THE CASE $S = \{2, 3, p\}$, p PRIME, $11 \leq p < 100$ AND $n = 2$ OR 3

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For each prime number p with $11 \leq p < 100$, we have listed below all the triples (x, y, z) such that

$$x + y = z^n, \quad n = 2 \text{ or } 3$$

where z is a non-zero integer and x, y are $\{2, 3, p\}$-units with $\gcd(x, y)$ nth-powerfree, $x \geq |y| > 0$ and $z > 0$.

The solutions in bold correspond to those "exceptional" solutions stated in Propositions A.1 and A.2.

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1. The case $p = 11$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	27	-2	5	132	-11	11	1728	121	43
2	2	2	27	22	7	192	33	15	2187	22	47
3	-2	1	33	-32	1	198	-2	14	2662	-162	50
3	1	2	33	-24	3	243	-242	1	2673	-2048	25
4	-3	1	33	-8	5	256	33	17	2673	352	55
6	-2	2	33	3	6	288	1	17	3993	-512	59
6	3	3	33	16	7	297	-176	11	3993	-24	63
8	1	3	36	-11	5	297	-128	13	3993	768	69
9	-8	1	48	1	7	297	-8	17	4224	1	65
11	-2	3	48	33	9	297	64	19	5632	297	77
12	-11	1	66	-2	8	352	9	19	12288	33	111
12	-3	3	81	-32	7	363	-2	19	14641	243	122
16	9	5	88	33	11	486	-2	22	19602	-2	140
18	-2	4	88	81	13	528	1	23	24057	-32	155
22	-18	2	99	1	10	726	3	27	59049	-968	241
22	-6	4	99	22	11	729	-704	5	235224	1	485
22	3	5	121	-96	5	972	-11	31	483153	-128	695
24	1	5	121	-72	7	1056	33	33			
27	-11	4	121	48	13	1089	-128	31			

Table 1: Solutions to $x + y = z^2$ with $S = \{2, 3, 11\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	12	-4	2	81	44	5	968	363	11
3	-2	1	16	11	3	99	-72	3	1089	242	11
4	-3	1	18	9	3	108	-44	4	1452	-121	11
4	4	2	24	3	3	121	4	5	3267	-1936	11
6	2	2	33	-32	1	128	-3	5	10692	-44	22
9	-8	1	33	-6	3	198	18	6	12288	-121	23
9	-1	2	36	-9	3	243	-242	1	34848	1089	33
11	-3	2	44	-36	2	352	-9	7			
12	-11	1	66	-2	4	726	3	9			

Table 2: Solutions to $x + y = z^3$ with $S = \{2, 3, 11\}$

2. The case $p = 13$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	18	-2	4	117	52	13	2028	-3	45
2	2	2	24	1	5	156	13	13	2187	-338	43
3	-2	1	26	-1	5	169	-144	5	2197	-972	35
3	1	2	27	-26	1	169	-48	11	2197	-81	46
4	-3	1	27	-2	5	169	27	14	2197	12	47
6	-2	2	36	13	7	192	169	19	2808	1	53
6	3	3	39	-3	6	208	-39	13	3328	-3159	13
8	1	3	48	-39	3	208	81	17	4212	13	65
9	-8	1	48	1	7	243	13	16	16384	-3159	115
12	-3	3	52	-27	5	288	1	17	57122	-1	239
13	-12	1	52	-3	7	486	-2	22	59049	13312	269
13	-9	2	64	-39	5	624	1	25	248832	169	499
13	-4	3	78	3	9	702	-26	26	851968	-39	923
13	3	4	81	-32	7	729	-104	25			
13	12	5	108	13	11	768	-39	27			
16	9	5	117	4	11	832	9	29			

Table 3: Solutions to $x + y = z^2$ with $S = \{2, 3, 13\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	13	-12	1	52	12	4	676	324	10
3	-2	1	18	9	3	117	8	5	768	-39	9
4	-3	1	24	3	3	128	-3	5	1521	676	13
4	4	2	26	-18	2	144	-117	3	2028	169	13
6	2	2	26	1	3	234	-18	6	2704	-507	13
9	-8	1	27	-26	1	351	-8	7	12168	-1	23
9	-1	2	36	-9	3	486	26	8	18252	-676	26
12	-4	2	39	-12	3	512	-169	7	43264	-41067	13

Table 4: Solutions to $x + y = z^3$ with $S = \{2, 3, 13\}$

3. The case $p = 17$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	18	-17	1	153	-32	11	1224	1	35
2	2	2	18	-2	4	153	16	13	1377	-1088	17
3	-2	1	24	1	5	153	136	17	1377	-8	37
3	1	2	27	-2	5	162	34	14	1377	1024	49
4	-3	1	32	17	7	272	17	17	4913	128	71
6	-2	2	34	-18	4	288	1	17	6561	-4352	47
6	3	3	34	-9	5	289	-288	1	12393	-512	109
8	1	3	34	2	6	289	-64	15	13122	-578	112
9	-8	1	48	1	7	289	72	19	20736	289	145
12	-3	3	51	-2	7	306	-17	17	34816	153	187
16	9	5	64	17	9	486	-2	22	44217	-32768	107
17	-16	1	81	-32	7	512	17	23	70227	-2	265
17	-8	3	81	-17	8	544	81	25	83521	-1152	287
17	-1	4	102	-2	10	578	-2	24	332928	1	577
17	8	5	153	-128	5	1088	1	33	1003833	2176	1003

Table 5: Solutions to $x + y = z^2$ with $S = \{2, 3, 17\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	17	-16	1	81	-17	4	1734	-6	12
3	-2	1	17	-9	2	108	17	5	2312	-2187	5
4	-3	1	18	-17	1	128	-3	5	2601	2312	17
4	4	2	18	9	3	204	12	6	4624	289	17
6	2	2	24	3	3	289	-288	1	4896	17	17
9	-8	1	36	-9	3	289	54	7	5202	-289	17
9	-1	2	51	-24	3	576	153	9	23409	-18496	17
12	-4	2	68	-4	4	1088	243	11	1685448	-289	119

Table 6: Solutions to $x + y = z^3$ with $S = \{2, 3, 17\}$

4. The case $p = 19$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	19	6	5	171	-2	13	1026	-2	32
2	2	2	24	1	5	228	-3	15	1083	6	33
3	-2	1	27	-2	5	288	1	17	1216	9	35
3	1	2	38	-2	6	304	57	19	1368	1	37
4	-3	1	48	1	7	342	19	19	4617	-128	67
6	-2	2	54	-38	4	361	-192	13	20577	-128	143
6	3	3	57	-48	3	361	-72	17	29184	57	171
8	1	3	57	-32	5	384	57	21	39366	6859	215
9	-8	1	57	-8	7	486	-2	22	41553	19456	247
12	-3	3	57	24	9	513	-512	1	65536	513	257
16	9	5	64	57	11	513	-152	19	263169	-2048	511
18	-2	4	76	-27	7	513	16	23	1050624	1	1025
19	-18	1	81	-32	7	729	-608	11			
19	-3	4	81	19	10	864	361	35			

Table 7: Solutions to $x + y = z^2$ with $S = \{2, 3, 19\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	18	9	3	144	-19	5	513	-512	1
3	-2	1	19	-18	1	152	-27	5	513	-1	8
4	-3	1	19	8	3	171	-144	3	5776	1083	19
4	4	2	24	3	3	228	-12	6	6498	361	19
6	2	2	27	-19	2	324	19	7	9747	-2888	19
9	-8	1	36	-9	3	342	1	7	184832	361	57
9	-1	2	76	-12	4	361	-18	7			
12	-4	2	128	-3	5	486	-361	5			

Table 8: Solutions to $x + y = z^3$ with $S = \{2, 3, 19\}$

5. The case $p = 23$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	24	1	5	192	-23	13	864	-23	29
2	2	2	27	-23	2	243	46	17	1152	529	41
3	-2	1	27	-2	5	256	-207	7	1458	-1058	20
3	1	2	32	-23	3	288	1	17	2048	-23	45
4	-3	1	46	3	7	384	-23	19	2208	1	47
6	-2	2	46	18	8	486	-2	22	2944	81	55
6	3	3	48	-23	5	529	96	25	6912	-23	83
8	1	3	48	1	7	529	432	31	12288	-12167	11
9	-8	1	54	46	10	552	-23	23	13248	-23	115
12	-3	3	69	12	9	621	-92	23	17496	-12167	73
16	9	5	72	-23	7	621	4	25	24334	2	156
18	-2	4	81	-32	7	648	-23	25	34992	-23	187
23	2	5	138	6	12	729	-368	19	89424	-23	299
24	-23	1	144	-23	11	736	-207	23			

Table 9: Solutions to $x + y = z^2$ with $S = \{2, 3, 23\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-1	2	36	-9	3	368	-243	5
3	-2	1	12	-4	2	46	18	4	1587	-256	11
4	-3	1	18	9	3	54	-46	2	2116	81	13
4	4	2	23	4	3	96	-69	3	12696	-529	23
6	2	2	24	-23	1	128	-3	5	14283	-2116	23
9	-8	1	24	3	3	207	9	6	16928	-4761	23

Table 10: Solutions to $x + y = z^3$ with $S = \{2, 3, 23\}$

6. The case $p = 29$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	12	-3	3	58	-9	7	783	1	28
2	2	2	16	9	5	58	6	8	783	58	29
3	-2	1	18	-2	4	81	-32	7	841	-216	25
3	1	2	24	1	5	87	-6	9	841	384	35
4	-3	1	27	-2	5	96	-87	3	928	-87	29
6	-2	2	29	-4	5	256	-87	13	1682	-1	41
6	3	3	48	1	7	288	1	17	3712	9	61
8	1	3	54	-29	5	486	-2	22	21141	-116	145
9	-8	1	58	-54	2	729	232	31	393216	-87	627

Table 11: Solutions to $x + y = z^2$ with $S = \{2, 3, 29\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-1	2	58	6	4	288	-261	3
3	-2	1	12	-4	2	96	29	5	22707	1682	29
4	-3	1	18	9	3	116	-108	2	26912	-2523	29
4	4	2	24	3	3	116	9	5	59392	-59049	7
6	2	2	29	-2	3	128	-3	5			
9	-8	1	36	-9	3	256	87	7			

Table 12: Solutions to $x + y = z^3$ with $S = \{2, 3, 29\}$

7. The case $p = 31$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	12	-3	3	48	1	7	729	496	35
2	2	2	16	9	5	62	2	8	837	4	29
3	-2	1	18	-2	4	81	-32	7	837	124	31
3	1	2	24	1	5	93	-12	9	961	-432	23
4	-3	1	27	-2	5	124	-3	11	961	128	33
6	-2	2	31	-27	2	162	-62	10	992	-31	31
6	3	3	31	-6	5	256	-31	15	3968	1	63
8	1	3	31	18	7	288	1	17			
9	-8	1	32	-31	1	486	-2	22			

Table 13: Solutions to $x + y = z^2$ with $S = \{2, 3, 31\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-1	2	36	-9	3	279	64	7
3	-2	1	12	-4	2	62	-54	2	25947	3844	31
4	-3	1	18	9	3	62	2	4	30752	-961	31
4	4	2	24	3	3	93	32	5	39366	-62	34
6	2	2	31	-4	3	124	1	5			
9	-8	1	32	-31	1	128	-3	5			

Table 14: Solutions to $x + y = z^3$ with $S = \{2, 3, 31\}$

8. The case $p = 37$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	16	9	5	81	-32	7	1332	37	37
2	2	2	18	-2	4	148	-27	11	1369	-144	35
3	-2	1	24	1	5	192	-111	9	2368	-999	37
3	1	2	27	-2	5	222	3	15	5328	1	73
4	-3	1	37	-36	1	243	-74	13	6912	1369	91
6	-2	2	37	-12	5	288	1	17	9472	729	101
6	3	3	37	-1	6	324	37	19	14348907	37	3788
8	1	3	37	12	7	444	-3	21			
9	-8	1	37	27	8	486	-2	22			
12	-3	3	48	1	7	1024	-999	5			

Table 15: Solutions to $x+y=z^2$ with $S=\{2,3,37\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-1	2	37	27	4	1332	-1	11
3	-2	1	12	-4	2	64	-37	3	2738	6	14
4	-3	1	18	9	3	128	-3	5	49284	1369	37
4	4	2	24	3	3	162	-37	5	87616	-36963	37
6	2	2	36	-9	3	222	-6	6			
9	-8	1	37	-36	1	999	1	10			

Table 16: Solutions to $x+y=z^3$ with $S=\{2,3,37\}$

9. The case $p = 41$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	18	-2	4	82	18	10	3321	-512	53
2	2	2	24	1	5	123	-2	11	3362	2	58
3	-2	1	27	-2	5	128	41	13	5043	-2	71
3	1	2	41	-32	3	162	-41	11	5248	81	73
4	-3	1	41	-16	5	288	1	17	6561	328	83
6	-2	2	41	8	7	369	-8	19	**6642**	**82**	**82**
6	3	3	48	1	7	369	256	25	26568	1	163
8	1	3	81	-32	7	486	-2	22	41984	41	205
9	-8	1	82	-81	1	1312	369	41	7263027	-2	2695
12	-3	3	82	-18	8	1681	-1152	23	21789081	-8192	4667
16	9	5	82	-1	9	1681	-81	40			

Table 17: Solutions to $x + y = z^2$ with $S = \{2, 3, 41\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	36	-9	3	128	-3	5
3	-2	1	9	-1	2	82	-81	1	384	-41	7
4	-3	1	12	-4	2	82	-18	4	738	-9	9
4	4	2	18	9	3	123	-96	3	53792	15129	41
6	2	2	24	3	3	123	2	5	544644	6724	82

Table 18: Solutions to $x + y = z^3$ with $S = \{2, 3, 41\}$

10. The case $p = 43$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	16	9	5	129	-48	9	1161	688	43
2	2	2	18	-2	4	129	-8	11	1849	-1728	11
3	-2	1	24	1	5	129	96	15	2752	729	59
3	1	2	27	-2	5	172	-3	13	3483	-2	59
4	-3	1	43	-27	4	258	-2	16	4096	129	65
6	-2	2	43	-18	5	288	1	17	4374	-3698	26
6	3	3	43	6	7	486	-86	20	16512	129	129
8	1	3	48	1	7	486	-2	22	16641	-512	127
9	-8	1	81	-32	7	486	43	23	66048	1	257
12	-3	3	129	-128	1	1161	64	35	94041	-8192	293

Table 19: Solutions to $x + y = z^2$ with $S = \{2, 3, 43\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-1	2	128	-3	5	516	-4	8
3	-2	1	12	-4	2	129	-128	1	2916	-172	14
4	-3	1	18	9	3	129	-4	5	49923	29584	43
4	4	2	24	3	3	172	-108	4	**2130048**	**16641**	**129**
6	2	2	36	-9	3	344	-1	7			
9	-8	1	43	-16	3	512	-387	5			

Table 20: Solutions to $x + y = z^3$ with $S = \{2, 3, 43\}$

11. The case $p = 47$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	18	-2	4	128	-47	9	3072	-47	55
2	2	2	24	1	5	141	3	12	4096	-3807	17
3	-2	1	27	-2	5	162	94	16	6016	-3807	47
3	1	2	47	2	7	216	-47	13	9024	1	95
4	-3	1	48	-47	1	243	-47	14	24064	6561	175
6	-2	2	48	1	7	288	1	17	41472	2209	209
6	3	3	72	-47	5	486	-2	22	52488	-47	229
8	1	3	81	-32	7	576	-47	23	108288	-47	329
9	-8	1	94	6	10	1728	-47	41	177147	94	421
12	-3	3	94	27	11	2209	192	49	5668704	-103823	2359
16	9	5	96	-47	7	2256	-47	47			

Table 21: Solutions to $x + y = z^2$ with $S = \{2, 3, 47\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-1	2	128	-3	5	3384	-9	15
3	-2	1	12	-4	2	141	-16	5	**106032**	**-2209**	**47**
4	-3	1	18	9	3	324	188	8	282752	-178929	47
4	4	2	24	3	3	1152	-423	9			
6	2	2	36	-9	3	1692	36	12			
9	-8	1	48	-47	1	2209	-12	13			

Table 22: Solutions to $x + y = z^3$ with $S = \{2, 3, 47\}$

12. The case $p = 53$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	54	-53	1	729	-53	26
2	2	2	12	-3	3	81	-32	7	2809	216	55
3	-2	1	16	9	5	106	-81	5	**2862**	**-53**	**53**
3	1	2	18	-2	4	106	-6	10	11448	1	107
4	-3	1	24	1	5	288	1	17	297754	-729	545
6	-2	2	27	-2	5	318	6	18			
6	3	3	48	1	7	384	-159	15			
8	1	3	53	-4	7	486	-2	22			

Table 23: Solutions to $x + y = z^2$ with $S = \{2, 3, 53\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	36	-9	3	424	-81	7
3	-2	1	9	-1	2	54	-53	1	6912	-53	19
4	-3	1	12	-4	2	72	53	5	151686	-2809	53
4	4	2	18	9	3	128	-3	5	18429849	179776	265
6	2	2	24	3	3	212	4	6			

Table 24: Solutions to $x + y = z^3$ with $S = \{2, 3, 53\}$

13. The case $p = 59$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	12	-3	3	118	-18	10	531	-2	23
2	2	2	16	9	5	118	3	11	1593	256	43
3	-2	1	18	-2	4	177	-128	7	1888	1593	59
3	1	2	24	1	5	177	-96	9	3072	177	57
4	-3	1	27	-2	5	177	-8	13	3481	-3456	5
6	-2	2	48	1	7	177	48	15	7552	729	91
6	3	3	81	-32	7	243	118	19	258066	-2	508
8	1	3	108	-59	7	288	1	17	1161297	-131072	1015
9	-8	1	118	-54	8	486	-2	22			

Table 25: Solutions to $x + y = z^2$ with $S = \{2, 3, 59\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	36	-9	3	31329	4608	33
3	-2	1	9	-1	2	59	-32	3	111392	93987	59
4	-3	1	12	-4	2	118	-54	4			
4	4	2	18	9	3	128	-3	5			
6	2	2	24	3	3	243	-118	5			

Table 26: Solutions to $x + y = z^3$ with $S = \{2, 3, 59\}$

14. The case $p = 61$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	16	9	5	122	-1	11	**14823**	**61**	**122**
2	2	2	18	-2	4	192	-183	3	14884	-243	121
3	-2	1	24	1	5	243	-122	11	15616	9	125
3	1	2	27	-2	5	244	-243	1	59049	976	245
4	-3	1	48	1	7	288	1	17	237168	1	487
6	-2	2	61	-36	5	486	-2	22	1361886	3	1167
6	3	3	61	-12	7	732	-3	27	**7203978**	**-122**	**2684**
8	1	3	61	3	8	1024	-183	29			
9	-8	1	81	-32	7	3721	768	67			
12	-3	3	108	61	13	3904	-183	61			

Table 27: Solutions to $x + y = z^2$ with $S = \{2, 3, 61\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-1	2	64	61	5	2196	1	13
3	-2	1	12	-4	2	122	3	5	15616	9	25
4	-3	1	18	9	3	128	-3	5	36864	-33489	15
4	4	2	24	3	3	244	-243	1	**238144**	**-11163**	**61**
6	2	2	36	-9	3	576	-549	3	1808406	7442	122
9	-8	1	61	3	4	732	-3	9			

Table 28: Solutions to $x + y = z^3$ with $S = \{2, 3, 61\}$

15. The case $p = 67$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	12	-3	3	81	-32	7	1809	-128	41
2	2	2	16	9	5	201	-192	3	4288	201	67
3	-2	1	18	-2	4	201	-32	13	4489	-768	61
3	1	2	24	1	5	201	24	15	17152	9	131
4	-3	1	27	-2	5	268	-243	5	48843	-2	221
6	-2	2	48	1	7	288	1	17	146529	-34304	335
6	3	3	67	-18	7	402	-2	20	73085409	-8	8549
8	1	3	67	-3	8	486	-2	22			
9	-8	1	67	54	11	1024	201	35			

Table 29: Solutions to $x + y = z^2$ with $S = \{2, 3, 67\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	36	-9	3	603	-576	3
3	-2	1	9	-1	2	67	-3	4	1072	-729	7
4	-3	1	12	-4	2	128	-3	5	5427	-4096	11
4	4	2	18	9	3	134	-9	5	**287296**	**13467**	**67**
6	2	2	24	3	3	192	-67	5			

Table 30: Solutions to $x + y = z^3$ with $S = \{2, 3, 67\}$

16. The case $p = 71$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	12	-3	3	142	2	12	1296	-71	35
2	2	2	16	9	5	142	27	13	1458	142	40
3	-2	1	18	-2	4	142	54	14	2187	-71	46
3	1	2	24	1	5	192	-71	11	5041	288	73
4	-3	1	27	-2	5	213	12	15	**5112**	**-71**	**71**
6	-2	2	48	1	7	288	1	17	15123	6	123
6	3	3	72	-71	1	432	-71	19	20448	1	143
8	1	3	81	-32	7	486	-2	22	55296	-71	235
9	-8	1	96	-71	5	512	-71	21			

Table 31: Solutions to $x + y = z^2$ with $S = \{2, 3, 71\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	36	-9	3	5041	-128	17
3	-2	1	9	-1	2	71	54	5	36352	-6561	31
4	-3	1	12	-4	2	72	-71	1	362952	-5041	71
4	4	2	18	9	3	128	-3	5			
6	2	2	24	3	3	213	3	6			

Table 32: Solutions to $x + y = z^3$ with $S = \{2, 3, 71\}$

17. The case $p = 73$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	73	-72	1	288	73	19	5329	2592	89
2	2	2	73	-64	3	292	-243	7	**5913**	**-584**	**73**
3	-2	1	73	-48	5	292	-3	17	5913	16	77
3	1	2	73	-24	7	438	3	21	6561	-2336	65
4	-3	1	73	-9	8	486	-2	22	15552	73	125
6	-2	2	73	8	9	657	-128	23	18688	81	137
6	3	3	73	27	10	657	-32	25	19683	-10658	95
8	1	3	73	48	11	768	73	29	21024	1	145
9	-8	1	81	-32	7	1024	657	41	24576	73	157
12	-3	3	96	73	13	1152	73	35	53217	-32768	143
16	9	5	146	-2	12	1296	73	37	373248	73	611
18	-2	4	162	-146	4	4672	657	73	389017	-34992	595
24	1	5	216	73	17	**5256**	**73**	**73**	389017	294912	827
27	-2	5	219	6	15	5329	-2304	55	3501153	-512	1871
48	1	7	288	1	17	5329	-288	71			

Table 33: Solutions to $x + y = z^2$ with $S = \{2, 3, 73\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	12	-4	2	128	-3	5	42632	243	35
3	-2	1	18	9	3	219	-192	3	341056	47961	73
4	-3	1	24	3	3	219	-3	6	383688	5329	73
4	4	2	36	-9	3	657	72	9	431649	-42632	73
6	2	2	73	-72	1	4672	2187	19			
9	-8	1	73	-9	4	15552	73	25			
9	-1	2	81	-73	2	17496	-5329	23			

Table 34: Solutions to $x + y = z^3$ with $S = \{2, 3, 73\}$

18. The case $p = 79$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	79	2	9	6241	648	83
2	2	2	12	-3	3	81	-32	7	**6399**	**-158**	**79**
3	-2	1	16	9	5	128	-79	7	6399	1	80
3	1	2	18	-2	4	162	-158	2	6561	-632	77
4	-3	1	24	1	5	237	-12	15	1062882	79	1031
6	-2	2	27	-2	5	288	1	17			
6	3	3	48	1	7	316	-27	17			
8	1	3	79	-54	5	486	-2	22			

Table 35: Solutions to $x+y=z^2$ with $S=\{2,3,79\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	36	-9	3	2133	64	13
3	-2	1	9	-1	2	128	-3	5	505521	-12482	79
4	-3	1	12	-4	2	316	27	7			
4	4	2	18	9	3	324	-316	2			
6	2	2	24	3	3	711	18	9			

Table 36: Solutions to $x+y=z^3$ with $S=\{2,3,79\}$

19. The case $p = 83$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	83	-2	9	486	-2	22
2	2	2	12	-3	3	108	-83	5	2241	-32	47
3	-2	1	16	9	5	166	-162	2	6561	664	85
3	1	2	18	-2	4	166	3	13	6723	1	82
4	-3	1	24	1	5	249	-128	11	**6723**	**166**	**83**
6	-2	2	27	-2	5	249	-24	15	6889	-648	79
6	3	3	48	1	7	249	192	21	19683	-83	140
8	1	3	81	-32	7	288	1	17	169984	2241	415

Table 37: Solutions to $x + y = z^2$ with $S = \{2, 3, 83\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	36	-9	3	1328	3	11
3	-2	1	9	-1	2	128	-3	5	558009	13778	83
4	-3	1	12	-4	2	332	-324	2			
4	4	2	18	9	3	747	-18	9			
6	2	2	24	3	3	996	4	10			

Table 38: Solutions to $x + y = z^3$ with $S = \{2, 3, 83\}$

20. The case $p = 89$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	12	-3	3	89	32	11	2403	-2	49
2	2	2	16	9	5	178	-162	4	6561	2848	97
3	-2	1	18	-2	4	178	-9	13	7209	16	85
3	1	2	24	1	5	178	18	14	**7209**	**712**	**89**
4	-3	1	27	-2	5	288	1	17	7921	-2592	73
6	-2	2	48	1	7	486	-2	22	8192	89	91
6	3	3	81	-32	7	801	-512	17	47526	-2	218
8	1	3	89	-64	5	1458	-89	37			
9	-8	1	89	-8	9	1602	-2	40			

Table 39: Solutions to $x + y = z^2$ with $S = \{2, 3, 89\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	36	-9	3	801	-72	9
3	-2	1	9	-1	2	89	-81	2	641601	63368	89
4	-3	1	12	-4	2	89	36	5			
4	4	2	18	9	3	128	-3	5			
6	2	2	24	3	3	432	-89	7			

Table 40: Solutions to $x + y = z^3$ with $S = \{2, 3, 89\}$

21. The case $p = 97$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	27	-2	5	194	2	14	6144	97	79
2	2	2	48	1	7	243	-194	7	6561	6208	113
3	-2	1	81	-32	7	288	1	17	7857	64	89
3	1	2	97	-96	1	291	-2	17	7857	1552	97
4	-3	1	97	-81	4	388	-27	19	9312	97	97
6	-2	2	97	-72	5	432	97	23	9409	-5184	65
6	3	3	97	-48	7	486	-2	22	9409	-384	95
8	1	3	97	-16	9	582	-6	24	28227	-3	168
9	-8	1	97	3	10	864	97	31	37248	1	193
12	-3	3	97	24	11	873	-512	19	912673	-648	955
16	9	5	97	72	13	873	-32	29	1589248	873	1261
18	-2	4	128	97	15	2304	97	49	8957952	97	2993
24	1	5	192	97	17	5832	97	77			

Table 41: Solutions to $x + y = z^2$ with $S = \{2, 3, 97\}$

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	9	-8	1	36	-9	3	1746	-18	12
3	-2	1	9	-1	2	97	-96	1	762129	150544	97
4	-3	1	12	-4	2	128	-3	5	**903264**	**9409**	**97**
4	4	2	18	9	3	388	-324	4			
6	2	2	24	3	3	873	-144	9			

Table 42: Solutions to $x + y = z^3$ with $S = \{2, 3, 97\}$

Appendix A.

In this appendix, we prove the following proposition which is stated without proof in [BB15] (we keep the notation of the paper).

Proposition A.1. *We have* $N(E) \leq 350000$ *unless* (x, y, w, z) *corresponds to a one of the following equations*

$$2^7 + 1 = 3 \cdot 43, \quad 3 \cdot 2^4 - 1 = 47, \quad 3 \cdot 2^5 + 1 = 97,$$

$$2^6 + 3 = 67 \quad \textit{and} \quad 2^6 - 3 = 61.$$

Proof. Write $N(E) = 2^{\delta_2} \cdot 3^{\alpha} \cdot p^{\delta_p}$ with $\delta_i = \mathrm{ord}_i(N) \in \{0, 1, 2\}$, $i = 2, p$ and $\alpha = \mathrm{ord}_3(N) \in \{0, 1, 2, 3, 4, 5\}$. It is easy to check that we have $N(E) \leq 350000$ unless we are in one of the following situations :

$(\alpha, \delta_2, \delta_p)$	$(5, 2, 2)$	$(5, 1, 2)$	$(4, 2, 2)$	$(5, 0, 2)$
p	≥ 19	≥ 29	≥ 37	≥ 41
$(\alpha, \delta_2, \delta_p)$	$(4, 1, 2)$	$(3, 2, 2)$	$(4, 0, 2)$	$(3, 1, 2)$
p	≥ 47	≥ 59	≥ 67	≥ 83

Of these, only the cases $(\alpha, \delta_2, \delta_p) \in \{(5,1,2), (4,1,2), (3,1,2)\}$ may actually occur. Therefore, we end up solving the following equations where $\epsilon = \pm 1$ and k is a positive integer

$$2^k + \epsilon = 3^{\gamma_3} p^{\gamma_p} z^3 \quad \text{for } p \ge 29,$$
$$3 \cdot 2^k + \epsilon = p^{\gamma_p} z^3, \quad 2^k + 3\epsilon = p^{\gamma_p} z^3 \quad \text{for } p \ge 47$$

and

$$2^k + \epsilon = p^{\gamma_p} z^2, \quad 3^2 \cdot 2^k + \epsilon = p^{\gamma_p} z^3, \quad 2^k + 3^2\epsilon = p^{\gamma_p} z^2 \quad \text{for } p \ge 83$$

where $\gamma_3, \gamma_p \in \{1,2\}$.

We therefore deal with each of these equations in turn. Considering first the case $2^k + 1 = 3^{\gamma_3} p^{\gamma_p} z^3$, we easily check that it has no solution modulo $24p$ unless $p \in \{43, 59, 67, 83\}$. Besides, if $p = 59$, then it has no solution modulo pq where

$$q = \begin{cases} 349 & \text{if } (\gamma_3, \gamma_p) = (1,2) \text{ or } (2,1) \\ 2089 & \text{if } (\gamma_3, \gamma_p) = (1,1) \text{ or } (2,2) \end{cases}.$$

Similarly, it has no solution modulo pq where

$$q = \begin{cases} 199 & \text{if } (\gamma_3, \gamma_p) = (2,1) \text{ or } (2,2) \\ 397 & \text{if } (\gamma_3, \gamma_p) = (1,1) \\ 3301 & \text{if } (\gamma_3, \gamma_p) = (1,2) \end{cases}.$$

If $p = 43$ or 83, write $w = 3^{\gamma_3} p^{\gamma_p}$ and $2^k = 2^\beta x^2$. Then, $(X, Y) = (2^\beta wz, 2^{2\beta} wx)$ is an integral point on the elliptic curve

$$Y^2 = X^3 - w^2 2^{3\beta}.$$

Computing its integral points using magma ([BCP97]) leads to a unique solution, namely $2^7 + 1 = 3 \cdot 43$. We now turn our attention to $2^k - 1 = wz^3$, where $w = 3^{\gamma_3} p^{\gamma_p}$. Since $3 \mid w$, k is even. Write $k = 2k_0$. Then, for some $w_1 \mid 3^2$, we have $2^{k_0} \pm 1 = w_1 z_1^3$ where z_1 is a non-zero integer. If $2^{k_0} = 2^\beta x^2$ with $\beta = 0, 1$, then $(X, Y) = (2^\beta w_1 z_1, 2^{2\beta} w_1 x)$ is an integral point on

$$Y^2 = X^3 \mp w_1^2 2^{3\beta}.$$

Computing again the integral points on these curves using magma, we find that the equation $2^k - 1 = wz^3$ has no solution for $p \ge 29$.

We now turn our attention to the equation $3 \cdot 2^k + \epsilon = wz^3$, where $w = p^{\gamma_p}$. Sieving mod $72p$ we find ourselves in one of the situations in Table 43.

p	w	k
59	p^2	odd
61	p^2	even
67	p	even
79	p	odd
79	p^2	even
83	p	odd
97	p	odd

p	w	k
47	p	even
47	p^2	odd
59	p^2	even
61	p	even
83	p	even
97	p^2	even

TABLE 43. Remaining values for $\epsilon = +1$ and $\epsilon = -1$ respectively

In each situation, write $2^k = 2^\beta x^2$ with $\beta = 0, 1$. Then, $(X, Y) = (3 \cdot 2^\beta wz, 3^2 \cdot 2^{2\beta} wx)$ is an integral point on

$$Y^2 = X^3 - \epsilon w^2 3^3 \cdot 2^{3\beta}.$$

Computing the integral points on these curves using magma, we find that the equation $3 \cdot 2^k + \epsilon = wz^3$ has precisely two solutions for $47 \le p < 100$, that are $3 \cdot 2^5 + 1 = 97$ and $3 \cdot 2^4 - 1 = 47$.

The next equation to consider is $2^k + 3\epsilon = wz^3$, where $w = p^{\gamma_p}$. Sieving mod $72p$ we find ourselves in one of the situations in Table 44.

p	w	k
53	p or p^2	odd
59	p or p^2	odd
61	p^2	even
67	p	even
79	p^2	odd
83	p or p^2	odd
97	p^2	odd

p	w	k
47	p or p^2	arbitrary
53	p or p^2	odd
59	p or p^2	even
61	p	even
67	p	odd
71	p or p^2	arbitrary
83	p or p^2	even

TABLE 44. Remaining values for $\epsilon = +1$ and $\epsilon = -1$ respectively

In each situation, write $2^k = 2^\beta x^2$ with $\beta = 0, 1$. Then, $(X, Y) = (2^\beta wz, 2^{2\beta} wx)$ is an integral point on

$$Y^2 = X^3 - \epsilon w^2 3 \cdot 2^{3\beta}.$$

Computing the integral points on these curves using magma, we find that the equation $3 \cdot 2^k + \epsilon = wz^3$ has precisely two solutions for $47 \le p < 100$, that are $2^6 + 3 = 67$ and $2^6 - 3 = 61$.

We now turn our attention to $2^k - 1 = wz^3$, where $w = p^{\gamma_p}$. If k is even, write $k = 2k_0$. Then, for some non-zero integer z_1, we have $2^{k_0} \pm 1 = z_1^3$ and if we write $2^{k_0} = 2^\beta x^2$, then $(X, Y) = (2^\beta z_1, 2^{2\beta} x)$ is an integral point on

$$Y^2 = X^3 \mp 2^{3\beta}.$$

Therefore we get that there is no solution to our equation for $83 \le p < 100$ and k even. If however k is odd, then reducing modulo p we end up to the case $p = 89$ and if we write $k = 2k_0 + 1$, then $(X, Y) = (2wz, 2^{k_0+2})$ is an integral point on

$$Y^2 = X^3 + 2^3 w^2.$$

Therefore we obtain that there is no solution to our equation for k odd as well.

For equation $2^k + 1 = wz^3$, sieving modulo p leads to either $p = 83$ and k odd, or $p = 97$ and k even. In both situation, write $2^k = 2^\beta x^2$ with $\beta = 0$ or 1. Then, $(X, Y) = (2^\beta wz, 2^{2\beta} wx)$ is an integral point on the elliptic curve

$$Y^2 = X^3 - 2^{3\beta} w^2.$$

Computing its integral on magma yields to no solution to our equation.

The equation $3^2 \cdot 2^k + \epsilon = p^{\gamma_p} z^3$ has no solution modulo $72p$. Finally, it remains to solve $2^k + 3^2\epsilon = wz^3$, where $w = p^{\gamma_p}$. Sieving modulo $72p$ we find ourselves in one of the situations below

$$(p, \epsilon) = (97, +1),\ (83, -1),\ (97, -1)$$

with in each case, k even. Write $k = 2k_0$. Then, $(X, Y) = (wz, 2^{k_0} w)$ is an integral point on the elliptic curve

$$Y^2 = X^3 - \epsilon(3w)^2.$$

Computing its integral on magma yields again to no solution to our equation. This ends the proof of the proposition.

□

For the reader's convenience, here is the corresponding statement (proved in the paper) in the case $n = 2$.

Proposition A.2. *We have* $N(E) \le 350000$ *unless* (x, y, w, z) *corresponds to a one of the following equations*

$$3^4+1 = 2\cdot 41, \quad 2\cdot 3^3-1 = 53, \quad 3^{10}-1 = 2\cdot 61\cdot 22^2, \quad 3^4+2 = 83, \quad 3^4-2 = 79,$$
$$3^5+1 = 61\cdot 2^2, \quad 2^3\cdot 3^2+1 = 73, \quad 2^3\cdot 3^2-1 = 71, \quad 3^4+2^3 = 89, \quad 3^4-2^3 = 73.$$

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