An unconditional existence result for elastohydrodynamic piezoviscous lubrication problems with Elrod-Adams model of cavitation[1]

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Abstract

An unconditional existence result of a solution for a steady fluid-structure problem is stated. More precisely, we consider an incompressible fluid in a thin film, ruled by the Reynolds equation coupled with a surface deformation modelled by a non-linear non local Hertz law. The viscosity is supposed to depend non-linearly on the fluid pressure. Due to the apparition of a mushy region, the two-phase flow satisfies a free boundary problem defined by a pressure-saturation model.

Such a problem has been studied with simpler free boundaries models (variational inequality), or with boundary conditions imposing small data assumptions. We show that up to a realistic hypothesis on the asymptotic pressure-viscosity behaviour it is possible to obtain an unconditional solution of the problem.

1 Introduction

The knowledge of the pressure in a lubricated device is a key problem to compute operational characteristic of such devices such as bearings, seals, magnetic recorder heads... Mathematically speaking, it means to solve the Reynolds equation ([10]). At first glance, it is a classical elliptic equation in which coefficients are related to the viscosity μ of the fluid, the gap h between the surrounding surfaces and some velocities data. However it is well known that in real operational condition, the pressure inside the fluid is so high that

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the viscosity is no longer constant while the surrounding elastic surfaces are deformed. This fluid-structure interaction is often described by the Hertz integral model ([18]).

Moreover, the fluid cannot be considered as an homogeneous one. Thus a free boundary between a full film area and the mushy region made by a mixture of oil and air (the cavitation region) must be included in the model. The most usual one in the mathematical literature is based upon a first kind of variational inequality ([11], [16]). Considering all these aspects leads to a much more complicated Reynolds EHD (elastohydrodynamic) "equation" which is a quasi-variational non local non linear inequality. Existence theorem and uniqueness results have been obtained by Oden an Wu ([15]), Rodriguez ([17]), Hu ([12]). Most often, the proof of the existence is obtained by a fixed point approach using both L^∞ and H^1 estimates, using a small data assumption to obtain compactness results.

More recently, it has been observed ([5], Bayada and Bellout [2]) that such small data assumption can be avoided if a specific viscosity-pressure behaviour is assumed. From a practical point of view, this behaviour is much more reasonable than the small data assumption: satisfactory numerical computation results are obtained for a very large range of data while the specific viscosity-pressure behaviour retained does not contradict any experiments ([19]).

Another step in the complexity of the model was introduced as it was observed ([8], [9]) that the previously used variational inequality model describing the cavitation does not fulfill a mass flow conservation property. Moreover, it cannot be used to describe some phenomena like starvation since only data on the pressure can be used in the variational inequality model in a satisfactory way. Based upon a generalization of the free boundary in the dam problem ([7], [6]), the new mathematical model addresses a two-unknown system (pressure and saturation) and a hyperbolic-elliptic Reynolds equation. This model is a full conservative one and allows both data on the pressure and input flow to be dealt with. Existence theorem and uniqueness properties have been obtained in [3], [1] for basic isoviscous fluid and rigid surfaces. Generalization to the full piezoviscous EHD problem appears in [4] in which an existence theorem using a small data assumption has been obtained considering only data for the pressure.

The purpose of the present paper is to prove that for this new cavitation model, the small data assumption can be avoided while boundary data both on input flow and pressure can be introduced. To be observed also is the fact that while small data assumption allows various approaches to be used (see [4]), the present work relies strongly on the Grübin transform (see section 2) and does not seem to be generalisable to other approaches.

In section 2 a precise statement of the problem is given and some related regularized systems are introduced. Section 3 is devoted to the obtaining of some estimates. Some of them are very close, although different, from the one used for the small data case. At last in section 4 new estimates and the introduction of a specific viscosity-pressure relation allow to prove the existence of a solution to the problem (Theorem 4.7).

2 Formulation and regularization of the problem

2.1 Statement of the problem

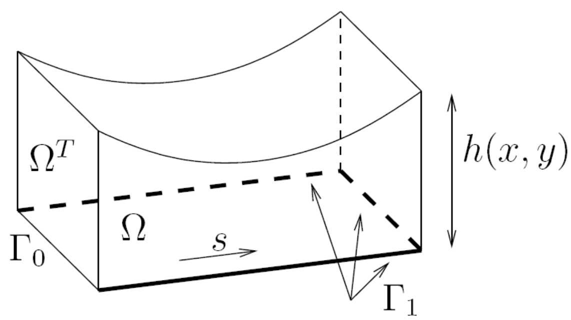

Figure 1: Domain Ω^T

Let $\Omega = (0, L) \times (0, 1)$ a rectangular two-dimensional domain, with its boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\Gamma_0 = \{(0, y), y \in (0, 1)\}$. Let Q be the three-dimensional domain given by $Q = \Omega \times (0, h(x, y))$ (see Figure 1).

We consider a newtonian fluid in the domain Q, with a given input parameter $\mathcal{G}_0$ on $\Gamma_0 \times (0, h(x, y))$, and a given velocity $\mathbf{s} = (s, 0)$ on Ω. Moreover, let us introduce $G_0(y) = \displaystyle\int_0^{h(0,y)} \mathcal{G}_0(0, y, z)\mathrm{d}z$.

In a thin domain (i.e. h small with respect to the other dimensions), it is possible to reduce the Navier-Stokes equations to the Reynolds equation, which is an equation in Ω on the pressure only. In order to take into account the phenomenon of cavitation, we introduce the Elrod-Adams model.

This model considers that the cavitation zone is characterized by :

- a constant pressure, supposed to be equal to zero,
- an homogeneous blend of air and fluid, for example oil.

It introduces a function θ, defined in Ω, corresponding to the local proportion of the fluid in an elementary domain around the point $M(x, y)$, for $(x, y) \in \Omega$ (see Figure 2).

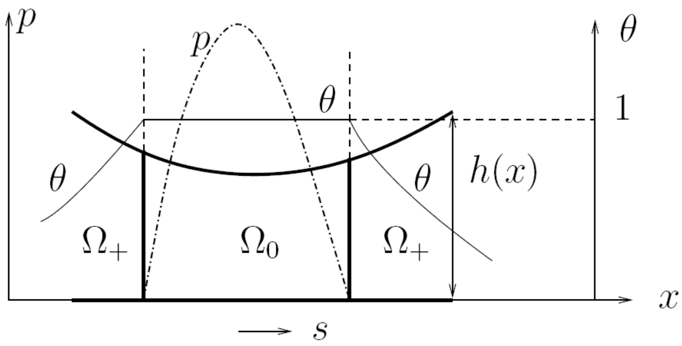

Figure 2: Partition of Ω and profiles of p and θ in the one-dimensional case

If the pressure p is equal to the saturation pressure, p must be positive, so that it is possible to define an unknown partition of Ω into a part Ω_+ where $p > 0$, and a part Ω_0 where $p = 0$ (cavitation zone, with a blend of air and oil). Therefore, the function $\theta \in L^\infty(\Omega)$ satisfies natural conditions :

$$\begin{cases} \theta = 1 & \text{in } \Omega_+ \\ 0 \leq \theta \leq 1 & \text{in } \Omega_0 \end{cases}$$

Physically, for high pressures, the viscosity of the fluid depends on the pressure p. Let us denote it by $\mu(p)$. In all generality, we suppose μ to be a positive continuous function of p.

Moreover, we consider the height of the fluid $h(p, x, y)$ to be given by:

$$h(p, x, y) = h_0(x, y) + \int_\Omega k(x - s, y - t)\, p(s, t)\, \mathrm{d}s\, \mathrm{d}t, \quad (x, y) \in \Omega,\ p \in L^2(\Omega),$$

where the kernel k is defined by $k(x, y) = \dfrac{k_0}{\sqrt{x^2 + y^2}}$, with k_0 a non-negative constant. This kernel corresponds physically to a point contact. The function h_0 is supposed to be regular and positive, such that $h_0 \geq m > 0$, where m is a constant.

Let us impose the following boundary condition: $p|_{\Gamma_1} = 0$. On Γ_0, the flow G_0 is supposed to be given as a positive function, with $G_0 \in L^\infty(\Gamma_0)$. It is now natural to define the following functional spaces:

$$\begin{aligned} V &= \left\{\phi \in H^1(\Omega),\, \phi|_{\Gamma_1} = 0\right\}, \\ V^+ &= \{\phi \in V,\, \phi \geq 0\}. \end{aligned}$$

When h tends to zero, it has been proved that the three-dimensional equations reduce to an equation on p in Ω. The strong formulation of the problem can be written as follows (see [3] for more details): *Find p and θ such that:*

$$\begin{cases} \operatorname{div}\left(\dfrac{h^3(p)}{\mu(p)}\nabla p\right) = 6s\dfrac{\partial(\theta(p)\,h(p))}{\partial x}, & \text{in } \mathcal{D}'(\Omega). \\ \theta \in H(p), \end{cases}$$

where $H(p)$ is the Heaviside function.

Thus the weak formulation of this problem is: *Find $p \in V^+$ and $\theta \in L^\infty(\Omega)$ such that:*

$$(\mathcal{P}) \begin{cases} \displaystyle\int_\Omega \frac{h^3(p)}{\mu(p)}\nabla p \cdot \nabla\varphi = 6s\int_\Omega h(p)\,\theta(p)\,\partial_x\varphi + \int_{\Gamma_0} G_0\,\varphi\,, \quad \forall\varphi \in V, \\ \theta \in H(p). \end{cases}$$

Remark 2.1 *It is possible to interpret physically the local input flow G_0 as follows. The weak formulation $(\mathcal{P})$ implicitly contains the following relation between the input flow G_0 and the pressure:*

$$-G_0(y) = \begin{cases} 6s\,\theta_0\,h(p,0,y) & \text{if } p(0,y) = 0, \\ 6s\,h(p,0,y) - \dfrac{h^3(p,0,y)}{\mu(p)}\dfrac{\partial p}{\partial n} & \text{if } p(0,y) \neq 0. \end{cases}$$

It is to be noticed that if $s < 0$, since θ_0, h and G_0 are positive, only the second case can occur, and thus $G_0(y) = 6s\,h(p,0,y) - \dfrac{h^3(p,0,y)}{\mu(p)}\dfrac{\partial p}{\partial n}$.

2.2 The problem for the reduced pressure

A classical approach consists in introducing a change of functions that reduces the problem to one close to an isoviscous case (see [4], in which this approach and the direct one without such change of functions are compared. It is shown that similar results are obtained in both cases).

Thus let us use the following change of functions (*Grübin transform*):

$$P(x,y) = a(p(x,y)) = \int_0^{p(x,y)} \frac{\mathrm{d}s}{\mu(s)}, \; (x,y) \in \Omega.$$

P is called *reduced pressure* (see Figure 3).

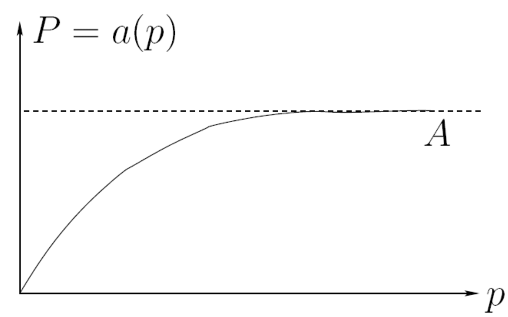

Figure 3: Profile of the reduced pressure P

Let A be defined by:

$$A = \int_0^{+\infty} \frac{\mathrm{d}s}{\mu(s)}.$$

The case $A = +\infty$ has already been treated in [17]. However, it has been proved experimentally that A has a finite value. In particular, for a viscosity given by Barus law:

$$\mu(p) = \mu_0 e^{\alpha p}, \quad \text{with } \mu_0 > 0,\ \alpha > 0,$$

the quantity A is finite ($A = \dfrac{\mu_0}{\alpha}$).

Therefore, we are concerned in this paper with fluids with a viscosity satisfying $A < +\infty$.
Furthermore, let the function γ be the inverse of the function a (as shown in Figure 4). Thus

$$p(x, y) = \gamma(P(x, y))$$

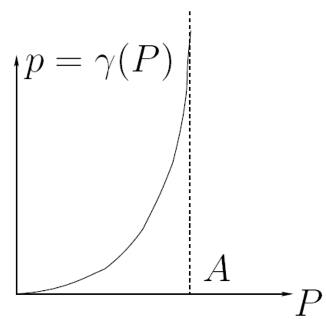

Figure 4: Profile of $\gamma(P)$

The weak formulation becomes: *Find $P \in V^+$ and $\theta \in L^\infty(\Omega)$ such that:*

$$(\mathcal{P}') \begin{cases} \displaystyle\int_\Omega H^3(P) \nabla P \cdot \nabla\varphi = 6s \int_\Omega H(P)\, \theta(P)\, \partial_x \varphi + \int_{\Gamma_0} G_0\, \varphi\,, \\ \forall \varphi \in V, \\ \theta \in H(P), \end{cases}$$

with

$$H(P, x, y) = h_0(x, y) + \int_\Omega k(x - s, y - t)\, \gamma(P(s, t))\, \mathrm{d}s\, \mathrm{d}t.$$

The purpose of this paper will be to prove an existence theorem (Theorem 4.7) for the weak formulation ($\mathcal{P}'$).

2.3 Introduction of a regularized problem

First, in order to regularize the Heaviside function, let us introduce Z_η a continuous approximation of θ (Figure 5) such that, for $\eta > 0$:

$$Z_\eta(t) = \begin{cases} 1 & \text{if } t > \eta, \\ 0 & \text{if } t < 0, \\ \dfrac{t}{\eta} & \text{if } 0 \le t \le \eta. \end{cases}$$

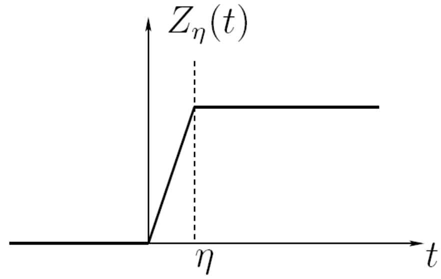

Figure 5: Regularization of θ

Now it remains to regularize the function γ, which is done by truncation (Figure 6). For $\varepsilon > 0$:

$$\gamma_\varepsilon(s) = \begin{cases} \gamma(s) & \text{if } 0 \le s \le A - \varepsilon, \\ \gamma(A-\varepsilon) & \text{if } s \ge A - \varepsilon. \end{cases}$$

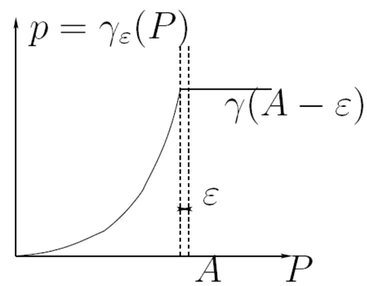

Figure 6: Regularization of γ

The regularized problem becomes: *Find* $P_{\eta\varepsilon} \in V^+$ *such that:*

$$(\mathcal{P}_{\eta\varepsilon}) \qquad \int_\Omega H_\varepsilon^3(P_{\eta\varepsilon}) \nabla P_{\eta\varepsilon} \cdot \nabla\varphi = 6s \int_\Omega H_\varepsilon(P_{\eta\varepsilon})\, Z_\eta(P_{\eta\varepsilon})\, \partial_x \varphi + \int_{\Gamma_0} G_0\, \varphi\,, \quad \forall \varphi \in V,$$

with

$$H_\varepsilon(q,x,y) = h_0(x,y) + \int_\Omega k(x-s, y-t)\, \gamma_\varepsilon(q(s,t))\, \mathrm{d}s\, \mathrm{d}t. \tag{1}$$

3 Existence result and first estimates for the regularized problem

3.1 Existence of a solution

In this section, the existence of a solution for the problem $(\mathcal{P}_{\eta\varepsilon})$ is established, for fixed η and ε.

Theorem 3.1 *For fixed $\eta > 0$ and $\varepsilon > 0$, there exists $P_{\eta\varepsilon} \in V^+$ solution of $(\mathcal{P}_{\eta\varepsilon})$. Moreover, $P_{\eta\varepsilon}$ satisfies:*

$$\|P_{\eta\varepsilon}\|_{H^1(\Omega)} \leq R,$$

where R is a constant independent of η and ε.

Proof. Let us emphasize that this result will be shown without any condition on the data, by means of a fixed point method.

For fixed $P_{\eta\varepsilon} \in V^+$, let us introduce the following problem: *Find $q \in V$ such that:*

$$(\mathcal{Q}) \qquad \int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})\nabla q\,\nabla\varphi = 6s\int_\Omega H_\varepsilon(P_{\eta\varepsilon})\,Z_\eta(q)\,\partial_x\varphi + \int_{\Gamma_0} G_0\,\varphi\,, \quad \forall\varphi \in V.$$

Step 1: Since $H_\varepsilon \geq h_0(x,y) > 0$, it is a classical mixed Dirichlet-Neumann problem, for which the existence and uniqueness of a solution is well known.

Step 2: Let us now derive estimates for the solution q of $(\mathcal{Q})$. Choosing $\varphi = q \in V$ in the weak formulation, it follows:

$$\int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla q|^2 = 6s\int_\Omega H_\varepsilon(P_{\eta\varepsilon})\,Z_\eta(q)\,\partial_x q + \int_{\Gamma_0} G_0\,q\,. \tag{2}$$

Let $h_{0m} = \min\limits_{(x,y)\in\Omega} h_0(x,y) > 0$, therefore $H_\varepsilon(P_{\eta\varepsilon}) \geq h_{0m}$. The left-hand side term can be estimated in the following way:

$$h_{0m}\,\|H_\varepsilon(P_{\eta\varepsilon})\nabla q\|^2_{L^2(\Omega)} \leq \int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla q|^2.$$

Moreover, using that $\|Z_\eta\|_{L^\infty} \leq 1$, and applying Cauchy-Schwarz inequality to the first right hand-side term in (2), it follows:

$$6s\int_\Omega H_\varepsilon(P_{\eta\varepsilon})\,Z_\eta(q)\,\partial_x q \leq 6|s|\int_\Omega |H_\varepsilon(P_{\eta\varepsilon})|\,|\partial_x q| \leq 6|s|\,|\Omega|^{1/2}\,\|H_\varepsilon(P_{\eta\varepsilon})\nabla q\|_{L^2(\Omega)}.$$

It remains the second right-hand side term. Let $G = \|G_0(y)\|_{L^2(\Gamma_0)}$, hence the following holds:

$$\int_{\Gamma_0} G_0\,q \leq G\,\|q\|_{L^2(\Gamma_0)} \leq G\,\|q\|_{L^2(\Gamma)} \leq G\,\|q\|_{H^{1/2}(\Gamma)} \leq C\,G\,\|q\|_{H^1(\Omega)}$$
$$\leq C\,G\,\|\nabla q\|_{L^2(\Omega)} \leq \frac{C}{h_{0m}}\,G\,\|H_\varepsilon(P_{\eta\varepsilon})\nabla q\|_{L^2(\Omega)},$$

where C denotes several constants obtained from trace theorems in Sobolev spaces and from Poincaré inequality. These constants are independent of both η and ε.

At last, equation (2) becomes:

$$h_{0m}\,\|H_\varepsilon(P_{\eta\varepsilon})\nabla q\|^2_{L^2(\Omega)} \leq 6|s|\,|\Omega|^{1/2}\,\|H_\varepsilon(P_{\eta\varepsilon})\nabla q\|_{L^2(\Omega)} + \frac{C\,G}{h_{0m}}\|H_\varepsilon(P_{\eta\varepsilon})\nabla q\|_{L^2(\Omega)}.$$

This implies that:

$$h_{0m}\,\|H_\varepsilon(P_{\eta\varepsilon})\nabla q\|_{L^2(\Omega)} \leq 6|s|\,|\Omega|^{1/2} + \frac{C\,G}{h_{0m}},$$

where $|\Omega|$ denotes the measure of Ω. The last estimate means that

$$\|\nabla q\|_{L^2(\Omega)} \leq \frac{6|s|\,|\Omega|^{1/2}\,h_{0m} + C\,G}{h^3_{0m}}.$$

Let us emphasize that this estimate is independent of η and ε.

Step 3: It remains to check out the positivity of q.

Let us choose $\varphi = q^- \in V$ (since $q \in H^1(\Omega)$, $q^- \in H^1(\Omega)$).We obtain

$$\int_\Omega H^3_\varepsilon(P_{\eta\varepsilon})\nabla(q^+ - q^-)\cdot\nabla q^- = 6s\int_\Omega H_\varepsilon(P_{\eta\varepsilon})\,Z_\eta(q)\,\partial_x q^- + \int_{\Gamma_0} G_0\,q^-.$$

The term $\nabla q^+\,\nabla q^-$ is zero almost everywhere, and so does the term $Z_\eta(q)\,\partial_x q^-$. Indeed, if $q \geq 0$, $q^- = 0$, and if $q < 0$, $Z_\eta(q) = 0$. It remains:

$$-\int_\Omega H^3_\varepsilon(P_{\eta\varepsilon})|\nabla q^-|^2 = \int_{\Gamma_0} G_0\,q^-.$$

Since $G_0(y) \geq 0$, we have $\displaystyle\int_\Omega H^3_\varepsilon(P_{\eta\varepsilon})|\nabla q^-|^2 \leq 0$, and thus $\displaystyle\int_\Omega H^3_\varepsilon(P_{\eta\varepsilon})|\nabla q^-|^2 = 0$. Hence q^- is constant almost everywhere. Furthermore $q|_{\Gamma_1} = 0$, therefore $q^- = 0$, which proves that $q \geq 0$.

Step 4: In order to finish the proof of Theorem 3.1, it remains to apply Schauder fixed point theorem. Now, let

$$R = \frac{6|s|\,|\Omega|^{1/2}\,h_{0m} + C\,G}{h^3_{0m}} \quad \text{and} \quad B_R = \left\{f \in V^+,\, 0 \leq \|f\|_{H^1} \leq R\right\},$$

and let us define $T : V^+ \longrightarrow V^+$ by $T(P_{\eta\varepsilon}) = q$, where q is solution of $(\mathcal{Q})$. T is well defined, and we proved that $T(B_R) \subset B_R$. Moreover, T is continuous, since the function $P_{\eta\varepsilon} \mapsto H_\varepsilon(P_{\eta\varepsilon})$ is continuous. Let $(P^n_{\eta\varepsilon})_{n\in\mathbb{N}}$ be a sequence of V^+-functions converging weakly to $P_{\eta\varepsilon}$, let $(q^n)_{n\in\mathbb{N}}$ be the sequence of solutions of $(\mathcal{Q})$ to $P^n_{\eta\varepsilon}$, and q the solution corresponding to $P_{\eta\varepsilon}$. It holds:

$$\int_\Omega H^3_\varepsilon(P^n_{\eta\varepsilon})\nabla q^n\,\nabla\varphi = 6s\int_\Omega H_\varepsilon(P^n_{\eta\varepsilon})\,Z_\eta(q^n)\,\partial_x\varphi + \int_{\Gamma_0} G_0\,\varphi$$

$$\int_\Omega H^3_\varepsilon(P_{\eta\varepsilon})\nabla q\,\nabla\varphi = 6s\int_\Omega H_\varepsilon(P_{\eta\varepsilon})\,Z_\eta(q)\,\partial_x\varphi + \int_{\Gamma_0} G_0\,\varphi$$

Subtracting the two equations, we write:

$$\begin{aligned}&H_\varepsilon^3(P_{\eta\varepsilon}^n)\nabla q^n - H_\varepsilon^3(P_{\eta\varepsilon})\nabla q\\ &= (H_\varepsilon^3(P_{\eta\varepsilon}^n)\nabla q^n - H_\varepsilon^3(P_{\eta\varepsilon})\nabla q^n) + (H_\varepsilon^3(P_{\eta\varepsilon})\nabla q^n - H_\varepsilon^3(P_{\eta\varepsilon})\nabla q).\end{aligned}$$

Using that $H_\varepsilon^3(P_{\eta\varepsilon}^n) - H_\varepsilon^3(P_{\eta\varepsilon})$ tends to zero, it remains the term $H_\varepsilon^3(P_{\eta\varepsilon})\,\nabla(q^n - q)$. In a similar way:

$$\begin{aligned}&H_\varepsilon(P_{\eta\varepsilon}^n)\,Z_\eta(q^n) - H_\varepsilon(P_{\eta\varepsilon})\,Z_\eta(q)\\ &= (H_\varepsilon(P_{\eta\varepsilon}^n)\,Z_\eta(q^n) - H_\varepsilon(P_{\eta\varepsilon})\,Z_\eta(q^n)) + (H_\varepsilon(P_{\eta\varepsilon})\,Z_\eta(q^n) - H_\varepsilon(P_{\eta\varepsilon})\,Z_\eta(q)).\end{aligned}$$

Again, $H_\varepsilon(P_{\eta\varepsilon}^n) - H_\varepsilon(P_{\eta\varepsilon})$ converges to zero. It follows, choosing $\varphi = q^n - q$ and using the definition of Z_η:

$$\begin{aligned}\int\limits_\Omega H_\varepsilon^3(P_{\eta\varepsilon})\,\nabla(q^n - q) &= 6s\int\limits_\Omega H_\varepsilon(P_{\eta\varepsilon})\,(Z_\eta(q^n) - Z_\eta(q))\,\partial_x(q^n - q)\\ &= \frac{6s}{\eta}\int\limits_\Omega H_\varepsilon(P_{\eta\varepsilon})\,(q^n - q)\,\partial_x(q^n - q)\end{aligned}$$

The first term is bounded . On the other hand, since $H_\varepsilon(P_{\eta\varepsilon}) \geq H_{m\eta\varepsilon} > 0$, and since the equation above holds for any η, it follows that $q^n - q$ tends to zero, and thus that $\|q^n - q\|_{H^1(\Omega)}$ tends to zero, thus q^n converges strongly to q in V^+. Finally, applying Schauder fixed point theorem, it follows that the problem $(\mathcal{P}_{\eta\varepsilon})$ admits a solution $P_{\eta\varepsilon}$ satisfying

$$\|P_{\eta\varepsilon}\|_{H^1(\Omega)} \leq R.$$

This finishes the proof. □

Remark 3.2 *Let us emphasize that in previous works (in particular [4]), similar H^1-bounds have been obtained provided some smallness assumption on the data. In the present paper, the constant R is not supposed to satisfy any smallness condition, and in particular is not supposed to be less than A. Therefore it will be shown separately that $P_{\eta\varepsilon}$ remains bounded by A.*

3.2 Classical estimates

In this section, we will obtain first estimates on $P_{\eta\varepsilon}$ and $\gamma_\varepsilon(P_{\eta\varepsilon})$. These estimates will be useful in order to prove the convergence of $\gamma_\varepsilon(P_{\eta\varepsilon})$ toward the expected function. However, it will not be enough to pass to the limit, and better estimates will be obtained in the next section.

Let us start with an L^∞ bound for $P_{\eta\varepsilon}$.

Proposition 3.3 *The solution $P_{\eta\varepsilon}$ of the problem $(\mathcal{P}_{\eta\varepsilon})$ satisfies the following inequality:*

$$\|P_{\eta\varepsilon}\|_{L^\infty(\Omega)} \leq \frac{8|\Omega|^{1/6}C}{H^2_{m\eta\varepsilon}} \left(6|s| + \frac{G'\,C}{H_{m\eta\varepsilon}}\right) \leq \frac{8|\Omega|^{1/6}C}{h^2_{0m}} \left(6|s| + \frac{G'\,C}{h_{0m}}\right)$$

where $H_{m\eta\varepsilon} = \min\limits_{(x,y)\in\Omega} H_\varepsilon(P_{\eta\varepsilon}(x,y))$ *and* $G' = \|G_0\|_{L^\infty(\Gamma_0)}$.

Proof. The key point in the proof is to use a lemma by Kinderlehrer-Stampacchia [13]. However, due to the boundary term on Γ_0, a specific treatment is to be used.
Let $k > 0$. Let $P^{(k)}_{\eta\varepsilon}$ be the function defined by

$$P^{(k)}_{\eta\varepsilon} = \begin{cases} P_{\eta\varepsilon} - k & \text{if } P_{\eta\varepsilon} \geq k \\ 0 & \text{if } P_{\eta\varepsilon} \leq k \end{cases}$$

and A_k the set $A_k = \{(x,y) \in \bar{\Omega}, P_{\eta\varepsilon}(x,y) \geq k\}$. It is easy to check that $P^{(k)}_{\eta\varepsilon}$ lies in V^+ and that

$$\nabla P^{(k)}_{\eta\varepsilon} = \begin{cases} \nabla P_{\eta\varepsilon} & \text{in } \mathring{A}_k, \\ 0 & \text{in } \bar{\Omega} \setminus A_k. \end{cases} \tag{3}$$

Obviously, we have a similar relation for $\partial_x P^{(k)}_{\eta\varepsilon}$.
Choosing $\varphi = P^{(k)}_{\eta\varepsilon}$ as a test function in $(\mathcal{P}_{\eta\varepsilon})$, we have:

$$\int_{A_k} H^3_\varepsilon(P^{(k)}_{\eta\varepsilon})|\nabla P^{(k)}_{\eta\varepsilon}|^2 = 6s \int_{A_k} H_\varepsilon(P^{(k)}_{\eta\varepsilon})\, Z_\eta(P^{(k)}_{\eta\varepsilon})\, \partial_x P^{(k)}_{\eta\varepsilon} + \int_{\Gamma_0} G_0\, P^{(k)}_{\eta\varepsilon}.$$

Now, the last term can be estimated in the following way:

$$\begin{aligned} \int_{\Gamma_0} G_0\, P^{(k)}_{\eta\varepsilon} &\leq \|G_0\|_{L^\infty(\Gamma_0)} \int_{\Gamma_0\cap A_k} P^{(k)}_{\eta\varepsilon} \leq \|G_0\|_{L^\infty(\Gamma_0)} \int_{A_k} P^{(k)}_{\eta\varepsilon} \\ &\leq \|G_0\|_{L^\infty(\Gamma_0)} \int_{A_k} \nabla P^{(k)}_{\eta\varepsilon}, \end{aligned}$$

using Poincaré inequality in $L^1(A_k)$. Thus, since $\|G_0\|_{L^\infty(\Gamma_0)}$ is a constant independent of ε and η we can conclude that

$$\begin{aligned} \int_{A_k} H^3_\varepsilon(P^{(k)}_{\eta\varepsilon})|\nabla P^{(k)}_{\eta\varepsilon}|^2 \leq 6|s| &\int_{A_k} \frac{1}{H^{1/2}_{m\eta\varepsilon}} \left(H^3_\varepsilon(P^{(k)}_{\eta\varepsilon})\,|\nabla P^{(k)}_{\eta\varepsilon}|^2\right)^{1/2} \\ &+ C \int_{A_k} \frac{1}{H^{3/2}_{m\eta\varepsilon}} \left(H^3_\varepsilon(P^{(k)}_{\eta\varepsilon})|\nabla P^{(k)}_{\eta\varepsilon}|^2\right)^{1/2}. \end{aligned}$$

Hence, using Cauchy-Schwarz inequality, and since $H_\varepsilon(P_{\eta\varepsilon}^{(k)}) \geq H_{m\eta\varepsilon}$:

$$\int_{A_k} H_\varepsilon^3(P_{\eta\varepsilon}^{(k)})|\nabla P_{\eta\varepsilon}^{(k)}|^2 \leq \left(\frac{6|s|}{H_{m\eta\varepsilon}^{1/2}} + \frac{C}{H_{m\eta\varepsilon}^{3/2}}\right) |A_k|^{1/2} \left(\int_{A_k} H_\varepsilon^3(P_{\eta\varepsilon}^{(k)})|\nabla P_{\eta\varepsilon}^{(k)}|^2\right)^{1/2}.$$

It follows that:

$$\left(\int_{A_k} H_\varepsilon^3(P_{\eta\varepsilon}^{(k)})|\nabla P_{\eta\varepsilon}^{(k)}|^2\right)^{1/2} \leq \left(\frac{6|s|}{H_{m\eta\varepsilon}^{1/2}} + \frac{C\,G'}{H_{m\eta\varepsilon}^{3/2}}\right) |A_k|^{1/2}.$$

Finally we get that:

$$\int_{A_k} |\nabla P_{\eta\varepsilon}^{(k)}|^2 \leq \frac{1}{H_{m\eta\varepsilon}^3} \left(\frac{6|s|}{H_{m\eta\varepsilon}^{1/2}} + \frac{C\,G'}{H_{m\eta\varepsilon}^{3/2}}\right)^2 |A_k| \leq \frac{1}{H_{m\eta\varepsilon}^4} \left(6|s| + \frac{C\,G'}{H_{m\eta\varepsilon}}\right)^2 |A_k|.$$

Moreover, classical Sobolev embeddings ($H^1(\Omega) \subset L^3(\Omega)$) imply that:

$$\int_{A_k} |\nabla P_{\eta\varepsilon}^{(k)}|^2 = \int_{\Omega} |\nabla P_{\eta\varepsilon}^{(k)}|^2 \geq C \left(\int_{\Omega} |P_{\eta\varepsilon}^{(k)}|^3\right)^{2/3} = C \left(\int_{A_k} |P_{\eta\varepsilon}^{(k)}|^3\right)^{2/3},$$

where C depends only on Ω, and thus does not depend on k. Now, let $\ell > k$. Clearly $A_\ell \subset A_k$, therefore

$$\left(\int_{A_k} |P_{\eta\varepsilon}^{(k)}|^3\right)^{2/3} \geq \left(\int_{A_\ell} |P_{\eta\varepsilon}^{(k)}|^3\right)^{2/3} \geq \left(\int_{A_\ell} |\ell - k|^3\right)^{2/3} \geq (\ell - k)^2 |A_\ell|^{2/3},$$

since in A_ℓ, $P \geq \ell$, thus $P_{\eta\varepsilon}^{(k)} = P - k \geq \ell - k$.

Let us denote $\phi(\ell) = |A_\ell|$. Precedent computations imply that:

$$\phi(\ell)^{2/3} \leq \frac{1}{(\ell - k)^2} \frac{C}{H_{m\eta\varepsilon}^4} \left(6|s| + \frac{C\,G'}{H_{m\eta\varepsilon}}\right)^2 \phi(k),$$

hence

$$\phi(\ell) \leq \frac{1}{(\ell - k)^3} \frac{C}{H_{m\eta\varepsilon}^6} \left(6|s| + \frac{C\,G'}{H_{m\eta\varepsilon}}\right)^3 \phi(k)^{3/2},$$

where C denotes different constants independent of ε and η. Applying a lemma by Kinderlehrer and Stampacchia, given for example in [13, Lemma B.1], we conclude that:

$$\phi(d) = 0 \quad \text{for} \quad d^3 = \frac{2^9 |\Omega|^{1/2} C}{H_{m\eta\varepsilon}^6} \left(6|s| + \frac{C\,G'}{H_{m\eta\varepsilon}}\right)^3.$$

Now, since $\phi(d) = 0 \longleftrightarrow |A_d| = 0 \longleftrightarrow P_{\eta\varepsilon} < d$ in Ω, the precedent relation implies that:

$$\|P_{\eta\varepsilon}\|_{L^\infty(\Omega)} \le \frac{8|\Omega|^{1/6}C}{H^2_{m\eta\varepsilon}} \left(6|s| + \frac{C\,G'}{H_{m\eta\varepsilon}}\right). \tag{4}$$

Moreover, since $H_{m\eta\varepsilon}$ is bounded from below by h_{0m}, the second part of the desired inequality follows immediately from the first one. □

To end this section, let us prove an L^1 estimate uniformly with respect to both ε and η for $p_{\eta\varepsilon}$. This estimate will be used in order to show an L^1 bound on p.

Theorem 3.4 *There exists a constant C independent of ε and η such that*

$$\int_\Omega p_{\eta\varepsilon} = \int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon}) \le C.$$

Proof. Because of the definition of the kernel k, we know that $k(x-s, y-t) \ge \dfrac{1}{2\sqrt{2}|\Omega|}$.Thus $H_{m\eta\varepsilon}$ satisfies

$$H_{m\eta\varepsilon} \ge (2\sqrt{2}|\Omega|)^{-1} \int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon}).$$

From (4), and using the fact that $H_{m\eta\varepsilon} \ge h_{0m}$, it follows that

$$\|P_{\eta\varepsilon}\|_{L^\infty(\Omega)} \le \frac{C}{H^2_{m\eta\varepsilon}},$$

thus

$$\int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon}) \le C\|P_{\eta\varepsilon}\|^{-2}_{L^\infty(\Omega)},$$

where C denotes some constants independent of ε and η.
Now, if we suppose that $\int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon})$ tends to infinity when ε tends to zero, $\|P_{\eta\varepsilon}\|_{L^\infty(\Omega)}$ would tend to zero. Thus $\|P_{\eta\varepsilon}\|_{L^\infty(\Omega)} \le \dfrac{A}{2}$ when ε tends to zero. But, from the definition of γ_ε and the monotonicity of γ we have, for ε small enough

$$\int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon}) \le \int_\Omega \gamma(P_{\eta\varepsilon}) \le \int_\Omega \gamma\left(\frac{A}{2}\right).$$

This leads to a contradiction and finishes the proof. □

4 Passing to the limit - Additional estimates

4.1 First estimates for the reduced problem

The following theorem states some immediate estimates on the limit of $P_{\eta\varepsilon}$.

Theorem 4.1 *Let P_η be the limit of the sequence $P_{\eta\varepsilon}$ for $\varepsilon \to 0$, and let P be the limit of the sequence P_η for $\eta \to 0$. P satisfies:*

$$\|P\|_{L^\infty(\Omega)} \leq A.$$

Moreover there exists a constant C independent of ε and η such that

$$\|p\|_{L^1(\Omega)} = \int_\Omega \gamma(P) \leq C.$$

Before proving this theorem, let us state the following lemma, whose proof can be find in Bayada and Bellout [2, Lemma 6].

Lemma 4.2 *Let $E =]-M, M[\times]-M, M[\subset \mathbb{R}^2$ and let v_n be a sequence of functions $L^2(E)$ which converges almost everywhere to v. If*

$$\Omega_\tau = \{(x,y) \in E,\, v(x,y) \geq A + \tau\},$$
$$\Omega_\tau^n = \{(x,y) \in E,\, v_n(x,y) \geq A\}$$

and $|\Omega_\tau| \neq 0$ then there exists $n_0 > 0$ such that $\forall n \geq n_0$, $|\Omega_\tau^n| \geq \dfrac{1}{2}|\Omega_\tau|$.

Proof. (of Theorem 4.1) The first estimate is obtained by contradiction. Let us assume that there exists $\tau > 0$ such that $\Omega_\tau = \{(x,y) \in \Omega,\, P(x,y) \geq A + \tau\}$ has a non-zero measure. Then, applying Lemma 4.2 to $P_{\eta\varepsilon}$, it follows that $\int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon}) \geq \frac{1}{2}|\Omega_\tau|\gamma(A - \varepsilon) \xrightarrow[\varepsilon,\eta\to 0]{} +\infty$, which is a contradiction with Theorem 3.4.

For the second estimate, let $\tau > 0$, and $P_{\eta\varepsilon}^\tau(x,y) = \inf(P_{\eta\varepsilon}(x,y), A - \tau)$. Since $P_{\eta\varepsilon}$ converges strongly to P in $L^2(\Omega)$, $P_{\eta\varepsilon}^\tau$ converges strongly to $P^\tau(x,y) = \inf(P(x,y), A-\tau)$ in $L^2(\Omega)$. Now, it is clear that $\gamma_\varepsilon(P_{\eta\varepsilon}^\tau) \leq \gamma_\varepsilon(P_{\eta\varepsilon})$, since γ_ε is increasing, thus $\int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon}^\tau) \leq C$ by Theorem 3.4. Moreover, for ε small enough, $\gamma_\varepsilon(P_{\eta\varepsilon}^\tau) = \gamma(P_{\eta\varepsilon}^\tau) \leq \gamma(A - \tau) \leq C$, with C a constant independent of ε and η, but which depends on τ. Thus for fixed τ, $\gamma_\varepsilon(P_{\eta\varepsilon}^\tau)$ converges to $\gamma(P^\tau)$ in $L^1(\Omega)$, and $\int_\Omega \gamma(P^\tau) \leq C$. Now, letting τ go to zero, we obtain from the monotone convergence theorem that $\int_\Omega \gamma(P) \leq C$, since $\gamma(P^\tau) \xrightarrow[\tau\to 0]{} \gamma(P)$. $\square$

4.2 Additional estimates

It remains to pass to the limit in the non-linear terms of $(\mathcal{P}_{\eta\varepsilon})$. Let us explain in the following the main steps of the proof.

4.2.1 Main idea for passing to the limit

It is well-known ([4]) that the estimates obtained in the precedent section are not enough to prove an unconditional existence result for the problem $(\mathcal{P})$.

In order to treat the non-linear term when passing to the limit in the equation when $\varepsilon \to 0$ and $\eta \to 0$, stronger estimates on H_ε have to be proved. For the term $H_\varepsilon^3 \nabla P_{\eta\varepsilon}$, since $\nabla P_{\eta\varepsilon}$ converges weakly in $L^2(\Omega)$, it suffices that H_ε converges strongly in $L^6(\Omega)$. To this purpose, we will show that H_ε is bounded in $W^{1,6}(\Omega)$.

To this end, we will see that it is enough to show that $\gamma_\varepsilon(P_{\eta\varepsilon})$ is bounded in $L^6(\Omega)$, since the convolution kernel k in H_ε has a regularizing effect. To prove this, we will introduce the function $\gamma_\varepsilon(P_{\eta\varepsilon})^\sigma$, for some $\sigma > 0$, and prove that this function is bounded in $H^1(\Omega)$. Thus, we will be able to conclude that $\gamma_\varepsilon(P_{\eta\varepsilon})$ is bounded in $L^{\sigma r}(\Omega)$, for any $r \geq 2$, and therefore at least in $L^6(\Omega)$ (see Proposition 4.3).

However, since $\gamma_\varepsilon(P_{\eta\varepsilon})^\sigma$ will be used as a test function in the weak formulation, it will be necessary to introduce a cut-off function ψ and consider $\gamma_\varepsilon(P_{\eta\varepsilon})^\sigma \psi(P_{\eta\varepsilon})$.

4.2.2 Detailed estimates

In order to obtain the needed estimates, let us introduce an additional hypothesis on the asymptotic behavior of the piezoviscosity law. More precisely, we suppose that:

$$\exists p^* > 0, \quad \mu(p) = (p+Q)^\beta \quad \text{for } p \geq p^*, \quad \text{with} \quad \beta > 1, \quad Q \geq 0, \tag{5}$$

where Q and β are constants. Actually we could suppose only that $\mu(p) \underset{+\infty}{\sim} (p+Q)^\beta$, which in particular allows to consider Barus law for finite values of p and an asymptotic behavior of this sort (see Introduction for physical explanation).

The following proposition is the key of the needed estimate on H_ε.

Proposition 4.3 *Let $\mu(p)$ satisfy the condition (5). For $1 < \beta < \frac{3}{2}$, $\gamma_\varepsilon(P_{\eta\varepsilon})$ satisfies*

$$\|\gamma_\varepsilon(P_{\eta\varepsilon})\|_{L^6(\Omega)} \leq C,$$

where C is independent of ε and η.

Before starting the proof, let us introduce the following functions and notations. Defining

$a_1 = \int\limits_0^{p^*} \frac{\mathrm{d}s}{\mu(s)}$, the hypothesis (5) means that $A = a_1 + \frac{(p^*+Q)^{1-\beta}}{\beta - 1}$. Let us denote $A = a_1 + a_2$. Moreover, let ε be small enough, so that $\varepsilon < \frac{a_2}{3}$.

Then we introduce the function

$$\delta_\varepsilon(P_{\eta\varepsilon}) = (Q + \gamma_\varepsilon(P_{\eta\varepsilon}))^\alpha\, \psi(P_{\eta\varepsilon}), \tag{6}$$

where α will be chosen below and where $\psi(t) \in C^2(\mathbb{R})$ is a cut-off function defined by $\psi'(t) \geq 0$ and

$$\psi(t) = \begin{cases} 0 & \text{for } t < a_1 + \dfrac{a_2}{3}, \\ 1 & \text{for } a_1 + \dfrac{2a_2}{3} < t. \end{cases} \tag{7}$$

Let us observe that the function $\delta_\varepsilon \in V$ defined in this way is an admissible test function for the problem $(\mathcal{P}_{\eta\varepsilon})$, since $P_{\eta\varepsilon}|_{\Gamma_1} = 0$, and thus on Γ_1 we have $\psi(P_{\eta\varepsilon}) = 0$.

Proof. The result of Proposition 4.3 will be proved under the following condition on the parameters :

$$2 - \alpha - \beta \geq 0, \quad 1 < \beta < \frac{3}{2}, \quad 1 + \alpha - \beta > 0. \tag{8}$$

Let us observe that the set of all α and β satisfying the condition (8) is non-empty. In particular for any $\beta \in \left]1, \frac{3}{2}\right[$, there exists an α such that (α, β) satisfies the condition (8).

Introducing the function $g_{\eta\varepsilon} = \gamma_\varepsilon(P_{\eta\varepsilon})^\sigma\, \psi(P_{\eta\varepsilon})$, for $\sigma > 0$, we will show that $\|g_{\eta\varepsilon}\|_{H^1(\Omega)}$ is bounded. Moreover, let us denote (see Figure 7)

$$\Omega_1 = \{(x,y) \in \Omega,\ P_{\eta\varepsilon}(x,y) \leq a_1 + \frac{2a_2}{3}\}$$
$$\Omega_2 = \{(x,y) \in \Omega,\ a_1 + \frac{2a_2}{3} < P_{\eta\varepsilon}(x,y) < A - \varepsilon\}$$
$$\Omega_3 = \{(x,y) \in \Omega,\ A - \varepsilon \leq P_{\eta\varepsilon}(x,y)\}$$

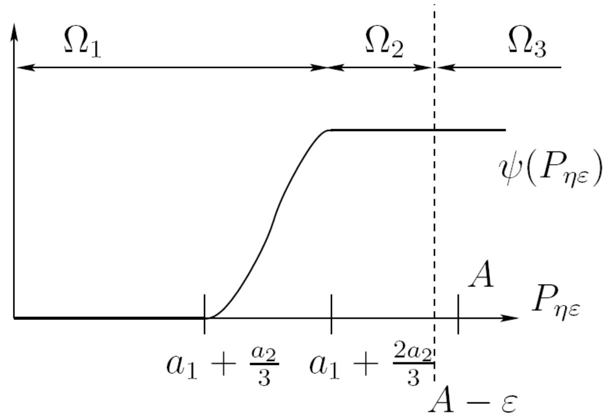

Figure 7: Partition of Ω and profile of $\psi(P_{\eta\varepsilon})$

Then $\bar{\Omega} = \bar{\Omega}_1 \amalg \bar{\Omega}_2 \amalg \bar{\Omega}_3$, since these three sets are pairwise disjoints.

Now, expanding $|\nabla g_{\eta\varepsilon}(P_{\eta\varepsilon})|^2$, it follows that

$$\int_\Omega |\nabla g_{\eta\varepsilon}(P_{\eta\varepsilon})|^2 = \sigma^2 \int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon})^{2(\sigma-1)}(\gamma'_\varepsilon)^2 |\nabla P_{\eta\varepsilon}|^2 \psi(P_{\eta\varepsilon})^2$$
$$+ 2\sigma \int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon})^{2\sigma-1}(\gamma'_\varepsilon)^2 |\nabla P_{\eta\varepsilon}|^2 \psi'(P_{\eta\varepsilon})\, \psi(P_{\eta\varepsilon}) + \int_\Omega \gamma_\varepsilon(P_{\eta\varepsilon})^{2\sigma} |\nabla P_{\eta\varepsilon}|^2 \psi'(P_{\eta\varepsilon})^2.$$

- In Ω_1, each of these three terms are bounded independently of ε and η, since far from $A-\varepsilon$, $P_{\eta\varepsilon}$ is bounded, and so is $\gamma_\varepsilon(P_{\eta\varepsilon})$.
- In Ω_3, $\gamma_\varepsilon(P_{\eta\varepsilon}) = \gamma(A-\varepsilon)$ is constant, hence $\gamma'_\varepsilon = 0$. Moreover $\psi \equiv 1$, thus $\psi' \equiv 0$. Therefore $\int_{\Omega_3} |\nabla g_{\eta\varepsilon}(P_{\eta\varepsilon})|^2 = 0$.
- In Ω_2, we have again $\psi \equiv 1$. It remains

$$\int_\Omega |\nabla g_{\eta\varepsilon}(P_{\eta\varepsilon})|^2 \leq C + \sigma^2 \int_{\Omega_2} \gamma_\varepsilon(P_{\eta\varepsilon})^{2(\sigma-1)}(\gamma'_\varepsilon(P_{\eta\varepsilon}))^2 |\nabla P_{\eta\varepsilon}|^2.$$

Now, since $\delta_\varepsilon = (Q + \gamma_\varepsilon(P_{\eta\varepsilon}))^\alpha\ \psi(P_{\eta\varepsilon}) = (Q + \gamma_\varepsilon(P_{\eta\varepsilon}))^\alpha$ on Ω_2, we have $\gamma_\varepsilon(P_{\eta\varepsilon}) = \delta_\varepsilon^{1/\alpha} - Q$, and thus

$$\gamma'_\varepsilon(P_{\eta\varepsilon}) = \frac{1}{\alpha}\delta_\varepsilon^{\frac{1-\alpha}{\alpha}}\ \delta'_\varepsilon = \frac{1}{\alpha}\left(Q + \gamma_\varepsilon(P_{\eta\varepsilon})\right)^{1-\alpha}\ \delta'_\varepsilon \tag{9}$$

On the other hand, the hypothesis (5) implies that

$$a(p) = a_1 + \int_{p^*}^{p} (s+Q)^{-\beta} \mathrm{d}s = A + \frac{(p+Q)^{1-\beta}}{1-\beta}.$$

Therefore

$$\gamma_\varepsilon(P_{\eta\varepsilon}) = ((1-\beta)(P_{\eta\varepsilon} - A))^{\frac{1}{1-\beta}} - Q,$$

hence

$$\gamma'_\varepsilon(P_{\eta\varepsilon}) = ((1-\beta)(P_{\eta\varepsilon} - A))^{\frac{\beta}{1-\beta}} = (\gamma_\varepsilon(P_{\eta\varepsilon}) + Q)^\beta \tag{10}$$

Using the two expressions of $\gamma'_\varepsilon(P_{\eta\varepsilon})$ obtained in (9) and (10), we get that

$$(\gamma'_\varepsilon(P_{\eta\varepsilon}))^2 \leq \frac{1}{\alpha}\left(Q + \gamma_\varepsilon(P_{\eta\varepsilon})\right)^{1-\alpha+\beta}\ \delta'_\varepsilon(P_{\eta\varepsilon}),$$

and conclude that

$$\int_\Omega |\nabla g_{\eta\varepsilon}(P_{\eta\varepsilon})|^2 \leq C + \sigma^2 \int_{\Omega_2} \frac{1}{\alpha}\left(Q + \gamma_\varepsilon(P_{\eta\varepsilon})\right)^{1-\alpha+\beta+2(\sigma-1)}\ \delta'_\varepsilon(P_{\eta\varepsilon})|\nabla P_{\eta\varepsilon}|^2.$$

Choosing $\sigma = \dfrac{1+\alpha-\beta}{2} > 0$, it follows that

$$\int_\Omega |\nabla g_{\eta\varepsilon}(P_{\eta\varepsilon})|^2 \leq C + \sigma^2 \int_{\Omega_2} \frac{1}{\alpha}\delta'_\varepsilon|\nabla P_{\eta\varepsilon}|^2.$$

Now, using Proposition 4.4 below, we can conclude that

$$\int_\Omega |\nabla g_{\eta\varepsilon}(P_{\eta\varepsilon})|^2 \leq C, \tag{11}$$

thus $\|g_{\eta\varepsilon}\|_{H^1(\Omega)}$ is bounded, and this implies that

$$\|\gamma_\varepsilon(P_{\eta\varepsilon})^\sigma\|_{H^1(\Omega)} \leq C.$$

Finally, using Sobolev embeddings, it follows that $\gamma_\varepsilon(P_{\eta\varepsilon})$ is bounded in $L^{\sigma r}(\Omega)$ for any $r \geq 2$, thus in $L^6(\Omega)$ for r big enough ($r = 6/\sigma$). Therefore, we proved that

$$\|\gamma_\varepsilon(P_{\eta\varepsilon})\|_{L^6(\Omega)} \leq C.$$

□

Now, let us present the proof of the following result, which has been used in the precedent proof in order to establish (11).

Proposition 4.4 *Suppose that there exists a constant $c^* > 0$ such that ψ satisfies*

$$\psi'(t) \leq c^*\psi(t), \quad \forall t > a_1 + \frac{a_2}{2}, \tag{12}$$

and suppose that $0 < \alpha < 1$ and $2 - \alpha - \beta \geq 0$. Then the following inequality holds:

$$\int_\Omega |\nabla P_{\eta\varepsilon}|^2 \delta'_\varepsilon(P_{\eta\varepsilon}) \leq C,$$

where C is a constant independent of ε and η.

Remark 4.5 *Let us observe that any C^2-function ψ satisfying the condition* (7) *satisfies also the condition* (12)*. Indeed, ψ' is a C^1-function on $\left[a_1 + \frac{a_2}{2}, A\right]$, thus bounded. Moreover, for $t \in \left[a_1 + \frac{a_2}{2}, A\right]$, $\psi(t) \geq \psi(a_1 + \frac{a_2}{2})$.*

Proof. *Step 1:* Let us obtain a bound for the term $\int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla P_{\eta\varepsilon}|^2\,\delta_\varepsilon'(P_{\eta\varepsilon})$ independent of η and ε. To this end, choosing $\varphi = \delta_\varepsilon(P_{\eta\varepsilon}) \in V$ as a test function in $(\mathcal{P}_{\eta\varepsilon})$, we obtain

$$\int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla P_{\eta\varepsilon}|^2\,\delta_\varepsilon'(P_{\eta\varepsilon}) = 6s\int_\Omega H_\varepsilon(P_{\eta\varepsilon})Z_\eta(P_{\eta\varepsilon})\delta_\varepsilon'(P_{\eta\varepsilon})\partial_x P_{\eta\varepsilon} + \int_{\Gamma_0} G_0\delta_\varepsilon(P_{\eta\varepsilon})$$

and, using Young inequality,

$$\int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla P_{\eta\varepsilon}|^2\,\delta_\varepsilon'(P_{\eta\varepsilon}) \leq \frac{1}{2}\int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla P_{\eta\varepsilon}|^2\,\delta_\varepsilon'(P_{\eta\varepsilon})$$
$$+ 18s^2\int_\Omega \frac{1}{H_\varepsilon(P_{\eta\varepsilon})}\delta_\varepsilon'(P_{\eta\varepsilon}) + \int_{\Gamma_0} G_0\delta_\varepsilon(P_{\eta\varepsilon}).$$

Thus the following estimate holds:

$$\int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla P_{\eta\varepsilon}|^2\,\delta_\varepsilon'(P_{\eta\varepsilon}) \leq 36s^2\int_\Omega \frac{1}{H_\varepsilon(P_{\eta\varepsilon})}\delta_\varepsilon'(P_{\eta\varepsilon}) + 2\int_{\Gamma_0} G_0\delta_\varepsilon(P_{\eta\varepsilon}). \tag{13}$$

On the other hand, the trace operator is continuous from $W^{1,1}(\Omega)$ to $L^1(\Gamma)$ (see for example[14]). Let us denote $\bar{G}_0$ the extension of G_0 to Ω such that $\bar{G}_0(x,y) = G_0(y)$. Thus, using Poincaré inequality, since $\delta_\varepsilon(P_{\eta\varepsilon})|_{\Gamma_1} = 0$,

$$2\int_\Gamma G_0\delta_\varepsilon(P_{\eta\varepsilon}) = 2\int_{\Gamma_0} G_0\delta_\varepsilon(P_{\eta\varepsilon}) \leq C2\int_\Omega \bar{G}_0\delta_\varepsilon(P_{\eta\varepsilon})$$
$$\leq \|\bar{G}_0\|_{L^\infty(\Omega)}\int_\Omega |\delta_\varepsilon(P_{\eta\varepsilon})| \leq \|\bar{G}_0\|_{L^\infty(\Omega)}\int_\Omega |\delta_\varepsilon'(P_{\eta\varepsilon})||\nabla P_{\eta\varepsilon}|.$$

Now, using Cauchy-Schwarz and Young inequalities, we have:

$$2\int_{\Gamma_0} G_0\delta_\varepsilon(P_{\eta\varepsilon}) \leq C\left(\frac{1}{2}\int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla P_{\eta\varepsilon}|^2\,\delta_\varepsilon'(P_{\eta\varepsilon}) + \frac{1}{2}\int_\Omega \frac{1}{H_\varepsilon^3(P_{\eta\varepsilon})}\delta_\varepsilon'(P_{\eta\varepsilon})\right),$$

where C denotes a constant independent of ε and η. Using this relation in (13), together with the definition of h_{0m}, it follows that

$$\int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla P_{\eta\varepsilon}|^2\,\delta_\varepsilon'(P_{\eta\varepsilon}) \leq \frac{C}{h_{0m}}\int_\Omega \delta_\varepsilon'(P_{\eta\varepsilon}) + \frac{C\|\bar{G}_0\|_{L^\infty}^2}{h_{0m}^3}\int_\Omega \bar{\delta_\varepsilon}'(P_{\eta\varepsilon})$$

and thus

$$\int_\Omega H_\varepsilon^3(P_{\eta\varepsilon})|\nabla P_{\eta\varepsilon}|^2\,\delta'_\varepsilon(P_{\eta\varepsilon}) \le C\int_\Omega \delta'_\varepsilon(P_{\eta\varepsilon}). \tag{14}$$

Step 2: Let us recall the following lemma (see [2, p. 147]).

Lemma 4.6 *Suppose that $\mu(p)$ satisfies (5). Let δ_ε be defined as in (6) and ψ as in (7). Then there exist some constants C and M independent of ε and η such that*

$$\delta'_\varepsilon(t) \le M + C(Q+\gamma_\varepsilon(t))\psi(t) \quad \forall\, t>0. \tag{15}$$

Now, using (14) and (15):

$$\int_\Omega |\nabla P_{\eta\varepsilon}|^2\,\delta'_\varepsilon(P_{\eta\varepsilon}) \le C\int_\Omega M + C(Q+\gamma_\varepsilon(P_{\eta\varepsilon}))\psi(P_{\eta\varepsilon})$$

where we used again that $H_\varepsilon \ge h_{0m}$ in order to get rid of the term $H_\varepsilon^3(P_{\eta\varepsilon})$. Therefore, using the fact that $\gamma_\varepsilon(P_{\eta\varepsilon})$ is bounded in $L^1(\Omega)$ (Theorem 3.4), and that $\psi(t)$ is a function in $C^2(\mathbb{R})$, we obtain

$$\int_\Omega |\nabla P_{\eta\varepsilon}|^2\,\delta'_\varepsilon(P_{\eta\varepsilon}) \le C, \tag{16}$$

which finishes the proof. □

4.3 Passing to the limit

In this section, we state the existence theorem. In the proof, it is shown that Proposition 4.3 is the key estimate in order to pass to the limit.

Theorem 4.7 *Let P be defined as the limit of $P_{\eta\varepsilon}$ as in Theorem 4.1 and θ be the limit of $Z_\eta(P_\eta)$ for $\eta \to 0$. Under the hypothesis (5) and (8), (P,θ) solves the following problem:*

$$(\mathcal{P}')\begin{cases} \displaystyle\int_\Omega H^3(P)\nabla P\cdot\nabla\varphi = 6s\int_\Omega H(P)\,\theta(P)\,\partial_x\varphi + \int_{\Gamma_0} G_0\,\varphi\,, \quad \forall\varphi\in V,\\ \theta\in H(P),\end{cases}$$

with

$$H(P,x,y) = h_0(x) + \int_\Omega k(x-s,y-t)\,\gamma(P(s,t))\,ds\,dt.$$

Proof. Since the Fourier transform of the kernel k is given by $K(\xi) = \dfrac{1}{\sqrt{|\xi|^2}}$, it follows from the properties of the convolution (see [20, Satz V.2.14] for example) that if $\gamma_\varepsilon(P_{\eta\varepsilon})$

is bounded in $L^6(\Omega)$, then $H_\varepsilon(P_{\eta\varepsilon})$ given by (1) is bounded in $W^{1,6}(\Omega)$, and thus $H^3_\varepsilon(P_{\eta\varepsilon})$ is bounded in $H^1(\Omega)$.

Hence we have the following convergences, for $\varepsilon \to 0$ and then $\eta \to 0$:

$$P_{\eta\varepsilon} \to P \text{ in } L^2(\Omega), \qquad \text{and} \qquad Z_\eta(P_{\eta\varepsilon}) \rightharpoonup \theta(P) \text{ in } L^\infty(\Omega),$$
$$H^3_\varepsilon(P_{\eta\varepsilon}) \rightharpoonup J \text{ in } H^1(\Omega), \qquad \text{and thus} \quad H^3_\varepsilon(P_{\eta\varepsilon}) \to J \text{ in } L^2(\Omega).$$

Now, we showed in the proof of Theorem 4.1 that $\gamma_\varepsilon(P_{\eta\varepsilon})$ converges to $\gamma(P)$ in $L^1(\Omega)$. Thus, by the uniqueness of the limit, it follows that $J = H^3(P)$.

Therefore, it is possible to pass to the limit in every term of problem $(\mathcal{P}_{\eta\varepsilon})$. Thus, (P, θ) is a solution of the problem $(\mathcal{P}')$. This finishes the proof. □

Remark 4.8 *Now, if we consider the problem in one dimension, it is possible to prove that problems $(\mathcal{P})$ and $(\mathcal{P}')$ are equivalent. Indeed, since the embedding $H^1 \hookrightarrow C^0$ is compact in one-dimensional space, the weak convergence of $P_{\eta\varepsilon}$ implies actually that $P_{\eta\varepsilon}$ converges uniformly towards P. Thus, $\gamma_\varepsilon(P_{\eta\varepsilon})$ also converges uniformly towards $\gamma(P)$, thus $P < A$ and problems $(\mathcal{P})$ and $(\mathcal{P}')$ are equivalent.*

However, in the two-dimensional case, we are not able to prove that problems $(\mathcal{P})$ and $(\mathcal{P}')$ are equivalent. The estimate $\|P\|_{L^\infty(\Omega)} \leq A$ we obtained previously is not enough to prove the existence of a solution of $(\mathcal{P})$, since p can be infinite and thus does not lie in $H^1(\Omega)$. In fact, since physically the pressure p cannot be infinite, it is relevant to have studied problem $(\mathcal{P}')$.

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References

[1] S. J. Alvarez and J. Carrillo, *A free boundary problem in theory of lubrication*, Comm. Partial Differential Equations, Vol. 19, 11-12 (1994), 1743–1761.

[2] G. Bayada, H. Bellout, *An unconditional existence result for the quasi-variational elastohydrodynamic free boundary value problem*, J. Differential Equations, Vol. 216, 1 (2005), 134–152.

[3] G. Bayada and M. Chambat, *Sur quelques modélisations de la zone de cavitation en lubrification hydrodynamique*, J. Méc. Théor. Appl., Vol. 5(5) (1986),703–729.

[4] G. Bayada, M. El Alaoui Talibi, and C. Vázquez, *Existence of solutions for elastohydrodynamic piezoviscous lubrication problems with a new model of cavitation*, European J. Appl. Math., Vol. 7(1) (1996), 63–73.

[5] H. Bellout, *Existence of a solution to the line contact problem of elastohydrodynamic lubrication*, European J. Appl. Math., Vol. 14(3) (2003), 279–290.

[6] H. Brézis, D. Kinderlehrer, and G. Stampacchia, *Sur une nouvelle formulation du probléme de l'écoulement à travers une digue*, C. R. Acad. Sci. Paris Sér. A-B, Vol. 287(9) (1978), A711–A714.

[7] M. Chipot, *Variational inequalities and flow in porous media*, Vol. 52 of "Applied Mathematical Sciences", Springer-Verlag, New York, 1984.

[8] H. G. Elrod and M. L. Adams, *A computer program for cavitation*, "Cavitation and related phenomena in lubrication - Proceedings -Mech. Eng. Publ. ltd" 37–42, 1975.

[9] L. Floberg, *Lubrication of two rotating cylinders at variable lubricant supply with reference to the tensile strength of the liquid lubricant*, ASME J. Lub. Technol., Vol. 95 (1973), 155–165.

[10] J. Frene, D. Nicolas, B. Deugueurce, D. Berthe, and M. Godet, "Hydordynamic lubrication: bearings and thrust bearings", Elsevier Science, 1997.

[11] R. Glowinski, J.-L. Lions, and R. Trémolières, "Analyse numérique des inéquations variationnelles. Tome 1", Dunod, Paris, 1976.

[12] B. Hu, *A quasi-variational inequality arising in elastohydrodynamics*, SIAM J. Math. Anal., Vol. 21(1) (1960), 18–36.

[13] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, volume 88 of "Pure and Applied Mathematics", Academic Press Inc., New York, 1980.

[14] J. Nečas, Les méthodes "directes en théorie des équations elliptiques", Masson et Cie, Éditeurs, Paris, 1967.

[15] J. T. Oden and S. R. Wu, *Existence of solutions to the reynolds equation of eslastohydrodynamic lubrication*, Int. Jour. Eng. Sci., Vol.31 (1985), 207–215.

[16] J.-F. Rodrigues, *Obstacle problems in mathematical physics*, Vol. 134 of "North-Holland Mathematics Studies", North-Holland Publishing Co., Amsterdam, 1987.

[17] J.-F. Rodrigues, *Remarks on the Reynolds problem of elastohydrodynamic lubrication*, European J. Appl. Math., Vol. 4(1) (1993), 83–96.

[18] A. Z. Szeri, "Fluid Film Lubrication: Theory and Design", Cambridge University Press, 1998.

[19] P. Vergne, "Comportement Rhéologique des Lubrifiants et Lubrification, Approches expérimentales", PhD thesis, INSA de Lyon, 2002.

[20] D. Werner, "Funktionalanalysis", Springer-Verlag, Berlin, extended edition, 2000.