

Weak solutions to the equations of stationary magnetohydrodynamic flows in porous media

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Abstract

In this paper we study the differential system which describes the steady flow of an electrically conducting fluid in a saturated porous medium, when the fluid is subjected to the action of a magnetic field. The system consists of the stationary Brinkman-Forchheimer equations and the stationary magnetic induction equation. We prove existence of weak solutions to the system posed in a bounded domain of \mathbb{R}^3 and equipped with boundary conditions. We also prove uniqueness in the class of small solutions, and regularity of weak solutions. Then we establish a convergence result, as the Brinkman coefficient (viscosity) tends to 0, of the weak solutions to a solution of the system formed by the Darcy-Forchheimer equations and the magnetic induction equation.

Keywords: magnetohydrodynamic flows in porous media, Brinkman-Forchheimer equations, Darcy-Forchheimer equations, Lorentz force, weak solutions.

1 Introduction and main results

Magnetohydrodynamic (MHD) flows in porous media arise in many applications such as in chemical technology, geophysical energy systems, flow control processes in mechanical engineering. In metallurgy, many alloy generally solidify with a dendritic columnar or equiaxial structure. The region where dendrites and the liquid phase coexist, referred to as *mushy zone* during the solidification process is heterogeneous and can be assimilated to a porous medium. The use of magnetic field is a tool for controlling the melt flow and thus can influence the solidification process, mostly to create a finely disposed equiaxed growth morphology, see for instance [18, 19, 33] and the references therein. In crystal growth applications in porous media, the applied external magnetic field has been successfully exploited to suppress unsteady flows and also to reduce the non-uniformity of composition [26].

The purpose of the present work is the mathematical analysis of the differential system which describes the steady flow of an electrically conducting fluid in a bounded domain $\Omega \subset \mathbb{R}^3$ representing a saturated porous medium, when the fluid is subjected to the action of a magnetic field.

1.1 Problem formulation

Let us consider we are given a source electric current density \mathbf{J}_0 with compact support in Ω , and denote by \mathbf{B} , \mathbf{J} , \mathbf{H} and \mathbf{E} , respectively, the magnetic induction, electric current density, magnetic field and electric field. We look for stationary electromagnetic fields that satisfy the stationary Maxwell equations where displacement currents are neglected (see [16, 23]):

$$\mathbf{curl} \mathbf{H} = \mathbf{J} + \mathbf{J}_0, \quad (1)$$

$$\mathbf{curl} \mathbf{E} = 0, \quad (2)$$

$$\mathbf{J} = \sigma 1_\Omega (\mathbf{E} + \mathbf{u} \times \mathbf{B}), \quad (3)$$

$$\mathbf{B} = \mu \mathbf{H}, \quad (4)$$

$$\operatorname{div} \mathbf{B} = 0. \quad (5)$$

Equations (1)–(5) are considered in \mathbb{R}^3 , 1_Ω is the characteristic function of Ω , σ is the electric conductivity, μ is the magnetic permeability, and \mathbf{u} is the fluid velocity. In the sequel σ and μ are assumed positive constants for simplicity. In other words, we assume that the fluid is homogeneous with constant electric conductivity and nonferromagnetic.

The motion of the fluid in Ω is governed by the so-called Brinkman-Forchheimer equations (see [8, 11, 14, 32]):

$$\operatorname{div} \mathbf{u} = h \quad \text{in } \Omega, \quad (6)$$

$$a\mathbf{u} + b|\mathbf{u}|\mathbf{u} - \gamma\Delta\mathbf{u} + \nabla p = (\mathbf{J} + \mathbf{J}_0) \times \mathbf{B} \quad \text{in } \Omega, \quad (7)$$

where p is the pressure, $a > 0$ (viscosity divided by permeability) and $b > 0$ are the Darcy and Forchheimer coefficients, respectively, and γ is the Brinkman coefficient (viscosity). We note that in (6) we assumed, for generality, a nonzero mass source h as this is frequently assumed in porous media applications. The term $(\mathbf{J} + \mathbf{J}_0) \times \mathbf{B}$ in (7) represents the Lorentz force due to the induced electric current of MHD.

In order to derive a well-posed problem in Ω we first note that using (3) and (4) we obtain \mathbf{J} and \mathbf{H} with respect to \mathbf{B} , \mathbf{E} and \mathbf{u} . Then, from (1) we deduce that we have

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{\sigma\mu} \mathbf{curl} \mathbf{B} - \frac{1}{\sigma} \mathbf{J}_0 \quad \text{in } \Omega.$$

Taking the curl of the above equation and using (2) we obtain

$$\frac{1}{\sigma\mu} \mathbf{curl}^2 \mathbf{B} = \mathbf{curl}(\mathbf{u} \times \mathbf{B}) + \frac{1}{\sigma} \mathbf{curl} \mathbf{J}_0 \quad \text{in } \Omega.$$

We require the functions \mathbf{u} , \mathbf{B} and \mathbf{E} to satisfy the following boundary conditions. For the velocity we specify

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{8}$$

where \mathbf{g} is a given function on $\partial\Omega$. We should notice here that naturally the given boundary velocity must be compatible with the conservation mass equation (6) by requiring the condition

$$\int_{\Omega} h \, dx = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, ds. \tag{9}$$

We also impose the perfect conductor boundary condition

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{10}$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{11}$$

\mathbf{n} being the unit outward normal vector to $\partial\Omega$. It results from (1), (3), (4), (8) and (10) that the field \mathbf{B} satisfies the condition

$$\frac{1}{\mu} \mathbf{curl} \mathbf{B} \times \mathbf{n} = \sigma(\mathbf{g} \times \mathbf{B}) \times \mathbf{n} \quad \text{on } \partial\Omega. \tag{12}$$

Using (11) and the vector identity

$$(\mathbf{g} \times \mathbf{B}) \times \mathbf{n} = (\mathbf{g} \cdot \mathbf{n}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{n}) \mathbf{g},$$

the boundary condition (12) turns into

$$\frac{1}{\mu} \mathbf{curl} \mathbf{B} \times \mathbf{n} = \sigma(\mathbf{g} \cdot \mathbf{n}) \mathbf{B} \quad \text{on } \partial\Omega.$$

Thus we obtain a boundary-value problem for the velocity \mathbf{u} , the pressure p and the magnetic induction \mathbf{B} , in the domain Ω , formed by the equations

$$\begin{cases} \operatorname{div} \mathbf{u} = h, & \operatorname{div} \mathbf{B} = 0, \\ a\mathbf{u} + b|\mathbf{u}|\mathbf{u} - \gamma\Delta\mathbf{u} + \nabla p = \frac{1}{\mu} \operatorname{curl} \mathbf{B} \times \mathbf{B}, \\ \frac{1}{\sigma\mu} \operatorname{curl}^2 \mathbf{B} = \operatorname{curl}(\mathbf{u} \times \mathbf{B}) + \frac{1}{\sigma} \operatorname{curl} \mathbf{J}_0, \end{cases} \quad (13)$$

and the boundary conditions

$$\mathbf{u} = \mathbf{g}, \quad \frac{1}{\mu} \operatorname{curl} \mathbf{B} \times \mathbf{n} = \sigma(\mathbf{g} \cdot \mathbf{n}) \mathbf{B}, \quad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad (14)$$

where h and \mathbf{g} are given functions defined on Ω and $\partial\Omega$, respectively, and satisfy (9).

Once Problem (13), (14) is solved, *i.e.* in fact when the fluid velocity \mathbf{u} and the magnetic field \mathbf{B} are known in Ω , the electric field \mathbf{E} can simply be deduced in Ω from \mathbf{B} using Equations (1), (3) and (4):

$$\mathbf{E} = \frac{1}{\sigma\mu} \operatorname{curl} \mathbf{B} - \mathbf{u} \times \mathbf{B} - \frac{1}{\sigma} \mathbf{J}_0 \quad \text{in } \Omega.$$

The aim of this paper is to investigate the solvability of problem (13), (14) and to study the limit, as $\gamma \rightarrow 0$, of the corresponding solutions. Formally, the limit problem is formed by the magnetic induction equation and the Darcy-Forchheimer equations that is the system

$$\begin{cases} \operatorname{div} \mathbf{u} = h, & \operatorname{div} \mathbf{B} = 0 \quad \text{in } \Omega, \\ a\mathbf{u} + b|\mathbf{u}|\mathbf{u} + \nabla p = \frac{1}{\mu} \operatorname{curl} \mathbf{B} \times \mathbf{B} \quad \text{in } \Omega, \\ \frac{1}{\sigma\mu} \operatorname{curl}^2 \mathbf{B} = \operatorname{curl}(\mathbf{u} \times \mathbf{B}) + \frac{1}{\sigma} \operatorname{curl} \mathbf{J}_0 \quad \text{in } \Omega, \end{cases} \quad (15)$$

supplemented with the boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n}, \quad \frac{1}{\mu} \operatorname{curl} \mathbf{B} \times \mathbf{n} = \sigma(\mathbf{g} \cdot \mathbf{n}) \mathbf{B}, \quad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \quad (16)$$

The Brinkman-Forchheimer equations, with a given body force \mathbf{f} ,

$$\operatorname{div} \mathbf{u} = h, \quad a\mathbf{u} + b|\mathbf{u}|\mathbf{u} - \gamma\Delta\mathbf{u} + \nabla p = \mathbf{f},$$

were studied by many authors, particularly for continuous dependence on changes in Brinkman and Forchheimer coefficients and convergence of solutions of Brinkman-Forchheimer equations to the solution of the Darcy-Forchheimer equations

$$\operatorname{div} \mathbf{u} = h, \quad a\mathbf{u} + b|\mathbf{u}|\mathbf{u} + \nabla p = \mathbf{f},$$

as the viscosity γ tends to zero, see [3, 9, 24, 25] and the references therein. For the derivation and justification of the Brinkman and Forchheimer equations we refer to [8, 14, 32]. The regularity of the solution to the Darcy-Forchheimer equations is discussed in [7]. A nonstationary model of gas flow through a porous medium, which obeys to the Darcy-Forchheimer law, is studied in [1]. A system coupling the linearized Darcy-Brinkman equations (with $b = 0$) to the magnetic induction is considered in [11]; a macroscopic tensorial filtration law in rigid porous media is derived by using the method of multiple scale expansions. However, to our knowledge, the system coupling the Darcy-Brinkman (or Darcy-Forchheimer) equations with the magnetic induction equation has not been studied. There have been extensive mathematical studies on the solutions of the equations of MHD viscous and resistive incompressible fluids. In particular, global weak solutions and local strong solutions have been constructed in [6]. Properties of weak and strong solutions have been examined in [28]. Some sufficient conditions for regularity of weak solutions to the MHD equations were obtained in [15]. A magnetohydrodynamic system consisting of the stationary Maxwell equations coupled with the transient Navier-Stokes equations is examined in [13]. Stationary systems coupling the MHD equations and the heat equation are investigated in [4] and [22].

1.2 Main results

Before we can formulate our main results we need to introduce some notations. We assume that Ω is a simply-connected bounded domain in \mathbb{R}^3 , with smooth boundary $\partial\Omega$. Let $L^q(\Omega)$ and $W^{s,q}(\Omega)$ ($1 \leq q \leq \infty$, $s \in \mathbb{R}$) be the usual Lebesgue and Sobolev spaces of scalar-valued functions, respectively. When $q = 2$, $W^{s,q}(\Omega)$ is denoted by $H^s(\Omega)$. By $\|\cdot\|$ and (\cdot, \cdot) we denote the L^2 -norm and its scalar product, respectively. The Hölder spaces $C^{k,\alpha}(\bar{\Omega})$ ($k \in \mathbb{N}$, $0 < \alpha < 1$) are defined as the subspaces of $C^k(\bar{\Omega})$ consisting of functions whose k -th order partial derivatives are Hölder continuous with exponent α . We set $\mathbb{L}^q(\Omega) = (L^q(\Omega))^3$, $\mathbb{W}^{s,q}(\Omega) = (W^{s,q}(\Omega))^3$, $\mathbb{H}^s(\Omega) = (H^s(\Omega))^3$ and $C^{k,\alpha}(\bar{\Omega}, \mathbb{R}^3) = (C^{k,\alpha}(\bar{\Omega}))^3$. We introduce the classical function spaces in the theory of the Navier-Stokes equations (see [10, 17, 20, 21, 30, 31]):

$$\begin{aligned} \mathcal{D}_s(\Omega) &= \{ \mathbf{v} \in \mathcal{D}(\Omega, \mathbb{R}^3) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \}, \\ V &= \text{closure of } \mathcal{D}_s(\Omega) \text{ in } \mathbb{H}^1(\Omega), \\ V_{0q} &= \text{closure of } \mathcal{D}_s(\Omega) \text{ in } \mathbb{L}^q(\Omega), \quad 1 < q < \infty. \end{aligned}$$

Here $\mathcal{D}(\Omega, \mathbb{R}^3)$ is the space of infinitely differentiable functions with compact support in Ω , and valued in \mathbb{R}^3 . As is well known,

$$\begin{aligned} V &= \{ \mathbf{v} \in \mathbb{H}_0^1(\Omega) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \}, \\ V_{0q} &= \{ \mathbf{v} \in \mathbb{L}^q(\Omega) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \}, \\ V &\subset V_{02} \subset V' = \text{dual space of } V \text{ when } V_{02} \text{ is identified with its dual.} \end{aligned}$$

We also introduce the spaces

$$\begin{aligned} \mathcal{D}_s(\bar{\Omega}) &= \{ \mathbf{C} \in \mathcal{D}(\bar{\Omega}, \mathbb{R}^3) : \operatorname{div} \mathbf{C} = 0 \text{ in } \Omega, \mathbf{C} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \}, \\ W &= \text{closure of } \mathcal{D}_s(\bar{\Omega}) \text{ in } \mathbb{H}^1(\Omega), \\ W_0 &= \text{closure of } \mathcal{D}_s(\bar{\Omega}) \text{ in } \mathbb{L}^2(\Omega), \end{aligned}$$

where $\mathcal{D}(\bar{\Omega}, \mathbb{R}^3)$ is the space of infinitely differentiable functions with compact support in $\bar{\Omega}$, and valued in \mathbb{R}^3 . We have

$$\begin{aligned} W &= \{ \mathbf{C} \in \mathbb{H}^1(\Omega) : \operatorname{div} \mathbf{C} = 0 \text{ in } \Omega, \mathbf{C} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \}, \\ W_0 &= V_{02}, \\ W &\subset W_0 \subset W' = \text{dual space of } W. \end{aligned}$$

Note that $(\int_{\Omega} |\mathbf{curl} \mathbf{C}|^2 dx)^{1/2}$ defines a norm on W which is equivalent to that induced by $\mathbb{H}^1(\Omega)$ on W , see [5, Chap. 7, Theorem 6.1]. Recall that $\mathbf{v} \cdot \mathbf{n}$ makes sense in $W^{-\frac{1}{q}, q}(\partial\Omega)$ when \mathbf{v} belongs to the space

$$H_q(\operatorname{div}, \Omega) = \{ \mathbf{v} \in \mathbb{L}^q(\Omega) : \operatorname{div} \mathbf{v} \in L^q(\Omega) \},$$

and we have the Stokes formula: $\forall \mathbf{v} \in H_q(\operatorname{div}, \Omega), \forall \varphi \in W^{1, q'}(\Omega)$,

$$\int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, d\mathbf{x} = - \int_{\Omega} \varphi \operatorname{div} \mathbf{v} \, d\mathbf{x} + \langle \mathbf{v} \cdot \mathbf{n}, \varphi \rangle_{\partial\Omega},$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ is the duality pairing between $W^{-\frac{1}{q}, q}(\partial\Omega)$ and $W^{\frac{1}{q}, q'}(\partial\Omega)$. Similarly, if \mathbf{v} belongs to the space

$$H(\operatorname{curl}; \Omega) = \{ \mathbf{v} \in \mathbb{L}^2(\Omega) : \mathbf{curl} \mathbf{v} \in \mathbb{L}^2(\Omega) \},$$

then \mathbf{v} has a tangential component $\mathbf{v} \times \mathbf{n} \in \mathbb{H}^{-\frac{1}{2}}(\partial\Omega)$ and the following Green's formula holds

$$\forall \mathbf{w} \in \mathbb{H}^1(\Omega), \int_{\Omega} \mathbf{curl} \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x} = \int_{\Omega} \mathbf{v} \cdot \mathbf{curl} \mathbf{w} \, d\mathbf{x} + \langle \mathbf{v} \times \mathbf{n}, \mathbf{w} \rangle_{\partial\Omega}.$$

We assume that

$$\mathbf{J}_0 \in \mathbb{L}^2(\Omega), \quad h \in L^2(\Omega), \quad \mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\partial\Omega), \quad \text{with } \int_{\Omega} h \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, ds. \quad (17)$$

By virtue of Bogovskii's Theorem, see [10, Section III. 3], there exists $\mathbf{u}_\star \in \mathbb{H}^1(\Omega)$ satisfying the equations $\operatorname{div} \mathbf{u}_\star = h$ in Ω , $\mathbf{u}_\star = \mathbf{g}$ on $\partial\Omega$ and the estimate

$$\|\mathbf{u}_\star\|_{\mathbb{H}^1(\Omega)} \leq c \left(\|h\| + \|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)} \right), \quad (18)$$

where c is a constant that depends on Ω only. Without loss of generality, in the sequel we will suppose that $\sigma = \mu = a = b = 1$. For notational convenience, we refer to Problem (13), (14) (for $\gamma > 0$) to as Problem (\mathcal{P}_γ) , and Problem (15), (16) to as Problem (\mathcal{P}) . We denote by c a generic constant that depends on Ω only and by $c(\gamma)$ a generic constant that depends on Ω and on γ only.

Definition 1. *We say that (\mathbf{u}, \mathbf{B}) is a weak solution of Problem (\mathcal{P}_γ) if the following conditions are satisfied:*

(i) $\mathbf{u} - \mathbf{u}_\star \in V$, $\mathbf{B} \in W$.

(ii) Equations (13)₂, (13)₃ hold weakly: for every $(\mathbf{v}, \mathbf{C}) \in V \times W$, we have

$$\left\{ \begin{array}{l} \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx + \int_{\Omega} |\mathbf{u}| \mathbf{u} \cdot \mathbf{v} \, dx + \gamma \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx \\ \quad = \int_{\Omega} (\operatorname{curl} \mathbf{B} \times \mathbf{B}) \cdot \mathbf{v} \, dx, \\ \int_{\Omega} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \mathbf{C} \, dx \\ \quad = - \int_{\Omega} (\operatorname{curl} \mathbf{C} \times \mathbf{B}) \cdot \mathbf{u} \, dx + \int_{\Omega} \mathbf{J}_0 \cdot \operatorname{curl} \mathbf{C} \, dx. \end{array} \right. \quad (19)$$

(iii) There exists $p \in L^2(\Omega)$ such that Equation (13)₂ holds in $\mathcal{D}'(\Omega, \mathbb{R}^3)$.

We denote in the sequel $V_0 = V_{03}$ for simplicity.

Definition 2. *We say that (\mathbf{u}, \mathbf{B}) is a weak solution of Problem (\mathcal{P}) if the following conditions hold:*

(i) $\mathbf{u} - \mathbf{u}_\star \in V_0$, $\mathbf{B} \in W$.

(ii) Equations (15)₂, (15)₃ hold weakly: for every $(\mathbf{v}, \mathbf{C}) \in V_0 \times W$, we have

$$\left\{ \begin{array}{l} \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx + \int_{\Omega} |\mathbf{u}| \mathbf{u} \cdot \mathbf{v} \, dx = \int_{\Omega} (\operatorname{curl} \mathbf{B} \times \mathbf{B}) \cdot \mathbf{v} \, dx, \\ \int_{\Omega} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \mathbf{C} \, dx \\ \quad = - \int_{\Omega} (\operatorname{curl} \mathbf{C} \times \mathbf{B}) \cdot \mathbf{u} \, dx + \int_{\Omega} \mathbf{J}_0 \cdot \operatorname{curl} \mathbf{C} \, dx. \end{array} \right. \quad (20)$$

(iii) There exists $p \in W^{1, \frac{3}{2}}(\Omega)$ such that Equation (15)₂ holds in $\mathbb{W}^{1, \frac{3}{2}}(\Omega)$.

Our first main result is concerned with the solvability of Problem (\mathcal{P}_γ) .

Theorem 1. *Assume that the boundary $\partial\Omega$ is Lipschitz-continuous. Let h , \mathbf{g} and \mathbf{J}_0 denote three functions satisfying (17). Then for all $\gamma > 0$, Problem (\mathcal{P}_γ) admits a weak solution $(\mathbf{u}_\gamma, \mathbf{B}_\gamma)$ in the sense of Definition 1. Moreover:*

(i) Any weak solution $(\mathbf{u}_\gamma, \mathbf{B}_\gamma)$ satisfies the estimate

$$\begin{aligned} & \|\mathbf{u}_\gamma\|^2 + \gamma \|\nabla \mathbf{u}_\gamma\|^2 + \|\mathbf{curl} \mathbf{B}_\gamma\|^2 + \|\mathbf{u}_\gamma\|_{\mathbb{L}^3(\Omega)}^3 \\ & \leq c \|\mathbf{J}_0\|^2 + c(\gamma) \left(\|\mathbf{u}_\star\|_{\mathbb{H}^1(\Omega)}^2 + \|\mathbf{u}_\star\|_{\mathbb{H}^1(\Omega)}^3 \right) \end{aligned} \quad (21)$$

and $c(\gamma)$ is uniformly bounded with respect to γ when γ varies in a bounded interval.

(ii) There exists $\varepsilon > 0$, depending only on the domain Ω , such that if the data h , \mathbf{g} and \mathbf{J}_0 satisfy

$$\|h\| + \|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)} + \|\mathbf{J}_0\| \leq \varepsilon, \quad (22)$$

then the weak solution $(\mathbf{u}_\gamma, \mathbf{B}_\gamma)$ of Problem (\mathcal{P}_γ) is unique.

Our second main result is concerned with the regularity of weak solutions of Problem (\mathcal{P}_γ) .

Theorem 2. *Assume that the boundary $\partial\Omega$ is of class \mathcal{C}^2 . Let h , \mathbf{g} and \mathbf{J}_0 denote three functions satisfying (17). Assume in addition that $\mathbf{g} \cdot \mathbf{n} = 0$ on $\partial\Omega$, and $\mathbf{curl} \mathbf{J}_0 \in \mathbb{L}^q(\Omega)$, $(h, \mathbf{g}) \in W^{1, q}(\Omega) \times \mathbb{W}^{2 - \frac{1}{q}, q}(\partial\Omega)$, with $q \geq \frac{3}{2}$. Then, any weak solution $(\mathbf{u}_\gamma, \mathbf{B}_\gamma)$ of Problem (\mathcal{P}_γ) belongs to $\mathbb{W}^{2, q}(\Omega) \times \mathbb{W}^{2, q}(\Omega)$, p_γ belongs to $W^{1, q}(\Omega)$ and we have the estimates*

$$\begin{aligned} & \|\mathbf{u}_\gamma\|_{\mathbb{W}^{2, q}(\Omega)} + \|p_\gamma\|_{W^{1, q}(\Omega)} \leq c(\gamma) \left(D + \|h\|_{W^{1, q}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{2 - \frac{1}{q}, q}(\partial\Omega)} \right), \\ & \|\mathbf{B}_\gamma\|_{\mathbb{W}^{2, q}(\Omega)} \leq c(\gamma) \tilde{D} + c \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^q(\Omega)}. \end{aligned}$$

Here D and \tilde{D} are positive quantities depending only on q , $\|\mathbf{J}_0\|$, $\|\mathbf{curl} \mathbf{J}_0\|$, $\|h\|$, $\|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)}$ and $\|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{\bar{q}}(\Omega)}$, $\|h\|_{W^{1, \bar{q}}(\Omega)}$, $\|\mathbf{g}\|_{\mathbb{W}^{2 - \frac{1}{\bar{q}}, \bar{q}}(\partial\Omega)}$, for some values of $\bar{q} \leq q$ (see (42), (45), (48), (51)–(54)).

Our third main result deals with the limit, as $\gamma \rightarrow 0$, of the weak solutions $(\mathbf{u}_\gamma, \mathbf{B}_\gamma)$ of Problem (\mathcal{P}_γ) .

Theorem 3. *Assume that the boundary $\partial\Omega$ is Lipschitz-continuous. Let h , \mathbf{g} and \mathbf{J}_0 denote three functions satisfying (17). Let (γ_n) be a sequence of positive numbers which tends to 0. For each γ_n , let $(\mathbf{u}_{\gamma_n}, \mathbf{B}_{\gamma_n})$ be a weak solution (in the sense of Definition 1) of Problem (\mathcal{P}_{γ_n}) . Then the sequence $(\mathbf{u}_{\gamma_n}, \mathbf{B}_{\gamma_n})$ converges weakly in $\mathbb{L}^3(\Omega) \times W$ to a weak solution (\mathbf{u}, \mathbf{B}) of Problem (\mathcal{P}) , in the sense of Definition 2.*

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. The existence of weak solutions is proved by introducing an operator A which is pseudo-monotone and coercive. The coerciveness of A ensures the control (21) of the solutions. Estimate (21) is then used to establish uniqueness in the class of small solutions. Section 3 is devoted to the proof of Theorem 2: the regularity of weak solutions is obtained by successive applications of classical regularity results for Stokes and \mathbf{curl}^2 equations. In Section 4 we examine the limit, as γ tends to zero, of the sequence of approximate solutions $(\mathbf{u}_\gamma, \mathbf{B}_\gamma)$ of Problem (\mathcal{P}_γ) . The main difficulty is to pass to the limit on the quadratic term $|\mathbf{u}_\gamma|\mathbf{u}_\gamma$, knowing that \mathbf{u}_γ and $\mathbf{curl} \mathbf{B}_\gamma \times \mathbf{B}_\gamma$ (the right-hand side of equation (15)₂) converge only weakly in $\mathbb{L}^3(\Omega)$ and $\mathbb{L}^{\frac{3}{2}}(\Omega)$, respectively. We use monotony arguments and a key ingredient which consists in introducing the quantity X_γ by (61) in order to take into account of the strong coupling of the Brinkman-Forchheimer equations with the magnetic induction equation. Finally we give a direct proof of existence of weak solutions to Problem (\mathcal{P}) by introducing an operator A_0 which is pseudo-monotone and coercive.

2 Proof of Theorem 1

One of the difficulties in the proof is related with the quadratic term $|\mathbf{u}|\mathbf{u}$ appearing in Problems (\mathcal{P}_γ) and (\mathcal{P}) . The key ingredient to treat such a term is the following lemma (see [29] for the proof):

Lemma 1. *We have, for any $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$,*

$$\frac{1}{2}|\mathbf{v}_2 - \mathbf{v}_1|^3 \leq (|\mathbf{v}_2|\mathbf{v}_2 - |\mathbf{v}_1|\mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1), \quad (23)$$

$$||\mathbf{v}_2|\mathbf{v}_2 - |\mathbf{v}_1|\mathbf{v}_1| \leq (|\mathbf{v}_1| + |\mathbf{v}_2|)|\mathbf{v}_2 - \mathbf{v}_1|. \quad (24)$$

2.1 Existence of a weak solution

For any $\mathbf{u} \in V$ we set $\tilde{\mathbf{u}} = \mathbf{u} + \mathbf{u}_*$. We define the operator

$$A : V \times W \rightarrow V' \times W'$$

by

$$\begin{aligned} \langle A(\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C}) \rangle &= \int_{\Omega} \tilde{\mathbf{u}} \cdot \mathbf{v} \, dx + \int_{\Omega} |\tilde{\mathbf{u}}|\tilde{\mathbf{u}} \cdot \mathbf{v} \, dx + \gamma \int_{\Omega} \nabla \tilde{\mathbf{u}} : \nabla \mathbf{v} \, dx \\ &\quad - \int_{\Omega} (\mathbf{curl} \mathbf{B} \times \mathbf{B}) \cdot \mathbf{v} \, dx + \int_{\Omega} \mathbf{curl} \mathbf{B} \cdot \mathbf{curl} \mathbf{C} \, dx \\ &\quad + \int_{\Omega} (\mathbf{curl} \mathbf{C} \times \mathbf{B}) \cdot \tilde{\mathbf{u}} \, dx, \end{aligned} \quad (25)$$

for any (\mathbf{u}, \mathbf{B}) and (\mathbf{v}, \mathbf{C}) in $V \times W$.

a) Let us show that the operator A is pseudo-monotone [20, Chapter 2, Section 2]. Recall that A is said to be pseudo-monotone if A is a bounded operator and if whenever $(\mathbf{u}_n, \mathbf{B}_n)$ converges to (\mathbf{u}, \mathbf{B}) in $V \times W$ weak and

$$\limsup \langle A(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n, \mathbf{B}_n) - (\mathbf{u}, \mathbf{B}) \rangle \leq 0, \quad (26)$$

it follows that for any $(\mathbf{v}, \mathbf{C}) \in V \times W$,

$$\liminf \langle A(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n, \mathbf{B}_n) - (\mathbf{v}, \mathbf{C}) \rangle \geq \langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) - (\mathbf{v}, \mathbf{C}) \rangle. \quad (27)$$

Using the Hölder inequality and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ we have

$$\begin{aligned} \int_{\Omega} |\tilde{\mathbf{u}} \tilde{\mathbf{u}} \cdot \mathbf{v}| \, dx &\leq \|\tilde{\mathbf{u}}\|_{\mathbb{L}^3(\Omega)}^2 \|\mathbf{v}\|_{\mathbb{L}^3(\Omega)} \leq c \|\tilde{\mathbf{u}}\|_{\mathbb{H}^1(\Omega)}^2 \|\mathbf{v}\|_V, \\ \int_{\Omega} |(\mathbf{curl} \mathbf{C} \times \mathbf{B}) \cdot \tilde{\mathbf{u}}| \, dx &\leq \|\tilde{\mathbf{u}}\|_{\mathbb{L}^3(\Omega)} \|\mathbf{B}\|_{\mathbb{L}^6(\Omega)} \|\mathbf{curl} \mathbf{C}\| \\ &\leq c \|\tilde{\mathbf{u}}\|_{\mathbb{H}^1(\Omega)} \|\mathbf{B}\|_W \|\mathbf{C}\|_W. \end{aligned}$$

Estimating similarly the other terms in the right-hand side of (25) we deduce that

$$\begin{aligned} |\langle A(\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C}) \rangle| &\leq c \left(\|\tilde{\mathbf{u}}\|_{\mathbb{H}^1(\Omega)} + \|\tilde{\mathbf{u}}\|_{\mathbb{H}^1(\Omega)}^2 + \gamma \|\tilde{\mathbf{u}}\|_{\mathbb{H}^1(\Omega)} + \|\mathbf{B}\|_W^2 \right) \|\mathbf{v}\|_V \\ &\quad + c \left(\|\mathbf{B}\|_W + \|\mathbf{B}\|_W \|\tilde{\mathbf{u}}\|_{\mathbb{H}^1(\Omega)} \right) \|\mathbf{C}\|_W, \end{aligned}$$

which shows that the operator A is bounded.

Consider now a sequence $(\mathbf{u}_n, \mathbf{B}_n)$ which converges to (\mathbf{u}, \mathbf{B}) in $V \times W$ weak and so that (26) holds. By the compact Sobolev embedding there is a subsequence of $(\mathbf{u}_n, \mathbf{B}_n)$, still indexed by n , such that

$$\mathbf{u}_n \rightarrow \mathbf{u} \quad \text{and} \quad \mathbf{B}_n \rightarrow \mathbf{B} \quad \text{in } \mathbb{L}^2(\Omega) \text{ strong.} \quad (28)$$

Using the fact that $\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}} = \mathbf{u}_n - \mathbf{u}$, we have

$$\begin{aligned} &\langle A(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n - \mathbf{u}, \mathbf{B}_n - \mathbf{B}) \rangle \\ &= \int_{\Omega} \tilde{\mathbf{u}}_n \cdot (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx + \int_{\Omega} |\tilde{\mathbf{u}}_n| \tilde{\mathbf{u}}_n \cdot (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx \\ &\quad + \gamma \int_{\Omega} \nabla \tilde{\mathbf{u}}_n : \nabla (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx + \int_{\Omega} (\mathbf{curl} \mathbf{B}_n \times \mathbf{B}_n) \cdot \tilde{\mathbf{u}} \, dx \\ &\quad + \int_{\Omega} \mathbf{curl} \mathbf{B}_n \cdot \mathbf{curl}(\mathbf{B}_n - \mathbf{B}) \, dx - \int_{\Omega} \mathbf{curl} \mathbf{B} \times \mathbf{B}_n \cdot \tilde{\mathbf{u}}_n \, dx. \quad (29) \end{aligned}$$

From (26), (28) and (29) we find that

$$\limsup \left(\int_{\Omega} \nabla \tilde{\mathbf{u}}_n : \nabla (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx + \int_{\Omega} \mathbf{curl} \mathbf{B}_n \cdot \mathbf{curl}(\mathbf{B}_n - \mathbf{B}) \, dx \right) \leq 0,$$

which implies that

$$\limsup \left(\int_{\Omega} |\nabla(\mathbf{u}_n - \mathbf{u})|^2 dx + \int_{\Omega} |\mathbf{curl}(\mathbf{B}_n - \mathbf{B})|^2 dx \right) \leq 0.$$

Thus there is a subsequence of $(\mathbf{u}_n, \mathbf{B}_n)$, still indexed by n , such that

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ in } V \text{ strong, } \mathbf{B}_n \rightarrow \mathbf{B} \text{ in } W \text{ strong,}$$

and then

$$\liminf \langle A(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n, \mathbf{B}_n) - (\mathbf{v}, \mathbf{C}) \rangle = \langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) - (\mathbf{v}, \mathbf{C}) \rangle,$$

for any $(\mathbf{v}, \mathbf{C}) \in V \times W$. This proves (27), then A is pseudo-monotone.

b) Let us now check that the operator A satisfies the coerciveness property:

$$\frac{\langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle}{\|(\mathbf{u}, \mathbf{B})\|_{V \times W}} \rightarrow +\infty \text{ as } \|(\mathbf{u}, \mathbf{B})\|_{V \times W} \rightarrow +\infty. \quad (30)$$

We have

$$\begin{aligned} \langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle &= \int_{\Omega} \tilde{\mathbf{u}} \cdot \mathbf{u} dx + \int_{\Omega} |\tilde{\mathbf{u}}| \tilde{\mathbf{u}} \cdot \mathbf{u} dx \\ &\quad + \gamma \int_{\Omega} \nabla \tilde{\mathbf{u}} : \nabla \mathbf{u} dx + \int_{\Omega} |\mathbf{curl} \mathbf{B}|^2 dx \\ &= \int_{\Omega} (|\mathbf{u}|^2 + \mathbf{u}_* \cdot \mathbf{u}) dx + \int_{\Omega} (|\tilde{\mathbf{u}}| \tilde{\mathbf{u}} - |\mathbf{u}| \mathbf{u}) \cdot \mathbf{u} dx \\ &\quad + \int_{\Omega} |\mathbf{u}|^3 dx + \gamma \int_{\Omega} (|\nabla \mathbf{u}|^2 + \nabla \mathbf{u}_* : \nabla \mathbf{u}) dx \\ &\quad + \int_{\Omega} |\mathbf{curl} \mathbf{B}|^2 dx. \end{aligned}$$

Using the Hölder inequality and (24) we find that

$$\begin{aligned} \langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle &\geq \frac{1}{2} (\|\mathbf{u}\|^2 - \|\mathbf{u}_*\|^2) + \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^3 \\ &\quad - \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)} \|\mathbf{u}_*\|_{\mathbb{L}^3(\Omega)} (2\|\mathbf{u}\|_{\mathbb{L}^3(\Omega)} + \|\mathbf{u}_*\|_{\mathbb{L}^3(\Omega)}) \\ &\quad + \frac{\gamma}{2} (\|\nabla \mathbf{u}\|^2 - \|\nabla \mathbf{u}_*\|^2) + \|\mathbf{curl} \mathbf{B}\|^2, \end{aligned}$$

which is written as

$$\begin{aligned} &\langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle \\ &\geq \frac{1}{2} \|\mathbf{u}\|^2 + \frac{\gamma}{2} \|\nabla \mathbf{u}\|^2 + \|\mathbf{curl} \mathbf{B}\|^2 - \frac{1}{2} \|\mathbf{u}_*\|^2 - \frac{\gamma}{2} \|\nabla \mathbf{u}_*\|^2 \\ &\quad + \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^3 \left(1 - \|\mathbf{u}_*\|_{\mathbb{L}^3(\Omega)} \left(\frac{2}{\|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}} + \frac{\|\mathbf{u}_*\|_{\mathbb{L}^3(\Omega)}}{\|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^2} \right) \right). \quad (31) \end{aligned}$$

Then

$$\begin{aligned} & \langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle \\ & \geq \frac{1}{2} \|\mathbf{u}\|^2 + \frac{\gamma}{2} \|\nabla \mathbf{u}\|^2 + \|\mathbf{curl} \mathbf{B}\|^2 - \frac{1}{2} \|\mathbf{u}_\star\|^2 - \frac{\gamma}{2} \|\nabla \mathbf{u}_\star\|^2, \end{aligned} \quad (32)$$

for $\|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}$ large enough. Inequality (32) implies that (30) holds.

c) We conclude by Theorem 2.7 (Chapter 2, Section 2) in [20] that there is $(\mathbf{u}, \mathbf{B}) \in V \times W$ solution of $A(\mathbf{u}, \mathbf{B}) = (\mathbf{0}, \mathbf{curl} \mathbf{J}_0)$ in a weak sense, that is $(\tilde{\mathbf{u}}, \mathbf{B})$ satisfies (19). Then by De Rham's Theorem there exists $p \in L^2(\Omega)$ such that (13)₂ holds in $\mathcal{D}'(\Omega, \mathbb{R}^3)$. Hence Problem (\mathcal{P}_γ) has a weak solution $(\tilde{\mathbf{u}}, \mathbf{B})$.

2.2 Proof of Estimate (21)

Let $(\tilde{\mathbf{u}}, \mathbf{B})$ be a weak solution of Problem (\mathcal{P}_γ) . Since $\tilde{\mathbf{u}} = \mathbf{u} + \mathbf{u}_\star$, with $\mathbf{u} \in V$, it turns to estimate \mathbf{u} and \mathbf{B} . We have

$$\langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle = \int_{\Omega} \mathbf{J}_0 \cdot \mathbf{curl} \mathbf{B} \, dx,$$

then using (31) it holds that

$$\begin{aligned} & \frac{1}{2} \|\mathbf{u}\|^2 + \frac{\gamma}{2} \|\nabla \mathbf{u}\|^2 + \|\mathbf{curl} \mathbf{B}\|^2 + \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^3 \leq \int_{\Omega} |\mathbf{J}_0| |\mathbf{curl} \mathbf{B}| \, dx \\ & + \frac{1}{2} \|\mathbf{u}_\star\|^2 + \frac{\gamma}{2} \|\nabla \mathbf{u}_\star\|^2 + 2\|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^2 \|\mathbf{u}_\star\|_{\mathbb{L}^3(\Omega)} + \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)} \|\mathbf{u}_\star\|_{\mathbb{L}^3(\Omega)}^2. \end{aligned} \quad (33)$$

Using the Young inequality we have

$$\begin{aligned} & \int_{\Omega} |\mathbf{J}_0| |\mathbf{curl} \mathbf{B}| \, dx \leq \frac{1}{2} \|\mathbf{J}_0\|^2 + \frac{1}{2} \|\mathbf{curl} \mathbf{B}\|^2, \\ & \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^2 \|\mathbf{u}_\star\|_{\mathbb{L}^3(\Omega)} \leq \frac{1}{8} \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^3 + c \|\mathbf{u}_\star\|_{\mathbb{L}^3(\Omega)}^3, \\ & \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)} \|\mathbf{u}_\star\|_{\mathbb{L}^3(\Omega)}^2 \leq \frac{1}{4} \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^3 + c \|\mathbf{u}_\star\|_{\mathbb{L}^3(\Omega)}^3, \end{aligned}$$

then we deduce from (33) that

$$\begin{aligned} & \frac{1}{2} \|\mathbf{u}\|^2 + \frac{\gamma}{2} \|\nabla \mathbf{u}\|^2 + \frac{1}{2} \|\mathbf{curl} \mathbf{B}\|^2 + \frac{1}{2} \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^3 \\ & \leq \frac{1}{2} \|\mathbf{J}_0\|^2 + \frac{1}{2} \|\mathbf{u}_\star\|^2 + \frac{\gamma}{2} \|\nabla \mathbf{u}_\star\|^2 + c \|\mathbf{u}_\star\|_{\mathbb{L}^3(\Omega)}^3. \end{aligned}$$

Using the Sobolev embedding $H^1(\Omega) \hookrightarrow L^3(\Omega)$ and the equality $\tilde{\mathbf{u}} = \mathbf{u} + \mathbf{u}_\star$ we deduce (21).

2.3 Uniqueness of weak solutions

Let $(\mathbf{u}_1, \mathbf{B}_1)$ and $(\mathbf{u}_2, \mathbf{B}_2)$ be two weak solutions of Problem (\mathcal{P}_γ) . We set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $\mathbf{B} = \mathbf{B}_1 - \mathbf{B}_2$. Combining the equations satisfied by $(\mathbf{u}_i, \mathbf{B}_i)$ ($i = 1, 2$) we have, for any $(\mathbf{v}, \mathbf{C}) \in V \times W$,

$$\begin{aligned} & \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx + \int_{\Omega} (|\tilde{\mathbf{u}}_1|\tilde{\mathbf{u}}_1 - |\tilde{\mathbf{u}}_2|\tilde{\mathbf{u}}_2) \cdot \mathbf{v} \, dx + \gamma \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx \\ &= \int_{\Omega} (\mathbf{curl} \mathbf{B} \times \mathbf{B}_1) \cdot \mathbf{v} \, dx + \int_{\Omega} (\mathbf{curl} \mathbf{B}_2 \times \mathbf{B}) \cdot \mathbf{v} \, dx, \\ & \int_{\Omega} \mathbf{curl} \mathbf{B} \cdot \mathbf{curl} \mathbf{C} \, dx = \int_{\Omega} (\mathbf{u} \times \mathbf{B}_1) \cdot \mathbf{curl} \mathbf{C} \, dx + \int_{\Omega} (\mathbf{u}_2 \times \mathbf{B}) \cdot \mathbf{curl} \mathbf{C} \, dx. \end{aligned}$$

Taking $(\mathbf{v}, \mathbf{C}) = (\mathbf{u}, \mathbf{B})$ in the previous equations then adding the results and using the identity

$$\int_{\Omega} (\mathbf{curl} \mathbf{B} \times \mathbf{B}_1) \cdot \mathbf{u} \, dx + \int_{\Omega} (\mathbf{u} \times \mathbf{B}_1) \cdot \mathbf{curl} \mathbf{B} \, dx = 0,$$

we get

$$\begin{aligned} & \int_{\Omega} |\mathbf{u}|^2 \, dx + \int_{\Omega} (|\tilde{\mathbf{u}}_1|\tilde{\mathbf{u}}_1 - |\tilde{\mathbf{u}}_2|\tilde{\mathbf{u}}_2) \cdot \mathbf{u} \, dx + \gamma \int_{\Omega} |\nabla \mathbf{u}|^2 \, dx + \int_{\Omega} |\mathbf{curl} \mathbf{B}|^2 \, dx \\ &= \int_{\Omega} (\mathbf{curl} \mathbf{B}_2 \times \mathbf{B}) \cdot \mathbf{u} \, dx + \int_{\Omega} (\mathbf{u}_2 \times \mathbf{B}) \cdot \mathbf{curl} \mathbf{B} \, dx. \end{aligned}$$

According to (23) we have

$$\begin{aligned} & \int_{\Omega} |\mathbf{u}|^2 \, dx + \frac{1}{2} \int_{\Omega} |\mathbf{u}|^3 \, dx + \gamma \int_{\Omega} |\nabla \mathbf{u}|^2 \, dx + \int_{\Omega} |\mathbf{curl} \mathbf{B}|^2 \, dx \\ & \leq \int_{\Omega} (\mathbf{curl} \mathbf{B}_2 \times \mathbf{B}) \cdot \mathbf{u} \, dx + \int_{\Omega} (\mathbf{u}_2 \times \mathbf{B}) \cdot \mathbf{curl} \mathbf{B} \, dx, \end{aligned} \quad (34)$$

which implies that

$$\begin{aligned} \min(1, \gamma) \|\mathbf{u}\|_V^2 + \|\mathbf{curl} \mathbf{B}\|^2 & \leq \int_{\Omega} (\mathbf{curl} \mathbf{B}_2 \times \mathbf{B}) \cdot \mathbf{u} \, dx \\ & \quad + \int_{\Omega} (\mathbf{u}_2 \times \mathbf{B}) \cdot \mathbf{curl} \mathbf{B} \, dx. \end{aligned} \quad (35)$$

Using the Hölder inequality we have

$$\left| \int_{\Omega} (\mathbf{curl} \mathbf{B}_2 \times \mathbf{B}) \cdot \mathbf{u} \, dx \right| \leq \|\mathbf{curl} \mathbf{B}_2\| \|\mathbf{B}\|_{\mathbb{L}^6(\Omega)} \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}.$$

Moreover, using the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, the equivalence of norms on W and the Young inequality we deduce that

$$\begin{aligned} \left| \int_{\Omega} (\mathbf{curl} \mathbf{B}_2 \times \mathbf{B}) \cdot \mathbf{u} \, dx \right| & \leq c \|\mathbf{curl} \mathbf{B}_2\| \|\mathbf{curl} \mathbf{B}\| \|\mathbf{u}\|_V, \\ & \leq c(\gamma) \|\mathbf{curl} \mathbf{B}_2\|^2 \|\mathbf{curl} \mathbf{B}\|^2 + \frac{\min(1, \gamma)}{2} \|\mathbf{u}\|_V^2. \end{aligned} \quad (36)$$

In the same way we obtain

$$\begin{aligned} \left| \int_{\Omega} (\mathbf{u}_2 \times \mathbf{B}) \cdot \mathbf{curl} \mathbf{B} \, d\mathbf{x} \right| &\leq \|\mathbf{u}_2\|_{\mathbb{L}^3(\Omega)} \|\mathbf{B}\|_{\mathbb{L}^6(\Omega)} \|\mathbf{curl} \mathbf{B}\| \\ &\leq c \|\mathbf{u}_2\|_{\mathbb{L}^3(\Omega)} \|\mathbf{curl} \mathbf{B}\|^2. \end{aligned} \quad (37)$$

Combining (35)–(37) it results that

$$\frac{\min(1, \gamma)}{2} \|\mathbf{u}\|_V^2 + \left(1 - c(\gamma) \|\mathbf{curl} \mathbf{B}_2\|^2 - c \|\mathbf{u}_2\|_{\mathbb{L}^3(\Omega)}\right) \|\mathbf{curl} \mathbf{B}\|^2 \leq 0. \quad (38)$$

We note that, due to (18), (21) and (22), the norms $\|\mathbf{curl} \mathbf{B}_2\|$ and $\|\mathbf{u}_2\|_{\mathbb{L}^3(\Omega)}$ must be chosen as small as possible for ε is small enough. We deduce from (38) that for such $\varepsilon > 0$ we have $\mathbf{u} = \mathbf{B} = 0$ which proves the uniqueness. The proof of Theorem 1 is achieved.

3 Proof of Theorem 2

Let (\mathbf{u}, \mathbf{B}) be a weak solution of Problem (\mathcal{P}_γ) and let p denote the associated pressure. We will obtain the regularity of \mathbf{u} , \mathbf{B} and p by successive applications of classical regularity results for the Stokes and \mathbf{curl}^2 equations. For the \mathbf{curl}^2 equation we will use the following lemma (see [12] and [27, Proposition 2.1] for the proof):

Lemma 2. *Let m be a nonnegative integer and $1 < q < \infty$. Assume that Ω is a simply-connected bounded domain in \mathbb{R}^3 with boundary $\partial\Omega$ of class C^{m+2} . Let $\mathcal{G} \in \mathbb{W}^{m,q}(\Omega)$ with $\operatorname{div} \mathcal{G} = 0$ in Ω and $\mathcal{G} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Then there exists a unique $\mathbf{B} \in \mathbb{W}^{m+2,q}(\Omega)$ such that*

$$\begin{cases} \mathbf{curl}^2 \mathbf{B} = \mathcal{G}, & \operatorname{div} \mathbf{B} = 0 \text{ in } \Omega, \\ \mathbf{curl} \mathbf{B} \times \mathbf{n} = 0, & \mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \end{cases}$$

and

$$\|\mathbf{B}\|_{\mathbb{W}^{m+2,q}(\Omega)} \leq c \|\mathcal{G}\|_{\mathbb{W}^{m,q}(\Omega)}.$$

The proof of Theorem 2 consists in three steps: the first one concerns the case $\frac{3}{2} \leq q < 2$, the second deals with the case $2 \leq q < 6$ and the last step concerns the case $q \geq 6$.

Step 1. Assume first that $\frac{3}{2} \leq q < 2$ and set $q_1 = q$. The functions \mathbf{u} and p satisfy

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathcal{F}, & \operatorname{div} \mathbf{u} = h \text{ in } \Omega, \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \quad (39)$$

with $\mathcal{F} = -\mathbf{u} - |\mathbf{u}|\mathbf{u} + \mathbf{curl} \mathbf{B} \times \mathbf{B}$. Since $\mathcal{F} \in \mathbb{L}^{\frac{3}{2}}(\Omega)$, applying a classical regularity result for Stokes equations with nonhomogeneous boundary conditions, see [2], we get that $\mathbf{u} \in \mathbb{W}^{2,\frac{3}{2}}(\Omega)$, $p \in W^{1,\frac{3}{2}}(\Omega)$ and we have the

estimate

$$\|\mathbf{u}\|_{\mathbb{W}^{2, \frac{3}{2}}(\Omega)} + \|p\|_{W^{1, \frac{3}{2}}(\Omega)} \leq c(\gamma) \left(\|\mathcal{F}\|_{\mathbb{L}^{\frac{3}{2}}(\Omega)} + \|h\|_{W^{1, \frac{3}{2}}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{\frac{4}{3}, \frac{3}{2}}(\partial\Omega)} \right). \quad (40)$$

Since

$$\|\mathcal{F}\|_{\mathbb{L}^{\frac{3}{2}}(\Omega)} \leq c \left(\|\mathbf{u}\| + \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^2 + \|\mathbf{curl} \mathbf{B}\|^2 \right),$$

using (18) and (21) we find that

$$\begin{aligned} \|\mathcal{F}\|_{\mathbb{L}^{\frac{3}{2}}(\Omega)} &\leq c \left(\|\mathbf{J}_0\| + \|\mathbf{J}_0\|^{\frac{4}{3}} + \|\mathbf{J}_0\|^2 \right) + c(\gamma) \sum_{j=0}^1 \left(\|h\|_{\mathbb{W}^{\frac{8+j}{6}}}^{\frac{8+j}{6}} + \|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)}^{\frac{8+j}{6}} \right) \\ &\quad + c(\gamma) \sum_{j=1}^3 \left(\|h\|^j + \|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)}^j \right). \end{aligned} \quad (41)$$

Combining (40) and (41) we thus obtain

$$\|\mathbf{u}\|_{\mathbb{W}^{2, \frac{3}{2}}(\Omega)} + \|p\|_{W^{1, \frac{3}{2}}(\Omega)} \leq c(\gamma) \left(D_1 + \|h\|_{W^{1, \frac{3}{2}}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{\frac{4}{3}, \frac{3}{2}}(\partial\Omega)} \right), \quad (42)$$

with

$$\begin{aligned} D_1 &= \|\mathbf{J}_0\| + \|\mathbf{J}_0\|^{\frac{4}{3}} + \|\mathbf{J}_0\|^2 + \sum_{j=0}^1 \left(\|h\|_{\mathbb{W}^{\frac{8+j}{6}}}^{\frac{8+j}{6}} + \|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)}^{\frac{8+j}{6}} \right) \\ &\quad + \sum_{j=1}^3 \left(\|h\|^j + \|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)}^j \right). \end{aligned}$$

The function \mathbf{B} solves the problem

$$\begin{cases} \mathbf{curl}^2 \mathbf{B} = \mathcal{G}, & \operatorname{div} \mathbf{B} = 0 \text{ in } \Omega, \\ \mathbf{curl} \mathbf{B} \times \mathbf{n} = 0, & \mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \end{cases} \quad (43)$$

with $\mathcal{G} = \mathbf{curl}(\mathbf{u} \times \mathbf{B}) + \mathbf{curl} \mathbf{J}_0$. Using the vector identity

$$\mathbf{curl}(\mathbf{u} \times \mathbf{B}) = (\operatorname{div} \mathbf{B})\mathbf{u} - (\operatorname{div} \mathbf{u})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B},$$

it holds that

$$\mathbf{curl}(\mathbf{u} \times \mathbf{B}) = -h\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B}, \quad (44)$$

so that using the Sobolev embeddings $W^{1, \frac{3}{2}}(\Omega) \hookrightarrow L^3(\Omega)$, $W^{2, \frac{3}{2}}(\Omega) \hookrightarrow L^r(\Omega)$, for all $1 \leq r < \infty$, and the Hölder inequality we find that $\mathbf{curl}(\mathbf{u} \times \mathbf{B}) \in \mathbb{L}^{q_1}(\Omega)$, therefore $\mathcal{G} \in \mathbb{L}^{q_1}(\Omega)$. Clearly $\operatorname{div} \mathcal{G} = 0$. Let us check that $\mathcal{G} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Since \mathbf{J}_0 has its support contained in Ω it turns to show that $\mathbf{curl}(\mathbf{u} \times \mathbf{B}) \cdot \mathbf{n} = 0$ on $\partial\Omega$. Since

$$(\mathbf{u} \times \mathbf{B}) \times \mathbf{n} = (\mathbf{u} \cdot \mathbf{n})\mathbf{B} - (\mathbf{B} \cdot \mathbf{n})\mathbf{u},$$

using (14) and the assumption $\mathbf{g} \cdot \mathbf{n} = 0$ on $\partial\Omega$, it holds that $(\mathbf{u} \times \mathbf{B}) \times \mathbf{n} = 0$ on $\partial\Omega$. The latter relation implies $\mathbf{curl}(\mathbf{u} \times \mathbf{B}) \cdot \mathbf{n} = 0$ on $\partial\Omega$ since $\mathbf{curl} \mathbf{C} \cdot \mathbf{n}$ is a tangential differential operator on $\partial\Omega$ in the direction of the vector $\mathbf{C} \times \mathbf{n}$, see for instance [5, Chap. 7, Section 5]. Hence $\mathbf{g} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Now, applying Lemma 2 with $m = 0$ we find that $\mathbf{B} \in \mathbb{W}^{2,q_1}(\Omega)$ and

$$\|\mathbf{B}\|_{\mathbb{W}^{2,q_1}(\Omega)} \leq c \|\mathbf{g}\|_{\mathbb{L}^{q_1}(\Omega)}.$$

From (44) we have that

$$\|\mathbf{curl}(\mathbf{u} \times \mathbf{B})\|_{\mathbb{L}^{q_1}(\Omega)} \leq c \left(\|h\|_{L^3(\Omega)} + \|\mathbf{u}\|_{\mathbb{W}^{2,\frac{3}{2}}(\Omega)} \right) \|\mathbf{curl} \mathbf{B}\|,$$

then using (18), (21) and (42) we find that

$$\|\mathbf{g}\|_{\mathbb{L}^{q_1}(\Omega)} \leq c(\gamma) D_2 + \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{q_1}(\Omega)}$$

where

$$\begin{aligned} D_2 &= \left(D_1 + \|h\|_{W^{1,\frac{3}{2}}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{\frac{4}{3},\frac{3}{2}}(\partial\Omega)} \right) \times \\ &\quad \times \left(\|\mathbf{J}_0\| + \sum_{j=1}^2 \left(\|h\|_{\frac{j+1}{2}} + \|\mathbf{g}\|_{\mathbb{H}^{\frac{j}{2}}(\partial\Omega)} \right) \right). \end{aligned}$$

We thus have (for $\frac{3}{2} \leq q_1 < 2$)

$$\|\mathbf{B}\|_{\mathbb{W}^{2,q_1}(\Omega)} \leq c(\gamma) D_2 + c \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{q_1}(\Omega)}. \quad (45)$$

It remains to establish the regularity of \mathbf{u} and p when $q_1 > \frac{3}{2}$. Using the Sobolev embeddings $W^{2,q_1}(\Omega) \hookrightarrow C^{0,\mu}(\bar{\Omega})$, with $0 < \mu < 1$, it holds that $\mathbf{B} \in C^{0,\mu}(\bar{\Omega}, \mathbb{R}^3)$ so that now the function \mathcal{F} in (39) belongs to $\mathbb{L}^{q_1}(\Omega)$. Applying the regularity result for the solution to (39) we find that $\mathbf{u} \in \mathbb{W}^{2,q_1}(\Omega)$, $p \in W^{1,q_1}(\Omega)$ and we have the estimate

$$\begin{aligned} &\|\mathbf{u}\|_{\mathbb{W}^{2,q_1}(\Omega)} + \|p\|_{W^{1,q_1}(\Omega)} \\ &\leq c(\gamma) \left(\|\mathcal{F}\|_{\mathbb{L}^{q_1}(\Omega)} + \|h\|_{W^{1,q_1}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{2-\frac{1}{q_1},q_1}(\partial\Omega)} \right). \end{aligned} \quad (46)$$

Using also the Sobolev embeddings $W^{2,\frac{3}{2}}(\Omega) \hookrightarrow L^r(\Omega)$, for all $1 \leq r < \infty$, we have

$$\|\mathcal{F}\|_{\mathbb{L}^{q_1}(\Omega)} \leq c \left(\|\mathbf{u}\|_{\mathbb{W}^{2,\frac{3}{2}}(\Omega)} + \|\mathbf{u}\|_{\mathbb{W}^{2,\frac{3}{2}}(\Omega)}^2 + \|\mathbf{B}\|_{\mathbb{W}^{2,q_1}(\Omega)}^2 \right). \quad (47)$$

Combining (42), (45)–(47) we get the estimate, for $\frac{3}{2} < q_1 < 2$,

$$\begin{aligned} &\|\mathbf{u}\|_{\mathbb{W}^{2,q_1}(\Omega)} + \|p\|_{W^{1,q_1}(\Omega)} \\ &\leq c(\gamma) \left(D_3 + \|h\|_{W^{1,q_1}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{2-\frac{1}{q_1},q_1}(\partial\Omega)} \right), \end{aligned} \quad (48)$$

where

$$\begin{aligned} D_3 &= D_1 + \|h\|_{W^{1, \frac{3}{2}}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{\frac{4}{3}, \frac{3}{2}}(\partial\Omega)} \\ &\quad + D_1^2 + \|h\|_{W^{1, \frac{3}{2}}(\Omega)}^2 + \|\mathbf{g}\|_{\mathbb{W}^{\frac{4}{3}, \frac{3}{2}}(\partial\Omega)}^2 \\ &\quad + D_2^2 + \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{q_1}(\Omega)}^2. \end{aligned}$$

Step 2. Let us now assume $2 \leq q < 6$ and set $q_2 = q$. By the Sobolev embedding there is $q_1 \in (\frac{3}{2}, 2)$ such that $W^{1, q_1}(\Omega) \hookrightarrow L^{q_2}(\Omega)$. From the previous step it results that $\mathbf{B} \in \mathcal{C}^{0, \mu}(\bar{\Omega}, \mathbb{R}^3)$ ($0 < \mu < 1$) and $\mathbf{curl} \mathbf{B} \in \mathbb{L}^{q_2}(\Omega)$. Using again the Sobolev embedding $W^{2, \frac{3}{2}}(\Omega) \hookrightarrow L^r(\Omega)$, for all $1 \leq r < \infty$, we have that the function \mathcal{F} in (39) belongs to $\mathbb{L}^{q_2}(\Omega)$ and we have the estimate

$$\|\mathcal{F}\|_{\mathbb{L}^{q_2}(\Omega)} \leq c \left(\|\mathbf{u}\|_{\mathbb{W}^{2, \frac{3}{2}}(\Omega)} + \|\mathbf{u}\|_{\mathbb{W}^{2, \frac{3}{2}}(\Omega)}^2 + \|\mathbf{B}\|_{\mathbb{W}^{2, q_1}(\Omega)}^2 \right). \quad (49)$$

Applying the regularity result for the solution to (39) we find that $\mathbf{u} \in \mathbb{W}^{2, q_2}(\Omega)$, $p \in W^{1, q_2}(\Omega)$ and we have the estimate

$$\begin{aligned} &\|\mathbf{u}\|_{\mathbb{W}^{2, q_2}(\Omega)} + \|p\|_{W^{1, q_2}(\Omega)} \\ &\leq c(\gamma) \left(\|\mathcal{F}\|_{\mathbb{L}^{q_2}(\Omega)} + \|h\|_{W^{1, q_2}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{2 - \frac{1}{q_2}, q_2}(\partial\Omega)} \right). \end{aligned} \quad (50)$$

Combining (42), (45), (49) and (50) we get the estimate

$$\|\mathbf{u}\|_{\mathbb{W}^{2, q_2}(\Omega)} + \|p\|_{W^{1, q_2}(\Omega)} \leq c(\gamma) \left(D_3 + \|h\|_{W^{1, q_2}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{2 - \frac{1}{q_2}, q_2}(\partial\Omega)} \right). \quad (51)$$

Next we deduce from (44) that $\mathbf{curl}(\mathbf{u} \times \mathbf{B})$ belongs to $\mathbb{L}^{q_2}(\Omega)$ and we have the estimate

$$\|\mathbf{curl}(\mathbf{u} \times \mathbf{B})\|_{\mathbb{L}^{q_2}(\Omega)} \leq c \|\mathbf{B}\|_{\mathbb{W}^{2, q_1}(\Omega)} \left(\|h\|_{L^{q_2}(\Omega)} + \|\mathbf{u}\|_{\mathbb{W}^{2, q_1}(\Omega)} \right).$$

Thus the function \mathcal{G} in (43) belongs to $\mathbb{L}^{q_2}(\Omega)$ and satisfies

$$\|\mathcal{G}\|_{\mathbb{L}^{q_2}(\Omega)} \leq c(\gamma) D_4 + \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{q_2}(\Omega)}$$

with

$$D_4 = \left(D_3 + \|h\|_{W^{1, q_1}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{2 - \frac{1}{q_1}, q_1}(\partial\Omega)} \right) \left(D_2 + \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{q_1}(\Omega)} \right).$$

Applying the regularity result to the solution \mathbf{B} of (43) we obtain that \mathbf{B} belongs to $\mathbb{W}^{2, q_2}(\Omega)$ and satisfies the estimate

$$\|\mathbf{B}\|_{\mathbb{W}^{2, q_2}(\Omega)} \leq c(\gamma) D_4 + c \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{q_2}(\Omega)}. \quad (52)$$

Step 3. Let us finally assume $q \geq 6$ and set $q_3 = q$. Let $q_2 \in (3, 6)$ so that by the Sobolev embedding we have $W^{1, q_2}(\Omega) \hookrightarrow \mathcal{C}^{0, \mu}(\bar{\Omega})$ ($0 < \mu < 1$).

Therefore, the functions \mathbf{u} , \mathbf{B} and $\mathbf{curl} \mathbf{B}$ belong to $\mathcal{C}^{0,\mu}(\overline{\Omega}, \mathbb{R}^3)$ and then the function \mathcal{F} in (39) belongs to $\mathcal{C}^{0,\mu}(\overline{\Omega}, \mathbb{R}^3)$. Arguing as in Step 2, using the Sobolev embedding $W^{2,\frac{3}{2}}(\Omega) \hookrightarrow L^{q_3}(\Omega)$, we find that $\mathbf{u} \in \mathbb{W}^{2,q_3}(\Omega)$, $p \in W^{1,q_3}(\Omega)$ and we have the analogue of (51):

$$\|\mathbf{u}\|_{\mathbb{W}^{2,q_3}(\Omega)} + \|p\|_{W^{1,q_3}(\Omega)} \leq c(\gamma) \left(D_5 + \|h\|_{W^{1,q_3}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{2-\frac{1}{q_3},q_3}(\partial\Omega)} \right), \quad (53)$$

with

$$\begin{aligned} D_5 = & D_1 + \|h\|_{W^{1,\frac{3}{2}}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{\frac{4}{3},\frac{3}{2}}(\partial\Omega)} \\ & + D_1^2 + \|h\|_{W^{1,\frac{3}{2}}(\Omega)}^2 + \|\mathbf{g}\|_{\mathbb{W}^{\frac{4}{3},\frac{3}{2}}(\partial\Omega)}^2 \\ & + D_4^2 + \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{q_2}(\Omega)}^2. \end{aligned}$$

Since $\mathbf{curl}(\mathbf{u} \times \mathbf{B})$ belongs to $\mathcal{C}^{0,\mu}(\overline{\Omega}, \mathbb{R}^3)$ it holds that the function \mathcal{G} in (43) belongs to $\mathbb{L}^{q_3}(\Omega)$. Arguing as in Step 2 we find that \mathbf{B} belongs to $\mathbb{W}^{2,q_3}(\Omega)$ and we have the analogue of (52):

$$\|\mathbf{B}\|_{\mathbb{W}^{2,q_3}(\Omega)} \leq c(\gamma) D_6 + c \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{q_3}(\Omega)}, \quad (54)$$

with

$$D_6 = \left(D_3 + \|h\|_{W^{1,q_2}(\Omega)} + \|\mathbf{g}\|_{\mathbb{W}^{2-\frac{1}{q_2},q_2}(\partial\Omega)} \right) \left(D_4 + \|\mathbf{curl} \mathbf{J}_0\|_{\mathbb{L}^{q_2}(\Omega)} \right).$$

The proof of Theorem 2 is achieved.

4 The limit $\gamma \rightarrow 0$

4.1 Proof of Theorem 3

For notational convenience we drop here the subscript n ; we denote by $(\mathbf{u}_\gamma, \mathbf{B}_\gamma)$ the weak solution of Problem (\mathcal{P}_γ) , the existence of which was stated in Theorem 1. From (21) we deduce that

$$\begin{aligned} (\mathbf{u}_\gamma)_{\gamma>0} & \text{ is bounded in } \mathbb{L}^3(\Omega), \\ (\sqrt{\gamma} \mathbf{u}_\gamma)_{\gamma>0} & \text{ is bounded in } \mathbb{H}^1(\Omega), \\ (\mathbf{B}_\gamma)_{\gamma>0} & \text{ is bounded in } W. \end{aligned}$$

The sequences $(|\mathbf{u}_\gamma| \mathbf{u}_\gamma)_{\gamma>0}$ and $(\mathbf{curl} \mathbf{B}_\gamma \times \mathbf{B}_\gamma)_{\gamma>0}$ are then bounded in $\mathbb{L}^{\frac{3}{2}}(\Omega)$. It results from the previous estimates that there exist $\mathbf{u} \in \mathbb{L}^3(\Omega)$, $\mathbf{B} \in W$ and $\boldsymbol{\chi} \in \mathbb{L}^{\frac{3}{2}}(\Omega)$, and subsequences of $(\mathbf{u}_\gamma)_{\gamma>0}$ and $(\mathbf{B}_\gamma)_{\gamma>0}$, still indexed by γ , such that:

$$\mathbf{u}_\gamma \rightharpoonup \mathbf{u} \text{ in } \mathbb{L}^3(\Omega) \text{ weak}, \quad (55)$$

$$\mathbf{B}_\gamma \rightharpoonup \mathbf{B} \text{ in } W \text{ weak}, \quad (56)$$

$$\mathbf{B}_\gamma \rightarrow \mathbf{B} \text{ in } \mathbb{L}^2(\Omega) \text{ strong}, \quad (57)$$

$$|\mathbf{u}_\gamma| \mathbf{u}_\gamma \rightharpoonup \boldsymbol{\chi} \text{ in } \mathbb{L}^{\frac{3}{2}}(\Omega) \text{ weak}. \quad (58)$$

Clearly, $\operatorname{div} \mathbf{u} = h$. It results from the Stokes formula that

$$\begin{aligned} \langle \mathbf{u}_\gamma \cdot \mathbf{n}, w \rangle_{\partial\Omega} &= \langle \mathbf{g} \cdot \mathbf{n}, w \rangle_{\partial\Omega} \\ &= \int_{\Omega} \mathbf{u}_\gamma \cdot \nabla w \, dx + \int_{\Omega} h w \, dx, \quad \forall w \in H^1(\Omega). \end{aligned}$$

According to (17) we then have $\int_{\Omega} \mathbf{u}_\gamma \cdot \nabla w \, dx = 0$ for all $\gamma > 0$, then for the weak limit \mathbf{u} of \mathbf{u}_γ we have

$$\lim_{\gamma \rightarrow 0} \int_{\Omega} \mathbf{u}_\gamma \cdot \nabla w \, dx = \int_{\Omega} \mathbf{u} \cdot \nabla w \, dx = 0,$$

therefore

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \langle \mathbf{u}_\gamma \cdot \mathbf{n}, w \rangle_{\partial\Omega} &= \langle \mathbf{g} \cdot \mathbf{n}, w \rangle_{\partial\Omega} \\ &= \int_{\Omega} \mathbf{u} \cdot \nabla w \, dx + \int_{\Omega} h w \, dx, \quad \forall w \in H^1(\Omega), \end{aligned}$$

that is $\mathbf{u} \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n}$ in $H^{-\frac{1}{2}}(\partial\Omega)$. We conclude that $\mathbf{u} - \mathbf{u}_* \in V_0$.

Now, using (55)–(58) one can pass to the limit, as $\gamma \rightarrow 0$, in (19) and obtain that, for every $(\mathbf{v}, \mathbf{C}) \in V \times W$,

$$\begin{cases} \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx + \int_{\Omega} \boldsymbol{\chi} \cdot \mathbf{v} \, dx = \int_{\Omega} (\operatorname{curl} \mathbf{B} \times \mathbf{B}) \cdot \mathbf{v} \, dx, \\ \int_{\Omega} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \mathbf{C} \, dx \\ = - \int_{\Omega} (\operatorname{curl} \mathbf{C} \times \mathbf{B}) \cdot \mathbf{u} \, dx + \int_{\Omega} \mathbf{J}_0 \cdot \operatorname{curl} \mathbf{C} \, dx. \end{cases} \quad (59)$$

Note that, by density of V in V_0 , Equation (59)₁ holds for any $\mathbf{v} \in V_0$. Using De Rham's Theorem, there exists $p \in W^{1, \frac{3}{2}}(\Omega)$ such that

$$\mathbf{u} + \boldsymbol{\chi} - \operatorname{curl} \mathbf{B} \times \mathbf{B} = -\nabla p.$$

Consequently, the functions \mathbf{u} , \mathbf{B} and $\boldsymbol{\chi}$ satisfy the differential system

$$\begin{cases} \mathbf{u} + \boldsymbol{\chi} + \nabla p = \operatorname{curl} \mathbf{B} \times \mathbf{B}, \\ \operatorname{div} \mathbf{u} = h, \\ \operatorname{curl}^2 \mathbf{B} = \operatorname{curl}(\mathbf{u} \times \mathbf{B}) + \operatorname{curl} \mathbf{J}_0. \end{cases} \quad (60)$$

To prove Theorem 3 it remains to show that $\boldsymbol{\chi} = |\mathbf{u}|\mathbf{u}$ and this is an essential point in the proof. This will be shown by using monotony arguments. Introduce the variable

$$\begin{aligned} X_\gamma &= \int_{\Omega} (|\mathbf{u}_\gamma - \mathbf{u}|^2 + (|\mathbf{u}_\gamma|\mathbf{u}_\gamma - |\mathbf{w}|\mathbf{w}) \cdot (\mathbf{u}_\gamma - \mathbf{w}) + \gamma |\nabla(\mathbf{u}_\gamma - \mathbf{u})|^2 \\ &\quad + |\operatorname{curl}(\mathbf{B}_\gamma - \mathbf{B})|^2) \, dx, \end{aligned} \quad (61)$$

where $\mathbf{w} \in \mathbb{L}^3(\Omega)$. Since the map $\mathbf{v} \mapsto |\mathbf{v}|\mathbf{v}$ from $\mathbb{L}^3(\Omega)$ into $\mathbb{L}^{3/2}(\Omega)$ is monotone, the function $(|\mathbf{u}_\gamma|\mathbf{u}_\gamma - |\mathbf{w}|\mathbf{w}) \cdot (\mathbf{u}_\gamma - \mathbf{w})$ is nonnegative. It results that $X_\gamma \geq 0$. Taking $\mathbf{v} = \mathbf{u}_\gamma - \mathbf{u}_*$ and $\mathbf{C} = \mathbf{B}_\gamma$ in (19) then adding the results we obtain

$$\begin{aligned} & \int_{\Omega} (|\mathbf{u}_\gamma|^2 + |\mathbf{u}_\gamma|^3 + \gamma|\nabla\mathbf{u}_\gamma|^2 + |\mathbf{curl}\mathbf{B}_\gamma|^2) dx \\ &= \int_{\Omega} (\mathbf{u}_\gamma \cdot \mathbf{u}_* + |\mathbf{u}_\gamma|\mathbf{u}_\gamma \cdot \mathbf{u}_* + \gamma\nabla\mathbf{u}_\gamma : \nabla\mathbf{u}_*) dx \\ & \quad - \int_{\Omega} ((\mathbf{curl}\mathbf{B}_\gamma \times \mathbf{B}_\gamma) \cdot \mathbf{u}_* - \mathbf{J}_0 \cdot \mathbf{curl}\mathbf{B}_\gamma) dx. \end{aligned} \quad (62)$$

Similarly, taking $\mathbf{v} = \mathbf{u} - \mathbf{u}_*$ as a test function in equation (60)₁ and $\mathbf{C} = \mathbf{B}$ in equation (60)₃ and adding the results yields

$$\begin{aligned} & \int_{\Omega} (|\mathbf{u}|^2 + \chi\mathbf{u} + |\mathbf{curl}\mathbf{B}|^2) dx \\ &= \int_{\Omega} (\mathbf{u} \cdot \mathbf{u}_* + \chi \cdot \mathbf{u}_* - (\mathbf{curl}\mathbf{B} \times \mathbf{B}) \cdot \mathbf{u}_* + \mathbf{J}_0 \cdot \mathbf{curl}\mathbf{B}) dx. \end{aligned} \quad (63)$$

Expanding X_γ and using (62) it results that

$$\begin{aligned} X_\gamma &= \int_{\Omega} (|\mathbf{u}|^2 - 2\mathbf{u}_\gamma \cdot \mathbf{u} + \mathbf{u}_\gamma \cdot \mathbf{u}_*) dx \\ & \quad + \int_{\Omega} (|\mathbf{w}|^3 + |\mathbf{u}_\gamma|\mathbf{u}_\gamma \cdot (\mathbf{u}_* - \mathbf{w}) - |\mathbf{w}|\mathbf{w} \cdot \mathbf{u}_\gamma) dx \\ & \quad + \int_{\Omega} \gamma (|\nabla\mathbf{u}|^2 - 2\nabla\mathbf{u}_\gamma : \nabla\mathbf{u} + \nabla\mathbf{u} : \nabla\mathbf{u}_*) dx \\ & \quad + \int_{\Omega} (|\mathbf{curl}\mathbf{B}|^2 - 2\mathbf{curl}\mathbf{B} \cdot \mathbf{curl}\mathbf{B}_\gamma - (\mathbf{curl}\mathbf{B}_\gamma \times \mathbf{B}_\gamma) \cdot \mathbf{u}_*) dx \\ & \quad + \int_{\Omega} \mathbf{J}_0 \cdot \mathbf{curl}\mathbf{B}_\gamma dx. \end{aligned}$$

Passing to the limit, as $\gamma \rightarrow 0$, in the previous equality, using (55)–(58), it holds that

$$\begin{aligned} \lim_{\gamma \rightarrow 0} X_\gamma &= \int_{\Omega} (\mathbf{u} \cdot \mathbf{u}_* - |\mathbf{u}|^2 + |\mathbf{w}|^3 - |\mathbf{w}|\mathbf{w} \cdot \mathbf{u} + \chi(\mathbf{u}_* - \mathbf{w})) dx \\ & \quad - \int_{\Omega} (|\mathbf{curl}\mathbf{B}|^2 + (\mathbf{curl}\mathbf{B} \times \mathbf{B}) \cdot \mathbf{u}_* - \mathbf{J}_0 \cdot \mathbf{curl}\mathbf{B}) dx. \end{aligned}$$

Using (63) we obtain

$$\lim_{\gamma \rightarrow 0} X_\gamma = \int_{\Omega} (\chi - |\mathbf{w}|\mathbf{w}) \cdot (\mathbf{u} - \mathbf{w}) dx \geq 0.$$

Since $\mathbf{u} \in \mathbb{L}^3(\Omega)$ and \mathbf{w} is an arbitrary function in $\mathbb{L}^3(\Omega)$, a Minty's argument allows to conclude that $\chi = |\mathbf{u}|\mathbf{u}$. The proof of Theorem 3 is complete.

4.2 Existence of a weak solution to problem (\mathcal{P}) : a direct proof

One can prove directly the existence of a weak solution to problem (\mathcal{P}) , without passing through the limit, as $\gamma_n \rightarrow 0$, of the sequence of approximate solutions $(\mathbf{u}_{\gamma_n}, \mathbf{B}_{\gamma_n})$ of (\mathcal{P}_{γ_n}) . We first note that $V_0 \times W$ is a reflexive Banach space. Similarly as in Section 2 we set, for any $\mathbf{u} \in V_0$, $\tilde{\mathbf{u}} = \mathbf{u} + \mathbf{u}_*$. Consider the operator

$$A_0 : V_0 \times W \rightarrow V_0' \times W'$$

defined by

$$\begin{aligned} \langle A_0(\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C}) \rangle &= \int_{\Omega} \tilde{\mathbf{u}} \cdot \mathbf{v} \, dx + \int_{\Omega} |\tilde{\mathbf{u}}| \tilde{\mathbf{u}} \cdot \mathbf{v} \, dx \\ &\quad - \int_{\Omega} (\mathbf{curl} \mathbf{B} \times \mathbf{B}) \cdot \mathbf{v} \, dx + \int_{\Omega} \mathbf{curl} \mathbf{B} \cdot \mathbf{curl} \mathbf{C} \, dx \\ &\quad + \int_{\Omega} (\mathbf{curl} \mathbf{C} \times \mathbf{B}) \cdot \tilde{\mathbf{u}} \, dx, \end{aligned} \quad (64)$$

for any (\mathbf{u}, \mathbf{B}) and (\mathbf{v}, \mathbf{C}) in $V_0 \times W$.

a) Let us show that the operator A_0 is pseudo-monotone. Indeed, using the Hölder inequality and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ we have

$$\begin{aligned} \int_{\Omega} |\tilde{\mathbf{u}}| \tilde{\mathbf{u}} \cdot \mathbf{v} \, dx &\leq \|\tilde{\mathbf{u}}\|_{\mathbb{L}^3(\Omega)}^2 \|\mathbf{v}\|_{\mathbb{L}^3(\Omega)} = \|\tilde{\mathbf{u}}\|_{\mathbb{L}^3(\Omega)}^2 \|\mathbf{v}\|_{V_0}, \\ \int_{\Omega} |(\mathbf{curl} \mathbf{C} \times \mathbf{B}) \cdot \tilde{\mathbf{u}}| \, dx &\leq \|\mathbf{curl} \mathbf{C}\| \|\mathbf{B}\|_{\mathbb{L}^6(\Omega)} \|\tilde{\mathbf{u}}\|_{\mathbb{L}^3(\Omega)} \\ &\leq c \|\tilde{\mathbf{u}}\|_{\mathbb{L}^3(\Omega)} \|\mathbf{B}\|_W \|\mathbf{C}\|_W. \end{aligned}$$

Estimating similarly the other terms in the right-hand side of (64) we arrive at

$$\begin{aligned} |\langle A_0(\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C}) \rangle| &\leq c \left(\|\tilde{\mathbf{u}}\|_{\mathbb{L}^3(\Omega)} + \|\tilde{\mathbf{u}}\|_{\mathbb{L}^3(\Omega)}^2 + \|\mathbf{B}\|_W^2 \right) \|\mathbf{v}\|_{V_0} \\ &\quad + c (\|\mathbf{B}\|_W + \|\mathbf{B}\|_W \|\tilde{\mathbf{u}}\|_{\mathbb{L}^3(\Omega)}) \|\mathbf{C}\|_W, \end{aligned}$$

which shows that the operator A_0 is bounded.

Consider now a sequence $(\mathbf{u}_n, \mathbf{B}_n)$ which converges to (\mathbf{u}, \mathbf{B}) in $V_0 \times W$ weak and so that

$$\limsup \langle A_0(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n, \mathbf{B}_n) - (\mathbf{u}, \mathbf{B}) \rangle \leq 0. \quad (65)$$

By the compact Sobolev embedding there is a subsequence of (\mathbf{B}_n) , still indexed by n , such that

$$\mathbf{B}_n \rightarrow \mathbf{B} \text{ in } \mathbb{L}^2(\Omega) \text{ strong.} \quad (66)$$

Using the identity $\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}} = \mathbf{u}_n - \mathbf{u}$, we have

$$\begin{aligned}
& \langle A_0(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n - \mathbf{u}, \mathbf{B}_n - \mathbf{B}) \rangle \\
&= \int_{\Omega} \tilde{\mathbf{u}}_n \cdot (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx + \int_{\Omega} |\tilde{\mathbf{u}}_n| \tilde{\mathbf{u}}_n \cdot (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx \\
&\quad + \int_{\Omega} (\mathbf{curl} \mathbf{B}_n \times \mathbf{B}_n) \cdot \tilde{\mathbf{u}} \, dx + \int_{\Omega} \mathbf{curl} \mathbf{B}_n \cdot \mathbf{curl}(\mathbf{B}_n - \mathbf{B}) \, dx \\
&\quad - \int_{\Omega} (\mathbf{curl} \mathbf{B} \times \mathbf{B}_n) \cdot \tilde{\mathbf{u}}_n \, dx.
\end{aligned} \tag{67}$$

From (65)–(67) and the fact that

$$\lim \int_{\Omega} |\tilde{\mathbf{u}}| \tilde{\mathbf{u}} \cdot (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx = 0,$$

we find that

$$\begin{aligned}
& \limsup \left(\int_{\Omega} \tilde{\mathbf{u}}_n \cdot (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx + \int_{\Omega} (|\tilde{\mathbf{u}}_n| \tilde{\mathbf{u}}_n - |\tilde{\mathbf{u}}| \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx \right. \\
&\quad \left. + \int_{\Omega} |\mathbf{curl}(\mathbf{B}_n - \mathbf{B})|^2 \, dx \right) \leq 0.
\end{aligned}$$

Since the map $\mathbf{v} \mapsto |\mathbf{v}| \mathbf{v}$ from $\mathbb{L}^3(\Omega)$ into $\mathbb{L}^{\frac{3}{2}}(\Omega)$ is monotone we obtain

$$\limsup \left(\int_{\Omega} \tilde{\mathbf{u}}_n \cdot (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx + \int_{\Omega} |\mathbf{curl}(\mathbf{B}_n - \mathbf{B})|^2 \, dx \right) \leq 0. \tag{68}$$

By convexity of the quadratic function $x \rightarrow x^2$ we have

$$\lim \int_{\Omega} \tilde{\mathbf{u}}_n \cdot (\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \, dx \geq 0,$$

then we deduce from (68) that there is a subsequence of $(\mathbf{u}_n, \mathbf{B}_n)$, still indexed by n , such that

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ in } \mathbb{L}^2(\Omega) \text{ strong, } \mathbf{B}_n \rightarrow \mathbf{B} \text{ in } W \text{ strong.}$$

Now we easily verify that

$$\liminf \langle A_0(\mathbf{u}_n, \mathbf{B}_n), (\mathbf{u}_n, \mathbf{B}_n) - (\mathbf{v}, \mathbf{C}) \rangle = \langle A_0(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) - (\mathbf{v}, \mathbf{C}) \rangle$$

for any $(\mathbf{v}, \mathbf{C}) \in V_0 \times W$. We conclude that A is pseudo-monotone.

b) Let us now verify that the operator A_0 satisfies the coerciveness property:

$$\frac{\langle A_0(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle}{\|(\mathbf{u}, \mathbf{B})\|_{V_0 \times W}} \rightarrow +\infty \text{ as } \|(\mathbf{u}, \mathbf{B})\|_{V_0 \times W} \rightarrow +\infty. \tag{69}$$

We have

$$\begin{aligned} \langle A_0(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle &= \int_{\Omega} (|\mathbf{u}|^2 + \mathbf{u}_* \cdot \mathbf{u}) \, dx + \int_{\Omega} ((|\tilde{\mathbf{u}}|\tilde{\mathbf{u}} - |\mathbf{u}|\mathbf{u}) \cdot \mathbf{u}) \, dx \\ &\quad + \int_{\Omega} |\mathbf{u}|^3 \, dx + \int_{\Omega} |\mathbf{curl} \mathbf{B}|^2 \, dx. \end{aligned}$$

Using the Hölder inequality and (24) we find that

$$\begin{aligned} \langle A_0(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle &\geq \frac{1}{2} (\|\mathbf{u}\|^2 - \|\mathbf{u}_*\|^2) + \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^3 + \|\mathbf{curl} \mathbf{B}\|^2 \\ &\quad - \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)} \|\mathbf{u}_*\|_{\mathbb{L}^3(\Omega)} (2\|\mathbf{u}\|_{\mathbb{L}^3(\Omega)} + \|\mathbf{u}_*\|_{\mathbb{L}^3(\Omega)}), \end{aligned}$$

that implies

$$\begin{aligned} \langle A_0(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle &\geq \|\mathbf{curl} \mathbf{B}\|^2 - \frac{1}{2} \|\mathbf{u}_*\|^2 \\ &\quad + \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^3 \left(1 - \|\mathbf{u}_*\|_{\mathbb{L}^3(\Omega)} \left(\frac{2}{\|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}} + \frac{\|\mathbf{u}_*\|_{\mathbb{L}^3(\Omega)}}{\|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^2} \right) \right), \end{aligned} \quad (70)$$

then

$$\langle A(\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}) \rangle \geq \|\mathbf{curl} \mathbf{B}\|^2 + \frac{1}{2} \|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}^3 - \frac{1}{2} \|\mathbf{u}_*\|^2 \quad (71)$$

for $\|\mathbf{u}\|_{\mathbb{L}^3(\Omega)}$ large enough. From (70) and (71) we easily deduce (69).

c) We conclude by Theorem 2.7 (Chapter 2, Section 2) in [20], that there is $(\mathbf{u}, \mathbf{B}) \in V_0 \times W$ so that $A_0(\mathbf{u}, \mathbf{B}) = (\mathbf{0}, \mathbf{curl} \mathbf{J}_0)$ in a weak sense, that is $(\tilde{\mathbf{u}}, \mathbf{B})$ satisfies (20). Then De Rham's Theorem gives the existence of $p \in W^{1, \frac{3}{2}}(\Omega)$ such that (15)₂ holds in $\mathbb{L}^{\frac{3}{2}}(\Omega)$. We conclude that Problem (P) has a weak solution $(\tilde{\mathbf{u}}, \mathbf{B})$.

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