_1 - \nabla_X^\perp \operatorname{curl}_X u_1$ and $\nabla_X \nabla_X h_2 - \nabla_X^\perp \nabla_X^\perp h_2 = (\Delta_X h_2) \operatorname{Id}$; thus, using the $\operatorname{div}_X u_1$ and $\operatorname{curl}_X u_1$ expressions (12) and (13), we obtain

$$h_1^2 \Delta_X u_1 = h_1 \Delta_X h_2 Z \partial_Z u_0.$$

Hence, the first equation of the system (11) can be written as

$$\nabla_X p_1 = 0.$$

• *Dynamics for the main order of the horizontal component u_0 .* Start to write the terms of the order ε in (4):

$$(14) \quad h_1 \operatorname{div}_x u_0 + h_1 \operatorname{div}_X u_2 + h_2 \operatorname{div}_X u_1 - \nabla_x h_1 \cdot Z \partial_Z u_0 - \nabla_X h_2 \cdot Z \partial_Z u_1 + \partial_Z w_1 = 0.$$

We compute the mean value in X of each term of this equation. Because the function h_1 depends only on x and all the unknowns are periodic in X , we have

$$\int_{\mathbb{T}^{d-1}} h_1 \operatorname{div}_X u_2 dX = 0.$$

Using (12), we have

$$\begin{aligned} \int_{\mathbb{T}^{d-1}} h_2 \operatorname{div}_X u_1 dX &= \int_{\mathbb{T}^{d-1}} \frac{h_2}{h_1} \nabla_X h_2 \cdot Z \partial_Z u_0 dX = \frac{1}{h_1} \left(\int_{\mathbb{T}^{d-1}} h_2 \nabla_X h_2 dX \right) \cdot Z \partial_Z u_0 \\ &= \frac{1}{2h_1} \left(\int_{\mathbb{T}^{d-1}} \nabla_X (h_2^2) dX \right) \cdot Z \partial_Z u_0 = 0. \end{aligned}$$

Integrating by parts the term $\nabla_X h_2 \cdot Z \partial_Z u_1$, we get

$$\int_{\mathbb{T}^{d-1}} \nabla_X h_2 \cdot Z \partial_Z u_1 dX = - \int_{\mathbb{T}^{d-1}} h_2 Z \operatorname{div}_X \partial_Z u_1 dX.$$

Using again (12), we find

$$\int_{\mathbb{T}^{d-1}} \nabla_X h_2 \cdot Z \partial_Z u_1 dX = - \int_{\mathbb{T}^{d-1}} \frac{h_2}{h_1} \nabla_X h_2 \cdot \partial_Z u_0 dX - \int_{\mathbb{T}^{d-1}} \frac{h_2}{h_1} \nabla_X h_2 \cdot Z \partial_Z^2 u_0 dX.$$

As before, only the coefficients h_2 and $\nabla_X h_2$ depend on X , and h_2 is X -periodic. We conclude that

$$\int_{\mathbb{T}^{d-1}} \nabla_X h_2 \cdot Z \partial_Z u_1 dX = 0.$$

The other terms in (14) do not depend on X . We find

$$h_1 \operatorname{div}_x u_0 - \nabla_x h_1 \cdot Z \partial_Z u_0 + \partial_Z w_1 = 0,$$

which is, in conservative form,

$$\operatorname{div}_x (h_1 u_0) + \partial_Z (w_1 - \nabla_x h_1 \cdot Z u_0) = 0.$$

Now we look at (3) at order 1. Since $w_0 = 0$, it gives

$$-\eta h_1^2 \Delta_X w_2 + \eta h_1 \Delta_X h_2 Z \partial_Z w_1 + h_1 \partial_Z p_1 = 0.$$

Taking the mean value in X , since w_1 does not depend on X , except of the last term, all the terms are equal to zero. We find

$$\partial_Z p_1 = 0.$$

Now, writing (2) at order 1 we get

$$\begin{aligned} &-\eta h_1^2 \Delta_X u_2 - 2\eta h_1 h_2 \Delta_X u_1 + 2\eta h_1 \nabla_X h_2 \cdot Z \nabla_X \partial_Z u_1 + \eta h_1 \Delta_X h_2 Z \partial_Z u_1 \\ &+ \eta h_2 \Delta_X h_2 Z \partial_Z u_0 - \eta |\nabla_X h_2|^2 Z \partial_Z u_0 - \eta |\nabla_X h_2|^2 Z^2 \partial_Z^2 u_0 \\ &- \eta \partial_Z^2 u_0 + h_1^2 \nabla_x p_0 + h_1^2 \nabla_X p_2 = 0. \end{aligned}$$

We take the mean value in X of each term of this equation. The first average is zero since h_1 does not depend on X . For the second one, denoted by J , we use again the decomposition $\Delta_X u_1 = \nabla_X \operatorname{div}_X u_1 - \nabla_X^\perp \operatorname{curl}_X u_1$ and then integrate by parts:

$$\begin{aligned} J &= - \int_{\mathbb{T}^{d-1}} 2\eta h_1 h_2 \nabla_X \operatorname{div}_X u_1 dX + \int_{\mathbb{T}^{d-1}} 2\eta h_1 h_2 \nabla_X^\perp \operatorname{curl}_X u_1 dX \\ &= \int_{\mathbb{T}^{d-1}} 2\eta h_1 \operatorname{div}_X u_1 \nabla_X h_2 dX - \int_{\mathbb{T}^{d-1}} 2\eta h_1 \operatorname{curl}_X u_1 \nabla_X^\perp h_2 dX. \end{aligned}$$

Using relations (12) and (13), we obtain

$$\begin{aligned} J &= \int_{\mathbb{T}^{d-1}} 2\eta \left((\nabla_X h_2 \cdot Z \partial_Z u_0) \nabla_X h_2 - (\nabla_X^\perp h_2 \cdot Z \partial_Z u_0) \nabla_X^\perp h_2 \right) dX \\ &= \int_{\mathbb{T}^{d-1}} 2\eta |\nabla_X h_2|^2 Z \partial_Z u_0 dX \\ &= 2\eta M Z \partial_Z u_0, \end{aligned}$$

where the quantity M is defined by

$$M = \int_{\mathbb{T}^{d-1}} |\nabla_X h_2|^2 dX.$$

All the other terms can be expressed with the value of J . For the third term an integration by parts gives

$$\int_{\mathbb{T}^{d-1}} 2\eta h_1 \nabla_X h_2 \cdot Z \nabla_X \partial_Z u_1 dX = Z \partial_Z J = 2\eta M Z \partial_Z u_0 + 2\eta M Z^2 \partial_Z^2 u_0.$$

Using again integrations by parts, the next terms can be written as easily as the preceding one. Adding all contributions, we finally deduce

$$-\eta \partial_Z^2 u_0 + h_1^2 \nabla_x p_0 + \eta M Z \partial_Z u_0 = 0.$$

• *Conclusion of the formal development.* We have stated that the principal terms of expansion, that is, u_0 , w_1 , and p_0 , do not depend on X but do satisfy

$$(15) \quad \begin{cases} -\eta \partial_Z^2 u_0 + h_1^2 \nabla_x p_0 + \eta M Z \partial_Z u_0 = 0, \\ \partial_Z p_0 = 0, \\ \operatorname{div}_x (h_1 u_0) + \partial_Z (w_1 - Z \nabla_x h_1 \cdot u_0) = 0. \end{cases}$$

Note that such an extra term (here the term $\eta M Z \partial_Z u_0$) can be found in other homogenization processes such as porous media; see, for instance, [7]. In the case introduced here, the additional term comes expressly from rugosities. Its “presence” will be rigorously proved in the last section.

2.3.2. Modified Reynolds equation. System (15) corresponds to a “modified Reynolds” system. More precisely, since p_0 does not depend on Z , we can explicitly obtain the velocity u_0 with respect to the pressure p_0 . Let $U = \partial_Z u_0$. The first equation of (15) is

$$\eta \partial_Z U - \eta M Z U = h_1^2 \nabla_x p_0.$$

We integrate this differential equation in the variable Z (for each fixed x) and find

$$U(x, Z) = \frac{h_1(x)^2}{\eta} \left(\int_0^Z e^{M(Z-s)^2/2} ds \right) \nabla_x p_0(x) + e^{M Z^2/2} C_1(x), \quad C_1(x) \in \mathbb{R}^2.$$

Integrating again with respect to Z , we get

$$u_0(x, Z) = u_b + \int_0^Z U(x, t) dt.$$

Using the boundary condition $u_0(x, 1) = 0$, we deduce the value of the constant $C_1(x)$:

$$C_1(x) = - \left(u_b + \frac{h_1(x)^2}{\eta} \left(\int_0^1 \int_0^s e^{M(s^2-t^2)/2} dt ds \right) \nabla_x p_0(x) \right) / \left(\int_0^1 e^{Ms^2/2} ds \right).$$

Finally, we obtain the velocity as a function of the pressure

(16)

$$\begin{aligned} u_0(x, Z) &= \left(\int_0^Z \int_0^s e^{M(s^2-t^2)/2} dt ds - \int_0^1 \int_0^s e^{M(s^2-t^2)/2} dt ds \frac{\int_0^Z e^{Ms^2/2} ds}{\int_0^1 e^{Ms^2/2} ds} \right) \frac{h_1(x)^2}{\eta} \nabla_x p_0(x) \\ &\quad + \left(1 - \frac{\int_0^Z e^{Ms^2/2} ds}{\int_0^1 e^{Ms^2/2} ds} \right) u_b. \end{aligned}$$

Next, integrating the last equation in (15) with respect to Z , that is, the free-divergence condition

$$\operatorname{div}_x(h_1 u_0) + \partial_Z(w_1 - Z \nabla_x h_1 \cdot u_0) = 0,$$

and taking into account that the velocity $w_1 - Z \nabla_x h_1 \cdot u_0$ cancels for $Z = 0$ and for $Z = 1$, we obtain

$$\operatorname{div}_x \left(\int_0^1 h_1 u_0 dZ \right) = 0.$$

With the previous expression (16) for the velocity the following pressure equation is obtained:

$$\operatorname{div}_x \left(\frac{A h_1^3}{\eta} \nabla_x p_0 \right) = \operatorname{div}_x (B h_1 u_b),$$

where A and B are two constants defined by

$$\begin{aligned} A &= \frac{1}{M} \left(e^{M/2} \int_0^1 e^{-Mt^2/2} dt - 1 \right) - \frac{1}{M} (e^{M/2} - 1) \frac{\int_0^1 \int_0^s e^{M(s^2-t^2)/2} dt ds}{\int_0^1 e^{Ms^2/2} ds}, \\ B &= \frac{1}{M} (e^{M/2} - 1) \frac{1}{\int_0^1 e^{Ms^2/2} ds}. \end{aligned}$$

Since A and B do not depend on x , this equation can be rewritten as

(17)
$$\operatorname{div}_x \left(\frac{h_1^3}{12\eta} \nabla_x p_0 \right) = \operatorname{div}_x \left(\frac{h_1}{2} K u_b \right),$$

where K is a coefficient depending on the fluid viscosity fluid η and rugosity (more exactly, on the coefficient M): $K = B/(6A)$. Concerning the boundary conditions, they are the same as in the case without rugosity, namely

$$\left. \frac{\partial p_0}{\partial \mathbf{n}} \right|_{\partial \omega^q} = q_\ell \quad \text{and} \quad p_0|_{\partial \omega^p} = 0.$$

3. Numerical simulations. Let us remark that system (15) is energetically consistent for M small enough. We do the calculation assuming homogeneous Dirichlet conditions on the bottom and the top but adding an exterior force f on the right-hand side of the horizontal component of the momentum equation. Multiplying the horizontal component by u_0/h_1 and integrating over the domain, we find, using that $\operatorname{div}(h_1 u_0) = \partial_Z(w_1 - Z \nabla_x h_1 \cdot u_0)$ and $\partial_Z p_0 = 0$,

$$\int_{\Omega} \frac{\eta}{h_1} |\partial_Z u_0|^2 - \frac{\eta M}{2} \int_{\Omega} \frac{1}{h_1} |u_0|^2 = \int_{\Omega} \frac{f}{h_1} u_0.$$

Therefore, using Poincaré’s inequality over the vertical component $\int_0^1 |u_0|^2 \leq \int_0^1 |\partial_Z u_0|^2$ we find an energetically consistent model assuming $1 - M/2 > 0$, namely $M < 2$. To better understand the effect of the fast variable dependency of h^ε , we expand the expressions of A, B assuming M to be small enough, and we find

$$A = \frac{1}{12} + \frac{M}{240} + \mathcal{O}(M^2), \quad B = \frac{1}{2} + \frac{M}{24} + \mathcal{O}(M^2).$$

This allows us to define $K = B/(6A)$, for which we obtain the following expansion:

$$K = 1 + \frac{M}{30} + \mathcal{O}(M^2).$$

For the sake of completeness we also draw K with respect to M (see Figure 2). Now we provide various coefficients M depending on rugosity profiles (see Figure 3) around $h_1(x) = 2x^2 - 2x + 1$. The left figure obviously gives $M = 0$, and the middle one provides $M = 1.23 \cdot 10^{-4}$, $K = 1.0000041$. The right figure, presenting a setting with a more oscillating boundary, gives as values $M = 1.612$ and $K = 1.063$.

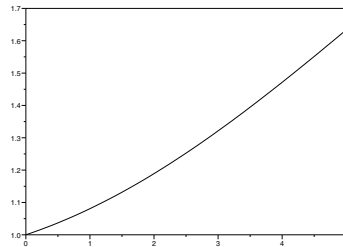


FIG. 2. K with respect to M .

The pressure corresponding to the rough domain (of characteristic coefficient K) with a prescribed velocity on the bottom u_b is linked to the pressure corresponding to the nonscillating domain with a prescribed velocity on the bottom Ku_b . In other words, the oscillations modify the flow at main order. Indeed if we are interested in the case where we impose homogeneous boundary conditions on the lateral sides (for instance, $p_0|_{\partial\omega^p} = 0$ and $\partial p/\partial \mathbf{n}|_{\partial\omega^q} = 0$), then the solution of

$$\operatorname{div}_x \left(\frac{h_1^3}{12\eta} \nabla_x p_{\text{rough}} \right) = \operatorname{div}_x \left(\frac{h_1}{2} K u_b \right)$$

satisfies $p_{\text{rough}} = K p_{\text{smooth}}$, where p_{smooth} is the solution of the Reynolds equation with rugosities:

$$\operatorname{div}_x \left(\frac{h_1^3}{12\eta} \nabla_x p_{\text{smooth}} \right) = \operatorname{div}_x \left(\frac{h_1}{2} u_b \right).$$

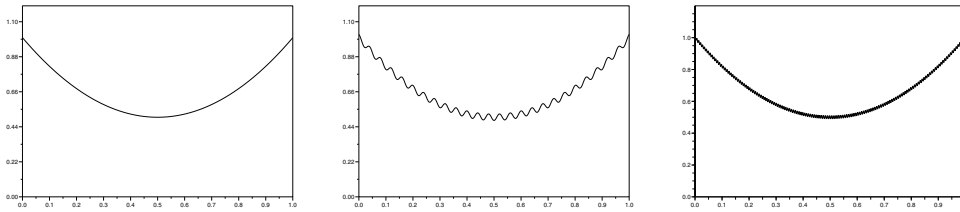


FIG. 3. $h(x) = h_1(x)$ (left), $h(x) = h_1(x) + 0.01 \cos(50\pi x)$ (middle), $h(x) = h_1(x) + 0.01 \cos(40000\pi x/7)$ (right).

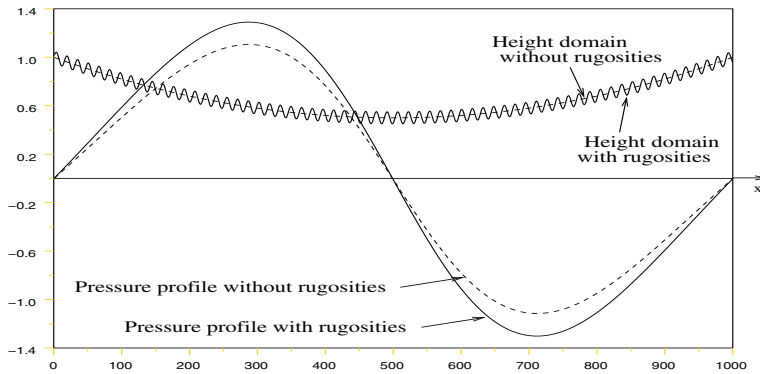


FIG. 4. Pressure in a thin domain with and without rugosities when homogeneous Dirichlet conditions are imposed.

Figures 4 and 5 are numerical results corresponding to such homogeneous conditions in the one-dimensional case. In Figure 4, homogeneous conditions on the pressure, that is, $p|_{x=0} = p|_{x=L} = 0$, are used. Moreover, two kinds of height surfaces are used, the first one corresponding to a smooth surface, $h(x) = \varepsilon h_1(x)$ with $h_1(x) = 2(x/L)^2 - 2(x/L) + 1$, and the second taking into account rugosities: $h(x) = \varepsilon h_1(x) + \varepsilon^2 h_2(x/\varepsilon^2)$ with $h_2(X) = \sin(X)$. In Figure 5, the left imposed lateral condition is a homogeneous flux condition $p'|_{x=0} = 0$, whereas the right lateral condition is a homogeneous pressure condition $p|_{x=L} = 0$.

An example with nonhomogeneous boundary conditions is given in Figure 6. This numerical test corresponds to an imposed flux $p'|_{x=0} = 2$ with an imposed pressure $p|_{x=L} = 0$.

4. Mathematical justification. We introduce the variable Z defined by

$$Z = \frac{z}{h^\varepsilon(x)} = \frac{z}{\varepsilon(h_1(x) + \varepsilon h_2(\frac{x}{\varepsilon^2}))}$$

to consider problem (1) in the fixed domain $\Omega = \omega \times (0, 1)$. According to this change of variables, we look at the unknown $\mathbf{u}^\varepsilon = (u^\varepsilon, w^\varepsilon)$ and p^ε as

$$u^\varepsilon(x, z) = \widetilde{u}^\varepsilon(x, Z), \quad w^\varepsilon(x, z) = \widetilde{w}^\varepsilon(x, Z), \quad \text{and} \quad p^\varepsilon(x, z) = \widetilde{p}^\varepsilon(x, Z).$$

For the sake of simplicity, we omit the tilde in the new unknowns, and we focus on standard Dirichlet boundary conditions $\mathbf{u}^\varepsilon = \mathbf{u}_b$ on $\partial\Omega$. Problem (1) reduces to the

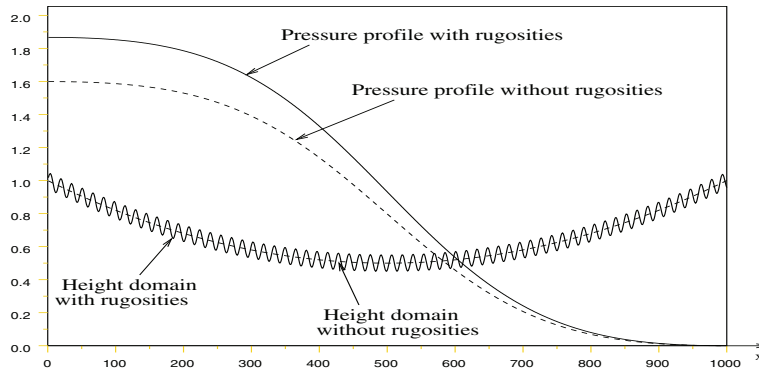


FIG. 5. Pressure in a thin domain with and without rugosities when homogeneous Neumann–Dirichlet conditions are imposed.

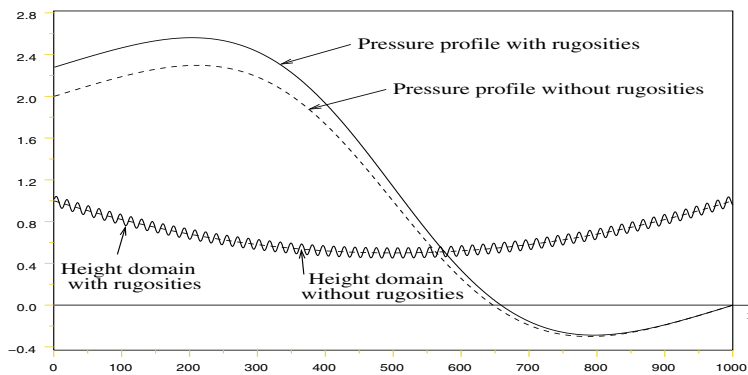


FIG. 6. Pressure in a thin domain with and without rugosities when the flux is imposed on the right.

following equations in Ω :

$$\begin{aligned}
 (18) \quad & -\eta \Delta_x u^\varepsilon + \eta \left(\frac{\Delta_x h^\varepsilon}{h^\varepsilon} - \frac{|\nabla_x h^\varepsilon|^2}{h^{\varepsilon 2}} \right) Z \partial_Z u^\varepsilon + 2\eta \frac{\nabla_x h^\varepsilon}{h^\varepsilon} \cdot Z \nabla_x \partial_Z u^\varepsilon \\
 & - \eta \frac{|\nabla_x h^\varepsilon|^2}{h^{\varepsilon 2}} Z^2 \partial_Z^2 u^\varepsilon - \eta \frac{1}{h^{\varepsilon 2}} \partial_Z^2 u^\varepsilon + \nabla_x p^\varepsilon - \frac{\nabla_x h^\varepsilon}{h^\varepsilon} Z \partial_Z p^\varepsilon = 0,
 \end{aligned}$$

$$\begin{aligned}
 (19) \quad & -\eta \Delta_x w^\varepsilon + \eta \left(\frac{\Delta_x h^\varepsilon}{h^\varepsilon} - \frac{|\nabla_x h^\varepsilon|^2}{h^{\varepsilon 2}} \right) Z \partial_Z w^\varepsilon + 2\eta \frac{\nabla_x h^\varepsilon}{h^\varepsilon} \cdot Z \nabla_x \partial_Z w^\varepsilon \\
 & - \eta \frac{|\nabla_x h^\varepsilon|^2}{h^{\varepsilon 2}} Z^2 \partial_Z^2 w^\varepsilon - \eta \frac{1}{h^{\varepsilon 2}} \partial_Z^2 w^\varepsilon + \frac{1}{h^\varepsilon} \partial_Z p^\varepsilon = 0,
 \end{aligned}$$

$$(20) \quad \operatorname{div}(u^\varepsilon) - \frac{\nabla_x h^\varepsilon}{h^\varepsilon} \cdot Z \partial_Z u^\varepsilon + \frac{1}{h^\varepsilon} \partial_Z w^\varepsilon = 0.$$

4.1. Estimates. We begin with some estimates for the velocity.

LEMMA 4.1. *The velocity components satisfy the following uniform estimates:*

$$(21) \quad \|\mathbf{u}^\varepsilon\|_{(L^2(\Omega))^d} \leq C,$$

$$(22) \quad \|\nabla \mathbf{u}^\varepsilon\|_{(L^2(\Omega))^{d \times d}} \leq \frac{C}{\varepsilon},$$

$$(23) \quad \|\partial_Z \mathbf{u}^\varepsilon\|_{(L^2(\Omega))^d} \leq C.$$

Proof. These estimates are directly derived from the original problem (1) after an adequate lifting of the nonhomogeneous boundary conditions. We write the standard energy estimate for a Stokes-type system and then use the change of variables to control the derivatives $\nabla_x \mathbf{u}^\varepsilon$ and $\partial_Z \mathbf{u}^\varepsilon$. Next we use the Poincaré inequality to control \mathbf{u}^ε . \square

We now state the following estimates for the pressure.

LEMMA 4.2. *The pressure is such that*

$$(24) \quad \|p^\varepsilon\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon^2},$$

$$(25) \quad \|\nabla_x p^\varepsilon\|_{H^{-1}(\Omega)} \leq \frac{C}{\varepsilon^2},$$

$$(26) \quad \|\partial_Z p^\varepsilon\|_{H^{-1}(\Omega)} \leq \frac{C}{\varepsilon}.$$

Proof. Such estimates come from (19) for $\partial_Z p^\varepsilon$ estimated in H^{-1} . Estimate (25) is deduced from (24). Concerning (24), we use the Bogovskii operator to get estimates for the pressure on the variable domain Ω^ε and then through the change of variables in z the L^2 -estimates for p^ε on Ω . The reader interested in such estimates is referred to [11]. \square

5. Proof of Theorem 1. In this section, we present a proof of Theorem 1 (see section 1.2). This proof is composed of three parts. First, in section 5.1, we use the previous estimates and two-scale convergence arguments to define a convenient limit (u_0, w_0, p_0) of $(u^\varepsilon, w^\varepsilon, \varepsilon^2 p^\varepsilon)$, and we obtain its main properties. Next, we pass to the limit in the divergence equation and in the momentum equation (sections 5.2 and 5.3, respectively), and we deduce that the limit (u_0, p_0) satisfies system (15).

5.1. Convergence results. In view of the previous estimates, we claim the following results.

There exist limit functions $p_0 \in L^2(\Omega; L^2(\mathbb{T}^{d-1}))$, $u_0 \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$, and $w_0 \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$ such that

$$\varepsilon^2 p^\varepsilon \rightharpoonup p_0, \quad u^\varepsilon \rightharpoonup u_0, \quad \text{and} \quad w^\varepsilon \rightharpoonup w_0.$$

Moreover, using Lemma 1.1, we assert the following.

LEMMA 5.1. *The two-scale limit for the velocity is such that $\nabla_X u_0 = 0$ and $\nabla_X w_0 = 0$.*

Before passing to the limit in the equations, we state some auxiliary results. We begin with the pressure function.

LEMMA 5.2. *The two-scale limit pressure is such that $\nabla_X p_0 = 0$ and $\partial_Z p_0 = 0$.*

Proof. Step 1. We first prove that $\nabla_X p_0 = 0$. We multiply (18) by an admissible test function in the form $\varepsilon^4 \phi(x, Z, x/\varepsilon^2) = \varepsilon^4 \phi^\varepsilon$, where $\phi \in \mathcal{D}(\Omega; \mathcal{C}^1(\mathbb{T}^{d-1}))$, and we integrate by parts. We get

$$\begin{aligned} & \int_\Omega \eta \nabla_x u^\varepsilon \cdot (\varepsilon^4 \nabla_x \phi^\varepsilon + \varepsilon^2 \nabla_X \phi^\varepsilon) - \int_\Omega \eta \varepsilon^4 \left(\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon \right) \cdot \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \nabla_x u^\varepsilon \partial_Z (Z \phi^\varepsilon) \\ & - \int_\Omega \eta \varepsilon^4 \left(\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon \right) \cdot \frac{1}{h_1 + \varepsilon h_2^\varepsilon} Z \partial_Z u^\varepsilon \left(\nabla_x \phi^\varepsilon + \frac{1}{\varepsilon^2} \nabla_X \phi^\varepsilon \right) \end{aligned}$$

$$\begin{aligned}
 &+ \int_{\Omega} \eta \varepsilon^4 \frac{|\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon|^2}{|h_1 + \varepsilon h_2^\varepsilon|^2} \partial_Z u^\varepsilon \cdot \partial_Z (Z^2 \phi^\varepsilon) + \int_{\Omega} \eta \varepsilon^4 \frac{1}{|h_1 + \varepsilon h_2^\varepsilon|^2} \partial_Z u^\varepsilon \cdot \partial_Z \phi^\varepsilon \\
 &- \int_{\Omega} \varepsilon^2 p^\varepsilon (\varepsilon^2 \operatorname{div}_x \phi^\varepsilon + \operatorname{div}_X \phi^\varepsilon) + \int_{\Omega} \varepsilon^2 \left(\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon \right) \cdot \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \varepsilon^2 p^\varepsilon \partial_Z (Z \phi^\varepsilon) = 0.
 \end{aligned}$$

Passing to the limit $\varepsilon \rightarrow 0$, we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varepsilon^2 p^\varepsilon \operatorname{div}_X \phi^\varepsilon \, dx dZ = \int_{\Omega} \int_{\mathbb{T}^{d-1}} p_0 \operatorname{div}_X \phi \, dx dZ dX = 0.$$

Thus $\nabla_X p_0 = 0$ and the two-scale limit and the weak L^2 limit of $(\varepsilon^2 p^\varepsilon)$ coincide.

Step 2. Next, we prove that $\partial_Z p_0 = 0$. Let $\phi \in L^2(\omega; H_0^1(0, 1))$. On the one hand, using the H^{-1} estimate on $\partial_Z p^\varepsilon$, we write

$$\int_{\Omega} \partial_Z (\varepsilon^2 p^\varepsilon) \phi = \varepsilon \langle \varepsilon \partial_Z p^\varepsilon, \phi \rangle_{H^{-1} \times H_0^1} \rightarrow 0,$$

and, on the other hand, using the L^2 weak limit on $\varepsilon^2 p^\varepsilon$,

$$\int_{\Omega} \partial_Z (\varepsilon^2 p^\varepsilon) \phi = - \int_{\Omega} \varepsilon^2 p^\varepsilon \partial_Z \phi \rightarrow - \int_{\Omega} p_0 \partial_Z \phi.$$

It follows that $\partial_Z p_0 = 0$. This ends the proof of the lemma. \square

We then study the two-scale limit of the vertical velocity component.

LEMMA 5.3. *The vertical velocity component is such that $w^\varepsilon \xrightarrow{2} 0$.*

Proof. We already mentioned that $\nabla_X w_0 = 0$. We now prove that $\partial_Z w_0 = 0$. We multiply the divergence equation (20) by $\varepsilon \phi(x, Z)$, with $\phi \in \mathcal{D}(\Omega)$. Integrating by parts, we obtain

$$\begin{aligned}
 &- \int_{\Omega} \varepsilon u^\varepsilon \cdot \nabla_x \phi + \int_{\Omega} \frac{\varepsilon}{h_1 + \varepsilon h_2^\varepsilon} \nabla_x h_1 \cdot u^\varepsilon \partial_Z (Z \phi) \\
 (27) \quad &+ \int_{\Omega} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \nabla_X h_2^\varepsilon \cdot u^\varepsilon \partial_Z (Z \phi) - \int_{\Omega} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} w^\varepsilon \partial_Z \phi = 0.
 \end{aligned}$$

We note that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \nabla_X h_2^\varepsilon \cdot u^\varepsilon \partial_Z (Z \phi) = \int_{\Omega} \frac{1}{h_1} \left(\int_{\mathbb{T}^{d-1}} \nabla_X h_2 \right) u_0 \partial_Z (Z \phi) = 0$$

because of the periodicity of h_2 . We thus infer from (27) that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} w^\varepsilon \partial_Z \phi = \int_{\Omega} \frac{1}{h_1} w_0 \partial_Z \phi = 0.$$

It follows that $\partial_Z w_0 = 0$. We conclude using the boundary conditions. \square

5.2. Divergence equation. We multiply the divergence equation (20) by $\phi(x, Z)$ with $\phi \in \mathcal{D}(\Omega)$:

$$\begin{aligned}
 &- \int_{\Omega} u^\varepsilon \cdot \nabla_x \phi - \int_{\Omega} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \nabla_x h_1 \cdot Z \partial_Z u^\varepsilon \phi \\
 &- \int_{\Omega} \frac{1}{\varepsilon} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \nabla_X h_2^\varepsilon \cdot Z \partial_Z u^\varepsilon \phi - \int_{\Omega} \frac{1}{\varepsilon} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} w^\varepsilon \partial_Z \phi = 0.
 \end{aligned}$$

It follows that

$$\int_{\Omega} -u_0 \cdot \nabla_x \phi - \frac{1}{h_1} \nabla_x h_1 \cdot Z \partial_Z u_0 \phi - \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} \frac{1}{\varepsilon} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \nabla_X h_2 \cdot Z \partial_Z u^\varepsilon \phi + \int_{\Omega} \frac{1}{\varepsilon} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} w^\varepsilon \partial_Z \phi \right) = 0.$$

We note that $1/(h_1 + \varepsilon h_2^\varepsilon)$ is only a perturbation of $1/h_1$ for our convergence needs:

$$\begin{aligned} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} - \frac{1}{h_1} &= \frac{-\varepsilon h_2^\varepsilon}{h_1(h_1 + \varepsilon h_2^\varepsilon)}, \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{\varepsilon} \frac{-\varepsilon h_2^\varepsilon}{h_1(h_1 + \varepsilon h_2^\varepsilon)} \nabla_X h_2 \cdot Z \partial_Z u^\varepsilon \phi &= - \int_{\Omega} \left(\int_{\mathbb{T}^{d-1}} h_2 \nabla_X h_2 \right) \cdot \frac{1}{h_1^2} Z \partial_Z u_0 \phi = 0, \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{\varepsilon} \frac{-\varepsilon h_2^\varepsilon}{h_1(h_1 + \varepsilon h_2^\varepsilon)} w^\varepsilon \partial_Z \phi &= - \int_{\Omega} \left(\int_{\mathbb{T}^{d-1}} h_2 \right) \frac{1}{h_1^2} w_0 \partial_Z \phi = 0. \end{aligned}$$

Notice that for this last equality we use the fact that h_2 is assumed to be free-average. We then have

$$(28) \quad \int_{\Omega} -u_0 \cdot \nabla_x \phi - \frac{1}{h_1} \nabla_x h_1 \cdot Z \partial_Z u_0 \phi - \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} \frac{1}{\varepsilon} \frac{1}{h_1} \nabla_X h_2 \cdot Z \partial_Z u^\varepsilon \phi + \int_{\Omega} \frac{1}{\varepsilon} \frac{1}{h_1} w^\varepsilon \partial_Z \phi \right) = 0.$$

We set

$$I_1 = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{\varepsilon} \frac{1}{h_1} \nabla_X h_2 \cdot Z \partial_Z u^\varepsilon \phi \quad \text{and} \quad I_2 = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{\varepsilon} \frac{1}{h_1} w^\varepsilon \partial_Z \phi.$$

Computation of I_1 . We write

$$(29) \quad \begin{aligned} I_1 &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{\varepsilon^2} \frac{1}{h_1} \nabla_X (\varepsilon h_2^\varepsilon) \cdot Z \partial_Z u^\varepsilon \phi \\ &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} \frac{\varepsilon h_2^\varepsilon}{h_1} \operatorname{div}(u^\varepsilon) \partial_Z (Z \phi) + \int_{\Omega} \frac{\varepsilon h_2^\varepsilon}{h_1} u^\varepsilon \cdot \nabla_x (\partial_Z (Z \phi)) \right) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{\varepsilon h_2^\varepsilon}{h_1} \operatorname{div}(u^\varepsilon) \partial_Z (Z \phi) = - \int_{\Omega} \left(\int_{\mathbb{T}^{d-1}} h_2(X) \partial_Z \xi(x, Z, X) dX \right) Z \frac{\phi}{h_1}, \end{aligned}$$

where ξ is the two-scale limit of the sequence $(\varepsilon \operatorname{div}(u^\varepsilon))$ which is bounded in $L^2(\Omega)$. Now we test the divergence equation by $\varepsilon h_2^\varepsilon \phi$, with $\phi \in L^2(\Omega)$. We pass to the limit $\varepsilon \rightarrow 0$. Bearing in mind that $\partial_Z u^\varepsilon$ is uniformly bounded in $L^2(\Omega)$ and $w_0 = 0$, we obtain

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \left(- \int_{\Omega} \varepsilon \operatorname{div}(u^\varepsilon) h_2^\varepsilon \phi + \int_{\Omega} \frac{h_2^\varepsilon}{h_1} \nabla_X h_2^\varepsilon \cdot Z \partial_Z u^\varepsilon \phi \right. \\ &\quad \left. + \int_{\Omega} \varepsilon \frac{h_2^\varepsilon}{h_1} \nabla_x h_1 \cdot Z \partial_Z u^\varepsilon \phi + \int_{\Omega} \frac{h_2^\varepsilon}{h_1 + \varepsilon h_2^\varepsilon} \partial_Z w^\varepsilon \phi \right) \\ &= - \int_{\Omega} \left(\int_{\mathbb{T}^{d-1}} \xi h_2 \right) \phi + \int_{\Omega} \left(\int_{\mathbb{T}^{d-1}} h_2 \nabla_X h_2 \right) \frac{1}{h_1} Z \partial_Z u_0 \phi = - \int_{\Omega} \left(\int_{\mathbb{T}^{d-1}} \xi h_2 \right) \phi. \end{aligned}$$

We infer from the latter relation that

$$\int_{\Omega} \left(\int_{\mathbb{T}^{d-1}} \xi h_2 \right) \phi = 0 \quad \forall \phi \in L^2(\Omega),$$

which gives in (29) the equality $I_1 = 0$, replacing ϕ by ϕ/h_1 .

Computation of I_2 . We write

$$\begin{aligned} \int_{\Omega} \frac{1}{\varepsilon} \frac{1}{h_1} w^\varepsilon \partial_Z \phi &= \int_{\Omega} \frac{1}{\varepsilon^2} \left(\varepsilon w^\varepsilon \frac{1}{h_1} \partial_Z \phi \right) = \int_{\Omega} \operatorname{div} \left(\frac{x}{\varepsilon^2} \right) \left(\varepsilon w^\varepsilon \frac{1}{h_1} \partial_Z \phi \right) \\ &= - \int_{\Omega} \left(\frac{x}{\varepsilon^2} \right) \cdot \nabla \left(\varepsilon w^\varepsilon \frac{1}{h_1} \partial_Z \phi \right) \\ &\rightarrow - \int_{\Omega} \left(\int_{\mathbb{T}^{d-1}} X \cdot \nabla_X \eta_1 \, dX \right) \frac{1}{h_1} \partial_Z \phi = \int_{\Omega} \left(\int_{\mathbb{T}^{d-1}} \eta_1 \right) \frac{1}{h_1} \partial_Z \phi, \end{aligned}$$

where $\eta_1 \in L^2(\Omega, H^1(\mathbb{T}^{d-1}))$ is defined by

$$\begin{cases} \varepsilon w^\varepsilon \xrightarrow{2} 0, \\ \nabla(\varepsilon w^\varepsilon) \xrightarrow{2} \nabla_x 0 + \nabla_X \eta_1. \end{cases}$$

We set

$$w^1 = \int_{\mathbb{T}^{d-1}} \eta_1 \, dX.$$

Then

$$I_2 = - \int_{\Omega} \frac{\partial_Z w^1}{h_1} \phi.$$

Relation (28) is thus the weak formulation corresponding to

$$h_1 \operatorname{div}_x(u_0) - \nabla_x h_1 \cdot Z \partial_Z u_0 + \partial_Z w^1 = 0;$$

thus the conservative form exactly corresponds to the last equation of (15) obtained in a formal way.

5.3. Momentum equation. Once again, we begin with some auxiliary results. In view of the estimates derived for u^ε , we can define the following ‘‘anisotropic’’ two-scale limit.

LEMMA 5.4. *There exists $u^1 \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$ such that $\nabla_x(\varepsilon u^\varepsilon) \xrightarrow{2} \nabla_X u^1$.*

LEMMA 5.5. *The function u^1 is such that $\Delta_X u^1 = \frac{\Delta_X h_2}{h_1} Z \partial_Z u_0$.*

Proof. On the one hand, we multiply the divergence equation by $\varepsilon \phi(x, Z, x/\varepsilon^2)$ with $\phi \in \mathcal{D}(\Omega; \mathcal{C}^1(\mathbb{T}^{d-1}))$. Integrating by parts, we obtain

$$\begin{aligned} \int_{\Omega} \varepsilon \operatorname{div}(u^\varepsilon) \phi^\varepsilon - \int_{\Omega} \nabla_X h_2^\varepsilon \cdot Z \partial_Z u^\varepsilon \frac{\phi^\varepsilon}{h_1 + \varepsilon h_2^\varepsilon} - \int_{\Omega} \varepsilon \nabla_x h_1 \cdot Z \partial_Z u^\varepsilon \frac{\phi^\varepsilon}{h_1 + \varepsilon h_2^\varepsilon} \\ + \int_{\Omega} \partial_Z w^\varepsilon \frac{\phi^\varepsilon}{h_1 + \varepsilon h_2^\varepsilon} = 0. \end{aligned}$$

We recall that $\varepsilon \partial_Z u^\varepsilon \xrightarrow{2} 0$ and $w^\varepsilon \xrightarrow{2} 0$. Passing to the limit in the latter relation, we get

$$\int_{\Omega} \int_{\mathbb{T}^{d-1}} \left(\operatorname{div}_X(u^1) - \frac{1}{h_1} \nabla_X h_2 \cdot Z \partial_Z u_0 \right) \phi \, dX \, dx \, dZ = 0,$$

that is,

$$(30) \quad \operatorname{div}_X(u^1) = \frac{1}{h_1} \nabla_X h_2 \cdot Z \partial_Z u_0.$$

On the other hand, we multiply (18) by $\varepsilon^3\phi(x, Z, x/\varepsilon^2)$, and we integrate by parts. We obtain

$$\begin{aligned}
 & \int_{\Omega} \eta \varepsilon \nabla u^\varepsilon \cdot (\varepsilon^2 \nabla_x \phi^\varepsilon + \nabla_X \phi^\varepsilon) + \int_{\Omega} \eta \varepsilon^3 \frac{\Delta_X h_1 + \frac{1}{\varepsilon^3} \Delta_X h_2^\varepsilon}{h_1 + \varepsilon h_2^\varepsilon} Z \partial_Z u^\varepsilon \cdot \phi^\varepsilon \\
 & - \int_{\Omega} \eta \varepsilon^2 \frac{|\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon|^2}{(h_1 + \varepsilon h_2^\varepsilon)^2} \varepsilon Z \partial_Z u^\varepsilon \cdot \phi^\varepsilon \\
 & - 2 \int_{\Omega} \frac{\eta \varepsilon^2}{h_1 + \varepsilon h_2^\varepsilon} \left(\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon \right) \cdot \varepsilon \nabla u^\varepsilon \partial_Z (Z \phi^\varepsilon) \\
 & + \int_{\Omega} \eta \varepsilon^2 \frac{|\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon|^2}{(h_1 + \varepsilon h_2^\varepsilon)^2} \varepsilon \partial_Z u^\varepsilon \cdot \partial_Z (Z^2 \phi^\varepsilon) + \int_{\Omega} \frac{\eta}{(h_1 + \varepsilon h_2^\varepsilon)^2} \varepsilon^3 \partial_Z u^\varepsilon \cdot \partial_Z \phi^\varepsilon \\
 & - \int_{\Omega} \varepsilon^3 p^\varepsilon \left(\operatorname{div}_x (\phi^\varepsilon) + \frac{1}{\varepsilon^2} \operatorname{div}_X (\phi_\varepsilon) \right) \\
 (31) \quad & + \int_{\Omega} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \varepsilon^2 p^\varepsilon \varepsilon \left(\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon \right) \cdot \partial_Z (Z \phi^\varepsilon) = 0.
 \end{aligned}$$

We choose ϕ in the form $\phi = \operatorname{curl}_X \psi$ to cancel the term containing $\operatorname{div}_X (\phi^\varepsilon)$. Passing to the limit in the other terms and using $\partial_Z p_0 = 0$, we get

$$\int_{\Omega} \int_{\mathbb{T}^{d-1}} \nabla_X u^1 \cdot \nabla_X \phi + \int_{\Omega} \int_{\mathbb{T}^{d-1}} \frac{\Delta_X h_2}{h_1} Z \partial_Z u_0 \phi = 0$$

for any $\phi = \operatorname{curl}_X \psi$. It follows that

$$\operatorname{curl}_X (\Delta_X u^1) = \frac{1}{h_1} \nabla_X^\perp (\Delta_X h_2) \cdot Z \partial_Z u_0$$

and then

$$(32) \quad \operatorname{curl}_X u^1 = \frac{1}{h_1} \nabla_X^\perp (h_2) \cdot Z \partial_Z u_0.$$

We infer the result of the lemma from (30)–(32) using the formula $\Delta_X u_1 = \nabla_X \operatorname{div}_X u_1 - \nabla_X^\perp \operatorname{curl}_X u_1$ and $\nabla_X \nabla_X h_2 - \nabla_X^\perp \nabla_X^\perp h_2 = (\Delta_X h_2) \operatorname{Id}$. \square

LEMMA 5.6. *The pressure is such that $\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varepsilon p^\varepsilon \operatorname{div}_X (\phi) = 0$ for any $\phi \in L^2(\Omega; H^1(\mathbb{T}^{d-1}))$.*

Proof. Going back to relation (31), passing to the limit $\varepsilon \rightarrow 0$, and then using the previous lemma, the result is obvious. \square

We now have sufficient tools to pass to the limit in the momentum equation. We multiply (18) by $\varepsilon^2 \phi(x, Z)$ with $\phi \in \mathcal{D}(\Omega)$. Since $\Delta h/h - |\nabla h|^2/h^2 = \operatorname{div}(\nabla h/h)$, we obtain

$$\begin{aligned}
 & \int_{\Omega} \eta \varepsilon^2 \nabla u^\varepsilon \cdot \nabla_x \phi^\varepsilon - \int_{\Omega} \frac{\eta \varepsilon}{h_1 + \varepsilon h_2^\varepsilon} \left(\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2 \right) \cdot \varepsilon \nabla u^\varepsilon \partial_Z (Z \phi) \\
 & + \int_{\Omega} \eta \varepsilon^2 \frac{|\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon|^2}{(h_1 + \varepsilon h_2^\varepsilon)^2} \partial_Z u^\varepsilon \cdot \partial_Z (Z^2 \phi) + \int_{\Omega} \frac{\eta}{(h_1 + \varepsilon h_2^\varepsilon)^2} \partial_Z u^\varepsilon \cdot \partial_Z \phi \\
 (33) \quad & - \int_{\Omega} \varepsilon^2 p^\varepsilon \operatorname{div}_x (\phi) + \int_{\Omega} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \varepsilon^2 p^\varepsilon \left(\nabla_x h_1 + \frac{1}{\varepsilon} \nabla_X h_2^\varepsilon \right) \cdot \partial_Z (Z \phi) = 0.
 \end{aligned}$$

With Lemma 5.6, we ensure that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{h_1 + \varepsilon h_2^\varepsilon} \varepsilon p^\varepsilon \nabla_X h_2^\varepsilon \cdot \partial_Z (Z \phi) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varepsilon p^\varepsilon \operatorname{div}_X \left(\frac{h_2^\varepsilon}{h_1} \partial_Z (Z \phi) \right) = 0.$$

Passing to the limit in (33), we get, using that $u_0 = u_0(x, Z)$ and $\partial_Z p_0 = 0$,

$$\begin{aligned} & -\eta \int_{\Omega} \int_{\mathbb{T}^{d-1}} \frac{1}{h_1} \nabla_X h_2 \cdot \nabla_X u_1 \partial_Z(Z\phi) + \eta \int_{\Omega} \int_{\mathbb{T}^{d-1}} \frac{|\nabla_X h_2|^2}{(h_1)^2} \partial_Z u_0 \partial_Z(Z^2\phi) \\ & + \eta \int_{\Omega} \int_{\mathbb{T}^{d-1}} \frac{1}{(h_1)^2} \partial_Z u_0 \cdot \partial_Z \phi - \int_{\Omega} \int_{\mathbb{T}^{d-1}} p_0 \operatorname{div}_x \phi = 0. \end{aligned}$$

Using Lemma 5.5, one easily checks that the former relation is the energy formulation corresponding to

$$\begin{aligned} & -\eta \left(\int_{\mathbb{T}^{d-1}} h_2 \Delta_X h_2 \right) \frac{1}{h_1} Z \partial_Z(Z \partial_Z u_0) - \eta \left(\int_{\mathbb{T}^{d-1}} |\nabla_X h_2|^2 \right) \frac{1}{h_1^2} Z^2 \partial_Z(\partial_Z u_0) \\ & - \frac{\eta}{h_1^2} \partial_{ZZ}^2 u_0 + \nabla_x p_0 = 0. \end{aligned}$$

Integrating by parts the integral terms, one finally gets

$$-\frac{\eta}{h_1^2} \partial_{ZZ}^2 u_0 + \nabla_x p_0 + \frac{\eta}{h_1^2} \left(\int_{\mathbb{T}^{d-1}} |\nabla_X h_2|^2 dX \right) Z \partial_Z u_0 = 0,$$

which corresponds to the first equation of (15) obtained in a formal way (recall that the quantity M is defined by $M = \int_{\mathbb{T}^{d-1}} |\nabla_X h_2|^2 dX$). Theorem 1 is proved.

REMARK 2. *In the proof of Theorem 1, we have kept the the periodicity assumption for the rugosity function h_2 , which is necessary for the formal derivation (section 2.3). Let us emphasize that this periodicity assumption is unnecessary for the rigorous derivation of the limit model. Actually, it is sufficient to assume the following:*

- (i) *The roughness is characterized by oscillations with minimal size of order ε^2 .*
- (ii) *The sequences $(h_2(x/\varepsilon^2))$ and $(\nabla h_2(x/\varepsilon^2))$ strongly two-scale converge to $h_2(X)$ and $\nabla_X h_2(X)$, respectively.*

The interested reader could follow the lines of section 5 and check that our arguments remain true under hypotheses (i)–(ii). Only the result of Lemma 5.3 would change: this has no influence on the limit model because $\nabla_X w_0 = 0$ remains true.

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