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VISCOELASTIC FLUIDS IN A THIN DOMAIN

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Abstract. The present paper deals with viscoelastic flows in a thin domain. In particular, we derive and analyse the asymptotic equations of the Stokes-Oldroyd system in thin films (including shear effects). We present a numerical method which solves the corresponding problem and present some related numerical tests which evidence the effects of the elastic contribution on the flow.

Contents

Introduction. A wide literature is devoted to non-Newtonian fluids in a thin domain in both mathematical aspects and applications. It is well known that numerous biological fluids, blood or physiological secretions like tears or synovial fluids, present such a non-Newtonian characteristic. In engineering applications, people are interested to control

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the characteristic of the flows in order to suit various requirements such as maintaining its qualities in a wide range of temperature and stresses. Commercial lubricants are then modified with different additives to be able to protect engines both in winter and in summer with the same product. This addition leads a non-Newtonan behavior of the actual lubricant. Another domain of applications is linked to the polymers, whose non-Newtonian characteristics appear in a wide range of applications as molding or injection process.

It is to be noticed that, in most of the practical applications, the geometry of the flow to be considered is anistropic. This is the case in lubrication studies which are mainly devoted to thin film flows, in the study of the spreading of tears or in the description of polymers through thin dies. If such anisotropy can induce some numerical problems in 3D computations, especially as the ratio-aspect of the geometry is big enough, it has however the advantage to allow some simplification in the equations. So if this approximation process can lead to 2D equations, it could be thought that such simplified equations are easier to solve than the original 3D ones. This explains the amount of work devoted to this topic.

Some particular classes of non-Newtonian models are often considered. This includes the Bingham flow or the quasi-Newtonian fluids (Carreau's law, power law or Williamson's law, in which various stresses-velocity relations are chosen, see [**?**]). For this kind of problems, it has been possible to give, in a rigorous way, some thin film approximation of the 3D equations by a so-called generalized Reynolds equation for the pressure. These models however considered the fluid as a viscous one and elasticity effects are neglected. Introducing such viscoelastic behavior is primilarly described by the Deborah number, denoted $\mathcal{D}e$ which can be viewed as a measure of the elasticity of the fluid and is related to the relaxation time. One of the laws which seems the most able to describe viscoelastic flows is the Olroyd-B model. This model is based on a constitutive equation which is an interpolation between purely viscous and purely elastic behaviors, thus introducing a supplementary parameter r describing the relative proportion of both behaviors (the solvant to solute ratio). Considering the Oldroyd model [**?**], the momentum, continuity and constitutive equations for an incompressible flow of such a non-Newtonian fluid are, respectively,

$$\rho\left(\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U}\cdot\nabla\boldsymbol{U}\right) - \eta(1-r)\Delta\boldsymbol{U} + \nabla p - \operatorname{div}\,\sigma = \boldsymbol{0}, \tag{1}$$

$$\operatorname{div}\,\boldsymbol{U} = 0, \tag{2}$$

$$\lambda\left(\frac{\partial \sigma}{\partial t} + \boldsymbol{U}\cdot\nabla\sigma + g_a(\nabla\boldsymbol{U},\sigma)\right) + f(\sigma)\;\sigma = 2\eta r D(\boldsymbol{U}). \tag{3}$$

In these equations, ρ, η and λ are positive constants which respectively correspond to the fluid density, the fluid viscosity and the relaxation time. Equations (1)–(3) compose a system of 10 equations with 10 unknowns: the lubricant velocity vector $\boldsymbol{U} = (u_1, u_2, w)$, the pressure p and the extra-stress symmeric tensor $\sigma = (\sigma_{i,j})_{1\le i,j\le 3}$. The bilinear application g_a, $-1 \le a \le 1$, is defined by

$$g_a(\nabla\boldsymbol{U},\sigma) = \sigma\cdot W(\boldsymbol{U}) - W(\boldsymbol{U})\cdot\sigma - a(\sigma\cdot D(\boldsymbol{U}) + D(\boldsymbol{U})\cdot\sigma)$$

whereas $D(\boldsymbol{U})$ and $W(\boldsymbol{U})$ are respectively the symmetric and skew-symmetric parts of the velocity gradient $\nabla\boldsymbol{U}$. Usually, $D(\boldsymbol{U})$ is called the rate of deformation tensor and $W(\boldsymbol{U})$ is called the vorticity tensor. Notice that the parameter a is considered to interpolate between upper convected ($a = 1$) and lower convective derivatives ($a = -1$), the case $a = 0$ being the corotationnal case [**?**]. To be noticed that taking $r = 1$ allows us to recover various form of the generalized Maxwell model. Then choosing f as the identity, this model is the classical Maxwell one while, by introducing a linearized form of f (see in particular [**?**]), Phan-Tein-Tanner laws [**?**] are obtained. Conversely, a Newtonian flow is described by choosing $r = 0$.

From the mathematical aspects, few results exist concerning existence or uniqueness of a solution for truly 3D or 2D viscoelastic models [**?**, **?**] and the way how to obtain the related thin film approximation is mainly heuristic. A first approach, which is often used in the engineering literature, is to take the parameter defining the (relative) thickness of the flow as a leading small parameter and to use the Deborah number as a pertubation parameter. This has been done in the lubrication field by Tichy [**?**] starting from the upper convected Maxwell model ($r = 1$, $f = \mathrm{Id}$, $a = 1$). The case of a Deborah number of the same order of magnitude than the relative thickness has been studied by Tichy and Huang from the UCM Maxwell model and by Bellout [**?**] from Phan-Thein-Tanner model. In all these works, a nonlinear Reynolds equation is gained, allowing to compute the pressure in the thin film. Same procedures can also include the free boundary upper surface of the flow (thin coating problem) or inertia [**?**, **?**, **?**]. However, the goal of these last studies are different as the primary unknown is not an equation for the pressure but an equation describing the evolution of the free boundary (a generalized shallow water equation).

The present paper addresses the mathematical and numerical study of a large class of viscoelastic thin film flows described by an Olroyd-B model in which the Deborah number has the same order of magnitude than the thickness of the fluid. This assumption allows to balance the order of Newtonian and non-Newtonian contribution (see [**?**] for mechanical comments). Boundary conditions are chosen in order to be applied to usual lubrication problems. After scaling both equations and stress tensor in an adequate way, we are able to obtain an asymptotic 2D problem. This problem generalizes the work of Bellout and Tichy, concerning not only the rheological model but also the dimension (2D instead of 1D for the pressure asymptotic problem). Obtaining the asymptotic problem is partly an heuristic process, so we have to rigorously prove the solvability of this problem. This is the goal of the second part of the paper. Interestingly, an existence and uniqueness result is obtained exactly for the same range of parameters r as the initial 3D problem. In numerous problems in the thin field, it is possible to eliminate the velocity in the limit problem, so retaining only a Reynolds equation with respect to the pressure. It is different in our case and we have to solve a nonlinear coupled problem in which a degenerate Stokes equation is still present. A new algorithm related to the Uzawa one is presented and convergence theorems are given. At last, numerical comparison between various model are given and the importance to get 2D and not only one 1D approximation is emphasized.

1. Mathematical formulation. The space coordinates are denoted by (x_1, x_2, z) or more simply by (x, z) with $x = (x_1, x_2)$. Let ω be a fixed bounded domain of the plane $z = 0$. We suppose that ω has a Lipschitz continuous boundary $\partial\omega$. The upper surface of the gap is defined by $z = H(x)$ with $H \in \mathcal{C}^1(\overline{\omega})$. Let us denote by Ω the following set (see FIG.1):

$$\Omega = \{(x, z) \in \mathbb{R}^3,\ x \in \omega \text{ and } 0 < z < H(x)\}.$$

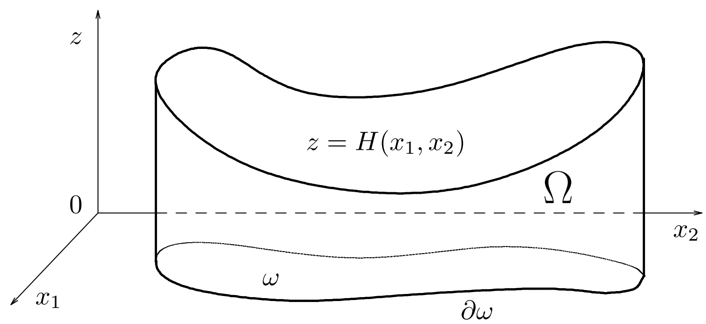

FIG. 1. The physical domain

1.1. *Thin film flow equations.* Introducing characteristic lengths $\mathcal{L}$ for the domain ω and $\mathcal{H}$ for the size of the gap, we can define the ratio

$$\varepsilon = \frac{\mathcal{H}}{\mathcal{L}}$$

which is, in the physical realistic case of lubrification, very small. The governing equations (1)–(3) can be expressed in dimensionless form in terms of the following dimensionless quantities :

$$x = x^*\mathcal{L}, \quad z = z^*\varepsilon\mathcal{L}, \quad u_i = u_i^*\mathcal{U}, \quad w = w^*\varepsilon\mathcal{U}, \tag{4}$$

$$p = p^*\frac{\eta\mathcal{L}\mathcal{U}}{\varepsilon^2\mathcal{L}^2}, \quad \sigma = \sigma^*\frac{\eta\mathcal{U}}{\varepsilon\mathcal{L}}, \quad t = t^*\frac{\mathcal{L}}{\mathcal{U}}. \tag{5}$$

We now introduce two classical numbers in viscoelasticity : the Reynolds number $\mathcal{R}e$ which characterises the viscous forces in front of the convective ones, and the Deborah number $\mathcal{D}e$ which highlights the elasticity of the fluid. They are defined by

$$\mathcal{R}e = \frac{\mathcal{U}\mathcal{L}}{\eta}, \qquad \mathcal{D}e = \frac{\lambda\mathcal{U}}{\mathcal{L}} = \varepsilon\mathcal{D}e^*. \tag{6}$$

REMARK 1.1. This scaling process is motivated by the following considerations:

- The length and velocity scaling (4) takes into account the thin film nature of lubrication flow.
- Classically, in lubrication theory, if the horizontal shear velocity is of order 1, then the real pressure is of order $1/\varepsilon^2$ (see [**?**] for a rigorous mathematical explanation).
- If we want to balance the order of Newtonian and non-Newtonian contribution, we must assume that the stress tensor is of order $1/\varepsilon$ and the Deborah number is of order ε (see [**?**] for further explanations).

Substituting these dimensionless variables (4)–(6) in Equations (1)–(3), and dropping the asterisks, we obtain the dimensionless governing equations.

- The three components of the momentum equation (1) write

$$\left\{\begin{array}{rcl}
\mathcal{R}e\rho\dfrac{d\,u_1}{dt}-\eta(1-r)\left(\dfrac{\partial^2 u_1}{\partial x_1^2}+\dfrac{\partial^2 u_1}{\partial x_2^2}+\dfrac{1}{\varepsilon^2}\dfrac{\partial^2 u_1}{\partial z^2}\right)+\dfrac{1}{\varepsilon^2}\dfrac{\partial p}{\partial x_1} & & \\
-\dfrac{1}{\varepsilon}\left(\dfrac{\partial\,\sigma_{1,1}}{\partial x_1}+\dfrac{\partial\,\sigma_{1,2}}{\partial x_2}+\dfrac{1}{\varepsilon}\dfrac{\partial\,\sigma_{1,3}}{\partial z}\right) & = & 0,\\
\mathcal{R}e\rho\dfrac{d\,u_2}{dt}-\eta(1-r)\left(\dfrac{\partial^2 u_2}{\partial x_1^2}+\dfrac{\partial^2 u_2}{\partial x_2^2}+\dfrac{1}{\varepsilon^2}\dfrac{\partial^2 u_2}{\partial z^2}\right)+\dfrac{1}{\varepsilon^2}\dfrac{\partial p}{\partial x_2} & & \\
-\dfrac{1}{\varepsilon}\left(\dfrac{\partial\,\sigma_{1,2}}{\partial x_1}+\dfrac{\partial\,\sigma_{2,2}}{\partial x_2}+\dfrac{1}{\varepsilon}\dfrac{\partial\,\sigma_{2,3}}{\partial z}\right) & = & 0,\\
\varepsilon\mathcal{R}e\rho\dfrac{d\,w}{dt}-\varepsilon\eta(1-r)\left(\dfrac{\partial^2 w}{\partial x_1^2}+\dfrac{\partial^2 w}{\partial x_2^2}+\dfrac{1}{\varepsilon^2}\dfrac{\partial^2 w}{\partial z^2}\right)+\dfrac{1}{\varepsilon^3}\dfrac{\partial p}{\partial z} & & \\
-\dfrac{1}{\varepsilon}\left(\dfrac{\partial\,\sigma_{1,3}}{\partial x_1}+\dfrac{\partial\,\sigma_{2,3}}{\partial x_2}+\dfrac{1}{\varepsilon}\dfrac{\partial\,\sigma_{3,3}}{\partial z}\right) & = & 0.
\end{array}\right.$$

When ε tends to zero, these equations formally reduce to the following set of equations:

$$\left\{\begin{array}{l}
-\eta(1-r)\dfrac{\partial^2 u_1}{\partial z^2}+\dfrac{\partial p}{\partial x_1}-\dfrac{\partial\,\sigma_{1,3}}{\partial z}=0,\\
-\eta(1-r)\dfrac{\partial^2 u_2}{\partial z^2}+\dfrac{\partial p}{\partial x_2}-\dfrac{\partial\,\sigma_{2,3}}{\partial z}=0,\\
\dfrac{\partial p}{\partial z}=0,
\end{array}\right. \tag{7}$$

- Due to the previous dimensionless, the free divergence condition is preserved for the dimensionless variables:

$$\frac{\partial\,u_1}{\partial x_1}+\frac{\partial\,u_2}{\partial x_2}+\frac{\partial\,w}{\partial z}=0, \tag{8}$$

- Concerning the constitutive law, the process is similar: equations are written for the quantities without dimension, then, passing formally to the limit $\varepsilon\to 0$, the following equations are obtained:

$$\left\{\begin{array}{l}
\sigma_{1,1}+\mathcal{D}e(1-a)\sigma_{1,3}\dfrac{\partial\,u_1}{\partial z}=0,\\
\sigma_{2,2}+\mathcal{D}e(1-a)\sigma_{2,3}\dfrac{\partial\,u_2}{\partial z}=0,\\
\sigma_{3,3}-\mathcal{D}e(1+a)(\sigma_{1,3}\dfrac{\partial\,u_1}{\partial z}+\sigma_{2,3}\dfrac{\partial\,u_2}{\partial z})=0,\\
\sigma_{1,2}+\dfrac{\mathcal{D}e}{2}(1-a)(\sigma_{2,3}\dfrac{\partial\,u_1}{\partial z}+\sigma_{1,3}\dfrac{\partial\,u_2}{\partial z})=0,\\
\sigma_{1,3}+\dfrac{\mathcal{D}e}{2}\left((1-a)\sigma_{3,3}\dfrac{\partial\,u_1}{\partial z}-(1+a)\sigma_{1,2}\dfrac{\partial\,u_2}{\partial z}-(1+a)\sigma_{1,1}\dfrac{\partial\,u_1}{\partial z}\right)=\eta\,r\dfrac{\partial\,u_1}{\partial z},\\
\sigma_{2,3}+\dfrac{\mathcal{D}e}{2}\left((1-a)\sigma_{3,3}\dfrac{\partial\,u_2}{\partial z}-(1+a)\sigma_{1,2}\dfrac{\partial\,u_1}{\partial z}-(1+a)\sigma_{2,2}\dfrac{\partial\,u_2}{\partial z}\right)=\eta\,r\dfrac{\partial\,u_2}{\partial z}.
\end{array}\right. \tag{9}$$

In this system, it is easy to see that coefficients $\sigma_{1,1}$, $\sigma_{2,2}$, $\sigma_{3,3}$ and $\sigma_{1,2}$ can be expressed according to $\sigma_{1,3}$, $\sigma_{2,3}$ and the velocity (u_1, u_2). In addition, using the last two equations, $\sigma_{1,3}$ and $\sigma_{2,3}$ are expressed with respect to the velocity:

$$\sigma_{1,3} = \frac{\eta\, r \dfrac{\partial u_1}{\partial z}}{1 + \mathcal{D}e^2(1-a^2)\left(\left(\dfrac{\partial u_1}{\partial z}\right)^2 + \left(\dfrac{\partial u_2}{\partial z}\right)^2\right)},$$

$$\sigma_{2,3} = \frac{\eta\, r \dfrac{\partial u_2}{\partial z}}{1 + \mathcal{D}e^2(1-a^2)\left(\left(\dfrac{\partial u_1}{\partial z}\right)^2 + \left(\dfrac{\partial u_2}{\partial z}\right)^2\right)}.$$

For the sake of simplicity, let us denote by $\boldsymbol{u}$ the first two coordinates of the velocity vector : $\boldsymbol{u} = (u_1, u_2)$ and by $\boldsymbol{\beta}$ the following two components of the stress tensor : $\boldsymbol{\beta} = (\sigma_{1,3}, \sigma_{2,3})$. The system obtained can be written on the following form:

$$\begin{cases} -\eta(1-r)\dfrac{\partial^2 \boldsymbol{u}}{\partial z^2} - \dfrac{\partial \boldsymbol{\beta}}{\partial z} + \nabla_x p = \boldsymbol{0}, \quad \text{with} \quad \boldsymbol{\beta} = \dfrac{\eta\, r \dfrac{\partial \boldsymbol{u}}{\partial z}}{1 + \mathcal{D}e^2(1-a^2)\left|\dfrac{\partial \boldsymbol{u}}{\partial z}\right|^2}, \\ \dfrac{\partial p}{\partial z} = 0, \\ \mathrm{div}_x \boldsymbol{u} + \dfrac{\partial w}{\partial z} = 0, \end{cases} \tag{10}$$

all the other components of the stress tensor being directly deduced from equations (9).

The vertical velocity w can be deduced from the horizontal velocity $\boldsymbol{u}$ by the free divergence condition. More clearly, problem (10) is equivalent to the following one:

$$\begin{cases} -\eta(1-r)\dfrac{\partial^2 \boldsymbol{u}}{\partial z^2} - \eta\, r \dfrac{\partial}{\partial z}\left(\dfrac{\dfrac{\partial \boldsymbol{u}}{\partial z}}{1 + \mathcal{D}e^2(1-a^2)\left|\dfrac{\partial \boldsymbol{u}}{\partial z}\right|^2}\right) + \nabla_x p = \boldsymbol{0}, \\ \dfrac{\partial p}{\partial z} = 0, \\ \mathrm{div}_x\left(\displaystyle\int_0^h \boldsymbol{u}\, dz\right) = w(\cdot, 0) - w(\cdot, h). \end{cases} \tag{11}$$

1.2. *Boundary conditions.* System (10) will be the subject of the forthcoming theoretical study, as it allows the knowledge of the pressure p (the primary factor of interest in lubrication problems) and the horizontal velocity $\boldsymbol{u}$ (while the vertical one w is in the real variables of order ε). Let us now introduce the boundary conditions. As it is well-known (see [**?**]), passing from 3D problems to 2D ones may induce boundary layer phenomena on the lateral parts of Ω. Then, only a part of the boundary condition for the initial problem have to be considered in the study of (10). We have to retain the following typical (no-slip) boundary conditions at $z = 0$ and $z = h$:

- $\boldsymbol{u}(\cdot, 0) = \boldsymbol{s}$ and $\boldsymbol{u}(\cdot, h) = \boldsymbol{0}$ on ω,
- $w(\cdot, 0) = 0$ and $w(\cdot, h) = 0$ on ω.

Moreover, two kinds of boundary conditions can be considered along this lateral boundary, one associated to the data of the pressure, the other one to the data of the average flux. The choice of the conditions highly depends on the devices to be considered. In most of the physical problems, two types of boundary conditions are simultaneously used: Neumann-type conditions and Dirichlet conditions. Thus, in the general case, the set of equations (11) has to be considered with the following boundary conditions:

$$p = p_0 \quad \text{on } \partial\omega^p, \qquad \int_0^h \boldsymbol{u}\, dz \cdot \boldsymbol{n} = q_0 \quad \text{on } \partial\omega^q, \tag{12}$$

where $\partial\omega^p$ and $\partial\omega^q$ define a partition of the boundary $\partial\omega$. Notice that $\partial\omega^p$ (resp. $\partial\omega^q$) may be the union of a finite number of connected components denoted $\partial\omega_i^p$ (resp. $\partial\omega_i^q$) (see FIG.2). Let us notice that a compatibility condition on the total flux is needed if $\partial\omega^p = \emptyset$:

$$\int_{\partial\omega} q_0 = 0.$$

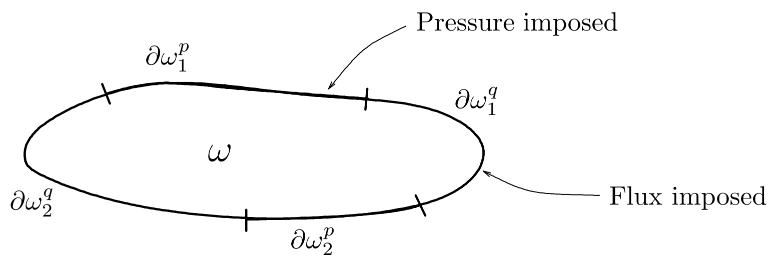

FIG. 2. Mixed boundary conditions

2. Theoretical analysis.

2.1. *The newtonian case.* The Newtonian case corresponds to the case where the stress tensor σ is zero. In the limit equations (11), this means that $\boldsymbol{\beta} = 0$. In this subsection, we first state the strong and weak formulations of the problem. Then, we do not only provide a rigorous mathematical study, but also establish the relevance of the weak formulation with respect to the physical (strong) formulation. Thus, let us introduce the formulations in the purely Newtonian case.

■ *Strong formulation:*

The problem deals with boundary conditions of two types: Neumann conditions and non-homogeneous Dirichlet conditions. Still, by introducing some kind of source-term, it is possible to get an equivalent problem with homogeneous Dirichlet conditions. Indeed, let $\widetilde{p_0}$ be an extension of p_0 on the closed set $\overline{\omega}$. It is obviously equivalent to work with a reduced pressure $\widetilde{p} = p - \widetilde{p_0}$ instead of the effective pressure p. For this, the strong formulation $(\mathcal{P}_s)$ is slightly modified by the introduction of a non-zero right-hand side

$\boldsymbol{F} = -\nabla_x \widetilde{p_0}$ (instead of 0), which takes into account the translation of the pressure. In the whole study, we will consider that the following assumptions on the data hold:

Assumption 1 (Regularity of the data).

- $h \in \mathcal{C}^0(\omega)$, $h \geq h_0 > 0$,
- $\boldsymbol{s} \in L^2(\Gamma_-)$, where Γ_- (resp. Γ_+) denotes the lower (resp. upper) boundary of Ω, i.e. $\Gamma_- = \{(x,0),\ x \in \omega\}$, $\Gamma_+ = \{(x,h(x)),\ x \in \omega\}$.
- $\boldsymbol{F} \in L^2(\Omega)$,
- $q_0 \in L^2(\partial\omega^q)$.

Now, the strong formulation is the following one:

$$
(\mathcal{P}_s)\ \left\{
\begin{array}{lll}
-\eta \dfrac{\partial^2 \boldsymbol{u}}{\partial z^2} + \nabla_x p = \boldsymbol{F}, & \text{in } L^2(\Omega), & (13)\\[2mm]
\dfrac{\partial p}{\partial z} = 0, & \text{in } L^2(\Omega), & (14)\\[2mm]
\mathrm{div}_x \left(\displaystyle\int_0^h \boldsymbol{u}(\cdot,z)\, dz \right) = 0, & \text{in } L^2(\omega), & (15)\\[2mm]
\boldsymbol{u} = \boldsymbol{s}, & \text{in } L^2(\Gamma_-), & (16)\\
\boldsymbol{u} = \boldsymbol{0}, & \text{in } L^2(\Gamma_+), & (17)\\
p = 0, & \text{in } L^2(\partial\omega^p), & (18)\\
\displaystyle\int_0^h \boldsymbol{u}(\cdot,z)\, dz \cdot \boldsymbol{n} = q_0, & \text{in } L^2(\partial\omega^q). & (19)
\end{array}
\right.
$$

To be noticed is the fact that this set of equations can be reduced to the classical Reynolds equation (see in particular [?]). Indeed, integrating twice Equation (13) with respect to z (and taking into account the velocity boundary conditions (16)-(17)), we obtain the velocity $\boldsymbol{u}$ as a function of the pressure p. Then, putting this expression into Equation (15) gives:

$$
\mathrm{div} \left(\frac{h^3}{6\eta} \nabla p \right) = \mathrm{div}\,(\boldsymbol{s}h)\,.
$$

In the purely Newtonian case, the Reynolds formulation allows to give a straightforward existence and uniqueness result (via elliptic theory). Here, we propose an alternate approach which will be easily adapted to the viscoelastic case (although the Reynolds approach could not be easily extended to this nonlinear case).

■ *Weak formulation:*

First, let us introduce the functional space which is used in the weak formulation. For $\boldsymbol{s} \in \mathbb{R}^2$ and $q_0 \in L^1(\partial\omega^q)$, we define

$$
K(\boldsymbol{s}, q_0) = \Bigg\{ \boldsymbol{\varphi} \in L^2(\Omega),\ \frac{\partial \boldsymbol{\varphi}}{\partial z} \in L^2(\Omega),\ \boldsymbol{\varphi} = \boldsymbol{s} \text{ in } L^2(\Gamma_-),\ \boldsymbol{\varphi} = \boldsymbol{0} \text{ in } L^2(\Gamma_+),
$$
$$
\forall \theta \in \{\theta \in \mathcal{D}(\overline{\omega}) \text{ s.t. } \exists \zeta \in \mathbb{R},\ \theta|_{\partial\omega^p} = \zeta\},\ \iint_\omega \nabla_x \theta \cdot \left(\int_0^h \boldsymbol{\varphi}(\cdot,z)\, dz \right) = \int_{\partial\omega^q} (\theta - \zeta)\, q_0 \Bigg\}.
$$

The space $K(\boldsymbol{s}, q_0)$ is equipped with the norm:

$$\left\|\boldsymbol{\varphi}\right\|_z = \left(\iiint_\Omega \left|\frac{\partial \boldsymbol{\varphi}}{\partial z}\right|^2\right)^{1/2}. \tag{20}$$

Remark 2.1. To be noticed that for every function in $\boldsymbol{\varphi} \in L^2(\Omega)$ such that $\dfrac{\partial \boldsymbol{\varphi}}{\partial z} \in L^2(\Omega)$, it is possible to define its trace $\gamma_-(\boldsymbol{\varphi}) \in L^2(\Gamma_-)$ on Γ_- (resp. $\gamma_+(\boldsymbol{\varphi}) \in L^2(\Gamma_+)$). Thus, the boundary conditions on Γ_- and Γ_+ in the definition of $K(\boldsymbol{s}, q_0)$ make sense. However, the indexes $\gamma_\pm$ will be dropped for the sake of simplicity.

The following proposition will enable us to better understand the interest of this set:

Proposition 2.1. Let $\boldsymbol{\varphi} \in H^1(\Omega)$. We have the following equivalence

$$\boldsymbol{\varphi} \in K(\boldsymbol{s}, q_0) \iff \boldsymbol{\varphi} \text{ satifies Equations (15)–(17) and (19).}$$

Proof. It is clear that if $\boldsymbol{\varphi} \in K(\boldsymbol{s}, q_0)$, then Equations (16)–(17) hold (see the definition of the functional space). Now, using an integration by parts, if $\boldsymbol{\varphi} \in K(\boldsymbol{s}, q_0) \cap H^1(\Omega)$ then for all $\theta \in \mathcal{D}(\overline{\omega})$, θ being constant on $\partial\omega^p$, we have

$$-\iint_\omega \theta \operatorname{div}_x\left(\int_0^h \boldsymbol{\varphi}(\cdot, z)\, dz\right) + \int_{\partial\omega^q} \theta\left(\int_0^h \boldsymbol{\varphi}(\cdot, z)\, dz\right) \cdot \boldsymbol{n} = \int_{\partial\omega^q} \theta\, q_0.$$

In particular, for all $\theta \in \mathcal{D}(\omega)$, we find

$$\iint_\omega \theta \operatorname{div}_x\left(\int_0^h \boldsymbol{\varphi}(\cdot, z)\, dz\right) = 0$$

that is Equation (15) holds. Then, for all $\widetilde{\theta} \in \mathcal{D}(\partial\omega^q)$, extended on ω such that $\widetilde{\theta} \in \mathcal{D}(\overline{\omega})$ and $\widetilde{\theta}|_{\partial\omega^p} = 0$, we obtain

$$\int_{\partial\omega^q} \widetilde{\theta}\left(\left(\int_0^h \boldsymbol{\varphi}(\cdot, z)\, dz\right) \cdot \boldsymbol{n} - q_0\right) = 0,$$

i.e. Equation (19) holds. This concludes the proof of the necessary condition. This condition is clearly sufficient. $\square$

We provide the way to attain the weak formulation of the problem. Let $(\boldsymbol{u}, p)$ be a regular solution of (13)–(19), and let $\boldsymbol{\varphi} \in K(\boldsymbol{s}, q_0)$. Multiplying Equation (13) by $\boldsymbol{u} - \boldsymbol{\varphi}$ and integrating over Ω, we obtain

$$\iiint_\Omega -\eta \frac{\partial^2 \boldsymbol{u}}{\partial z^2} \cdot (\boldsymbol{u} - \boldsymbol{\varphi}) + \iiint_\Omega \nabla_x p \cdot (\boldsymbol{u} - \boldsymbol{\varphi}) = \iiint_\Omega \boldsymbol{F} \cdot (\boldsymbol{u} - \boldsymbol{\varphi}).$$

Since $\boldsymbol{u} - \boldsymbol{\varphi} \in K(\boldsymbol{0}, 0)$, we can integrate by parts the first integral, and use p as a test function θ to cancel the second integral (let us recall here that p does not depend on z). In particular, we deduce the weak formulation of the problem:

$$(\mathcal{P}_w) \begin{cases} \text{Find } \boldsymbol{u} \in K(\boldsymbol{s}, q_0) \text{ such that} \\ \displaystyle\iiint_\Omega \eta \frac{\partial \boldsymbol{u}}{\partial z} \cdot \frac{\partial (\boldsymbol{u} - \boldsymbol{\varphi})}{\partial z} \leq \iiint_\Omega \boldsymbol{F} \cdot (\boldsymbol{u} - \boldsymbol{\varphi}), \quad \forall\, \boldsymbol{\varphi} \in K(\boldsymbol{s}, q_0). \end{cases} \tag{21}$$

Now, this subsection is concluded with two major results: we first give an existence and uniqueness result for the weak problem and then we describe the link between the two formulations.

THEOREM 1 (Newtonian case). Problem $(\mathcal{P}_w)$ admits a unique solution.

Proof. The proof is based on the theory of variational inequalities [**?**]. Obviously, the space $(K(\boldsymbol{s}, q_0), \|\cdot\|_z)$ is closed. Moreover, linearity of the boundary conditions leads to the affine property of the space so that it is convex. Thus, it remains to prove that the space is non-empty. Using Proposition 2.1, we look for a function satisfying Equations (15)–(17) and (19). It is obvious that the function

$$\overline{\phi} = \frac{\boldsymbol{a}}{2\,\eta} z(z-h)) + \boldsymbol{s}\frac{h-z}{h}, \tag{22}$$

satisfies (16) and (17). Here, $\boldsymbol{a}$ is any vector only depending on x (to be further detailed). In order to ensure that $\overline{\phi}$ satisfies Equations (16), (17) and (19), $\boldsymbol{a}$ has to satisfy:

$$\begin{cases} \operatorname{div}\left(\dfrac{h^3}{12\eta}\boldsymbol{a}\right) = \operatorname{div}\left(\dfrac{\boldsymbol{s}h}{2}\right) & \text{on } \omega, \\ \left(\dfrac{\boldsymbol{s}h}{2} - \dfrac{h^3}{12\eta}\boldsymbol{a}\right)\cdot\boldsymbol{n} = q_0 & \text{on } \partial\omega^q. \end{cases} \tag{23}$$

In order to state that there exists some $\boldsymbol{a}$ satisfying the earlier set of equations, we consider the following Reynolds problem (as an auxiliary problem):

$$\begin{cases} \operatorname{div}\left(\dfrac{h^3}{12\eta}\nabla\pi\right) = \operatorname{div}\left(\dfrac{\boldsymbol{s}h}{2}\right) & \text{on } \omega, \\ \left(\dfrac{\boldsymbol{s}h}{2} - \dfrac{h^3}{12\eta}\nabla\pi\right)\cdot\boldsymbol{n} = q_0 & \text{on } \partial\omega. \end{cases} \tag{24}$$

Obviously, there exists a unique $\pi \in H^1(\omega)/\mathbb{R}$ satisfying (24). Then, choosing $\boldsymbol{a} = \nabla\pi \in L^2(\omega)$, the proof is concluded: by means of construction $\boldsymbol{a}$ satisfies Equations (16), (17) and (19). Thus, the function $\overline{\phi}$ defined by Equation (22), with the previous choice for $\boldsymbol{a}$, belongs to $K(\boldsymbol{s}, q_0)$ which is consequently non-empty. □

The link between $(\mathcal{P}_w)$ and $(\mathcal{P}_s)$ is given by the following theorem.

THEOREM 2 (Newtonian case). Let $\boldsymbol{u}$ be the unique solution of $(\mathcal{P}_w)$.

(i) There exists a unique $p \in H^1(\omega)$ such that $(\boldsymbol{u}, p)$ satisfies (13), (14), (16)–(18).

(ii) Moreover, if $\boldsymbol{u} \in H^1(\Omega)$, then (15) and (19) hold. In particular, $(\boldsymbol{u}, p)$ is the unique solution of $(\mathcal{P}_s)$.

Proof. The result is checked in three steps:

- *Step 1:* Let us state that Equations (13) and (14) hold.

For this, we use the de Rham theorem in order to ensure the existence of a pressure p. Choosing $\boldsymbol{\varphi} = \boldsymbol{u} \pm \overline{\boldsymbol{\varphi}}$ with $\overline{\boldsymbol{\varphi}} \in K(\boldsymbol{0}, 0) \cap \mathcal{D}(\Omega)$ as a test function in Equation (21), we deduce that

$$\forall\overline{\boldsymbol{\varphi}} \in K(\boldsymbol{0}, 0) \cap \mathcal{D}(\Omega), \qquad \iiint_\Omega \eta\frac{\partial\,\boldsymbol{u}}{\partial z}\cdot\frac{\partial\,\overline{\boldsymbol{\varphi}}}{\partial z} = \iiint_\Omega \boldsymbol{F}\cdot\overline{\boldsymbol{\varphi}}. \tag{25}$$

Then, as $\boldsymbol{u}$ belongs to $K(\boldsymbol{s}, q_0)$, we find

$$\forall \overline{\boldsymbol{\varphi}} \in K(\mathbf{0}, 0) \cap \mathcal{D}(\Omega), \quad \left\langle -\eta \frac{\partial^2 \boldsymbol{u}}{\partial z^2} - \boldsymbol{F}, \overline{\boldsymbol{\varphi}} \right\rangle = 0, \tag{26}$$

in the sense of distributions. The next lemma allows us to use the classical De Rham theorem to find a pressure:

Lemma 2.1. For $\overline{\boldsymbol{\varphi}} = (\varphi_1, \varphi_2) \in K(\mathbf{0}, 0) \cap \mathcal{D}(\Omega)$, there exists $\varphi_3 \in \mathcal{D}(\Omega)$ such that $\operatorname{div}(\varphi_1, \varphi_2, \varphi_3) = 0$. Conversely, if $\boldsymbol{\Phi} = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{D}(\Omega)$ is such that $\operatorname{div} \boldsymbol{\Phi} = 0$, then $(\varphi_1, \varphi_2) \in K(\mathbf{0}, 0)$.

Proof. For $\overline{\boldsymbol{\varphi}} \in K(\mathbf{0}, 0) \cap \mathcal{D}(\Omega)$ it is enough to define

$$\varphi_3(x, z) = -\int_0^z \operatorname{div}_x \overline{\boldsymbol{\varphi}}(x, \xi) \, d\xi$$

so that

$$\operatorname{div}_x \overline{\boldsymbol{\varphi}} + \frac{\partial \varphi_3}{\partial z} = 0, \tag{27}$$

with $\varphi_3 \in \mathcal{D}(\Omega)$. Conversely, define $\overline{\boldsymbol{\varphi}} = (\varphi_1, \varphi_2)$, if Equation (27) holds, then using the fact that φ_3 is zero at the boundaries $z = 0$ and $z = h$, we find

$$\operatorname{div}_x \left(\int_0^h \overline{\boldsymbol{\varphi}}(\cdot, z) \, dz \right) = 0.$$

Moreover, if $\overline{\boldsymbol{\varphi}} \in \mathcal{D}(\Omega)$ then

$$\begin{aligned} \overline{\boldsymbol{\varphi}} &= \mathbf{0}, && \text{on } \partial\omega^q \\ \left(\int_0^h \overline{\boldsymbol{\varphi}}(\cdot, z) \, dz \right) \cdot \boldsymbol{n} &= 0, && \text{on } \partial\omega^q, \end{aligned}$$

i.e. $\overline{\boldsymbol{\varphi}} \in K(\mathbf{0}, 0)$. □

Let us define

$$\boldsymbol{\mathcal{F}} = \left(-\eta \frac{\partial^2 \boldsymbol{u}}{\partial z^2} + \nabla_x \widetilde{p}_0, 0 \right),$$

and using the previous lemma, Equation (26) is rewritten as:

$$\forall \boldsymbol{\Phi} \in \mathcal{D}(\Omega) \text{ such that } \operatorname{div} \boldsymbol{\Phi} = 0, \quad \langle \boldsymbol{\mathcal{F}}, \boldsymbol{\Phi} \rangle = 0.$$

With the De Rham theorem, we deduce that there exists a unique pressure $p \in \mathcal{D}'(\Omega)/\mathbb{R}$ such that $\boldsymbol{\mathcal{F}} = \nabla p$, with

$$\nabla p = \left(\nabla_x p, \frac{\partial p}{\partial z} \right),$$

so that

$$-\eta \frac{\partial^2 \boldsymbol{u}}{\partial z^2} + \nabla_x p = \boldsymbol{F}, \qquad \text{in } \mathcal{D}'(\Omega), \tag{28}$$

$$\frac{\partial p}{\partial z} = 0, \qquad \text{in } \mathcal{D}'(\Omega), \tag{29}$$

Now, let us discuss the regularity of $\boldsymbol{u}$ and p: as $\boldsymbol{u}$ is a solution of problem $(\mathcal{P}_w)$, then

$$\boldsymbol{u}, \ \frac{\partial \boldsymbol{u}}{\partial z} \in L^2(\Omega).$$

In particular, if $\boldsymbol{u}$ is extended by $\mathbf{0}$ on $\{(x,z) \in \Omega,\ z \geq h(x)\}$, and denoting $h_\infty = \|h\|_{L^\infty(\omega)}$, we have

$$\boldsymbol{u} \in \mathcal{C}([0,h_\infty]; L^2(\omega)).$$

Now, by Equation (28), as $\boldsymbol{F}$ and p do not depend on z, one has

$$\frac{\partial}{\partial z}\left(\frac{\partial^2 \boldsymbol{u}}{\partial z^2}\right) \in L^2((0,h_\infty); L^2(\omega)),$$

and since $\dfrac{\partial^2 \boldsymbol{u}}{\partial z^2} \in \mathcal{D}'((0,h_\infty); L^2(\omega))$, then

$$\frac{\partial^2 \boldsymbol{u}}{\partial z^2} \in \mathcal{C}([0,h_\infty]; L^2(\omega)).$$

Moreover, by Equation (28),

$$\nabla p \in \mathcal{C}([0,h_\infty]; L^2(\omega))$$

and by Equation (29) (p does not depend on z), we conclude that $\nabla p \in L^2(\omega)$, i.e. $(\boldsymbol{u},p)$ satisfies (13) and (14). In particular, boundary conditions for the pressure on $\partial\omega$ make sense.

- *Step 2:* Let us state that Equation (18) holds.

The last point to be checked consists in showing that the pressure p is constant along the curve $\partial\omega^p$. Since $\boldsymbol{u}$ is a weak solution (that is solution of (26)) and $(\boldsymbol{u},p)$ satisfies (13), we immediately deduce by difference that the pressure p satisfies:

$$\forall \boldsymbol{\varphi} \in K(\mathbf{0},0) \cap \mathcal{D}(\Omega) \qquad \iiint_\Omega \nabla_x p \cdot \boldsymbol{\varphi} = 0. \tag{30}$$

The end of this part thus will be devoted to show that this condition (30) implies that p is constant on $\partial\omega^p$. The proof is realised in three sub-steps:

▷ Step 2-1. (Technical lemma)

Lemma 2.2. The following application is surjective:

$$\begin{array}{cccc} \Phi : K(\mathbf{0},0) \cap \mathcal{D}(\overline{\Omega}) & \longrightarrow & X = \{\boldsymbol{f} \in \mathcal{D}(\overline{\omega}),\ \mathrm{div}_x \boldsymbol{f} = 0,\ \boldsymbol{f}\cdot\boldsymbol{n} = 0 \text{ on } \partial\omega^q\} \\ \boldsymbol{\varphi} & \longmapsto & \displaystyle\int_0^h \boldsymbol{\varphi}(\cdot,z)\, dz \end{array}$$

Proof. Using Proposition 2.1, we show that this application is well defined and with values in X. For $\boldsymbol{f} \in X$, we define

$$\boldsymbol{\varphi}(x,z) = \frac{12}{h(x)^3} z(z-h(x))\boldsymbol{f}(x)$$

and we verify that $\boldsymbol{\varphi} \in K(\mathbf{0},0) \cap \mathcal{D}(\overline{\Omega})$ and $\Phi(\boldsymbol{\varphi}) = \boldsymbol{f}$. □

▷ Step 2-2. (Constant pressure on each connected component of $\partial\omega^p$) - Let us define

$$Z = \{\varphi \in \mathcal{D}(\partial\omega) \text{ such that } \frac{\partial\varphi}{\partial\tau} = 0 \text{ on } \partial\omega^q\}$$

(τ being the tangent vector to the boundary $\partial\omega$). For all $\varphi \in Z$, we extend φ on ω and define $\boldsymbol{f} = \text{rot } \varphi$. Since $\text{div(rot)} = 0$ and $\boldsymbol{n} \cdot \text{rot} = \partial/\partial\tau$, we deduce that $\boldsymbol{f} \in X$. There exists $\boldsymbol{\psi} \in K(\mathbf{0}, 0) \cap \mathcal{D}(\overline{\Omega})$ such that

$$\text{rot } \varphi = \int_0^h \boldsymbol{\psi}(\cdot, z)\ dz.$$

From (30), we deduce

$$\iint_\omega \nabla_x p \cdot \text{rot } \varphi = 0.$$

After integrating by parts, we obtain

$$\forall \varphi \in Z, \quad \int_{\partial\omega} p\ \frac{\partial\varphi}{\partial\tau} = 0. \tag{31}$$

Then, for $\widetilde{\varphi} \in \mathcal{D}(\partial\omega^p)$, extended by zero on $\partial\omega^q$, we have $\varphi + \widetilde{\varphi} \in Z$, so that

$$\int_{\partial\omega} p\ \frac{\partial(\varphi + \widetilde{\varphi})}{\partial\tau} = 0.$$

By difference with Equation (31), we find

$$\forall \widetilde{\varphi} \in \mathcal{D}(\partial\omega^p), \quad \int_{\partial\omega^p} p\ \frac{\partial\widetilde{\varphi}}{\partial\tau} = 0,$$

that is p is constant along each connected component of $\partial\omega^p$.

▷ Step 2-3. (The value of the boundary pressure is the same on each connected component of $\partial\omega^p$) - If $\partial\omega^p$ is composed of n connected components $\partial\omega_i^p$, $i \in \{1, ..., n\}$, then its complementary subset is also composed of n connected components: $\partial\omega_i^q$, $i \in \{1, ..., n\}$ (see Fig.2). For all $(a_1, ..., a_n) \in \mathbb{R}^n$, we define a function $a \in \mathcal{D}(\partial\omega)$ such that $a|_{\partial\omega_i^q} = a_i$. We have $\varphi + \widetilde{\varphi} + a \in Z$, so that

$$\int_{\partial\omega} p\ \frac{\partial(\varphi + \widetilde{\varphi} + a)}{\partial\tau} = 0.$$

By difference, we find

$$\int_{\partial\omega} p\ \frac{\partial a}{\partial\tau} = 0.$$

Since p is constant on each connected component $\partial\omega_i^p$ of $\partial\omega^p$ (with the value p_i), this equality may be also written as

$$\begin{aligned} 0 &= \int_{\partial\omega} p\ \frac{\partial a}{\partial\tau} = \int_{\partial\omega^p} p\ \frac{\partial a}{\partial\tau} = \sum_{i=1}^n p_i \int_{\partial\omega_i^p} \frac{\partial a}{\partial\tau} \\ &= \sum_{i=1}^n p_i(a_i - a_{i-1}) = \sum_{i=1}^n a_i(p_i - p_{i+1}), \end{aligned} \tag{32}$$

with the convention $p_1 = p_{n+1}$ and $a_0 = a_n$. As Equation (32) must be satisfied for all constants $a_i \in \mathbb{R}$, we find that all the p_i have the same value.

• *Step 3:* Since $\boldsymbol{u} \in K(\boldsymbol{s}, q_0)$, Equations (16) and (17) hold. If furthermore $\boldsymbol{u} \in H^1(\Omega)$ then, by Proposition 2.1, Equations (15) and (19) hold. □

2.2. *The viscoelastic case.* The introduction of viscoelastic phenomena differs from the purely Newtonian case by the effect of nonlinear additive terms. However, we show in this subsection that the approach developped earlier allows to state a rigorous analysis of the complete problem. Due to the introduction of the nonlinear terms, the mathematical analysis of the corresponding weak formulation (to be further detailed) has to be adapted in order to ensure the existence and uniqueness of the (weak) solution. Thus, let us first introduce the strong and weak formulations of the viscoelastic problem in a thin domain.

■ *Strong formulation:*

$$(\mathcal{Q}_s) \begin{cases} -\eta(1-r)\dfrac{\partial^2 \boldsymbol{u}}{\partial z^2} - \eta\, r\dfrac{\partial}{\partial z}\left(\dfrac{\dfrac{\partial \boldsymbol{u}}{\partial z}}{1 + C^2\left|\dfrac{\partial \boldsymbol{u}}{\partial z}\right|^2}\right) + \nabla_x p = \boldsymbol{F}, & \text{in } L^2(\Omega), \quad (33) \\ \dfrac{\partial p}{\partial z} = 0, & \text{in } L^2(\Omega), \quad (34) \\ \operatorname{div}_x\left(\displaystyle\int_0^h \boldsymbol{u}(\cdot, z)\, dz\right) = 0, & \text{in } L^2(\omega), \quad (35) \\ \boldsymbol{u} = \boldsymbol{s}, & \text{in } L^2(\Gamma_-), \quad (36) \\ \boldsymbol{u} = \boldsymbol{0}, & \text{in } L^2(\Gamma_+), \quad (37) \\ p = 0, & \text{in } L^2(\partial\omega^p), \quad (38) \\ \displaystyle\int_0^h \boldsymbol{u}(\cdot, z)\, dz \cdot \boldsymbol{n} = q_0, & \text{in } L^2(\partial\omega^q). \quad (39) \end{cases}$$

where the constant $C \geq 0$ includes viscoelastic parameters, namely $C^2 = \mathcal{D}e^2(1-a^2)$.

■ *Weak formulation:*

Following the same idea as before, nonlinear terms due to the viscoelasticity has to be taken into account, leading to a significant modification of the Newtonian case, so that the weak formulation of the problem is written as:

$$(\mathcal{Q}_w) \begin{cases} \text{Find } \boldsymbol{u} \in K(\boldsymbol{s}, q_0) \text{ such that} \\ \ll A\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{\varphi} \gg \,\leq\, \ll \boldsymbol{F}, \boldsymbol{u} - \boldsymbol{\varphi} \gg, \quad \forall\, \boldsymbol{\varphi} \in K(\boldsymbol{s}, q_0), \end{cases} \tag{40}$$

where $A : K(\boldsymbol{s}, q_0) \to (K(\boldsymbol{s}, q_0))'$ is the operator defined by

$$\ll A\boldsymbol{u}, \boldsymbol{v} \gg = \eta\,(1-r)\left(\frac{\partial\,\boldsymbol{u}}{\partial z}, \frac{\partial\,\boldsymbol{v}}{\partial z}\right)_{L^2(\Omega)} + \eta\, r\left(\frac{\dfrac{\partial\,\boldsymbol{u}}{\partial z}}{1+C^2\left|\dfrac{\partial\,\boldsymbol{u}}{\partial z}\right|^2}, \frac{\partial\,\boldsymbol{v}}{\partial z}\right)_{L^2(\Omega)}$$

and let us recall that $K(\boldsymbol{s}, q_0)$ is equipped with the norm $\|\cdot\|_z$ (see its definition given by (20)). Now, we give the following theorem, which is a generalisation of Theorem 1 taking into account the viscoelastic terms.

THEOREM 3 (Viscoelastic case). If $r < 8/9$, problem $(\mathcal{Q}_w)$ admits a unique solution.

Proof. The proof is based on a classical result on variational inequalities with monotone operators (see [**?**], page 247). It is obtained using three steps:

- *Step* 1*: boundary operator*

Obviously, since $r \geq 0$, we write

$$\ll A\boldsymbol{u}, \boldsymbol{u} \gg = \eta\,(1-r)\iiint_\Omega \left|\frac{\partial\,\boldsymbol{u}}{\partial z}\right|^2 + \eta\, r\iiint_\Omega \frac{\left|\dfrac{\partial\,\boldsymbol{u}}{\partial z}\right|^2}{1+C^2\left|\dfrac{\partial\,\boldsymbol{u}}{\partial z}\right|^2} \leq \eta\,\|\boldsymbol{u}\|_z^2,$$

which means that A is bounded.

- *Step* 2*: coercive operator*

Here, we use the fact that $r < 1$: indeed, $\dfrac{\ll A\boldsymbol{u}, \boldsymbol{u} \gg}{||\boldsymbol{u}||_z} \geq \eta\,(1-r)||\boldsymbol{u}||_z$, so that

$$\lim_{\|\boldsymbol{u}\|_z \to +\infty} \frac{\ll A\boldsymbol{u}, \boldsymbol{u} \gg}{||\boldsymbol{u}||_z} = +\infty.$$

- *Step* 3*: monotone operator*

We show here that the operator A is strictly monotone if and only if $r < 8/9$ (independently of the constant C). Thus let us compute $\ll A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \gg$:

$$\begin{aligned}
&\ll A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \gg \\
&\quad = \eta\,(1-r)\iiint_\Omega \left|\frac{\partial\,\boldsymbol{u}}{\partial z} - \frac{\partial\,\boldsymbol{v}}{\partial z}\right|^2 + \eta\, r\iiint_\Omega \frac{\left|\dfrac{\partial\,\boldsymbol{u}}{\partial z} - \dfrac{\partial\,\boldsymbol{v}}{\partial z}\right|^2\left(1 - C^2\dfrac{\partial\,\boldsymbol{u}}{\partial z}\cdot\dfrac{\partial\,\boldsymbol{v}}{\partial z}\right)}{\left(1+C^2\left|\dfrac{\partial\,\boldsymbol{u}}{\partial z}\right|^2\right)\left(1+C^2\left|\dfrac{\partial\,\boldsymbol{v}}{\partial z}\right|^2\right)} \\
&\quad = \iiint_\Omega \frac{\left|\dfrac{\partial\,\boldsymbol{u}}{\partial z} - \dfrac{\partial\,\boldsymbol{v}}{\partial z}\right|^2 \eta\, B\left(\dfrac{\partial\,\boldsymbol{u}}{\partial z}, \dfrac{\partial\,\boldsymbol{v}}{\partial z}\right)}{\left(1+C^2\left|\dfrac{\partial\,\boldsymbol{u}}{\partial z}\right|^2\right)\left(1+C^2\left|\dfrac{\partial\,\boldsymbol{v}}{\partial z}\right|^2\right)}
\end{aligned}$$

where $B(\boldsymbol{a},\boldsymbol{b}) = (1-r)\left(1+C^2\left|\boldsymbol{a}\right|^2\right)\left(1+C^2\left|\boldsymbol{b}\right|^2\right) + r\left(1-C^2\boldsymbol{a}\cdot\boldsymbol{b}\right)$. Rewriting this term as

$$\begin{aligned} B(\boldsymbol{a},\boldsymbol{b}) \quad = \quad & C^2(1-r)\left(1+C^2\left|\boldsymbol{b}\right|^2\right)\left|\boldsymbol{a}-\frac{r\boldsymbol{b}}{2(1-r)\left(1+C^2\left|\boldsymbol{b}\right|^2\right)}\right|^2 \\ & +\frac{1}{4(1-r)\left(1+C^2\left|\boldsymbol{b}\right|^2\right)}\left(2(1-r)C^2\left|\boldsymbol{b}\right|^2+\frac{3r^2-12r+8}{4(1-r)}\right)^2 \\ & +\frac{r^3}{64(1-r)^3\left(1+C^2\left|\boldsymbol{b}\right|^2\right)}(8-9r), \end{aligned}$$

we deduce the sign of $\ll A\boldsymbol{u}-A\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}\gg$. Indeed, studying the sign of $B(\boldsymbol{a},\boldsymbol{b})$ gives:

- ▷ if $r<8/9$, the operator A is stricly monotone.
- ▷ if $r=8/9$, the operator A is monotone.
- ▷ if $r>8/9$, the operator A is non monotone: we can find $\boldsymbol{u}$ and $\boldsymbol{v}$ such that

$$\ll A\boldsymbol{u}-A\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}\gg<0.$$

Now, the proof is concluded using the theory of monotone operators in variational inequalities (see [**?**], page 247). □

Remark 2.2 (A non-uniqueness result). Interestingly, we can prove that the problem is well-posed if $r<8/9$. In the case $r=8/9$, the proof of Theorem 3 ensures existence of a weak solution (but not necessarily uniqueness). In the case $r>8/9$, it does not even state an existence result. However, using a simple geometrical configuration ($h\equiv 1$), a counter-example for uniqueness can be established (see [**?**] for further details).

The link between $(\mathcal{Q}_w)$ and $(\mathcal{Q}_s)$ is given by the following theorem.

Theorem 4 (Viscoelastic case). Let $\boldsymbol{u}$ be the unique solution of $(\mathcal{Q}_w)$.

(*i*) There exists a unique $p\in H^1(\omega)$ such that $(\boldsymbol{u},p)$ satisfies (33), (34), (36)–(38).

(*ii*) Moreover, if $\boldsymbol{u}\in H^1(\Omega)$, then (35) and (39) hold. In particular, $(\boldsymbol{u},p)$ is the unique solution of $(\mathcal{Q}_s)$.

Proof. The result is stated using the same arguments that have been developped in the proof of Theorem 2. □

In the next section, we provide some tools which allow to solve the asymptotic equations of a viscoelastic flow in a thin domain. We present and analyse an algorithm and, then, we focus on some applications which are related to lubrication theory: in particular, we illustrate boundary effects thus showing that the infinite journal bearing approximation, which is widely used in tribology, may lack relevance in viscoelastic regimes.

3. Numerical results and discusion. As it has been mentionned before, introducing viscoelastic effects leads to add a non linear term into the classical newtonian problem in pressure-velocity. This prevents us to follow the classical way to obtain only a problem in pressure both for the full continuous problem than to the numerical discretised one. Then we proposed a new method which will be presented in detail in section 3.3. This method is based on a two-step fixed point procedure. Actually, we are not able to rigously prove the convergence of this method in the general case. However, we give in section 3.2 a convergence result of each sub-step which can be considered as a new way of solving a near-newtonian problem presented in section 3.1.

The method has been developped for a domain ω which is supposed to be rectangular with dimensions $L \times D$.

3.1. *Numerical analysis for the Newtonian case.*
Let us recall the main equations of the Newtonian model:

$$(\mathcal{P}) \begin{cases} -\dfrac{\partial}{\partial z}\left(\eta \dfrac{\partial \boldsymbol{u}}{\partial z}\right) + \nabla_x p = \mathbf{0}, \\ \operatorname{div}_x\left(\displaystyle\int_0^h \boldsymbol{u}(\cdot, z)\, dz\right) = 0. \end{cases}$$

In order to solve $(\mathcal{P})$, a semi-discretized version of this problem, in the (x_1, x_2)-direction, is introduced. Thus, we use a centered structured grid based on a classical cell configuration (see FIG.3). This particular case corresponds to an imposed flux on the left boundary $x_1 = 0$ and Dirichlet conditions for the pressure on the other boundaries. A similar discretisation may be adapted to the case of Dirichlet conditions for the pressure on the whole boundary. Let us denote by $N = N_{x_1} \times N_{x_2}$ the overall number of un-

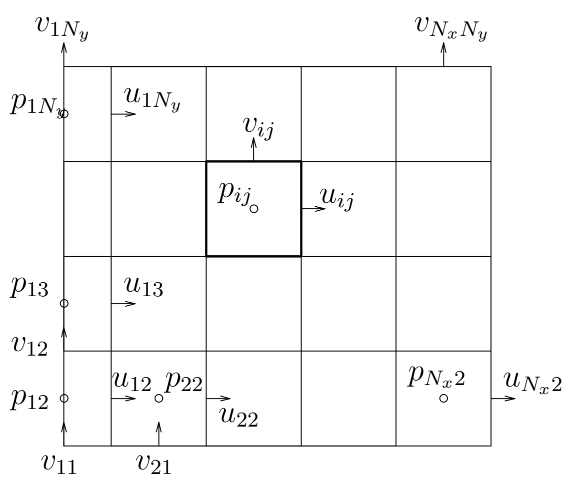

FIG. 3. Spatial discretisation and position of the unknowns

knowns corresponding to this discretisation, by δ_1 (resp. δ_2) the step in the x_1 (resp.

x_2) direction, by h_{ij} the value of h at a node (i,j). Furthermore, we denote

$$\begin{gathered}\boldsymbol{U}(z) = \left(\boldsymbol{u}_{ij}(z)\right)_{i,j} := \left(\boldsymbol{u}(i\delta_1, j\delta_2, z)\right)_{i,j}\\ P = \left(p_{ij}\right)_{i,j} := \left(p(i\delta_1, j\delta_2)\right)_{i,j}\end{gathered}$$

the semi-discretized horizontal velocity and discretized pressure.

Let A (resp. B) corresponds to the x-discretisation of the operator ∇ (resp. div). Moreover, we use the notation

$$\left(\widetilde{H\boldsymbol{U}}\right)_{ij} := \int_0^{h_{ij}} \boldsymbol{u}_{ij}(z)\, dz.$$

The problem $\mathcal{P}$ can be semi-discretised (i.e. discretised in the x-variable only) in the following one

$$(\mathcal{P}^*)\begin{cases} -\dfrac{\partial}{\partial z}\left(\eta\dfrac{\partial \boldsymbol{U}}{\partial z}\right) + A\circ P = \mathbf{0},\\ B\circ\left(\widetilde{H\boldsymbol{U}}\right) = 0.\end{cases}$$

Concerning the boundary conditions, we impose that, for each nodes (i,j),

$$u_{ij} \in H^1(0,h_{ij}) \quad \text{with } u_{ij}(0) = s_{ij} \text{ and } u_{ij}(h_{ij}) = 0,$$

the impose velocity $\boldsymbol{s}$ being discetised by s_{ij}. For the pressure, we imposed a dirichlet boundary condition which writes

$$p_{ij} = \widetilde{p_{ij}} \quad \text{for } (i,j) \text{ at the boundary of the discret domain,}$$

the imposed pressure being denoted by $\widetilde{p_{ij}}$.

Notice that it is possible to solve $(\mathcal{P}^*)$ in a near analytic way by two integrations in the z-direction the first equation in $(\mathcal{P}^*)$, taking into account the boundary condition on the velocity. We deduce

$$\boldsymbol{U} = ... \tag{41}$$

Then putting the corresponding value of $\boldsymbol{U}$ as a function of the pressure P in the last equation of $(\mathcal{P}^*)$, we get the equation satisfies by P:

$$B\circ\left(\frac{h^3}{12\eta}A\circ P\right) = B\circ\left(\frac{h}{2}\boldsymbol{s}\right).$$

This equation is the discretised finite difference formulation of the Reynolds equation whose solution P is unique and induces the knowledge of the velocity $\boldsymbol{U}$ by Equation (41).

As it has been mentionned before, this last approach can not be generalised in the viscoelastic case. Then, we proposed another algorithm which does not used the z-integration as the previous one. This algorithm is based on a fixed point formulation of the semi-discretised problem $(\mathcal{P}^*)$:

$$(\mathcal{P}^k)\begin{cases} -\dfrac{\partial}{\partial z}\left(\eta\dfrac{\partial \boldsymbol{U}^{k+1}(z)}{\partial z}\right) + A\circ P^k = \mathbf{0}, & (42)\\ P^{k+1} - P^k + \rho\, B\circ\left(\widetilde{H\boldsymbol{U}}^{k+1}\right) = 0. & (43)\end{cases}$$

The stopping test of this process is based on the pressure error $P^{k+1} - P^k$ and on the velocity error $\boldsymbol{U}^{k+1} - \boldsymbol{U}^k$. To note that the precision sought in pressure will induce a precision on the incompressibility condition via the parameter ρ. Indeed, the algorithm is stopped as soon as $P^{k+1} - P^k$ is smaller than a prescribed value, denoted r_p, in some sense (in the discrete ℓ^2 norm, for instance). This condition being satisfied, it means in particular that the divergence term satisfies

$$\max_{ij} \left| \left(B \circ \left(\widetilde{H\boldsymbol{U}}^{k+1} \right) \right)_{ij} \right| \leq \frac{r_p}{\rho},$$

i.e. the free divergence equality is satisfied with an order r_p/ρ. For this reason, r_p/ρ will be called "equilibrium parameter (for the free divergence condition)". In order to numerically attain the free divergence equality, we have to impose some r_p satisfying $r_p \ll \rho$.

3.2. *Convergence of the method.* We state the following theorem:

THEOREM 5 (Convergence result). Assume that

$$0 < \rho < \frac{3\,\eta}{2\|h\|^3_{L^\infty(\omega)} \left(\dfrac{1}{\delta_1^2} + \dfrac{1}{\delta_2^2} \right)}.$$

Then for all $k \in \mathbb{N}$, the problem $(\mathcal{P}^k)$ admits a solution such that

$$(\boldsymbol{U}^k, P^k) \in \left(\prod_{ij} H^1(]0, h_{ij}[) \right) \times \mathbb{R}^{N_{x_1} \times N_{x_2}}.$$

Moreover, there exists a subsequence (still denoted $\{k\}$) such that, for all (i,j)

$$\begin{array}{lcll} \boldsymbol{U}^k & \rightharpoonup & \boldsymbol{U} & \text{in } \prod_{ij} H^1(]0, h_{ij}[), \\ P^k & \to & P & \text{in } \mathbb{R}^{N_{x_1} \times N_{x_2}}. \end{array}$$

$\boldsymbol{U}$ and P being the solution of the problem $(\mathcal{P}^*)$.

Proof. First of all, using the linearity of problems $(\mathcal{P}^k)$ and $(\mathcal{P}^*)$, we prefer to work with the quantities $\overline{\boldsymbol{U}^k} = \boldsymbol{U}^k - \boldsymbol{U}$ and $\overline{P^k} = P^k - P$ which satisfy the problem $(\mathcal{P}^k)$ with homogeneous boundary Dirichlet conditions. For the sake of simplicity, during this proof, we denote by $\boldsymbol{U}^k$ and P^k instead of $\overline{\boldsymbol{U}^k}$ and $\overline{P^k}$.

At that point, we want to obtain estimates on the sequence $(\boldsymbol{U}^k, P^k)$ and then prove that it converge to zero in appropriate space. For each i,j, multiplying each component $(42)_{ij}$ by $\boldsymbol{u}_{ij}^{k+1}$, integrating over $[0, h_{ij}]$ and then making the sum for all i,j, we have, using an integration by parts,

$$\left(\eta \frac{\partial \boldsymbol{U}^{k+1}}{\partial z}, \frac{\partial \boldsymbol{U}^{k+1}}{\partial z} \right)_* + \left(A \circ P^k, \boldsymbol{U}^{k+1} \right)_* = 0, \tag{44}$$

where $(\cdot,\cdot)_*$ indicates the $*$-scalar product

$$(\boldsymbol{U}, \boldsymbol{V})_* = \sum_{i,j} \int_0^{h_{ij}} \boldsymbol{u}_{ij}(z) \boldsymbol{v}_{ij}(z)\; dz$$

and, in the same way, $(\cdot,\cdot)_\#$ is the scalar product defined by

$$(P,Q)_\# = \sum_{i,j} p_{ij}\, q_{ij}.$$

Now, $|\cdot|_\#$ being the associated norm, we deduce from Equation (43), after taking the #-scalar product by P^{k+1}, that

$$|P^{k+1}|^2_\# - |P^k|^2_\# + |P^{k+1}-P^k|^2_\# + 2\rho\left(B\circ\left(\widetilde{H\boldsymbol{U}}^{k+1}\right), P^{k+1}\right)_\# = 0. \qquad (45)$$

Moreover, one has (first using a discrete integration by parts and then observing that P^{k+1} does not depend on z):

$$\left(B\circ\widetilde{H\boldsymbol{U}}^{k+1}, P^{k+1}\right)_\# = -\left(A\circ P^{k+1}, \widetilde{H\boldsymbol{U}}^{k+1}\right)_\# = -\left(A\circ P^{k+1}, \boldsymbol{U}^{k+1}\right)_* .$$

Using the previous equality, adding Equations (44) and (45) (with a multiplier 2ρ for (44)), we obtain the following estimate

$$\begin{aligned}|P^{k+1}|^2_\# - |P^k|^2_\# + |P^{k+1}-P^k|^2_\# + 2\rho\left|\sqrt{\eta}\frac{\partial \boldsymbol{U}^{k+1}}{\partial z}\right|^2_* \\ = 2\rho\left(A\circ(P^{k+1}-P^k), \widetilde{H\boldsymbol{U}}^{k+1}\right)_\# .\end{aligned} \qquad (46)$$

Now, we state estimates for the second member of this equality, denoted I_1:
The operator A is bounded[1] by $\sqrt{\lambda_{xy}}$ defined by

$$\lambda_{xy} = 2\left(\frac{1}{\delta_1^2}+\frac{1}{\delta_2^2}\right).$$

We obtain that

$$I_1 \le 2\rho\sqrt{\lambda_{xy}}\; |P^{k+1}-P^k|_\# \;\left|\widetilde{H\boldsymbol{U}}^{k+1}\right|_\#,$$

and using the fact that for all $(a,b)\in\mathbb{R}^2$ and $\alpha>0$, we have $2ab \le \dfrac{a^2}{\alpha} + \alpha b^2$ we find that, for all $\alpha>0$

$$I_1 \le \frac{\rho\lambda_{xy}}{\alpha}\; |P^{k+1}-P^k|^2_\# + \alpha\rho\left|\widetilde{H\boldsymbol{U}}^{k+1}\right|^2_\#.$$

Moreover, for a regular function $g:[0,h]\to\mathbb{R}$ such that $g(h)=0$, we note that

$$g(z) = \int_h^z \frac{\partial g}{\partial z}(\xi)\; d\xi$$

so that, integrating over $[0,h]$,

$$\int_0^h g(z)\; dz = \int_0^h \left(\int_h^z \frac{\partial g}{\partial z}(\xi)\; d\xi\right) dz = \int_0^h z\frac{\partial g}{\partial z}(z)\; dz,$$

[1] Indeed, we have for instance

$$|A\circ P|^2_\# = \sum_{ij}\left(\frac{p_{i+1,j}-p_{ij}}{\delta_1}\right)^2 + \sum_{ij}\left(\frac{p_{i,j+1}-p_{ij}}{\delta_2}\right)^2 \le 2\left(\frac{1}{\delta_1^2}+\frac{1}{\delta_2^2}\right)|P|^2_\#.$$

and using the Cauchy-Schwarz inequality,

$$\left(\int_0^h g(z)\,dz\right)^2 \le \left(\int_0^h \frac{z^2}{\eta}\,dz\right)\left(\int_0^h \eta\Big(\frac{\partial g}{\partial z}(z)\Big)^2\,dz\right).$$

Thus, we deduce that

$$\begin{aligned}\left|\widetilde{H\boldsymbol{U}}^{k+1}\right|_{\#}^2 &= \sum_{ij}\left(\int_0^{h_{ij}} \boldsymbol{u}_{ij}^{k+1}(z)\,dz\right)^2 \\ &\le C(h,\eta)\sum_{ij}\int_0^{h_{ij}} \eta\Big(\frac{\partial\,\boldsymbol{u}_{ij}^{k+1}}{\partial z}(z)\Big)^2\,dz,\end{aligned}$$

where $C(h,\eta) = \max\limits_{i,j}\displaystyle\int_0^{h_{ij}} \frac{z^2}{\eta}\,dz = \frac{\|h\|^3_{L^\infty(\omega)}}{3\eta}$. That is,

$$\left|\widetilde{H\boldsymbol{U}}^{k+1}\right|_{\#}^2 \le C(h,\eta)\left|\sqrt{\eta}\frac{\partial \boldsymbol{U}^{k+1}}{\partial z}\right|_*^2.$$

We obtain

$$I_1 \le \frac{\rho\lambda_{xy}}{\alpha}\,|P^{k+1}-P^k|_{\#}^2 + \alpha\rho C(h,\eta)\left|\sqrt{\eta}\frac{\partial \boldsymbol{U}^{k+1}}{\partial z}\right|_*^2. \tag{47}$$

Putting Inequality (47) into (46), with an appropriate choice for α and ρ (to be detailed later), we can define two constants $c_1 > 0$ and $c_2 > 0$ such that:

$$|P^{k+1}|_{\#}^2 - |P^k|_{\#}^2 + \underbrace{\left(1-\frac{\rho\lambda_{xy}}{\alpha}\right)}_{c_1>0}|P^{k+1}-P^k|_{\#}^2 + \underbrace{\rho\Big(2-\alpha\,C(h,\eta)\Big)}_{c_2>0}\left|\sqrt{\eta}\frac{\partial \boldsymbol{U}^{k+1}}{\partial z}\right|_*^2 \le 0. \tag{48}$$

The sign conditions on the two constants are clearly satisfied if

$$0 < \rho < \frac{2}{C(h,\eta)\,\lambda_{xy}} := \rho_{\text{crit.}},$$

and α being arbitrarily chosen in the set $]\lambda_{xy}\,\rho,\,\lambda_{xy}\,\rho_{\text{crit.}}[$. Notice that $\rho_{\text{crit.}}$ is a critical value of the parameter ρ allowing the above estimates.
Summing the estimates (48) for $k = 0$ to $k = K$, we find bounds for the sequences $\boldsymbol{U}^k$ and P^k which allows us to perform the limite $k \to +\infty$ in the problem $(\mathcal{P}^k)$. The limite being the unique solution of the problem $(\mathcal{P}^*)$ (which is the zero solution since the boundary condition are zero), we obtain the desired result. □

3.3. *Numerical analysis for the viscoelastic case.*

3.3.1. *Algorithm.* The nonlinear problem (33)–(39) is solved using a fixed-point method at different levels of the resolution. Let us define a continuous fixed-point procedure: the idea of the general algorithm relies on the possibility to attain solution of the nonlinear problem (33)–(39) as the limit ($n \to +\infty$) of the following problem:

$$(\mathcal{P}_n) \begin{cases} -\dfrac{\partial}{\partial z}\left(f(\boldsymbol{u}^n)\dfrac{\partial\,\boldsymbol{u}^{n+1}}{\partial z}\right) + \nabla_x p^{n+1} = \mathbf{0}, \\ \operatorname{div}_x\left(\displaystyle\int_0^h \boldsymbol{u}^{n+1}(\cdot, z)\,dz\right) = 0. \end{cases}$$

with

$$f(\boldsymbol{u}^n) = \eta\,(1-r) + \frac{\eta\, r}{1 + \mathcal{D}e^2(1-a^2)\left|\dfrac{\partial\,\boldsymbol{u}^n}{\partial z}\right|^2}$$

In order to solve $(\mathcal{P}_n)$, the same semi-discretisation, in the (x_1, x_2)-direction, is used.

Now, we present the algorithm which solves the semi-discrete version of $(\mathcal{P}_n)$. The way to compute $\boldsymbol{U}^{n+1}$ and P^{n+1} is provided by the algorithm presented in the Newtonian case :

$$\begin{cases} \text{Input:} & \boldsymbol{U}^{n,0} = \boldsymbol{U}^n, \quad P^{n,0} = P^n, \\ \text{Loops on } k: & (\mathcal{P}_n^k) \begin{cases} -\dfrac{\partial}{\partial z}\left(f(\boldsymbol{U}^n)\dfrac{\partial\,\boldsymbol{U}^{n,k+1}}{\partial z}\right) + A \circ P^{n,k} = \boldsymbol{G}, \\ P^{n,k+1} - P^{n,k} + \rho\, B \circ \left(\widetilde{H\boldsymbol{U}}^{n,k+1}\right) = 0, \end{cases} \\ \text{Output:} & \boldsymbol{U}^{n+1} = \boldsymbol{U}^{n,\infty}, \quad P^{n+1} = P^{n,\infty}. \end{cases}$$

The algorithm is stopped as soon as $P^{n+1} - P^n$ is smaller than a prescribed value in some sense (in the discrete ℓ^2 norm, for instance).

3.3.2. *Remarks on the method.* The algorithm that we propose views the viscoelastic problem as a sequence of Newtonian-type problems. Formally, the numerical solution which is attained is a fixed-point solution of the semi-discretized version of $(\mathcal{P}_n)$.

Following the same idea as in the Newtonian case, the theoretical study establishes the boundedness of the sequence, provided some constraints (which do note depend on k and n) are respected. More precisely, we can notice that, since the function f satisfies $f \geq \eta\,(1-r)$, then we obtain estimates which do not depend on n and k under the condition

$$0 < \rho < \frac{3\,\eta}{2\|h\|^3_{L^\infty(\omega)}\left(\dfrac{1}{\delta_1^2} + \dfrac{1}{\delta_2^2}\right)(1-r)}.$$

This condition is more restrictive than the preceeding one, but sufficient for all n-step. Unfortunately, it is not so obvious that the sequence of solutions $(\boldsymbol{U}^n, P^n)$ converges to a fixed-point solution of the semi-discretised version of problem $\mathcal{P}_n$, because of the nonlinearity which leads to a lack of compactness.

However, in practical situations, we observe the following phenomena:

(i) Under the constraint $r < 8/9$, we observe that the algorithm converges to a numerical viscoelastic solution under the previous condition.

(ii) Under the constraint $r > 8/9$, it is observed that the sequence of the numerical Newtonian-type solutions does not converge to a viscoelastic one, which may be related to the non-uniqueness result (see [?] for similar observations in a Stokes-Oldroyd flow).

3.4. *Numerical results.* In this subsection, we propose three series of numerical tests:

- Test 1: we study the influence of the numerical parameters on the solution. In particular, the control of ρ with respect to the stopping error may have some influence on the numerical solution. However, at least in the Newtonian case, we illustrate the behaviour of the solution with respect to ρ and show that it converges to the solution of the Reynolds equation as ρ tends to 0.
- Test 2: we study the influence of the Deborah number.
- Test 3: we show that three-dimensional effects may occur. In particular, the approximation of the "journal bearing of infinite width", which is valid (and widely used) in the Newtonian case, cannot be considered due to viscoelastic effects.

For this, the following data have been used

	Test 1	Test 2	Test 3
Domain ω	$[0,1]\times[0,5]$	$[0,1]\times[0,5]$	$[0,1]\times[0,5]$
Gap $h(x)$	$(2x_1-1)^2+0.5$	$1-0.3x_1+0.5x_1^2$	$(2x_1-1)^2+0.5$
Shear velocity $\boldsymbol{s}$	$(1,0)$	$(1,0)$	$(1,0)$
Deborah De	0	$0.1\sim 3$	0.8
Retardation r	0	0.8	$0\sim 0.8$
Conditions at $x_1=0$	flux	pressure	flux
Conditions at $\partial\omega\setminus\{x_1=0\}$	pressure	pressure	pressure
Mesh size	$40\times40\times20$	$40\times20\times20$	$40\times80\times20$
Artificial time step ρ	10^{-3}	10^{-3}	8.10^{-4}
Equilibrium parameter r_p/ρ	$10^{-2}\sim 10^{-4}$	10^{-4}	10^{-4}

TABLE 1. Numerical data

3.4.1. *Test 1: influence of the numerical parameters.* In this setting, we study the purely Newtonian case, which allows us to compare our numerical pressure to the theoretical one: the solution of the classical Reynolds equation. In particular, we focus on the role of the equilibrium parameter r_p/ρ (corresponding to the error on the free divergence condition). Since our goal is to get simultaneously the convergence of the pressure and the equilibrium of the free divergence condition, we first impose an artificial time step $\rho = 10^{-3}$, which ensures the convergence of the method. Then, we choose different values of r_p/ρ in order to observe its numerical influence over the corresponding solution: in particular, it is sufficient to compare our numerical solution (for different values of r_p/ρ) to the Reynolds one. Thus, we consider numerical data given in TABLE 1. Let us precise the values for the boundary conditions: at the (left) boundary $x_1 = 0$, the normalized flux is given by $q_0 = 0.3 s_x h_{|x_1=0}$ while, at other boundaries, the pressure is $p = 0$.

Now, the influence of the ratio r_p/ρ is illustrated on FIG.4: the left-hand side figure is the Reynolds pressure distribution, in the full domain. The right-hand side figure allows to observe in the x_1 direction (at a fixed x_2, namely $x_2 = x_2^0 = 2.5$) the solutions corresponding to different values of r_p/ρ. It can be observed that the numerical pressure tends to the Reynolds one as the value of r_p/ρ decreases. At $r_p/\rho = 10^{-4}$, numerical and Reynolds solutions even coincide.

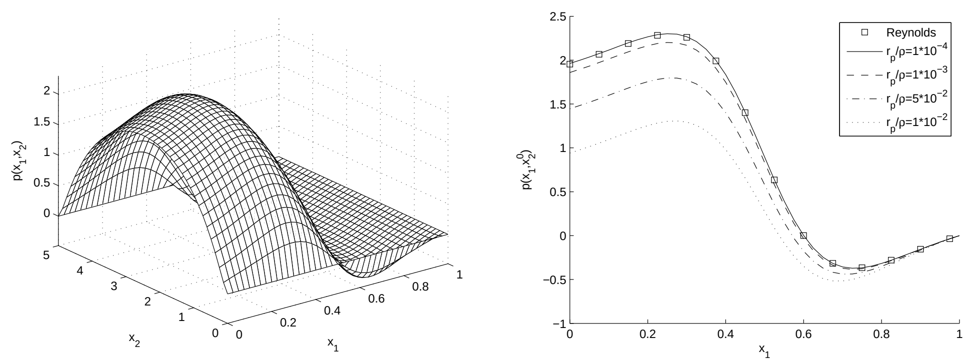

FIG. 4. Influence of the equilibrium parameter

3.4.2. *Test 2: influence of the Deborah number.* In this subsection, we compare our model to the ones developped by F.T. Akyildiz and H. Bellout [**?**] and J.A. Tichy [**?**]. Notice that, unlike our model, these previous works only deal with two-dimensional flows, corresponding for example to journal bearings with an infinite width (i.e. devices whose size satisfy $D/L > 4$). This assumption allows to consider that, up to boundary effects localized at $x_2 = 0$ and $x_2 = D$, the flow is mainly described by its behaviour at a cross section ($x_2 = D/2$ for instance) and that it remains the same at another cross section (as long as it is far from the boundaries). Following the work of F.T. Akyildiz and H. Bellout [**?**], we choose the physical data given at TABLE 1. To complete the scope of the boundary conditions, let us metion that $p = 0$ is imposed on the whole boundary $\partial\omega$.

More precisely, in order to observe the effects of the Deborah number over the pressure distribution, we used the same values as in the paper of F.T. Akyildiz and H. Bellout [**?**]: $\mathcal{D}e = 0.1$, $\mathcal{D}e = 0.2$,..., $\mathcal{D}e = 3$. We may observe the behaviour of the solution, as $\mathcal{D}e$ increases, on FIG.5, corresponding to the pressure profiles at a fixed $x_2 = 2.5$.

For small values of $\mathcal{D}e$ (0, 0.1, 0.2, 0.3), the results are similar to the ones of F.T. Akyildiz and H. Bellout, except that they have been generalized to a three-dimensional flow: viscoelastic effects tend to damp the peak pressure.

For large values of $\mathcal{D}e$ (1, 2,...), the results differ from the ones of F.T. Akyildiz and H. Bellout: this is due to the fact that our initial models are different (and so are the corresponding asymptotic analyses). Here, we may observe that the viscoelastic solution, as $\mathcal{D}e \to +\infty$ converges to the solution of the purely viscous solution with an effective viscosity parameter $\eta(1-r)$ (instead of η). The viscoelastic nonlinear contribution formally tends to vanish for large values of $\mathcal{D}e$. However, our model is not relevant for large

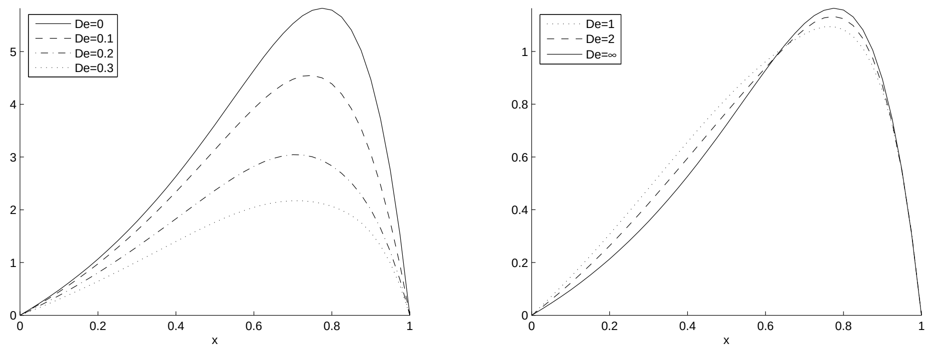

FIG. 5. Influence of the Deborah number

values of $\mathcal{D}e$, since $\mathcal{D}e$ is assumed to be of order ε.

3.4.3. *Test 3: three-dimensional effects.* As it was pointed out in the previous subsection, when the length and width of a device satisfy $D/L > 4$, a classical approximation is used in lubrication theory: the one-dimensional Reynolds equation is used to describe the behaviour of the flow at any cross-section which is not located at boundaries $x_2 = 0$ or $x_2 = D$. This assumption allows to reduce the space dimension in the analysis of such phenomena. This is well understood in the Newtonian case but numerical tests illustrate the fact that such an assumption is not necessarily relevant when viscoelastic effects occur: indeed, three dimensional boundary layers are induced by viscoelastic effects.

We have used the physical and numerical data given at TABLE 1. Let us precise the values for the boundary conditions: at the (left) boundary $x_1 = 0$, the normalized flux is given by $q_0 = 0.2 s_x h_{|x_1=0}$ while, at other boundaries, the pressure is $p = 0$.

On FIG.6, (from left to right, top to bottom), we have the pressure profiles corresponding to $r = 0$ (Newtonian case), $r = 0.2$, $r = 0.5$ and $r = 0.8$. Obviously, the one-dimensional flow assumption in the Newtonian case is valid as long as the cross-section is not located at the boundaries, but we can see that this assumption does not hold anymore for increasing values of r.

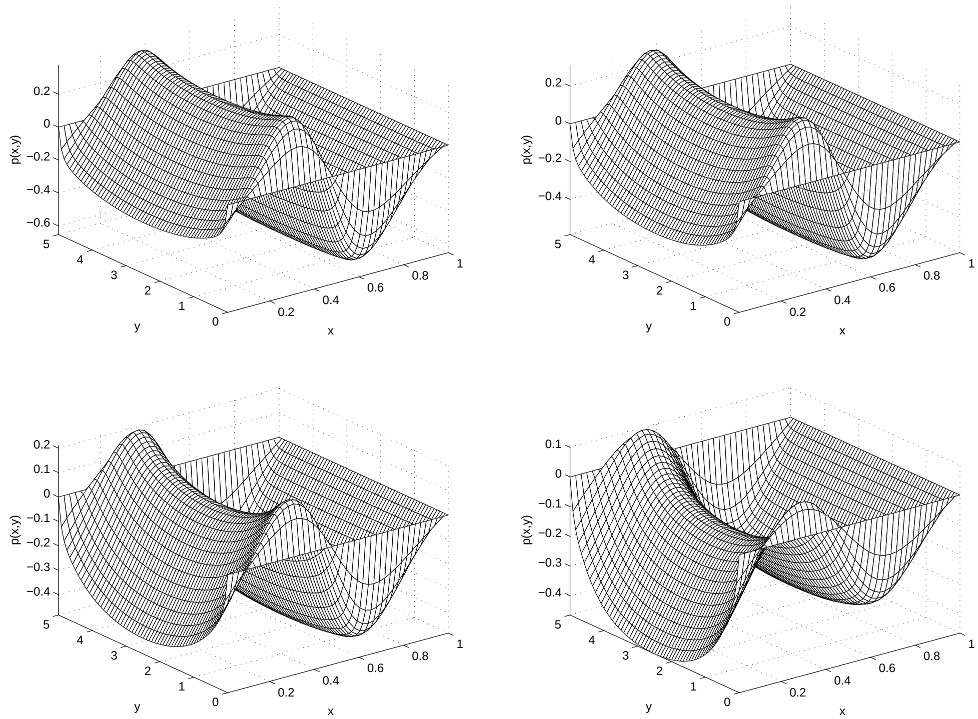

FIG. 6. Pressure profiles for $r = 0.0$, $r = 0.2$, $r = 0.5$ and $r = 0.8$