THE CASE $S = \{2, 3, 5, 7\}$ AND $n = 2$ OR 3

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In Table 1 below, we have listed all the triples (x, y, z) such that

$$x + y = z^2$$

where z is a non-zero integer and x, y are $\{2, 3, 5, 7\}$-units with $\gcd(x, y)$ squarefree, $x \geq |y| > 0$ and $z > 0$. There are exactly 388 such triples.

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	75	6	9	486	-2	22	4480	9	67
2	2	2	80	1	9	490	-486	2	4704	625	73
3	-2	1	81	-80	1	490	-90	20	5040	1	71
3	1	2	81	-56	5	490	-6	22	5145	-5120	5
4	-3	1	81	-32	7	490	135	25	5145	-384	69
5	-4	1	81	40	11	504	25	23	5145	480	75
5	-1	2	84	-75	3	512	-343	13	6250	-9	79
5	4	3	84	-35	7	525	-84	21	6561	-6272	17
6	-5	1	84	-3	9	525	4	23	6561	-5600	31
6	-2	2	90	10	10	576	49	25	6561	-320	79
6	3	3	96	-15	9	625	-576	7	6804	-875	77
7	-6	1	96	25	11	625	-504	11	7840	81	89
7	-3	2	98	2	10	625	-336	17	8505	-1280	85
7	2	3	100	21	11	625	-96	23	8505	-224	91
8	-7	1	105	-96	3	625	-49	24	9375	1029	102
8	1	3	105	-80	5	625	216	29	9408	1	97
9	-8	1	105	-56	7	625	336	31	9800	1	99
9	-5	2	105	-24	9	630	-5	25	10206	-5	101
9	7	4	105	-5	10	640	-15	25	10240	-1215	95
10	-9	1	105	16	11	675	1	26	10368	2401	113
10	-6	2	105	64	13	686	-10	26	10584	25	103
10	-1	3	112	-63	7	729	-560	13	11250	-14	106
10	6	4	112	9	11	729	-245	22	12544	225	113
12	-3	3	120	1	11	729	-200	23	13230	-5	115
14	-10	2	120	49	13	729	112	29	14175	-14	119
14	-5	3	120	105	15	729	640	37	14336	-175	119
14	2	4	125	-4	11	735	-6	27	14406	-6	120
15	-14	1	126	-125	1	750	-686	8	15309	-12500	53
15	-6	3	126	-5	11	750	-21	27	15625	-10584	71
15	1	4	126	70	14	840	1	29	15625	-1701	118
15	10	5	128	-7	11	896	625	39	15625	504	127

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16	-15	1	135	-35	10	945	-896	7	15625	1536	131
16	-7	3	135	-14	11	945	-320	25	16128	1	127
16	9	5	144	25	13	945	16	31	16384	-14175	47
18	-14	2	147	-3	12	945	280	35	16807	13122	173
18	-2	4	150	-6	12	960	-735	15	17500	189	133
18	7	5	160	-135	5	960	1	31	18144	625	137
20	5	5	160	9	13	1024	-735	17	18225	-3584	121
21	-20	1	162	-98	8	1024	-63	31	18750	294	138
21	-12	3	162	7	13	1029	-500	23	21504	105	147
21	-5	4	168	1	13	1029	-300	27	23625	1024	157
21	4	5	175	-126	7	1029	-5	32	24010	15	155
21	15	6	175	-54	11	1029	60	33	25920	1	161
24	-15	3	175	-6	13	1120	105	35	26250	-6	162
24	1	5	175	21	14	1120	729	43	30618	7	175
25	-24	1	175	81	16	1134	-350	28	32768	-7	181
25	-21	2	189	-140	7	1215	-686	23	33614	-125	183
25	-16	3	189	-125	8	1215	10	35	35721	-32000	61
25	-9	4	189	-20	13	1225	-864	19	43740	-1715	205
25	24	7	189	7	14	1225	-384	29	43750	-486	208
27	-2	5	189	100	17	1225	144	37	46305	-80	215
28	-27	1	196	-75	11	1250	686	44	48384	15625	253
28	-3	5	196	-27	13	1260	-35	35	49000	729	223
28	21	7	210	-14	14	1344	25	37	50625	-896	223
30	-21	3	210	15	15	1372	-3	37	55566	-3125	229
30	-14	4	224	-175	7	1458	-14	38	59049	1960	247
30	-5	5	224	1	15	1500	21	39	59049	7000	257
30	6	6	225	-224	1	1536	-15	39	59535	1	244
32	-7	5	225	-56	13	1600	81	41	60025	-1944	241
35	-10	5	225	64	17	1680	1	41	63000	1	251
35	1	6	240	-15	15	1701	-20	41	64000	9	253
35	14	7	240	49	17	1701	700	49	65625	-57344	91
36	-35	1	245	-20	15	1750	-1701	7	69120	49	263
40	-15	5	250	-81	13	1750	14	42	76545	71680	385
40	9	7	250	-54	14	1792	-567	35	82944	30625	337
42	-6	6	250	6	16	1800	49	43	83349	2500	293
42	7	7	256	-175	9	1920	105	45	85750	-486	292
45	-20	5	256	-135	11	2025	784	53	109375	-1134	329
45	4	7	256	105	19	2160	49	47	117649	-97200	143
48	1	7	270	-245	5	2205	4	47	117649	48000	407
49	-48	1	270	-14	16	2400	1	49	128625	256	359
49	-45	2	280	9	17	2401	-2400	1	129654	-78125	227
49	-40	3	280	81	19	2401	-1440	31	137200	59049	443
49	-24	5	288	1	17	2401	-720	41	137781	-140	371
49	15	8	294	-150	12	2401	-192	47	140625	-43904	311
49	32	9	294	-125	13	2401	200	51	168070	30	410
50	-49	1	294	-5	17	2401	1080	59	179200	6561	431
50	-14	6	294	30	18	2430	70	50	196830	-33614	404
50	-1	7	324	-35	17	2625	-1536	33	201600	1	449

50	14	8	336	25	19	2625	-224	49	201684	-1875	447
54	-50	2	336	105	21	2625	-24	51	214375	-6	463
54	-5	7	343	-243	10	2625	1344	63	243000	49	493
54	10	8	343	-54	17	2800	9	53	245760	-735	495
56	-7	7	343	18	19	3024	1	55	252105	-24576	477
56	25	9	360	1	19	3150	-14	56	262144	5145	517
60	-35	5	375	-294	9	3200	49	57	390625	-112896	527
60	21	9	375	-14	19	3375	2401	76	688905	-5	830
63	-14	7	384	-375	3	3430	-405	55	1058841	-20480	1019
63	1	8	405	-5	20	3456	25	59	1440000	2401	1201
64	-63	1	420	21	21	3840	2401	79	1640625	336	1281
64	-15	7	441	-320	11	3969	-125	62	4214784	25	2053
70	-54	4	441	-80	19	3969	256	65	4782969	4375	2188
70	-45	5	441	400	29	4050	-686	58	5764801	-9600	2399
70	-21	7	448	-7	21	4096	-375	61	19140625	-17496	4373
70	-6	8	448	81	23	4096	945	71	23049600	1	4801
70	30	10	480	49	23	4374	250	68	76545000	1	8749
72	49	11	486	-125	19	4375	-4374	1	199290375	-686	14117

Table 1: Solutions to $x + y = z^2$ with $S = \{2,3,5,7\}$

We similarly list in Table 2 below all the triples (x, y, z) such that

$$x + y = z^3$$

where z is a non-zero integer and x, y are $\{2,3,5,7\}$-units with $\gcd(x, y)$ cubefree, $x \geq |y| > 0$ and $z > 0$. There are exactly 207 such triples.

x	y	z	x	y	z	x	y	z	x	y	z
2	-1	1	63	1	4	336	7	7	2304	-1575	9
3	-2	1	64	-63	1	350	-225	5	2401	-2400	1
4	-3	1	70	-6	4	350	-7	7	2450	294	14
4	4	2	72	-45	3	350	162	8	2646	98	14
5	-4	1	75	-48	3	375	-32	7	2940	-196	14
5	3	2	75	50	5	378	-35	7	3072	-875	13
6	-5	1	80	45	5	392	-49	7	3087	288	15
6	2	2	81	-80	1	405	-280	5	3150	225	15
7	-6	1	84	-20	4	441	-225	6	3200	175	15
7	1	2	90	-63	3	441	-98	7	3360	15	15
8	-7	1	90	35	5	441	288	9	3430	-2430	10
9	-8	1	98	-90	2	448	-105	7	3500	-756	14
9	-1	2	98	27	5	490	-147	7	3600	-225	15
10	-9	1	100	-36	4	500	12	8	3675	-300	15
10	-2	2	100	25	5	504	225	9	3969	-1225	14
12	-4	2	105	20	5	525	-400	5	4116	-20	16
14	-6	2	108	-100	2	540	-28	8	4375	-4374	1
15	-14	1	120	5	5	567	-224	7	4375	-4032	7
15	-7	2	125	-98	3	588	-245	7	6075	784	19
15	12	3	126	-125	1	700	300	10	6125	3136	21

16	-15	1	126	-1	5	720	9	9	6615	-6272	7
18	-10	2	126	90	6	735	-392	7	6860	-1	19
18	9	3	128	-3	5	735	-6	9	7056	2205	21
20	-12	2	135	-10	5	750	-686	4	8000	-3087	17
20	7	3	140	-15	5	750	-21	9	8100	-100	20
21	-20	1	147	-120	3	784	-441	7	8820	441	21
21	6	3	150	-25	5	800	-675	5	9216	45	21
24	3	3	160	-35	5	900	100	10	9408	-147	21
25	-24	1	162	-98	4	972	28	10	10935	-9604	11
25	2	3	175	-50	5	980	-972	2	11025	-1764	21
28	-27	1	175	168	7	980	20	10	12005	162	23
28	-20	2	180	36	6	1029	-300	9	12250	-11907	7
28	-1	3	189	-125	4	1050	-50	10	14400	-11025	15
30	-3	3	189	-64	5	1152	-1125	3	14400	1225	25
32	-5	3	196	20	6	1225	-882	7	22050	-98	28
35	-27	2	196	147	7	1225	-225	10	25725	-25600	5
35	-8	3	200	-75	5	1225	972	13	33075	9800	35
36	-35	1	210	6	6	1250	81	11	36000	-63	33
36	-28	2	225	-224	1	1296	35	11	39200	3675	35
36	-9	3	225	-100	5	1323	-980	7	42525	350	35
36	28	4	225	-9	6	1323	8	11	44100	-1225	35
42	-15	3	243	100	7	1350	-1225	5	50625	28	37
45	-18	3	245	-120	5	1350	-350	10	62720	-19845	35
48	-21	3	245	98	7	1568	-1225	7	73500	588	42
49	-48	1	252	-225	3	1715	-384	11	117600	49	49
49	15	4	252	-36	6	1764	-36	12	157500	-36	54
50	-49	1	280	63	7	1764	980	14	216090	-90	60
50	-42	2	294	49	7	1792	405	13	274400	225	65
50	14	4	300	-175	5	1800	1575	15	328125	384	69
54	10	4	300	-84	6	1875	-147	12	401408	55125	77
60	4	4	315	-288	3	2187	10	13	5358150	1225	175
63	-36	3	315	28	7	2205	-8	13			

Table 2: Solutions to $x + y = z^3$ with $S = \{2, 3, 5, 7\}$

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