Sums of two S-units via Frey-Hellegouarch curves
(joint work with Michael A. Bennett)

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Notation and history

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- De Weger (90's) has developped on algorithm based on linear forms in complex and p-adic logarithms to solve such equations.
- This led him to a complete characterization of the solutions when $S = \{2, 3, 5, 7\}$.

A more general equation and a different approach

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Can we compute the set of primitive solutions?

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Putting $x = u/d$ and $y = v/d$, we get a solution to

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Remark

Note that for a fixed value of n, there are only finitely many possible values of d and w as (u, v, c, n) ranges over all primitive solutions.

A finiteness result for a fixed value of $n \geq 3$

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Defining x, y, w and z as before and writing $y = y_0 y_1^n$ where y_0 is n-th powerfree (whereby, there are at most $2n^{|S|}$ choices for y_0), we end up with a **Thue-Mahler equation** of degree n :

$$wz^n - y_0 y_1^n = x$$

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$$wz^n - y_0 y_1^n = x$$

Hence, the set of primitive solutions is **finite** for a fixed value of $n \geq 3$.

A Frey curve approach for $n = 2$

Writing $x + y = wz^2$ as

$$x1^p + y1^p = wz^2$$

one may pretend that $(1, 1, z)$ is a solution to the generalized Fermat equation $xX^p + yY^p = wZ^2$.

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An associated Frey-Hellegouarch curve for this equation is

$$E\colon\ Y^2 = X^3 + 2wzX^2 + ywX$$

whose standard invariants are

$$\Delta(E) = 2^6 xy^2 w^3 \quad \text{and} \quad c_4(E) = 2^4 w(4x + y).$$

A Frey curve approach for $n = 2$ (continued)

Therefore, E has good reduction outside $S \cup \{2\}$ and by Shafarevich theorem,

$$j(E) = j(x, y) = 2^6 \cdot \frac{(4x + y)^3}{xy^2}$$

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Since $j(x, y) = j(x/y, 1)$ and x, y are coprime, this in turn forces x and y to be bounded.

As a conclusion, we get the following result.

Proposition

For any **fixed value** of $n \geq 2$, there are only finitely many primitive solutions to $u + v = c^n$.

A practical approach ?

- Solving a Thue-Mahler equation is a quite challenging task.

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- Computing representatives of elliptic curves with good reduction outside a given set of primes has been done for a rather restricted list of sets (Cremona *et al.*, Bennett-Rechnitzer, von Känel-Matschke,...).

Making the modular approach effective for $n = 2$

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For a specific set S, we wish to understand which elliptic Frey curves E as before have conductor **not** within Cremona's range.

Proposition

For $S = \{2, 3, 5, 7\}$ and $S = \{2, 3, p\}$ with $p < 100$ prime, the conductor $N(E)$ of E satisfies $N(E) < 350000$ unless the solution corresponds to :

$$3^4 + 1 = 2 \cdot 41, \quad 2 \cdot 3^3 - 1 = 53, \quad 3^{10} - 1 = 2 \cdot 61 \cdot 22^2, \quad 3^4 + 2 = 83,$$

$$3^4 - 2 = 79, \quad 3^5 + 1 = 61 \cdot 2^2, \quad 2^3 \cdot 3^2 + 1 = 73, \quad 2^3 \cdot 3^2 - 1 = 71,$$

$$3^4 + 2^3 = 89, \quad 3^4 - 2^3 = 73.$$

A sample result : $n = 2$ and $S = \{2, 3, 61\}$

u	v	c	u	v	c	u	v	c
2	−1	1	27	−2	5	732	−3	27
2	2	2	48	1	7	1024	−183	29
3	−2	1	61	−36	5	3721	768	67
3	1	2	61	−12	7	3904	−183	61
4	−3	1	61	3	8	**14823**	**61**	**122**
6	−2	2	81	−32	7	14884	−243	121
6	3	3	108	61	13	15616	9	125
8	1	3	122	−1	11	59049	976	245
9	−8	1	192	−183	3	237168	1	487
12	−3	3	243	−122	11	1361886	3	1167
16	9	5	244	−243	1	**7203978**	**−122**	**2684**
18	−2	4	288	1	17			
24	1	5	486	−2	22			

Table – Primitive solutions to $u + v = c^2$ with $S = \{2, 3, 61\}$

Another sample result : $n = 3$ and $S = \{2, 3, 61\}$

A similar approach using $(p, p, 3)$ Frey curves can be used for $n = 3$. Here is an example of the kind of results one obtains.

x	y	z	x	y	z	x	y	z
2	−1	1	18	9	3	576	−549	3
3	−2	1	24	3	3	732	−3	9
4	−3	1	36	−9	3	2196	1	13
4	4	2	61	3	4	15616	9	25
6	2	2	64	61	5	36864	−33489	15
9	−8	1	122	3	5	**238144**	**−11163**	**61**
9	−1	2	128	−3	5	1808406	7442	122
12	−4	2	244	−243	1			

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Assume $2 \notin S$ and $n \geq 5$ is prime. Recall that any primitive solution (with exponent n) gives rise to a solution of

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with x y and w **coprime** S-units, w positive and n-th powerfree and z nonzero.

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Writing $x + y = wz^n$ as

$$wz^n + y(-1)^n = x1^3$$

one may pretend that $(z, -1, 1)$ is a solution to the generalized Fermat equation $wX^n + yY^n = xZ^3$.

An associated Frey-Hellegouarch curve for this equation is

$$E\colon\ Y^2 + 3xXY - x^2yY = X^3$$

whose standard invariants are

$$\Delta(E) = -3^3x^8y^3wz^n \quad \text{and} \quad c_4(E) = 3^2x^3(9wz^n - y).$$

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Let $p \notin \{3\} \cup S$ be a prime dividing z. Then, E has multiplicative reduction at p with $\mathrm{val}_p(\Delta(E)) \equiv 0 \pmod{n}$.

It follows from the theory of Tate curves that the mod n Galois representation attached to E

$$\rho_{E,n} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(E[n]) \simeq \mathrm{GL}_2(\mathbb{F}_n)$$

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If, moreover, $n \notin S$ and $n > 163$, then by Ribet's level lowering theorem, the representation $\rho_{E,n}$ arises from a newform $f = \sum_{m\geq 1} a_m(f)q^m$ of weight 2, level N_0 and trivial character with

$$N_0 \mid 3^5 \cdot \prod_{\substack{p\in S \\ p\neq 3}} p^2.$$

...and level lowering

This means that there exists a prime ideal $\mathfrak{n}$ above n in the coefficient field K of f such that for every prime $p \nmid N_0 n$, we have

$$a_p(f) \equiv \begin{cases} a_p(E) \pmod{\mathfrak{n}} & \text{if } E \text{ has good reduction at } p\ ; \\ \pm(p+1) \pmod{\mathfrak{n}} & \text{if } E \text{ has bad multiplicative reduction at } p. \end{cases}$$

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Now since $2 \notin S$, then z is **even** and therefore

$$a_2(f) \equiv \pm(2+1) \pmod{\mathfrak{n}}.$$

The general finiteness result

By Deligne's bounds, we deduce

$$n \leq \mathrm{Norm}_{K/\mathbb{Q}}(a_2(f) \pm 3) \leq \left(1+\sqrt{2}\right)^{2g_0(N_0)}$$

where $g_0(N_0)$ is the number of newforms of weight 2 and level N_0.

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Theorem

The set of primitive solutions to $u + v = c^n$ is finite if and only if $2 \notin S$.

First complete result : $S = \{2, 3\}$

Combining this approach with similar reasonnings using (n, n, n) Frey curves enables us to completely determine **all primitive solutions** to $u + v = c^n$ when $S = \{2, 3\}$.

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Theorem

The only primitive solutions to equation $u + v = c^n$ with $S = \{2, 3\}$ and, say, $u \geq |v| > 0$ and $c > 0$ are given by the following infinite families

$$\begin{aligned}
&(u, v, c, n) = (2, -1, 1, n),\ (3, -2, 1, n),\ (4, -3, 1, n),\ (9, -8, 1, n),\\
&(2^{n-1}, 2^{n-1}, 2, n),\ (3 \cdot 2^{n-2}, 2^{n-2}, 2, n),\ (3 \cdot 2^{n-1}, -2^{n-1}, 2, n),\\
&(2 \cdot 3^{n-1}, 3^{n-1}, 3, n),\ (2^2 \cdot 3^{n-1}, -3^{n-1}, 3, n),\\
&(2^3 \cdot 3^{n-2}, 3^{n-2}, 3, n), \quad \textit{all with } n \geq 2,\\
&(u, v, c, n) = (3^2 \cdot 2^{n-3}, -2^{n-3}, 2, n) \quad \textit{for } n \geq 3
\end{aligned}$$

and by

$$\begin{aligned}
&(u, v, c, n) = (16, 9, 5, 2),\ (18, -2, 4, 2),\ (24, 1, 5, 2),\\
&(27, -2, 5, 2),\ (81, -32, 7, 2),\ (48, 1, 7, 2),\ (128, -3, 5, 3),\\
&(288, 1, 17, 2) \textit{ and } (486, -2, 22, 2).
\end{aligned}$$

Complete determination of the solutions for $S = \{3, 5, 7\}$

From now on, and until the end, we let $S = \{3, 5, 7\}$.

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Applying the $(n, n, 3)$ Frey curve approach introduced before indeed bounds n :

$$n \leq (1 + \sqrt{2})^{(N_0+1)/6}$$

where $N_0 = 3^5 \cdot 5^2 \cdot 7^2$.

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We have to lower this upper bound !

Lemma

Let a, b be coprime nonzero integers and $n \geq 5$ be a prime number. Then, the greatest prime divisor of $a^n + b^n$ is ≥ 11 unless we have $|a| = |b| = 1$.

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The following proposition uses the above lemma and results by Bennett, Győry, Mignotte and Pintér.

Proposition

Let x and w be coprime positive $\{3, 5, 7\}$-units, $y = \pm 1$ and let $n \geq 5$ be a prime number such that $x + y = wz^n$ for some positive integer z. Then, $y = -1$, $n = 5$, $z = 2$ and

$$(x, w) = (7^4, 3 \cdot 5^2) \quad \text{or} \quad (3^2 \cdot 5^2, 7).$$

Remaining equations

$$3^\alpha 5^\beta + (-1)^\delta 7^\gamma = z^n, \quad (\alpha, \beta) \not\equiv (0,0) \mod n, \quad \gamma \not\equiv 0 \mod n$$
$$3^\alpha 7^\gamma + (-1)^\delta 5^\beta = z^n, \quad (\alpha, \gamma) \not\equiv (0,0) \mod n, \quad \beta \not\equiv 0 \mod n$$
$$5^\beta 7^\gamma + (-1)^\delta 3^\alpha = z^n, \quad (\beta, \gamma) \not\equiv (0,0) \mod n, \quad \alpha \not\equiv 0 \mod n$$
$$3^\alpha + (-1)^\delta 5^\beta = 7^\gamma z^n, \quad \alpha, \beta > 0, \quad 0 < \gamma \leq n-1$$
$$3^\alpha + (-1)^\delta 7^\gamma = 5^\beta z^n, \quad \alpha, \gamma > 0 \quad 0 < \beta \leq n-1$$
$$5^\beta + (-1)^\delta 7^\gamma = 3^\alpha z^n, \quad \beta, \gamma > 0, \quad 0 < \alpha \leq n-1$$

where α, β and γ are nonnegative integers, $n \geq 5$ is prime and $\delta \in \{0, 1\}$.

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      1.a.ii If $\beta = 0$ and $n = 7$, then local sieve modulo 49 (and 43 if $\gamma = 1$)

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  $(n, n, n) \rightsquigarrow 23 \nmid z$ and $67 \nmid z$.
  Sieve modulo 23 and 67 $\rightsquigarrow$ contradiction.

- iii. If $n \geq 11$, then $(n, n, 3)$ (with cube $= (-1)^\delta 7^\gamma$) + level lowering (to $N_0 \mid 3 \cdot 5 \cdot 7^2$ or $N_0 = 3^4 \cdot N_1$ with $N_1 \mid 5 \cdot 7^2$, if $\alpha \geq 3$ or $\alpha = 1$ respectively) $\Rightarrow \alpha = 1$ and $n = 11$.
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2. Assume $n = 5$

A sample equation : $3^\alpha 5^\beta + (-1)^\delta 7^\gamma = z^n$ (continued)

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2. Assume $n = 5 \rightsquigarrow$ Thue-Mahler equation $z^5 - 3^a 5^b y^5 = 7^c$

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2. Assume $n = 5 \rightsquigarrow$ Thue-Mahler equation $z^5 - 3^a 5^b y^5 = 7^c$
   Hambrook's Magma code (2012) $\rightsquigarrow$ solutions :

$$3 \cdot 5^3 - 7^3 = 5^2 + 7 = 3^4 - 7^2 = 2^5.$$

Main result : $S = \{3, 5, 7\}$

Main Theorem 2

The only primitive solutions to equation $u + v = c^n$ with $S = \{3, 5, 7\}$ and $u > |v| > 0$ are given by

$$(u, v) = (3,1),\ (5,-1),\ (5,3),\ (7,-3),\ (7,1),\ (9,-5),\ (9,-1),$$
$$(9,7),\ (15,-7),\ (15,1),\ (21,-5),\ (21,15),\ (25,-21),\ (25,-9),$$
$$(25,7),\ (27,5),\ (35,-27),\ (35,-3),\ (35,1),\ (49,-45),\ (49,15),$$
$$(63,1),\ (81,-49),\ (105,-5),\ (125,3),\ (135,-35),\ (135,-7),$$
$$(147,-3),\ (175,21),\ (175,81),\ (189,-125),\ (189,7),\ (225,-9),$$
$$(343,-243),\ (375,-343),\ (405,-5),\ (441,-225),\ (625,-49),$$
$$(675,1),\ (729,-245),\ (1029,-5),\ (1225,-225),\ (1323,-27),$$
$$(1875,-147),\ (3375,2401),\ (3969,-1225),\ (3969,-125),$$
$$(9375,1029),\ (10125,-125),\ (15625,-1701),\ (50625,-3969),$$
$$(59535,1),\ (540225,-2401),\ (688905,-5),$$
$$(4782969,4375) \text{ and } (24310125,-10125).$$