CM modular forms and dihedral representations

joint work with F. A. E. Nuccio

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Ramanujan's congruences

Let

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n \in \mathbb{Z}[[q]]$$

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It was conjectured by Ramanujan in 1916 that for every prime $p \neq 23$, we have

$$\tau(p) \equiv \begin{cases} 0 \pmod{23} & \text{if } p \text{ is a quadratic non residue modulo 23 ;} \\ 2 \pmod{23} & \text{if } p = u^2 + 23v^2 \text{ ;} \\ -1 \pmod{23} & \text{if } p \text{ is a quadratic residue modulo 23,} \\ & \quad \text{but } p \neq u^2 + 23v^2. \end{cases}$$

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These congruences were first proved in 1930 by Wilton using the modularity of Δ.

CM forms

Recall the definition of a CM form.

Definition

Let ν be a non trivial Dirichlet character. A newform $g = \sum_{n \geq 1} c_n q^n$ has *complex multiplication by* ν if

$$\nu(p) c_p = c_p$$

for all primes p in a set of primes of density 1.

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Remarks

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Remarks

If a newform g has complex multiplication (CM) by a Dirichlet character ν, then the field K fixed by the kernel of ν is imaginary quadractic. We also say that g has CM by K.

A CM form modulo 23

Ramanujan's congruences imply in particular that

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Question - First version

Does there exist a CM form $g = \sum_{n \geq 1} c_n q^n$ such that '$g \equiv \Delta \pmod{23}$', i.e. there exists a prime ideal $\mathfrak{L}$ dividing 23 in a sufficiently large number field such that for all but finitely primes p we have

$$c_p \equiv \tau(p) \pmod{\mathfrak{L}} \ ?$$

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Galois representations

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Given a newform $g = \sum_{n\geq 1} c_n q^n$ of weight $k \geq 2$, level $N \geq 1$ and Nebentypus character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, denote by

$$\overline{\rho}_{g,\ell}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$$

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Recall that $\overline{\rho}_{g,\ell}$ is unramified outside $N\ell$ and for every prime $p \nmid N\ell$, the characteristic polynomial of $\overline{\rho}_{g,\ell}(\mathrm{Frob}_p)$ is

$$X^2 - c_p X + \chi(p)p^{k-1}.$$

Galois representation associated with Δ (mod 23)

In the case of $g = \Delta$ and $\ell = 23$, the construction of $\overline{\rho}_{\Delta,23}$ dates back to Serre and Swinnerton-Dyer.

Galois representation associated with $\Delta \pmod{23}$

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The quadratic field $K = \mathbb{Q}(\sqrt{-23})$ is unramified away from 23 and has class number 3. Let H be its Hilbert class field. The extension $H/\mathbb{Q}$ is Galois and $\mathrm{Gal}(H/\mathbb{Q}) \simeq D_3$.

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Consider the group morphism

$$\sigma\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(H/\mathbb{Q}) \simeq D_3 \to \mathrm{GL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{F}_{23})$$

where the second arrow is the unique 2-dimensional irreducible representation of D_3. It defines an irreducible representation, unramified outside 23 and with image isomorphic to D_3.

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Class Field Theory and Ramanujan's congruences then imply that

$$\sigma \simeq \overline{\rho}_{\Delta,23}.$$

Dihedral Galois representations

Definition

Let $\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be an irreducible Galois representation. We say that $\overline{\rho}$ is a *dihedral* Galois representation if its projective image is isomorphic to the dihedral group D_n of order $2n$ with $n \geq 3$ coprime to ℓ.

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Hence the following natural question.

Question - Second version

Let $\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be a dihedral Galois representation. Does there exist a newform g *with CM* such that

$$\overline{\rho} \simeq \overline{\rho}_{g,\ell}\ ?$$

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Serre's parameters

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- a character $\epsilon(\overline{\rho})\colon (\mathbb{Z}/N(\overline{\rho})\mathbb{Z})^\times \to \overline{\mathbb{F}}_\ell^\times$ such that

  $$\det \overline{\rho} = \epsilon(\overline{\rho})\chi_\ell^{k(\overline{\rho})-1}$$

  with χ_ℓ the mod ℓ cyclotomic character.

Lack of optimality

Recall Serre's famous modularity conjecture.

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Our choice was to seek for a CM form of 'optimal' weight $k(\overline{\rho})$.

The setting

Let $\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be a dihedral Galois representation. We denote by

$$\mathbb{P}\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\bar{\rho}} \mathrm{GL}_2(\overline{\mathbb{F}}_\ell) \twoheadrightarrow \mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$$

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By assumption $\mathbb{P}\overline{\rho}\left(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\right) \simeq D_n$ with D_n dihedral group of order $2n$ with $n \geq 3$ and coprime to ℓ. Denote by C_n the unique cyclic subgroup of order n in D_n.

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Let K be the quadratic field cut out by the kernel of the character

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It is unramified outside $N\ell$.

Main result

Theorem (B.-Nuccio)

With the previous notation, assume that

- the field K is quadratic *imaginary*;
- we have $2 \leq k(\overline{\rho}) \leq \ell - 1$ and $\ell \geq 5$.

Then, the representation $\overline{\rho}$ is modular and arises from a newform *with complex multiplication by* K of weight $k(\overline{\rho})$ and level

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Moreover, the following properties hold :

1. if ℓ is ramified in K, then $\ell \in \{2k(\overline{\rho}) - 1,\ 2k(\overline{\rho}) - 3\}$;
2. if $\epsilon(\overline{\rho})$ is trivial then, $\overline{\rho}$ arises from a newform with CM by K, weight $k(\overline{\rho})$, level dividing N' and trivial Nebentypus character.

Application to abelian varieties of GL_2-type

Let $A/\mathbb{Q}$ be a simple abelian variety. Denote by $\mathrm{End}_{\mathbb{Q}}(A)$ its ring of endomorphisms defined over $\mathbb{Q}$.

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Corollary

With the notation above, assume $\ell \geq 5$, $\ell \nmid N_A$ (the conductor of A) and the image of $\overline{\rho}_{A,\lambda}$ is included in the normalizer of a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\lambda)$. Then, $\overline{\rho}_{A,\lambda}$ arises from a CM newform of weight 2 and level $N(\overline{\rho}_{A,\lambda})$ (dividing N_A).

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- The main difference in the higher weight situation lies in the fact that K might ramify at ℓ ;
- Our result is optimal : cf. the case of $\Delta \pmod{23}$.

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Hecke-Shimura construction

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View such a δ as homomorphism

$$\{\text{fractional ideals of } K \text{ coprime to } \mathfrak{m}\} \to \mathbb{C}^\times$$

where $\mathfrak{m}$ is the conductor of δ and define

$$\eta(m) = \delta(m\mathcal{O}_K)/m^{k-1}, \quad \text{for } m \in \mathbb{Z}.$$

Then η induces a Dirichlet character modulo $M = \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{m})$.

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Theorem (Hecke-Shimura)

The q-series $g_\delta = \sum_{\substack{\mathfrak{a} \text{ integral} \\ (\mathfrak{a},\mathfrak{m})=1}} \delta(\mathfrak{a}) q^{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})}$ is a newform of weight k, level MD and Nebentypus character $\eta\left(\frac{-D}{\cdot}\right)$ with complex multiplication by K. Furthermore, all CM newforms arise this way.

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Hecke-Shimura construction

Theorem (Hecke-Shimura)

The q-series $g_\delta = \sum_{\substack{\mathfrak{a} \text{ integral} \\ (\mathfrak{a},\mathfrak{m})=1}} \delta(\mathfrak{a}) q^{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})}$ is a newform of weight k, level MD and Nebentypus character $\eta\left(\frac{-D}{\cdot}\right)$ with complex multiplication by K. Furthermore, all CM newforms arise this way.

In order to prove the main result we have to pin down a Grössencharacter with

- good reduction properties ;
- weight $k(\overline{\rho})$;
- controlled ramification.

Construction of the appropriate Grössencharacter

Let $\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be a dihedral Galois representation. With the notation of the theorem, we have

$$\overline{\rho} \simeq \mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}^{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}(\varphi), \quad \text{for some character } \varphi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \overline{\mathbb{F}}_\ell^\times.$$

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The composition map

$$\alpha\colon \mathbb{A}_K^\times \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(\overline{\mathbb{Q}}/K)^{\mathrm{ab}} \xrightarrow{\varphi} \overline{\mathbb{F}}_\ell^\times \xrightarrow{\mathrm{lift}} \mathbb{C}^\times$$

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Starting from α and using Class Field Theory, we construct a Grössencharacter δ of K such that

- has weight k;
- reduces to φ;
- has the same conductor as α away from λ
- has controlled ramification at λ with respect to that of φ and K.

Study of the ramification of φ and K at ℓ

Here we denote by I_ℓ a inertia group at ℓ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and by I_λ its inertia subgroup at λ in $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

Proposition

1. Assume that $\overline{\rho}_{|I_\ell}$ is reducible. Then,

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   - either $\ell\mathcal{O}_K = \lambda \cdot \overline{\lambda}$ is split in K, and in this case φ is unramified at $\overline{\lambda}$ and tamely ramified at λ with $\varphi_{|I_\lambda} = \chi_\ell^{k-1}$;

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   - or ℓ is ramified in K and in this case, $\ell = 2k-1$ and φ is unramified at λ.

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2. Assume that $\overline{\rho}_{|I_\ell}$ is irreducible. Then,
   - either ℓ is inert in K ;

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2. Assume that $\overline{\rho}_{|I_\ell}$ is irreducible. Then,
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2. Assume that $\overline{\rho}_{|I_\ell}$ is irreducible. Then,
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   - or ℓ is ramified in K and $\ell = 2k-3$.

   In both cases, φ is ramified at λ and $\varphi_{|I_\lambda} = \psi_2^{k-1}$ with ψ_2 a fundamental character of level 2.

End of the proof

Take δ to be the suitable modification of

$$\alpha\colon \mathbb{A}_K^\times \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(\overline{\mathbb{Q}}/K)^{\mathrm{ab}} \xrightarrow{\varphi} \overline{\mathbb{F}}_\ell^\times \xrightarrow{\mathrm{lift}} \mathbb{C}^\times$$

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Conclude that g_δ has the required properties using the formula for the conductor of an induced representation and the previous proposition.

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According to the theorem, there exists a newform g of weight 12, level $529 = 23^2$ and trivial Nebentypus with CM by $K = \mathbb{Q}(\sqrt{-23})$ such that

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We find that

$$g_\delta = q + (-21b^2 - 4b + 84)q^2 + (53b^2 + 251b - 212)q^3 + \cdots$$

where b a root of $X^3 - 6X - 3$.