On Darmon's program for the generalized Fermat equation of signature (r, r, p)

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Table of contents

Table of contents

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[CONSTRUCTION] (Hellegouarch, Frey)

- Consider
$$E : y^2 = x(x - a^p)(x + b^p).$$
The discriminant $\Delta = 2^4(abc)^{2p}$ of this model is non-zero, and hence it defines an elliptic curve over $\mathbf{Q}$ (with full 2-torsion).

- There is a 2-dimensional mod p representation attached to E
$$\overline{\rho}_{E,p} : G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}(E[p]) \simeq \mathrm{GL}_2(\mathbf{F}_p)$$
given by the action of $G_{\mathbf{Q}}$ on the group of p-torsion points on E.

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[MODULARITY] (Wiles)

- Without loss of generality, assume from now on that

$$a^p \equiv -1 \pmod 4 \quad \text{and} \quad b^p \equiv 0 \pmod{16}.$$

  Hence the curve E is semistable (at 2).

- Since $E/\mathbf{Q}$ is semistable, the elliptic curve $E/\mathbf{Q}$ is **modular**.

- Its conductor is $N_E = \mathrm{rad}(\Delta_{\min}(E)) = \mathrm{rad}\left(\frac{(abc)^p}{16}\right)$.

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[IRREDUCIBILITY] (Mazur)

- Since E has full 2-torsion over $\mathbf{Q}$ and is semistable, the representation
$$\overline{\rho}_{E,p} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$$
is **absolutely irreducible**.

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[Level lowering] (Ribet)

- By Tate's theory (recall: $E/\mathbf{Q}$ is semistable), the representation $\overline{\rho}_{E,p}$ has Serre's conductor $N(\overline{\rho}_{E,p}) = 2$.
- It has weight 2 in the sense of Edixhoven (or Serre).
- Since $E/\mathbf{Q}$ is modular and the representation $\overline{\rho}_{E,p}$ is absolutely irreducible, then $\overline{\rho}_{E,p}$ **arises from** a newform of weight 2 and level $N(\overline{\rho}_{E,p}) = 2$ (with trivial character).

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[Contradiction]

- There are no newforms of weight 2 and level 2!

The modular method

1. Construction
2. Modularity
3. Irreducibility
4. Level lowering
5. Contradiction

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Table of contents

Our Diophantine problem

We wish to extend the modular method to deal with generalized Fermat equations

$$Ax^r + By^q = Cz^p$$

where A, B, C are fixed non-zero coprime integers and p, q, r are non-negative integers.

In this work, we restrict ourselves to the case of

$$x^r + y^r = Cz^p$$

where $r \geq 3$ is a **fixed prime**, C is a fixed positive integer and p is a prime which is allowed to vary.

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Notation

$r \geq 3$ prime number

ζ_r primitive r-th root of unity

$\omega_i = \zeta_r^i + \zeta_r^{-i}$, for every $i \geq 0$

$$h(X) = \prod_{i=1}^{(r-1)/2} (X - \omega_i) \in \mathbf{Z}[X]$$

$K = \mathbf{Q}(\zeta_r)^+ = \mathbf{Q}(\omega_1)$ maximal totally real subfield of $\mathbf{Q}(\zeta_r)$

$\mathcal{O}_K$ integer ring of K

$\mathfrak{p}_r$ unique prime ideal above r in $\mathcal{O}_K$ (totally ramified)

Step 1 – Kraus' Frey hyperelliptic curve

Let a, b be non-zero coprime integers such that $a^r + b^r \neq 0$.

$$C_r(a,b) : y^2 = (ab)^{\frac{r-1}{2}} x h\left(\frac{x^2}{ab} + 2\right) + b^r - a^r.$$

The discriminant of this model is

$$\Delta_r(a,b) = (-1)^{\frac{r-1}{2}} 2^{2(r-1)} r^r (a^r + b^r)^{r-1} \neq 0.$$

In particular, it defines a hyperelliptic curve of genus $\frac{r-1}{2}$.

Examples

$$\begin{aligned}
r = 3 : &\quad y^2 = x^3 + 3abx + b^3 - a^3 \\
r = 5 : &\quad y^2 = x^5 + 5abx^3 + 5a^2b^2x + b^5 - a^5 \\
r = 7 : &\quad y^2 = x^7 + 7abx^5 + 14a^2b^2x^3 + 7a^3b^3x + b^7 - a^7.
\end{aligned}$$

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Frey representations

For a field M of characteristic 0, write $G_M = \mathrm{Gal}(\overline{M}/M)$ for its absolute Galois group.

Definition (Darmon)

A **Frey representation** of signature $(r, q, p) \in (\mathbf{Z}_{>0})^3$ over a number field L in characteristic $\ell > 0$ is a Galois representation

$$\overline{\rho} = \overline{\rho}(t) : G_{L(t)} \to \mathrm{GL}_2(\mathbf{F})$$

where $\mathbf{F}$ finite field of characteristic ℓ such that the following conditions hold.

1. The restriction of $\overline{\rho}$ to $G_{\overline{L}(t)}$ has trivial determinant and is irreducible.
2. The projectivization $\overline{\rho}^{\mathrm{geom}} : G_{\overline{L}(t)} \to \mathrm{PSL}_2(\mathbf{F})$ of this representation is unramified outside $\{0, 1, \infty\}$.
3. It maps the inertia groups at 0, 1, and ∞ to subgroups of $\mathrm{PSL}_2(\mathbf{F})$ of order r, q, and p respectively.

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Hecke–Darmon's classification theorem

Let p be a prime number.

Theorem (Hecke–Darmon)

Up to equivalence, there is only one Frey representation of signature (p, p, p). It occurs over $\mathbf{Q}$ in characteristic p and is associated with the Legendre family

$$L(t) : y^2 = x(x-1)(x-t).$$

The classical Frey–Hellegouarch curve

$$y^2 = x(x-a^p)(x+b^p)$$

is obtained from $L(t)$ after **specialization** at $t_0 = \frac{a^p}{a^p+b^p}$ and **quadratic twist** by $-(a^p+b^p)$.

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Abelian varieties of GL_2-type

Definition

Let A be an abelian variety over a field L of characteristic 0. We say that A/L is of GL_2-type (or $\mathrm{GL}_2(F)$-type) if there is an embedding $F \hookrightarrow \mathrm{End}_L(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ where F is a number field with $[F : \mathbf{Q}] = \dim A$.

Let A/L be an abelian variety of $\mathrm{GL}_2(F)$-type.

- For each prime ideal $\lambda \mid \ell$ in F, we have a λ-adic representation
$$\rho_{A,\lambda} : G_L \longrightarrow \mathrm{Aut}_{F_\lambda}(V_\lambda(A)) \simeq \mathrm{GL}_2(F_\lambda),$$
coming from the linear action of G_L on $V_\lambda(A) = V_\ell(A) \otimes_{F\otimes \mathbf{Q}_\ell} F_\lambda$.
- The representations $\{\rho_{A,\lambda}\}_\lambda$ form a strictly compatible system of F-integral representations.
- For each prime ideal $\lambda \mid \ell$ in F, we have a residual representation
$$\overline{\rho}_{A,\lambda} : G_L \longrightarrow \mathrm{GL}_2(\mathbf{F}_\lambda),$$
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Frey representations in signature (r, r, p)

Theorem (B.–Chen–Dieulefait–Freitas, 2022)

There exists a hyperelliptic curve $C_r'(t)$ over $K(t)$ of genus $\frac{r-1}{2}$ such that $J_r'(t) = \mathrm{Jac}(C_r'(t))$ is of $\mathrm{GL}_2(K)$-type, i.e. there is an embedding

$$K \hookrightarrow \mathrm{End}_{K(t)}(J_r'(t)) \otimes \mathbf{Q}.$$

Moreover, for every prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ above a rational prime p,

$$\overline{\rho}_{J_r'(t),\mathfrak{p}} : G_{K(t)} \to \mathrm{GL}_2(\mathcal{O}_K/\mathfrak{p})$$

is a Frey representation of signature (r, r, p).
The hyperelliptic curve $C_r(a, b)/K$ is obtained from $C_r'(t)$ after **specialization** at $t_0 = \frac{a^r}{a^r+b^r}$ and **quadratic twist** by $-\frac{(ab)^{\frac{r-1}{2}}}{a^r+b^r}$.

- The proof uses Darmon's construction of Frey representations of signature (p, p, r).

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- The proof uses Darmon's construction of Frey representations of signature (p, p, r).

Two-dimensional $\mathfrak{p}$-adic and mod $\mathfrak{p}$ representations

Write $J_r = \mathrm{Jac}(C_r(a,b))/K$ for the Jacobian of $C_r(a,b)$ base changed to K.

- There is a compatible system of K-rational Galois representations

$$\rho_{J_r,\mathfrak{p}} : G_K \to \mathrm{GL}_2(K_{\mathfrak{p}})$$

  indexed by the prime ideals $\mathfrak{p}$ in $\mathcal{O}_K$ associated with J_r.

- For $\mathfrak{p} = \mathfrak{p}_r$, the residual representation $\overline{\rho}_{J_r,\mathfrak{p}_r}$ arises after specialization and twisting from a Frey representation of signature (r,r,r).

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Step 2 – The representation $\overline{\rho}_{J_r,\mathfrak{p}_r}$ and modularity

Theorem (B.–Chen–Dieulefait–Freitas, 2022)

Assume $r \geq 5$. The representation $\overline{\rho}_{J_r,\mathfrak{p}_r} : G_K \to \mathrm{GL}_2(\mathbf{F}_r)$ is absolutely irreducible when restricted to $G_{\mathbf{Q}(\zeta_r)}$.

Corollary

The abelian variety J_r/K is modular (for any prime $r \geq 3$).

- Classification theorem of Frey representations with constant signature (Hecke–Darmon).
- New irreducibility results for Galois representations attached to elliptic curves over $\mathbf{Q}(\zeta_r)$ (Najman).
- Serre's modularity conjecture (Khare–Wintenberger).
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Step 4 – Refined level lowering

Assume that there exists a non-zero integer c such that $a^r + b^r = Cc^p$ for some fixed positive integer C.
Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$ above the rational prime p.

Theorem (B.–Chen–Dieulefait–Freitas, 2022)

Assume that $a \equiv 0 \pmod 2$ and $b \equiv 1 \pmod 4$. Suppose further that $\overline{\rho}_{J_r,\mathfrak{p}}$ is absolutely irreducible. Then, there is a Hilbert newform g over K of parallel weight 2, trivial character and level $2^2\mathfrak{p}_r^2\mathfrak{n}'$ such that

$$\overline{\rho}_{J_r,\mathfrak{p}} \simeq \overline{\rho}_{g,\mathfrak{P}}$$

for some $\mathfrak{P} \mid p$ in the coefficient field K_g of g.
Here, $\mathfrak{n}'$ denotes the product of ideals coprime to $2r$ dividing C.
Moreover, we have $K \subset K_g$.

- Uses a refined level lowering theorem of Breuil–Diamond.
- Various situations where the irreducibility assumption is satisfied.

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for some $\mathfrak{P} \mid p$ in the coefficient field K_g of g.
Here, $\mathfrak{n}'$ denotes the product of ideals coprime to $2r$ dividing C.
Moreover, we have $K \subset K_g$.

- Uses a refined level lowering theorem of Breuil–Diamond.
- Various situations where the irreducibility assumption is satisfied.

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Assume that there exists a non-zero integer c such that $a^r + b^r = Cc^p$ for some fixed positive integer C.
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Table of contents

Step 5 – Main obstacles

In applying the modular method to Fermat equations of the shape

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for specific values of r and C, we find that the **contradiction step** (and, to some extent, the irreducibility step) is the most problematic:

- Newform subspaces may not be accessible to computer softwares (as they are too large or by lack of efficient algorithms, for instance).
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The case $r = 7$ and $C = 3$

Theorem (B.–Chen–Dieulefait–Freitas, 2024)

For every integer $n \geq 2$, there are no integers a, b, c such that

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$$y^2 = x^7 + 7abx^5 + 14a^2b^2x^3 + 7a^3b^3x + b^7 - a^7$$

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A partial answer in the case $r = 11$ and $C = 1$

Theorem (B.–Chen–Dieulefait–Freitas, 2022)

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- Total running time in Magma: 7 hours = 6 hours (computation of the relevant Hilbert space) + 1 hour (elimination).
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