

Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups

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Abstract

We construct new examples of ergodic coactions of compact quantum groups, in which the multiplicity of an irreducible corepresentation can be strictly larger than the dimension of the latter. These examples are obtained using a bijective correspondence between certain ergodic coactions on C^* -algebras and unitary fiber functors on the representation category of a compact quantum group. We classify these unitary fiber functors on the universal orthogonal and unitary quantum groups. The associated C^* -algebras and von Neumann algebras can be defined by generators and relations, but are not yet well understood.

Introduction

By a well known theorem of Høegh-Krohn, Landstad and Størmer [16], compact groups only admit ergodic actions on tracial C^* -algebras. Indeed, the (unique) invariant state is necessarily a trace. Moreover, given a compact group G acting ergodically on a C^* -algebra B , one studies the so-called *spectral subspaces*: the action of G on B yields a unitary representation of G which can be decomposed into a direct sum of irreducible representations. One proves that the multiplicity of an irreducible representation is necessarily bounded by the dimension of this irreducible representation.

A deeper analysis of the spectral structure of ergodic actions of compact groups has been made by A. Wassermann [27, 28, 29]. In the culmination of his work, Wassermann shows that the compact group $SU(2)$ only admits ergodic actions on von Neumann algebras of finite type I.

In the 1980's, Woronowicz introduced the notion of a *compact quantum group* and generalized the classical Peter-Weyl representation theory. Many fascinating examples of compact quantum groups are available by now: Drinfel'd and Jimbo [14, 17] introduced the *q-deformations of compact semi-simple Lie groups* and Rosso [21] showed that they fit into the theory of Woronowicz. The *universal orthogonal and unitary quantum groups* were introduced by Van Daele and Wang [25] and studied in detail by Banica [2, 3]. Other examples of compact quantum groups, related with graphs and metric spaces have been constructed by the first author [9] and by Banica [4, 5].

The abstract theory of *ergodic coactions* of compact quantum groups on C^* -algebras has been initiated by Boca [10] and Landstad [18]. The major difference with the compact group case, is the following: the (unique) invariant state is no longer a trace. Indeed, Wang [26] gave examples of ergodic coactions of universal unitary quantum groups on type III factors.

Moreover, in the work of Boca, the multiplicity of an irreducible corepresentation is proved to be bounded by the *quantum dimension* rather than the ordinary dimension of the corepresentation. Nevertheless, in all the examples known up to now, the multiplicity is actually bounded by the ordinary dimension. In this paper, we provide examples of ergodic coactions where the multiplicity of an irreducible corepresentation is strictly larger than the ordinary dimension of the corepresentation.

In [29], A. Wassermann gives a complete classification of the ergodic actions of $SU(2)$, essentially labeling them by the finite subgroups of $SU(2)$. It would, of course, be great to give a complete classification of ergodic coactions of the deformed $SU_q(2)$. In [23], Tomatsu provides a first step in this direction: he computes all ergodic coactions of $SU_q(2)$ on 'virtual' quotient spaces $SU_q(2)/\Gamma$. (More precisely, he describes all the

coideals of the quantum group $SU_q(2)$.) By construction, the ergodic coactions of $SU_q(2)$ on its virtual quotient spaces are such that the multiplicity of an irreducible corepresentation is bounded by its dimension. The results of this paper imply in particular that there are much more ergodic coactions of $SU_q(2)$ than the ones studied by Tomatsu.

The major tool to produce our new examples of ergodic coactions of compact quantum groups, is the notion of *monoidal equivalence* of quantum groups. One can look at a compact quantum group with several degrees of precision. At first, we study only the *fusion rules* in the representation theory: we label the irreducible corepresentations and describe how a tensor product of irreducibles breaks up into irreducibles. Taking into account only these fusion rules, we loose a lot of information: for example, the q -deformed compact Lie groups have the same fusion rules as their classical counterparts. In a next approximation, one studies the corepresentation theory of a compact quantum group as a *monoidal category*, but without its concrete realization (the so-called *forgetful functor* to the category of Hilbert spaces). This is crucial: by the Tannaka-Krein reconstruction theorem [32], the concrete monoidal category of (finite-dimensional) corepresentations essentially determines the compact quantum group. Note that knowing the representation theory of a compact quantum group as a monoidal category, comes down to knowing the *fusion rules together with the 6j-symbols*, see Remark 3.3.

Closely related to the notion of monoidally equivalent quantum groups, is the notion of a *unitary fiber functor* on a compact quantum group. Essentially, a unitary fiber functor gives *another concrete realization*, different from the tautological realization, of the representation theory of a compact quantum group.

In this paper, we choose not to use the abstract language of categories. We give ‘down-to-earth definitions’ of monoidally equivalent quantum groups and unitary fiber functors, see 3.1 and 3.7. This makes the construction of associated C^* -algebras and coactions straightforward. This concrete approach is well adapted to the language of corepresentations of compact quantum groups. In this way, using previous results of Banica [2, 3], we can show very easily as well the monoidal equivalence of the universal orthogonal and universal unitary quantum groups.

The results in this paper can be summarized as follows.

- In Section 2, we recall the theory of spectral subspaces [10, 18] and provide a simple proof for the multiplicity bound. We also introduce the notion of *quantum multiplicity* of an irreducible corepresentation in an ergodic coaction and this can be strictly larger than the ordinary multiplicity.
- In Theorem 3.9, we show that there is a natural bijective correspondence between certain ergodic coactions of compact quantum groups and unitary fiber functors. These coactions are called *of full quantum multiplicity*. These are precisely the ergodic coactions for which the crossed product is isomorphic with the compact operators, see [18]. They can also be described as the ergodic coactions for which the inequality between quantum multiplicity and quantum dimension, becomes an equality.
- In Section 4, we study the special case of unitary fiber functors preserving the dimension. This leads to a bijective correspondence with *unitary 2-cocycles on the dual, discrete, quantum group*. The ideas for this section come from the work of Wassermann [28].
- In Sections 5, we establish the monoidal equivalence between the universal orthogonal quantum groups $A_o(F)$. Recall that, for any $F \in GL(n, \mathbb{C})$ satisfying $F\bar{F} = \pm 1$, one defines the compact quantum group $A_o(F)$ as the universal quantum group generated by the coefficients of a unitary n by n matrix U with relations $U = F\bar{U}F^{-1}$. The comultiplication on $A_o(F)$ is (uniquely) defined in such a way that U becomes a corepresentation. We show that $A_o(F_1)$ is monoidally equivalent with $A_o(F_2)$ if and only if the signs of the $F_i\bar{F}_i$ agree and $\text{Tr}(F_1^*F_1) = \text{Tr}(F_2^*F_2)$. In particular, if $0 < q \leq 2 - \sqrt{3}$, there is a continuous family of non-isomorphic $A_o(F)$ monoidally equivalent with $SU_q(2)$.
- In Section 6, we prove similar results for the universal unitary quantum groups $A_u(F)$, defined as the universal quantum group generated by the coefficients of a unitary n by n matrix U with the relation that $F\bar{U}F^{-1}$ is unitary. Again, the comultiplication is defined such that U becomes a corepresentation. We show that the quantum dimension of U , i.e. $\sqrt{\text{Tr}(F^*F)\text{Tr}((F^*F)^{-1})}$, is a complete invariant for the $A_u(F)$ up to monoidal equivalence.

- Using the previous results, we obtain a complete classification of the ergodic coactions of full quantum multiplicity of $A_o(F)$ and $A_u(F)$, as well as a computation of the 2-cohomology of their duals (Corollary 5.9). In particular, we construct ergodic coactions of $SU_q(2)$ such that the multiplicity of the fundamental corepresentations is arbitrarily large (Corollary 5.8).

In the theory of Hopf algebras, ergodic coactions of full quantum multiplicity correspond to *Hopf-Galois extensions*. In this algebraic setting, several results related to ours have been obtained. The relation between Hopf-Galois extensions and fiber functors is due to Ulbrich [24] and the relation between monoidal equivalence of Hopf algebras and Hopf-bi-Galois extensions has been established by Schauenburg [22]. Fiber functors preserving the dimension and 2-cocycles have been studied by Etingof and Gelaki [15]. The main difference between these Hopf algebraic results and our work, lies in dealing with the *-structure and positivity. In a sense, we are dealing with the real forms of (certain) Hopf algebras. This allows us to construct Hilbert space representations and C*-algebras. The compatibility of fiber functors with *-structures is a severe restriction. Indeed, there exist many fiber functors on the representation category of $SU(2)$ (see [11]), but the forgetful functor is the only one compatible with the *-structure.

After completion of a first version of this paper – signed by the last two authors – the first author joined the project and his preprint [6] was taken into account, yielding the current paper as a final result.

1 Preliminaries

Notation 1.1. Consider a subset S of a C*-algebra. We denote by $\langle S \rangle$ the linear span of S and by $[S]$ the closed linear span of S .

The symbol \otimes denotes *tensor products* of Hilbert spaces, *minimal* tensor products of C*-algebras as well as algebraic tensor products of *-algebras.

If A is a *-algebra and $U \in M_{n_1, n_2}(\mathbb{C}) \otimes A$, we denote by \bar{U} the matrix $\bar{U}_{ij} = U_{ij}^*$.

We make use of the *leg numbering notation*. For instance, if $v \in A \otimes B$, then v_{12} denotes the element in $A \otimes B \otimes C$ defined by $v_{12} = v \otimes 1$. We analogously use the notations v_{13} , v_{23} , etc.

Definition 1.2. A *compact quantum group* $\mathbb{G} = (A, \Delta)$ consists of a unital C*-algebra A together with a unital *-homomorphism $\Delta : A \rightarrow A \otimes A$ satisfying the coassociativity relation

$$(\iota \otimes \Delta)\Delta = (\Delta \otimes \iota)\Delta$$

and the cancellation properties

$$[\Delta(A)(A \otimes 1)] = A \otimes A = [\Delta(A)(1 \otimes A)].$$

If $\mathbb{G} = (A, \Delta)$ is a compact quantum group, there exists a unique state h on A which is invariant under the comultiplication:

$$(\iota \otimes h)\Delta(a) = (h \otimes \iota)\Delta(a) = h(a)1$$

for all $a \in A$. We call h the *Haar state* of \mathbb{G} .

Definition 1.3. Let H be a Hilbert space. A *unitary corepresentation* v of \mathbb{G} on H is a unitary element of $M(\mathcal{K}(H) \otimes A)$ satisfying $(\iota \otimes \Delta)(v) = v_{12}v_{13}$. The dimension of the underlying Hilbert space H is called the dimension of v and denoted by $\dim v$.

The tensor product of the unitary corepresentations v and w is defined by

$$v \oplus w := v_{13}w_{23}.$$

Note that the corepresentations $v \oplus w$ and $w \oplus v$ are in general not unitarily equivalent. This is a crucial feature of quantum groups.

Notation 1.4. Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. Given two unitary corepresentations $v \in M(\mathcal{K}(H) \otimes A)$ and $w \in M(\mathcal{K}(K) \otimes A)$, we denote by $\text{Mor}(v, w)$ the intertwiners between v and w :

$$\text{Mor}(v, w) = \{S \in B(H, K) \mid (S \otimes 1)v = w(S \otimes 1)\}.$$

Terminology 1.5. A unitary corepresentation $v \in M(\mathcal{K}(H) \otimes A)$ is called *irreducible* if $\text{Mor}(v, v) = \mathbb{C}1$. A unitary corepresentation $w \in M(\mathcal{K}(L) \otimes A)$ is called *unitarily equivalent* to v if $\text{Mor}(v, w)$ contains a unitary operator.

In this paper, all corepresentations are assumed to be unitary. Recall the following well known facts (see [30]). Every irreducible corepresentation of a compact quantum group is finite dimensional and every corepresentation decomposes as a direct sum of irreducible corepresentations.

Notation 1.6. Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. We denote by $\widehat{\mathbb{G}}$ the set of equivalence classes of irreducible corepresentations of \mathbb{G} and we choose unitary representatives $U^x \in B(H_x) \otimes A$ for all $x \in \widehat{\mathbb{G}}$. We denote by ϵ the (class of the) trivial corepresentation $1 \in A$.

For every $x \in \widehat{\mathbb{G}}$, there is a unique $\bar{x} \in \widehat{\mathbb{G}}$ such that $\text{Mor}(\epsilon, x \oplus \bar{x}) \neq 0$. The irreducible corepresentation \bar{x} is called the *adjoint* of x . Take now $t \in \text{Mor}(\epsilon, x \oplus \bar{x})$, $t \neq 0$ and define the antilinear map

$$S_t : H_x \rightarrow H_{\bar{x}} : \xi \mapsto (\xi^* \otimes 1)t.$$

Define $Q_x := S_t^* S_t$ and normalize t in such a way that $\text{Tr}(Q_x) = \text{Tr}(Q_x^{-1})$. This uniquely determines Q_x and fixes t up to a number of modulus 1. If we take the unique $\tilde{t} \in \text{Mor}(\epsilon, \bar{x} \oplus x)$ such that $(t^* \otimes 1)(1 \otimes \tilde{t}) = 1$, then $S_{\tilde{t}} = S_t^{-1}$ and $Q_{\bar{x}} = (S_t S_t^*)^{-1}$.

Notation 1.7. The *quantum dimension* of $x \in \widehat{\mathbb{G}}$ is defined as $\text{Tr}(Q_x)$ and denoted by $\dim_q(x)$.

Observe that $t^* t = \dim_q(x)1$ and that $\dim_q(\bar{x}) = \dim_q(x) \geq \dim(x)$. The orthogonality relations can then be written as follows (see [30]): for $\xi \in H_x$, $\eta \in H_y$,

$$(\iota \otimes h)(U^x(\xi \eta^* \otimes 1)(U^y)^*) = \frac{\delta_{x,y} 1}{\dim_q(x)} \langle Q_x \xi, \eta \rangle. \quad (1.1)$$

Notation 1.8. Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. We denote by \mathcal{A} the set of coefficients of finite dimensional corepresentations of \mathbb{G} . Hence,

$$\mathcal{A} = \langle (\omega_{\xi, \eta} \otimes \iota)(U^x) \mid x \in \widehat{\mathbb{G}}, \xi, \eta \in H_x \rangle$$

Then, \mathcal{A} is a unital dense $*$ -subalgebra of A . Restricting Δ to \mathcal{A} , \mathcal{A} becomes a Hopf $*$ -algebra.

Terminology 1.9. Let \mathcal{B} be a unital $*$ -algebra. A linear functional $\omega : \mathcal{B} \rightarrow \mathbb{C}$ is said to be a *faithful state* on \mathcal{B} if $\omega(1) = 1$ and if $\omega(aa^*) \geq 0$ for all $a \in \mathcal{B}$, with equality holding if and only if $a = 0$.

Observe that the Haar state of a compact quantum group (A, Δ) is a faithful state on the underlying Hopf $*$ -algebra \mathcal{A} .

The dual (discrete) quantum group of (A, Δ) , can be defined as

$$\hat{A} = \bigoplus_{x \in \widehat{\mathbb{G}}} B(H_x) \quad \text{with} \quad \hat{\Delta}(a)S = Sa \quad \text{for all } a \in \hat{A}, S \in \text{Mor}(x, y \oplus z).$$

There is, of course, another way to define $(\hat{A}, \hat{\Delta})$, using the left regular corepresentation of (A, Δ) and the theory of multiplicative unitaries. We only need this in Section 4, see Notation 4.6.

2 Spectral subspaces and quantum multiplicity

In this section, we give a brief overview of the general theory of ergodic coactions of compact quantum groups. We study spectral subspaces and prove in particular that they are finite dimensional. The results in this section are well known (see [16] for the classical case of compact groups and [10, 18, 23] for compact quantum groups). We give a short presentation for the convenience of the reader.

Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. Recall that $\widehat{\mathbb{G}}$ denotes the set of equivalence classes of irreducible corepresentations of \mathbb{G} and that we chose unitary representatives $U^x \in B(H_x) \otimes A$ for all $x \in \widehat{\mathbb{G}}$. We have $(\iota \otimes \Delta)(U^x) = U_{12}^x U_{13}^x$.

Definition 2.1. Let B be a unital C^* -algebra. A (right) coaction of (A, Δ) on B is a unital $*$ -homomorphism $\delta : B \rightarrow B \otimes A$ satisfying

$$(\delta \otimes \iota)\delta = (\iota \otimes \Delta)\delta \quad \text{and} \quad [\delta(B)(1 \otimes A)] = B \otimes A.$$

The coaction δ is said to be ergodic if the fixed point algebra $B^\delta := \{x \in B \mid \delta(x) = x \otimes 1\}$ equals $\mathbb{C}1$.

Remark 2.2. If $\delta : B \rightarrow B \otimes A$ is an ergodic coaction of (A, Δ) on B there is a unique state ω on B which is invariant under δ , given by $\omega(b)1 = (\iota \otimes h)\delta(b)$.

In what follows, we fix a coaction $\delta : B \rightarrow B \otimes A$ of a compact quantum group $\mathbb{G} = (A, \Delta)$.

Definition 2.3. Let $x \in \widehat{\mathbb{G}}$. We define the spectral subspace associated with x by

$$K_x = \{X \in \overline{H_x} \otimes B \mid (\iota \otimes \delta)(X) = X_{12} U_{13}^x\}.$$

Defining $\text{Hom}(H_x, B) = \{S : H_x \rightarrow B \mid S \text{ linear and } \delta(S\xi) = (S \otimes \iota)(U^x(\xi \otimes 1))\}$, we have $K_x \cong \text{Hom}(H_x, B)$, associating to every $X \in K_x$ the operator $S_X : H_x \rightarrow B : \xi \mapsto X(\xi \otimes 1)$.

Definition 2.4. We define \mathcal{B} as the subspace of B generated by the spectral subspaces, i.e.

$$\mathcal{B} := \langle X(\xi \otimes 1) \mid x \in \widehat{\mathbb{G}}, X \in K_x, \xi \in H_x \rangle.$$

Observe that \mathcal{B} is a dense unital $*$ -subalgebra of B and that the restriction $\delta : \mathcal{B} \rightarrow \mathcal{B} \otimes A$ defines a coaction of the Hopf $*$ -algebra (A, Δ) on \mathcal{B} .

Terminology 2.5. A coaction $\delta : B \rightarrow B \otimes A$ of (A, Δ) on B is said to be *universal* if B is the universal enveloping C^* -algebra of \mathcal{B} . It is said to be *reduced* if the conditional expectation $(\iota \otimes h)\delta$ of B on B^δ is faithful.

Remark 2.6. Observe that an ergodic coaction is reduced if and only if the unique invariant state is faithful.

In the special case where $B = A$ and $\delta = \Delta$, the $*$ -algebra \mathcal{B} coincides with the underlying Hopf $*$ -algebra $\mathcal{A} \subset A$ consisting of coefficients of finite-dimensional corepresentations. So, we obtain the usual notions: a compact quantum group (A, Δ) is said to be universal if A is the universal enveloping C^* -algebra of \mathcal{A} and reduced if the Haar state is faithful on A . Clearly, any compact quantum group has its universal and reduced companion. In the case where (A, Δ) is the dual of a discrete group, these notions coincide with the full, resp. reduced group C^* -algebra.

A compact quantum group is said to be *co-amenable* if its universal and reduced companion coincide. Equivalently, a compact quantum group is co-amenable if we have on the same C^* -algebra a bounded co-unit and a faithful Haar state. It is then clear that a coaction of a co-amenable compact quantum group is always both universal and reduced. Examples of co-amenable compact quantum groups include $SU_q(n)$ and other q -deformations of compact Lie groups.

Fix now an *ergodic* coaction $\delta : B \rightarrow B \otimes A$ of a compact quantum group (A, Δ) on a unital C^* -algebra B . Denote by ω the unique invariant state on B .

For every $X, Y \in K_x$, $XY^* \in B^\delta = \mathbb{C}1$. So, for $x \in \widehat{\mathbb{G}}$, the spectral subspace K_x is a Hilbert space with scalar product $\langle X, Y \rangle 1 = XY^*$.

Terminology 2.7. Let δ be an ergodic coaction of (A, Δ) on B . The dimension of the Hilbert space K_x is called the *multiplicity of x in δ* and denoted by $\text{mult}(x)$.

Define, for every $x \in \widehat{\mathbb{G}}$, the element $X^x \in B(H_x, \overline{K_x}) \otimes B$ such that $(X^x)^*(\overline{Y} \otimes 1) = Y^*$ for all $Y \in K_x$. Observe that $X^x(X^x)^* = 1$. Therefore, $(X^x)^*X^x \in B(H_x) \otimes B$ is a projection.

Let $x \in \widehat{\mathbb{G}}$. Take $t \in \text{Mor}(\epsilon, \overline{x} \oplus x)$, normalized in such a way that $t^*t = \dim_q(x)$. Define the antilinear map

$$T_t : K_x \rightarrow K_{\overline{x}} : T_t(Y) = (t^* \otimes 1)(1 \otimes Y^*).$$

Since t is fixed up to a number of modulus one, $L_x := T_t^*T_t$ is a well defined positive element of $B(K_x)$.

Definition 2.8. We put $\text{mult}_q(x) := \sqrt{\text{Tr}(L_x) \text{Tr}(L_{\overline{x}})}$ and call $\text{mult}_q(x)$ the *quantum multiplicity of x in δ* . We prove in Theorem 2.9 that $\text{mult}_q(x) \leq \dim_q(x)$ for all $x \in \widehat{\mathbb{G}}$. If equality holds for all $x \in \widehat{\mathbb{G}}$, we say that δ is of *full quantum multiplicity*.

Theorem 2.9. Let $\delta : B \rightarrow B \otimes A$ be an ergodic coaction of a compact quantum group $\mathbb{G} = (A, \Delta)$ on a unital C^* -algebra B . For every irreducible corepresentation $x \in \widehat{\mathbb{G}}$,

$$\text{mult}(x) \leq \text{mult}_q(x) \leq \dim_q(x).$$

With X^x defined as above, the ergodic coaction is of full quantum multiplicity if and only if X^x is unitary for all $x \in \widehat{\mathbb{G}}$.

Proof. Let $x \in \widehat{\mathbb{G}}$ and take $t \in \text{Mor}(\epsilon, \overline{x} \oplus x)$, normalized in such a way that $t^*t = \dim_q(x)$. If we take the unique $\tilde{t} \in \text{Mor}(\epsilon, x \oplus \overline{x})$ satisfying $(1 \otimes t^*)(\tilde{t} \otimes 1) = 1$, we have $\tilde{t}^*\tilde{t} = \dim_q(\overline{x})$ and hence, it is clear that $T_{\tilde{t}} = T_t^{-1}$ and $L_{\overline{x}} = (T_t T_t^*)^{-1}$. So,

$$\text{mult}_q(x) = \sqrt{\text{Tr}(T_t T_t^*) \text{Tr}((T_t T_t^*)^{-1})} \geq \dim(T_t) = \text{mult}(x),$$

for all $x \in \widehat{\mathbb{G}}$. By definition

$$\langle L_x X, Y \rangle 1 = \langle (t^* \otimes 1)(1 \otimes Y^*), (t^* \otimes 1)(1 \otimes X^*) \rangle 1 = (t^* \otimes 1)(1 \otimes Y^* X)(t \otimes 1). \quad (2.1)$$

Since $t^*(1 \otimes a)t = \text{Tr}(Q_x^{-1}a)$ for all $a \in B(H_x)$, we conclude that

$$\text{Tr}(L_x)1 = (\text{Tr}(Q_x^{-1} \cdot) \otimes \iota)((X^x)^*X^x).$$

Since $(X^x)^*X^x$ is a projection, it follows that $\text{Tr}(L_x) \leq \text{Tr}(Q_x^{-1}) = \dim_q(x)$ for all $x \in \widehat{\mathbb{G}}$. Applying this inequality to x and \overline{x} , we conclude that $\text{mult}_q(x) \leq \dim_q(x)$ for all $x \in \widehat{\mathbb{G}}$.

Moreover, $\text{mult}_q(x) = \dim_q(x)$ for all $x \in \widehat{\mathbb{G}}$ if and only if $(\text{Tr}(Q_x^{-1} \cdot) \otimes \iota)((X^x)^*X^x) = \text{Tr}(Q_x^{-1})1$ for all $x \in \widehat{\mathbb{G}}$. This last statement holds if and only if $(X^x)^*X^x = 1$ for all $x \in \widehat{\mathbb{G}}$, i.e. when X^x is unitary for all $x \in \widehat{\mathbb{G}}$. \square

It is also straightforward to show that the invariant state ω is a KMS state, at least for universal and for reduced ergodic coactions.

Proposition 2.10. If the ergodic coaction δ of (A, Δ) on B is either universal or reduced, the invariant state ω is a KMS state. The elements of the dense $*$ -subalgebra $\mathcal{B} \subset B$ (Definition 2.4) are analytic with respect to the modular group, given by

$$\sigma_t^\omega(Y(\xi \otimes 1)) = (L_x^{-it}Y)(Q_x^{it}\xi \otimes 1) \quad \text{for all } x \in \widehat{\mathbb{G}}, Y \in K_x, \xi \in H_x.$$

Proof. Let $y, z \in \widehat{\mathbb{G}}$ and $Y \in K_y, Z \in K_z$. One verifies that $(U^z)^*((\iota \otimes \omega)(Z^*Y) \otimes 1)U^y = (\iota \otimes \omega)(Z^*Y) \otimes 1$. Hence, $(\iota \otimes \omega)(Z^*Y)$ equals 0 if $y \neq z$ and is scalar if $y = z$. Applying ω to (2.1), we conclude that

$$(\iota \otimes \omega)(Z^*Y) = \frac{\delta_{y,z} 1}{\dim_q(y)} \langle L_y Y, Z \rangle.$$

If moreover $\xi \in H_y, \eta \in H_z$ and if we put $a = Y(\xi \otimes 1)$ and $b = Z(\eta \otimes 1)$, we get

$$\omega(b^*a) = \frac{\delta_{y,z}}{\dim_q(y)} \langle \xi, \eta \rangle \langle L_y Y, Z \rangle. \quad (2.2)$$

Using (1.1), one checks that

$$\omega(ab^*) = (\iota \otimes \omega)(Y(\xi \eta^* \otimes 1)Z^*) = \frac{\delta_{y,z} 1}{\dim_q(y)} \langle Q_y \xi, \eta \rangle \langle Y, Z \rangle.$$

As a linear space, $\mathcal{B} \cong \bigoplus_{x \in \widehat{\mathbb{G}}} (K_x \otimes H_x)$. So, we can define linear maps $\sigma_t^\omega : \mathcal{B} \rightarrow \mathcal{B}$ by the formula

$$\sigma_t^\omega(Y(\xi \otimes 1)) := L_y^{-it}(Y)(Q_y^{it} \xi \otimes 1).$$

It is clear that (σ_t^ω) is a one-parameter group of linear isomorphisms of \mathcal{B} . Observe that all elements of \mathcal{B} are analytic with respect to (σ_t^ω) and that $\omega(\sigma_t^\omega(a)b^*) = \omega(b^*a) = \omega(a\sigma_t^\omega(b)^*)$ for all $a, b \in \mathcal{B}$. Since (2.2) implies that ω is faithful on \mathcal{B} , it follows that $\sigma_t^\omega : \mathcal{B} \rightarrow \mathcal{B}$ is multiplicative and that $\sigma_t^\omega(a)^* = \sigma_{-t}^\omega(a^*)$ for all $a \in \mathcal{B}$. Standard complex analysis allows to conclude that the σ_t^ω are *-automorphisms of \mathcal{B} . It is also clear that ω is invariant under σ_t^ω .

If δ is a universal coaction, the one-parameter group (σ_t^ω) extends to B by universality. If δ is a reduced coaction, we can extend σ_t^ω to B because ω is invariant under σ_t^ω and ω is faithful on B . In both cases, it follows that (σ_t^ω) satisfies the KMS condition with respect to ω and so, ω is a KMS state. \square

Finally observe that ω is a trace if and only if $L_x = 1$ and $Q_x = 1$ for all $x \in \widehat{\mathbb{G}}$ with $K_x \neq 0$.

3 Commuting coactions and monoidal equivalence of quantum groups

Our main goal in this section is to show the relation between ergodic coactions of full quantum multiplicity and monoidal equivalence of compact quantum groups. In Sections 5 and 6, we shall give examples of monoidally equivalent compact quantum groups, giving rise to new examples of ergodic coactions.

The relation between ergodic coactions of full quantum multiplicity and monoidal equivalence allows us to classify completely such coactions for the unitary and orthogonal quantum groups $A_u(F)$ and $A_o(F)$, in particular for $SU_q(2)$.

Definition 3.1. Two compact quantum groups $\mathbb{G} = (A, \Delta)$ and $\mathbb{G}_2 = (A_2, \Delta_2)$ are said to be *monoidally equivalent* if there exists a bijection $\varphi : \widehat{\mathbb{G}} \rightarrow \widehat{\mathbb{G}}_2$ satisfying $\varphi(\epsilon) = \epsilon$, together with linear isomorphisms

$$\varphi : \text{Mor}(x_1 \oplus \cdots \oplus x_r, y_1 \oplus \cdots \oplus y_k) \rightarrow \text{Mor}(\varphi(x_1) \oplus \cdots \oplus \varphi(x_r), \varphi(y_1) \oplus \cdots \oplus \varphi(y_k))$$

satisfying the following conditions:

$$\begin{aligned} \varphi(1) &= 1 & \varphi(S \otimes T) &= \varphi(S) \otimes \varphi(T) \\ \varphi(S^*) &= \varphi(S)^* & \varphi(ST) &= \varphi(S)\varphi(T) \end{aligned} \quad (3.1)$$

whenever the formulas make sense. In the first formula, we consider $1 \in \text{Mor}(x, x) = \text{Mor}(x, x \oplus \epsilon) = \text{Mor}(x, \epsilon \oplus x)$. Such a collection of maps φ is called a *monoidal equivalence* between \mathbb{G} and \mathbb{G}_2 .

Remark 3.2. To define a monoidal equivalence between \mathbb{G} and \mathbb{G}_2 , it suffices to define a bijection $\varphi : \widehat{\mathbb{G}} \rightarrow \widehat{\mathbb{G}}_2$ satisfying $\varphi(\epsilon) = \epsilon$, together with linear isomorphisms $\varphi : \text{Mor}(x, y_1 \oplus \cdots \oplus y_k) \rightarrow \text{Mor}(\varphi(x), \varphi(y_1) \oplus \cdots \oplus \varphi(y_k))$ for $k = 1, 2, 3$, satisfying

$$\varphi(1) = 1 \quad (3.2)$$

$$\varphi(S)^* \varphi(T) = \varphi(S^*T) \quad \text{for all } S \in \text{Mor}(a, x \oplus y), T \in \text{Mor}(b, x \oplus y) \quad (3.3)$$

$$\varphi((S \otimes 1)T) = (\varphi(S) \otimes 1)\varphi(T) \quad \text{for all } T \in \text{Mor}(a, b \oplus z), S \in \text{Mor}(b, x \oplus y) \quad (3.4)$$

$$\varphi((1 \otimes S)T) = (1 \otimes \varphi(S))\varphi(T) \quad \text{for all } T \in \text{Mor}(a, x \oplus b), S \in \text{Mor}(b, y \oplus z) \quad (3.5)$$

Indeed, such a φ admits a unique extension to a monoidal equivalence. Again, (3.2) should be valid for $1 \in \text{Mor}(x, x) = \text{Mor}(x, x \oplus \epsilon) = \text{Mor}(x, \epsilon \oplus x)$.

Remark 3.3. Observe that the existence of the linear isomorphisms $\varphi : \text{Mor}(a, b \oplus c) \rightarrow \text{Mor}(\varphi(a), \varphi(b) \oplus \varphi(c))$ only says that \mathbb{G} and \mathbb{G}_2 have the *same fusion rules*. Adding (3.2)–(3.5) means that \mathbb{G} and \mathbb{G}_2 moreover have the *same 6j-symbols* (see [12]). Indeed, taking orthonormal bases for all $\text{Mor}(a, b \oplus c)$, we can write two natural orthonormal bases for $\text{Mor}(a, x \oplus y \oplus z)$, one given by elements $(S \otimes 1)T$, the other given by elements $(1 \otimes S)T$. The coefficients of the transition unitary between both orthonormal bases are called the 6j-symbols of \mathbb{G} .

Remark 3.4. As we shall see in Sections 5 and 6, there are natural examples of monoidal equivalences where $\dim(\varphi(x)) \neq \dim(x)$. On the other hand, it is clear that $\dim_q(\varphi(x)) = \dim_q(x)$ for all $x \in \mathbb{G}$ and all monoidal equivalences φ . We shall see in Section 6 that for a certain class of compact quantum groups (the universal unitary ones), this equality of quantum dimension is the only constraint for monoidal equivalence.

Remark 3.5. It is clear that a monoidal equivalence φ in the sense of Definition 3.1, defines a monoidal equivalence in the usual sense (preserving the $*$ -operation), between the monoidal categories of finite dimensional corepresentations of \mathbb{G} and \mathbb{G}_2 ([20]). Moreover, this monoidal equivalence is uniquely determined up to isomorphism. We prefer to work with the ‘concrete’ data of Definition 3.1, avoiding all kinds of identifications.

Notation 3.6. If two compact quantum groups $\mathbb{G} = (A, \Delta)$ and $\mathbb{G}_2 = (A_2, \Delta_2)$ are monoidally equivalent, we write $\mathbb{G} \underset{\text{mon}}{\sim} \mathbb{G}_2$

Closely related to the notion of monoidal equivalence, is the following notion of *unitary fiber functor* (see Proposition 3.12 for the relation between both notions).

Definition 3.7. Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group. A *unitary fiber functor* associates to every $x \in \widehat{\mathbb{G}}$ a finite dimensional Hilbert space $H_{\varphi(x)}$ and consists further of linear maps

$$\varphi : \text{Mor}(x_1 \oplus \cdots \oplus x_r, y_1 \oplus \cdots \oplus y_k) \rightarrow \text{B}(H_{\varphi(x_1)} \otimes \cdots \otimes H_{\varphi(x_r)}, H_{\varphi(y_1)} \otimes \cdots \otimes H_{\varphi(y_k)})$$

satisfying equations (3.1) in Definition 3.1.

Remark 3.8. We make a remark analogous to 3.2. To define a unitary fiber functor on \mathbb{G} , it suffices to associate to every $x \in \widehat{\mathbb{G}}$ a finite-dimensional Hilbert space $H_{\varphi(x)}$, with $H_{\varphi(\epsilon)} = \mathbb{C}$ and to define linear maps $\varphi : \text{Mor}(x, y_1 \oplus \cdots \oplus y_k) \rightarrow \text{B}(H_{\varphi(x)}, H_{\varphi(y_1)} \otimes \cdots \otimes H_{\varphi(y_k)})$ for $k = 1, 2, 3$, satisfying (3.2) – (3.5) as well as the non-degenerateness assumption

$$\{\varphi(S)\xi \mid a \in \widehat{\mathbb{G}}, S \in \text{Mor}(a, b \oplus c), \xi \in H_{\varphi(a)}\} \text{ is total in } H_{\varphi(b)} \otimes H_{\varphi(c)}$$

for all $b, c \in \widehat{\mathbb{G}}$.

Moreover, it follows from Proposition 3.12 below, that a unitary fiber functor φ on \mathbb{G} naturally defines a compact quantum group \mathbb{G}_2 such that φ becomes a monoidal equivalence between \mathbb{G} and \mathbb{G}_2 .

Theorem 3.9. Consider a compact quantum group $\mathbb{G} = (A, \Delta)$ and let φ be a unitary fiber functor on \mathbb{G} .

- There exists a unique unital $*$ -algebra \mathcal{B} equipped with a faithful state ω and unitary elements $X^x \in \text{B}(H_x, H_{\varphi(x)}) \otimes \mathcal{B}$ for all $x \in \widehat{\mathbb{G}}$, satisfying
 1. $X_{13}^y X_{23}^z (S \otimes 1) = (\varphi(S) \otimes 1) X^x$ for all $S \in \text{Mor}(x, y \oplus z)$,
 2. the matrix coefficients of the X^x form a linear basis of \mathcal{B} ,
 3. $(\iota \otimes \omega)(X^x) = 0$ if $x \neq \epsilon$.
- There exists a unique coaction $\delta : \mathcal{B} \rightarrow \mathcal{B} \otimes A$ satisfying

$$(\iota \otimes \delta)(X^x) = X_{12}^x U_{13}^x$$

for all $x \in \widehat{\mathbb{G}}$.

- The state ω is invariant under δ . Denoting by B_r the C^* -algebra generated by \mathcal{B} in the GNS-representation associated with ω and denoting by B_u the universal enveloping C^* -algebra of \mathcal{B} , the coaction δ admits a unique extension to a coaction on B_r , resp. B_u .

These coactions are reduced, resp. universal and they are ergodic and of full quantum multiplicity.

- Every reduced, resp. universal, ergodic coaction of full quantum multiplicity, arises in this way from a unitary fiber functor.

Proof. Let φ be a unitary fiber functor on $\mathbb{G} = (A, \Delta)$. Define the vector space $\mathcal{B} = \bigoplus_{x \in \widehat{\mathbb{G}}} B(H_x, H_{\varphi(x)})^*$. We shall turn this vector space into a $*$ -algebra.

Define natural elements $X^x \in B(H_x, H_{\varphi(x)}) \otimes \mathcal{B}$ by $(\omega_x \otimes \iota)(X^x) = (\delta_{x,y} \omega_x)_{y \in \widehat{\mathbb{G}}}$ for all $\omega_x \in B(H_x, H_{\varphi(x)})^*$. By definition, the coefficients of the X^x form a linear basis of \mathcal{B} . Hence, it suffices to define a product and an involution on the level of the X^x .

It is clear that there exists a unique bilinear multiplication map $\mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ such that

$$X_{13}^y X_{23}^z (S \otimes 1) = (\varphi(S) \otimes 1) X^x \quad \text{for all } S \in \text{Mor}(x, y \oplus z).$$

But then

$$(X_{14}^a X_{24}^b) X_{34}^c ((S \otimes 1) T \otimes 1) = (\varphi(S) \otimes 1 \otimes 1) X_{13}^y X_{23}^c (T \otimes 1) = ((\varphi(S) \otimes 1) \varphi(T) \otimes 1) X^x = (\varphi((S \otimes 1) T) \otimes 1) X^x$$

for all $S \in \text{Mor}(y, a \oplus b)$, $T \in \text{Mor}(x, y \oplus c)$. Since intertwiners of the form $(S \otimes 1) T$ linearly span $\text{Mor}(x, a \oplus b \oplus c)$, we conclude that

$$(X_{14}^a X_{24}^b) X_{34}^c (S \otimes 1) = (\varphi(S) \otimes 1) X^x$$

for all $S \in \text{Mor}(x, a \oplus b \oplus c)$. Analogously,

$$X_{14}^a (X_{24}^b X_{34}^c) (S \otimes 1) = (\varphi(S) \otimes 1) X^x$$

for all $S \in \text{Mor}(x, a \oplus b \oplus c)$. This proves the associativity of the product on \mathcal{B} . It is clear that X^ϵ provides the unit element of \mathcal{B} .

Observe also that

$$(\varphi(S)^* \otimes 1) X_{13}^x X_{23}^y = X^z (S^* \otimes 1) \tag{3.6}$$

for all $S \in \text{Mor}(z, x \oplus y)$.

We define an antilinear map $b \mapsto b^*$ on \mathcal{B} such that

$$(X^x)^*_{13} (\varphi(t) \otimes 1) = X_{23}^{\bar{x}} (t \otimes 1) \tag{3.7}$$

for all $x \in \widehat{\mathbb{G}}$, $t \in \text{Mor}(\epsilon, x \oplus \bar{x})$. This antilinear map is well defined: taking $t \in \text{Mor}(\epsilon, x \oplus \bar{x})$ and $\tilde{t} \in \text{Mor}(\epsilon, \bar{x} \oplus x)$, normalized in such a way that $(t^* \otimes 1)(1 \otimes \tilde{t}) = 1$, we define

$$((\omega_{\xi, \eta} \otimes \iota)(X^x))^* := (\omega_{(\xi^* \otimes 1)t, (1 \otimes \eta^*)\varphi(\tilde{t})} \otimes \iota)(X^{\bar{x}})$$

for all $\xi \in H_x$ and $\eta \in H_{\varphi(x)}$.

For $\xi \in H_x, \eta \in H_{\varphi(x)}$, we compute

$$\begin{aligned} (((\omega_{\xi, \eta} \otimes \iota)(X^x))^*)^* &= ((\omega_{(\xi^* \otimes 1)t, (1 \otimes \eta^*)\varphi(\tilde{t})} \otimes \iota)(X^{\bar{x}}))^* \\ &= (\omega_{(((\xi^* \otimes 1)t)^* \otimes 1)\tilde{t}, (1 \otimes ((1 \otimes \eta^*)\varphi(\tilde{t}))^*)\varphi(t)} \otimes \iota)(X^x) \\ &= (\omega_{\xi, \eta} \otimes \iota)(X^x), \end{aligned}$$

in the last step using our particular choice of t and \tilde{t} .

We also get

$$(t^* \otimes 1)(X_{23}^{\bar{x}})^* = (\varphi(t)^* \otimes 1) X_{13}^x.$$

Because

$$(X^x(X^x)^*)_{13}(\varphi(t) \otimes 1) = X_{13}^x X_{23}^{\bar{x}}(t \otimes 1) = \varphi(t) \otimes 1$$

and because, by (3.6),

$$(t^* \otimes 1)((X^{\bar{x}})^* X^{\bar{x}})_{23} = t^* \otimes 1,$$

the elements X^x are unitaries.

Since for all $x, y, z \in \widehat{\mathbb{G}}$ and $S \in \text{Mor}(z, x \oplus y)$,

$$(X_{13}^x X_{23}^y (S \otimes 1))^* = ((\varphi(S) \otimes 1) X^z)^* = (X^z)^* (\varphi(S)^* \otimes 1) = (S^* \otimes 1) (X_{23}^y)^* (X_{13}^x)^*$$

by (3.6) and the fact that the X^x are unitary, our involution is anti-multiplicative. We conclude that \mathcal{B} a *-algebra.

Denote by ω the linear functional $\omega : \mathcal{B} \rightarrow \mathbb{C}$ given by $\omega(1) = 1$ and $(\iota \otimes \omega)(X^x) = 0$ for all $x \neq \epsilon$. We show that ω is a faithful state on \mathcal{B} .

Let $x, y \in \widehat{\mathbb{G}}$. Take $t \in \text{Mor}(\epsilon, x \oplus \bar{x})$ such that $t^* t = \dim_q(x)$. Take $\tilde{t} \in \text{Mor}(\epsilon, \bar{x} \oplus x)$ such that $(t^* \otimes 1)(1 \otimes \tilde{t}) = 1$. Then,

$$(\omega_{\mu, \rho} \otimes \iota)(X^y)(\omega_{\xi, \eta} \otimes \iota)(X^x)^* = (\omega_{\mu \otimes (\xi^* \otimes 1)t, \rho \otimes (1 \otimes \eta^*)\varphi(\tilde{t})} \otimes \iota)(X_{13}^y X_{23}^{\bar{x}}).$$

We conclude that

$$\begin{aligned} \omega((\omega_{\mu, \rho} \otimes \iota)(X^y)(\omega_{\xi, \eta} \otimes \iota)(X^x)^*) &= \delta_{x, y} \frac{1}{\dim_q(x)} \langle \mu \otimes (\xi^* \otimes 1)t, t \rangle \langle \varphi(t), \rho \otimes (1 \otimes \eta^*)\varphi(\tilde{t}) \rangle \\ &= \delta_{x, y} \frac{1}{\dim_q(x)} \langle (\xi^* \otimes 1)t, (\mu^* \otimes 1)t \rangle \langle \rho, \eta \rangle \\ &= \delta_{x, y} \frac{1}{\dim_q(x)} \langle Q_x \mu, \xi \rangle \langle \rho, \eta \rangle. \end{aligned}$$

Choose orthonormal bases (f_i^x) for every space $H_{\varphi(x)}$. Any element $a \in \mathcal{B}$ admits a unique decomposition

$$a = \sum_{x, i} (\omega_{\xi_i^x, f_i^x} \otimes \iota)(X^x)$$

in terms of vectors $\xi_i^x \in H_x$. Then,

$$\omega(aa^*) = \sum_{x, i} \frac{1}{\dim_q(x)} \langle Q_x \xi_i^x, \xi_i^x \rangle.$$

It follows that $\omega(aa^*) \geq 0$ for all a and that $\omega(aa^*) = 0$ if and only if $\xi_i^x = 0$ for all x and i , i.e. if and only if $a = 0$.

The definition of the coaction $\delta : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ is obvious and it is clear that ω is invariant under δ . It follows that we can extend δ to coactions δ_r , resp. δ_u of (A, Δ) on B_r , resp. B_u . Moreover, $\omega(x)1 = (\iota \otimes h)\delta_r(x)$ for all $x \in B_r$ and analogously for $x \in B_u$. It follows that δ_r and δ_u are ergodic coactions. Given the unitary elements X^x and Theorem 2.9, it follows that δ_r and δ_u are of full quantum multiplicity.

By definition, the coaction δ_r on B_r is reduced and the coaction δ_u on B_u is universal. Indeed, the canonical *-subalgebra of B_u generated by the spectral subspaces for δ_u , is exactly \mathcal{B} .

It remains to show that any reduced, resp. universal ergodic coaction of full quantum multiplicity arises as above from a unitary fiber functor. Let $\delta : B \rightarrow B \otimes A$ be an ergodic coaction of full quantum multiplicity. Construct for $x \in \widehat{\mathbb{G}}$, the unitary elements $X^x \in B(H_x, \overline{K}_x) \otimes B$ as in Section 2. Define $H_{\varphi(x)} := \overline{K}_x$. Let $S \in \text{Mor}(x_1 \oplus \dots \oplus x_r, y_1 \oplus \dots \oplus y_k)$. The element

$$X_{1, k+1}^{y_1} \cdots X_{k, k+1}^{y_k} (S \otimes 1) (X_{1, r+1}^{x_1} \cdots X_{r, r+1}^{x_r})^*$$

is invariant under $\iota \otimes \delta$. So, we can define $\varphi(S)$ by the formula

$$X_{1, k+1}^{y_1} \cdots X_{k, k+1}^{y_k} (S \otimes 1) = (\varphi(S) \otimes 1) X_{1, r+1}^{x_1} \cdots X_{r, r+1}^{x_r}$$

for all $S \in \text{Mor}(x_1 \oplus \cdots \oplus x_r, y_1 \oplus \cdots \oplus y_k)$. It is clear that φ is a unitary fiber functor on \mathbb{G} .

Denote by \mathcal{B} the $*$ -subalgebra of B generated by the spectral subspaces of δ as defined in Section 2. By definition, \mathcal{B} is generated by the coefficients of the X^x . In order to show that \mathcal{B} is isomorphic to the $*$ -algebra defined by the unitary fiber functor φ , it suffices to show that the coefficients of the X^x form a linear basis of \mathcal{B} . But this follows immediately from (2.2). \square

Definition 3.10. Two unitary fiber functors φ and ψ on a compact quantum group \mathbb{G} are said to be *isomorphic* if there exist unitaries $u_x \in B(H_{\varphi(x)}, H_{\psi(x)})$ satisfying

$$\psi(S) = (u_{y_1} \otimes \cdots \otimes u_{y_k})\varphi(S)(u_{x_1}^* \otimes \cdots \otimes u_{x_r}^*)$$

for all $S \in \text{Mor}(x_1 \oplus \cdots \oplus x_r, y_1 \oplus \cdots \oplus y_k)$.

Proposition 3.11. *Let φ and ψ be unitary fiber functors on \mathbb{G} and denote by δ_φ , resp. δ_ψ the associated coactions on \mathcal{B}_φ , resp. \mathcal{B}_ψ . Then the following statements are equivalent.*

- *The fiber functors φ and ψ are isomorphic.*
- *There exists a $*$ -isomorphism $\pi : \mathcal{B}_\varphi \rightarrow \mathcal{B}_\psi$ satisfying $(\pi \otimes \iota)\delta_\varphi = \delta_\psi\pi$.*

Proof. Straightforward. \square

The following proposition follows immediately from the Tannaka-Krein reconstruction theorem, but we give a detailed statement for clarity. Its proof is completely analogous to the proof of Theorem 3.9.

Proposition 3.12. *Let φ be a unitary fiber functor on a compact quantum group $\mathbb{G} = (A, \Delta)$.*

- *There exists a unique universal compact quantum group (A_2, Δ_2) with underlying Hopf $*$ -algebra (A_2, Δ_2) and unitary corepresentations $U^{\varphi(x)} \in B(H_{\varphi(x)} \otimes A_2)$ satisfying*
 1. $U_{13}^{\varphi(y)} U_{23}^{\varphi(z)} (\varphi(S) \otimes 1) = (\varphi(S) \otimes 1) U^{\varphi(x)}$ for all $S \in \text{Mor}(x, y \oplus z)$,
 2. *the matrix coefficients of the $U^{\varphi(x)}$ form a linear basis of A_2 .*
- *$\{U^{\varphi(x)} \mid x \in \mathbb{G}\}$ is a complete set of irreducible corepresentations of (A_2, Δ_2) and φ is a monoidal equivalence of compact quantum groups.*

The following result is then a corollary of Theorem 3.9.

Proposition 3.13. *Consider two compact quantum groups $\mathbb{G} = (A, \Delta)$ and $\mathbb{G}_2 = (A_2, \Delta_2)$. Let $\varphi : \mathbb{G} \rightarrow \mathbb{G}_2$ be a monoidal equivalence. In particular, φ is a unitary fiber functor on \mathbb{G} .*

Denote by B_r , resp. B_u the C^ -algebras associated to φ as in Theorem 3.9, with dense $*$ -subalgebra \mathcal{B} . Denote by δ the corresponding coaction of (A, Δ) on \mathcal{B} . Denote by $X^x \in B(H_x, H_{\varphi(x)}) \otimes \mathcal{B}$ the unitaries generating \mathcal{B} .*

- *There is a unique coaction $\delta_2 : \mathcal{B} \rightarrow A_2 \otimes \mathcal{B}$ satisfying $(\iota \otimes \delta_2)(X^x) = U_{13}^{\varphi(x)} X_{23}^x$ for all $x \in \widehat{\mathbb{G}}$. The coaction δ_2 commutes with δ and extends to B_r , resp. B_u , yielding a reduced, resp. universal, ergodic coaction of full quantum multiplicity.*
- *Every pair of commuting reduced, resp. universal, ergodic coactions of full quantum multiplicity arises in this way from a monoidal equivalence.*

Proof. Given the monoidal equivalence φ , it is obvious to construct the coaction δ_2 .

It remains to show the second statement. Let $\delta : B \rightarrow B \otimes A$ and $\delta_2 : B \rightarrow A_2 \otimes B$ be commuting ergodic coactions of full quantum multiplicity. Denote by \mathcal{B} the unital $*$ -subalgebra of B generated by the spectral subspaces of δ . Using Theorem 3.9, we get a unitary fiber functor φ on \mathbb{G} and we may assume

that \mathcal{B} and δ are constructed from φ as in Theorem 3.9. In particular, \mathcal{B} is generated by the coefficients of $X^x \in B(H_x, H_{\varphi(x)}) \otimes \mathcal{B}$.

Because δ and δ_2 commute, the element $(\iota \otimes \delta_2)(X^x)(X^x)_{13}^*$ is invariant under $(\iota \otimes \iota \otimes \delta)$. Since δ is ergodic and X^x unitary, we get a unitary element $U^{\varphi(x)} \in B(H_{\varphi(x)}) \otimes A_2$ such that $(\iota \otimes \delta_2)(X^x) = U_{12}^{\varphi(x)} X_{13}^x$. Because δ_2 is a coaction, we easily compute that $U^{\varphi(x)}$ is a unitary corepresentation of \mathbb{G}_2 .

It remains to show that $\{U^{\varphi(x)} \mid x \in \widehat{\mathbb{G}}\}$ is a complete set of irreducible unitary corepresentations of \mathbb{G}_2 and that φ is a monoidal equivalence.

Assume that $S \in \text{Mor}(\varphi(x), \varphi(y))$. The element $(X^y)^*(S \otimes 1)X^x \in B(H_x, H_y) \otimes B$ is invariant under $\iota \otimes \delta_2$, so it has the form $T \otimes 1$, with $T \in B(H_x, H_y)$. It follows that $T \in \text{Mor}(x, y) = \delta_{x,y}\mathbb{C}$ and hence, $S \in \delta_{x,y}\mathbb{C}$. So, the $U^{\varphi(x)}$ are mutually inequivalent irreducible corepresentations of \mathbb{G}_2 .

In order to show that the set $\{U^{\varphi(x)} \mid x \in \widehat{\mathbb{G}}\}$ exhausts all irreducible corepresentations of \mathbb{G}_2 , it suffices to show, for all $a \in \mathcal{A}_2$, that $(\iota \otimes h_2)((1 \otimes a)U^{\varphi(x)}) = 0$ for all $x \in \widehat{\mathbb{G}}$, implies $a = 0$. But, given the formula for δ_2 , we get $(h_2 \otimes \iota)((a \otimes 1)\delta_2(x)) = 0$ for all $x \in B$. Since δ_2 is of full quantum multiplicity, this implies that $a = 0$.

It remains to show that φ is a monoidal equivalence. For this, it suffices to show that $\text{Mor}(\varphi(x), \varphi(y) \oplus \varphi(z)) = \varphi(\text{Mor}(x, y \oplus z))$. If $S \in \text{Mor}(x, y \oplus z)$, we use the multiplicativity of δ_2 to obtain

$$\begin{aligned} (\varphi(S) \otimes 1 \otimes 1)U_{12}^{\varphi(x)} X_{13}^x &= (\iota \otimes \iota \otimes \delta_2)((\varphi(S) \otimes 1)X^x) = (\iota \otimes \iota \otimes \delta_2)(X_{13}^y X_{23}^z (S \otimes 1)) \\ &= U_{13}^{\varphi(y)} X_{14}^y U_{23}^{\varphi(z)} X_{24}^z (S \otimes 1 \otimes 1) = U_{13}^{\varphi(y)} U_{23}^{\varphi(z)} (\varphi(S) \otimes 1 \otimes 1) X_{13}^x . \end{aligned}$$

It follows that $S \in \text{Mor}(\varphi(x), \varphi(y) \oplus \varphi(z))$. The converse inclusion is shown analogously. \square

4 Unitary fiber functors preserving the dimension

We study in this section unitary fiber functors φ on a compact quantum group \mathbb{G} preserving the dimension, i.e. satisfying $\dim H_{\varphi(x)} = \dim H_x$ for all $x \in \widehat{\mathbb{G}}$. Taking into account Theorem 3.9, this comes down to the study of ergodic coactions of full quantum multiplicity satisfying $\text{mult}(x) = \dim(x)$ for all $x \in \widehat{\mathbb{G}}$.

We establish a relation between unitary fiber functors preserving the dimension (up to isomorphism) and the 2-cohomology of the dual, discrete quantum group $(\hat{A}, \hat{\Delta})$. The following definition is due to Landstad [19] and Wassermann [28], who consider it for the dual of a compact group.

Definition 4.1. A unitary element $\Omega \in M(\hat{A} \otimes \hat{A})$ is said to be a 2-cocycle if it satisfies

$$(\hat{\Delta} \otimes \iota)(\Omega)(\Omega \otimes 1) = (\iota \otimes \hat{\Delta})(\Omega)(1 \otimes \Omega) . \quad (4.1)$$

Two 2-cocycles Ω_1 and Ω_2 are said to differ by a coboundary if there exists a unitary $u \in M(\hat{A})$ such that $\Omega_2 = \hat{\Delta}(u)\Omega_1(u^* \otimes u^*)$. We denote this relation by $\Omega_1 \sim \Omega_2$ and observe that \sim is an equivalence relation on the set of 2-cocycles.

Remark 4.2. In the quantum setting, there is no reason that the product of two 2-cocycles is again a 2-cocycle. So, although we could define the 2-cohomology of $(\hat{A}, \hat{\Delta})$ as the set of equivalence classes of 2-cocycles, this set has no natural group structure.

Notation 4.3. We denote by $p_x, x \in \widehat{\mathbb{G}}$, the minimal central projections of $\hat{A} = \bigoplus_x B(H_x)$.

Remark 4.4. Up to coboundary, we can and will assume that a unitary 2-cocycle is *normalized*, i.e.

$$(p_\epsilon \otimes 1)\Omega = p_\epsilon \otimes 1 \quad \text{and} \quad (1 \otimes p_\epsilon)\Omega = 1 \otimes p_\epsilon .$$

Let Ω be a normalized unitary 2-cocycle on $(\hat{A}, \hat{\Delta})$, the dual of $\mathbb{G} = (A, \Delta)$. Denote

$$\Omega_{(2)} := (\hat{\Delta} \otimes \iota)(\Omega)(\Omega \otimes 1) = (\iota \otimes \hat{\Delta})(\Omega)(1 \otimes \Omega) .$$

It follows from Remark 3.2 that there is a unique unitary fiber functor φ_Ω on \mathbb{G} satisfying

$$H_{\varphi_\Omega(x)} = H_x, \quad \varphi_\Omega(S) = \Omega^* S, \quad \varphi_\Omega(T) = \Omega_{(2)}^* T,$$

for all $S \in \text{Mor}(x, y \oplus z)$ and $T \in \text{Mor}(a, x \oplus y \oplus z)$. Observe that we implicitly used that $B(H_y \otimes H_z)$ is an ideal in $M(\hat{A} \otimes \hat{A})$ and hence, $\Omega^* S$ is a well defined element of $B(H_x, H_y \otimes H_z)$ whenever $S \in \text{Mor}(x, y \oplus z)$.

From Proposition 3.12, we get a compact quantum group $(A_\Omega, \Delta_\Omega)$, whose dual $(\hat{A}_\Omega, \hat{\Delta}_\Omega)$ is given by

$$\hat{A}_\Omega = \oplus_x B(H_x) = \hat{A} \quad \text{and} \quad \hat{\Delta}_\Omega(a)\varphi_\Omega(S) = \varphi_\Omega(S)a \quad \text{for all } a \in B(H_x), S \in \text{Mor}(x, y \oplus z).$$

Also, φ_Ω becomes a monoidal equivalence between (A, Δ) and $(A_\Omega, \Delta_\Omega)$. Observe that

$$\hat{\Delta}_\Omega(a) = \Omega^* \hat{\Delta}(a) \Omega \quad \text{for all } a \in \hat{A}_\Omega = \hat{A}.$$

Proposition 4.5. *Let φ be a unitary fiber functor on a compact quantum group $\mathbb{G} = (A, \Delta)$ such that $\dim H_{\varphi(x)} = \dim H_x$ for all $x \in \hat{\mathbb{G}}$. Then there exists a normalized unitary 2-cocycle Ω on $(\hat{A}, \hat{\Delta})$, uniquely determined up to coboundary, such that φ is isomorphic with φ_Ω .*

Proof. Denote by $\delta : B \rightarrow B \otimes A$ the reduced ergodic coaction associated with φ by Theorem 3.9. Consider the generating unitaries $X^x \in B(H_x, H_{\varphi(x)}) \otimes B$ satisfying $(\iota \otimes \delta)(X^x) = X_{12}^x U_{13}^x$ for all $x \in \hat{\mathbb{G}}$.

Since $\dim H_{\varphi(x)} = \dim H_x$, we can take unitary elements $u_x : H_{\varphi(x)} \rightarrow H_x$. Take $u_\epsilon = 1$. Define $Y^x = (u_x \otimes 1)X^x$ and consider $Y := \oplus_x Y^x \in M(\hat{A} \otimes B)$. Because the element $(\hat{\Delta} \otimes \iota)(Y)Y_{23}^* Y_{13}^*$ is invariant under $(\iota \otimes \iota \otimes \delta)$, we find a unitary element $\Omega \in M(\hat{A} \otimes \hat{A})$ such that

$$(\hat{\Delta} \otimes \iota)(Y) = (\Omega \otimes 1)Y_{13}Y_{23}.$$

Applying $\hat{\Delta} \otimes \iota \otimes \iota$ and $\iota \otimes \hat{\Delta} \otimes \iota$ to this equality, we obtain that Ω is a unitary 2-cocycle on $(\hat{A}, \hat{\Delta})$.

It remains to show that φ and φ_Ω are isomorphic. Let $S \in \text{Mor}(x, y \oplus z)$. Then,

$$\begin{aligned} (S \otimes 1)Y^x &= (\hat{\Delta} \otimes \iota)(Y)(S \otimes 1) = (\Omega \otimes 1)Y_{13}Y_{23}(S \otimes 1) = (\Omega(u_y \otimes u_z) \otimes 1)X_{13}^y X_{23}^z (S \otimes 1) \\ &= (\Omega(u_y \otimes u_z) \otimes 1)(\varphi(S) \otimes 1)X^x = (\Omega(u_y \otimes u_z)\varphi(S)u_x^* \otimes 1)Y^x. \end{aligned}$$

Hence, $\varphi_\Omega(S) = \Omega^* S = (u_y \otimes u_z)\varphi(S)u_x^*$ for all $S \in \text{Mor}(x, y \oplus z)$.

It is obvious that φ_{Ω_1} is isomorphic with φ_{Ω_2} if and only if the 2-cocycles Ω_1 and Ω_2 differ by a coboundary. \square

Fix a normalized unitary 2-cocycle Ω on $(\hat{A}, \hat{\Delta})$ and consider the unitary fiber functor φ_Ω . Theorem 3.9 yields C^* -algebras B_r^Ω and B_u^Ω with ergodic coactions δ_r and δ_u of full quantum multiplicity. It is, of course, possible to describe these C^* -algebras directly in terms of Ω : they correspond to the Ω -twisted group C^* -algebras of $(\hat{A}, \hat{\Delta})$. Before we can prove such a statement, we have to introduce a few notations and a bit of terminology. Such Ω -twisted group C^* -algebras have been studied by Landstad [19] and Wassermann [28] when Ω is a unitary 2-cocycle on the dual of a compact group.

Notation 4.6. Denote by H the L^2 -space of the Haar state h on (A, Δ) . We consider A and \hat{A} as being represented on H . Denote by V the right regular corepresentation of (A, Δ) . The unitary V is multiplicative in the sense of [1] and belongs to $M(\hat{A} \otimes A)$. There exists a canonical unitary $u \in B(H)$ with $u^2 = 1$ such that $\hat{V} := (u \otimes 1)\Sigma V \Sigma(u \otimes 1)$ is a multiplicative unitary. We know that $uAu \subset A'$, $u\hat{A}u \subset \hat{A}'$ and

$$\hat{\Delta}(x) = V^*(1 \otimes x)V = \hat{V}(x \otimes 1)\hat{V}^* \quad \text{and} \quad (\iota \otimes \Delta)(V) = V_{12}V_{13}, \quad (\iota \otimes \hat{\Delta})(\hat{V}) = \hat{V}_{12}\hat{V}_{13}.$$

The unitaries V and \hat{V} satisfy the pentagonal equation

$$V_{12}V_{13}V_{23} = V_{23}V_{12} \quad \text{and} \quad \hat{V}_{12}\hat{V}_{13}\hat{V}_{23} = \hat{V}_{23}\hat{V}_{12}$$

Definition 4.7. An Ω -corepresentation of $(\hat{A}, \hat{\Delta})$ on a Hilbert space K is a unitary $X \in M(\hat{A} \otimes \mathcal{K}(K))$ satisfying

$$(\hat{\Delta} \otimes \iota)(X) = (\Omega \otimes 1)X_{13}X_{23}.$$

The following lemma can be checked immediately using the formulas in Notation 4.6 and in particular, the commutation relation $V_{12}\hat{V}_{23} = \hat{V}_{23}V_{12}$.

Lemma 4.8. Denoting $\tilde{\Omega} := (1 \otimes u)\Sigma\Omega^*\Sigma(1 \otimes u)$, the unitary $\tilde{\Omega}V \in M(\hat{A} \otimes \mathcal{K}(H))$ is an Ω -corepresentation. It is called the right regular Ω -corepresentation.

The next lemma is crucial to define the twisted quantum group C^* -algebras.

Lemma 4.9. Let X be an Ω -corepresentation of $(\hat{A}, \hat{\Delta})$ on K . Then, $[(\mu \otimes \iota)(X) \mid \mu \in \hat{A}^*]$ is a unital C^* -algebra.

Proof. Write $B := [(\mu \otimes \iota)(X) \mid \mu \in \hat{A}^*]$. From the defining relation for an Ω -corepresentation, it follows that B is an algebra acting non-degenerately on K . Since $(\hat{\Delta} \otimes \iota)(X)X_{23}^* = (\Omega \otimes 1)X_{13}$, we have

$$\begin{aligned} B &= [(\mu \otimes \eta \otimes \iota)((\hat{\Delta} \otimes \iota)(X)X_{23}^*) \mid \mu, \eta \in \hat{A}^*] = [(\mu \otimes \eta \otimes \iota)(\hat{V}_{12}X_{13}\hat{V}_{12}^*X_{23}^*) \mid \mu, \eta \in \hat{A}^*] \\ &= [(\mu \otimes \eta \otimes \iota)(X_{13}\hat{V}_{12}^*X_{23}^*) \mid \mu, \eta \in \hat{A}^*] = [(\mu \otimes \eta \otimes \iota)(X_{13}((1 \otimes \mathcal{K}(H))\hat{V}^*(\mathcal{K}(H) \otimes 1))_{12}X_{23}^*) \mid \mu, \eta \in \hat{A}^*] \\ &= [(\mu \otimes \eta \otimes \iota)(X_{13}(\mathcal{K}(H) \otimes \mathcal{K}(H) \otimes 1)X_{23}^*) \mid \mu, \eta \in \hat{A}^*] \\ &= [(\mu \otimes \iota)(X)(\eta \otimes \iota)(X)^* \mid \mu, \eta \in \hat{A}^*] = [BB^*]. \end{aligned}$$

Here we used the regularity of the multiplicative unitary \hat{V} . It follows that B is a C^* -algebra. Since Ω is normalized, we have $(p_\epsilon \otimes 1)X = p_\epsilon \otimes 1$ and hence, B is unital. \square

Definition 4.10. We define the unital C^* -algebras

$$C_r^*(\mathbb{G}, \Omega) := [(\mu \otimes \iota)(\tilde{\Omega}V) \mid \mu \in \hat{A}^*] \quad \text{and} \quad C_u^*(\mathbb{G}, \Omega) := [(\mu \otimes \iota)(X) \mid \mu \in \hat{A}^*],$$

where X denotes a universal Ω -corepresentation.

Remark that an Ω -corepresentation X on K is said to be *universal* if for any Ω -corepresentation Y on K_1 , there exists an isometry $v : K_1 \rightarrow K$ such that $X(1 \otimes v) = (1 \otimes v)Y$. It is clear that a universal Ω -corepresentation exists and that the C^* -algebra $C_u^*(\mathbb{G}, \Omega)$ is well defined up to an isomorphism.

Proposition 4.11. Denote by \mathcal{B} the unital $*$ -algebra associated by Theorem 3.9 with the unitary fiber functor φ_Ω . Consider the unitaries $X^x \in B(H_x) \otimes \mathcal{B}$ generating \mathcal{B} . Denote by B_r^Ω and B_u^Ω the associated reduced and universal C^* -algebra.

For any Ω -corepresentation X of $(\hat{A}, \hat{\Delta})$ on a Hilbert space K , we obtain a representation π of \mathcal{B} on K given by $(\iota \otimes \pi)(X^x) = (p_x \otimes 1)X$ for all $x \in \hat{\mathbb{G}}$. Taking $X = \tilde{\Omega}V$, we get an isomorphism $B_r^\Omega \cong C_r^*(\mathbb{G}, \Omega)$. Taking X to be a universal Ω -corepresentation, we get an isomorphism $B_u^\Omega \cong C_u^*(\mathbb{G}, \Omega)$.

Proof. It is immediate that the formula $(\iota \otimes \pi)(X^x) = (p_x \otimes 1)X$ defines a one-to-one correspondence between representations of \mathcal{B} and Ω -corepresentations of $(\hat{A}, \hat{\Delta})$. This already shows the isomorphism $B_u^\Omega \cong C_u^*(\mathbb{G}, \Omega)$.

Consider $\pi : \mathcal{B} \rightarrow B(H)$ given by $(\iota \otimes \pi)(X^x) = (p_x \otimes 1)\tilde{\Omega}V$. Denote by ω the unique invariant state on \mathcal{B} . To prove the isomorphism $B_r^\Omega \cong C_r^*(\mathbb{G}, \Omega)$, it suffices to define a faithful state ω_1 on $C_r^*(\mathbb{G}, \Omega)$ such that $\omega_1\pi = \omega$. Define $\alpha : C_r^*(\mathbb{G}, \Omega) \rightarrow M(\mathcal{K}(H) \otimes A) : \alpha(a) = V(a \otimes 1)V^*$. One verifies that $(\iota \otimes \alpha)(\tilde{\Omega}V) = (\tilde{\Omega}V)_{12}V_{13}$. Hence, $\alpha : C_r^*(\mathbb{G}, \Omega) \rightarrow C_r^*(\mathbb{G}, \Omega) \otimes A$ is a coaction satisfying $\alpha\pi = (\pi \otimes \iota)\delta$. It follows that $(\iota \otimes h)\alpha(a) \in \mathbb{C}1$ for all $a \in C_r^*(\mathbb{G}, \Omega)$. So, we can define ω_1 by the formula $\omega_1(a)1 = (\iota \otimes h)\alpha(a)$. Clearly, $\omega_1\pi = \omega$. \square

5 Monoidal equivalence for $A_o(F)$

Recall the following definition of the compact quantum group $A_o(F)$ [25].

Definition 5.1. For all $n \in \mathbb{N}$ and $F \in \text{GL}(n, \mathbb{C})$ with $F\bar{F} = c1 \in \mathbb{R}1$, $A_o(F)$ is defined as the universal compact quantum group generated by the coefficients of the corepresentation $U \in M_n(\mathbb{C}) \otimes A_o(F)$ with relations

$$U \text{ is unitary and } U = (F \otimes 1)\bar{U}(F^{-1} \otimes 1).$$

Then, $(A_o(F), U)$ is a compact (matrix) quantum group.

When the matrix F has dimension 2, we precisely obtain the quantized versions of the classical Lie group $\text{SU}(2)$, as considered by Woronowicz.

Definition 5.2. ([31]) Let $q \in [-1, 1] \setminus \{0\}$. We define $\text{SU}_q(2)$ to be the universal C^* -algebra generated by 2 elements α, γ such that $U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$ is a unitary corepresentation. Then $(\text{SU}_q(2), U)$ is a compact (matrix) quantum group.

Observe that $\text{SU}_q(2) \cong A_o\left(\begin{smallmatrix} 0 & 1 \\ -q^{-1} & 0 \end{smallmatrix}\right)$ and that $\dim_q(U) = |q + \frac{1}{q}|$.

Banica [3] has shown that the irreducible corepresentations of $A_o(F)$ can be labelled by \mathbb{N} , in such a way that the fusion rules are identical to the fusion rules of the compact Lie group $\text{SU}(2)$. In particular, $\text{Mor}(\epsilon, U \bar{\otimes} U)$ is one-dimensional and generated by

$$t_F := \sum_i e_i \otimes F e_i,$$

where (e_i) is the standard basis of \mathbb{C}^n .

Theorem 5.3. Let $F \in \text{GL}(n, \mathbb{C})$ with $F\bar{F} = c1$ and $c \in \mathbb{R}$. Consider $\mathbb{G} = A_o(F)$.

- Take $F_1 \in \text{GL}(n_1, \mathbb{C})$ satisfying $F_1\bar{F}_1 = c_11$ and $\frac{c}{\text{Tr}(F^*F)} = \frac{c_1}{\text{Tr}(F_1^*F_1)}$. There exists a unitary fiber functor φ_{F_1} on \mathbb{G} , uniquely determined up to isomorphism, such that

$$\varphi\left(\frac{1}{\sqrt{\text{Tr}(F^*F)}}t_F\right) = \frac{1}{\sqrt{\text{Tr}(F_1^*F_1)}}t_{F_1}. \quad (5.1)$$

- Every unitary fiber functor φ on \mathbb{G} is isomorphic with one of the form φ_{F_1} . Moreover, φ_{F_1} is isomorphic with φ_{F_2} if and only if $n_1 = n_2$ and there exists a unitary $v \in \text{U}(n_1)$ and a $\lambda \neq 0$ such that $F_2 = \lambda v F_1 v^t$.

Proof. Take a parameter $\beta \in \mathbb{R} \setminus \{0\}$. Consider the universal graded C^* -algebra $(A(n, m))_{n, m \in \mathbb{N}}$ satisfying $A_{n, m} = \{0\}$ if $n - m$ is odd and generated by elements $v(r, s) \in A(r + s, r + s + 2)$ with relations (denoting 1_n the unit of the C^* -algebra $A(n) := A(n, n)$)

$$\begin{aligned} v(r, s)^*v(r, s) &= 1_{r+s} \\ v(r, s+1)^*v(r+1, s) &= \beta 1_{r+s+1} \\ v(r, k+l+2)v(r+k, l) &= v(r+k+2, l)v(r, k+l) \\ v(r, k+l+2)^*v(r+k+2, l) &= v(r+k, l)v(r, k+l)^* \end{aligned}$$

Take $F \in \text{GL}(n, \mathbb{C})$ with $F\bar{F} = c1$ and $c \in \mathbb{R}$. Put $\beta = \frac{c}{\text{Tr}(F^*F)}$. Let $\mathbb{G} = (A_o(F), \Delta)$ and denote by U^n the n -fold tensor product of the fundamental corepresentation, with the convention that $U^0 = \epsilon$. Take the isometric $t \in \text{Mor}(\epsilon, U^2)$ given by $t = \frac{1}{\sqrt{\text{Tr}(F^*F)}}t_F$. Then, $(t^* \otimes 1)(1 \otimes t) = \beta 1$.

We get a natural $*$ -homomorphism

$$\pi : (A(n, m))_{n, m \in \mathbb{N}} \rightarrow (\text{Mor}(U^n, U^m))_{n, m \in \mathbb{N}}$$

given by $\pi(v(r, s)) = 1_r \otimes t \otimes 1_s$. Because of Proposition 1 in [2], π is surjective. It follows from the comments after Théorème 4 in [3], that π is faithful on $A(n, n)$ for all n . But then, π is faithful on $A(n, m)$ because

$$\pi(T) = 0 \Leftrightarrow \pi(T^*T) = 0 \Leftrightarrow T^*T = 0 \Leftrightarrow T = 0.$$

We conclude that π is a *-isomorphism.

Take $F_1 \in \text{GL}(n_1, \mathbb{C})$ satisfying $F_1 \overline{F_1} = c_1 1$ and $\frac{c}{\text{Tr}(F^*F)} = \frac{c_1}{\text{Tr}(F_1^*F_1)}$. Write $K = \mathbb{C}^{n_1}$ and denote by K^n the n -fold tensor product of K , with the convention that $K^0 = \mathbb{C}$. From the preceding discussion, we obtain a faithful *-homomorphism

$$\pi : (\text{Mor}(U^n, U^m))_{n, m \in \mathbb{N}} \rightarrow (\text{B}(K^n, K^m))_{n, m \in \mathbb{N}}$$

satisfying $\pi(t) = \frac{1}{\sqrt{\text{Tr}(F_1^*F_1)}} t_{F_1}$, $\pi(1) = 1$ and $\pi(1 \otimes T \otimes 1) = 1 \otimes \pi(T) \otimes 1$ for all T .

We choose a concrete identification $\widehat{\mathbb{G}} = \mathbb{N}$ as follows. We define $P_n \in \text{Mor}(U^n, U^n)$ as the unique projection satisfying $P_n T = 0$ for all $r < n$ and all $T \in \text{Mor}(U^r, U^n)$. We define U_n as the restriction of U^n to the image of P_n . We then identify

$$\text{Mor}(n_1 \oplus \cdots \oplus n_r, m_1 \oplus \cdots \oplus m_k) = (P_{m_1} \otimes \cdots \otimes P_{m_k}) \text{Mor}(U^{n_1+\cdots+n_r}, U^{m_1+\cdots+m_k})(P_{n_1} \otimes \cdots \otimes P_{n_r}).$$

Define $H_{\varphi(n)} := \pi(P_n)K^n$ and define $\varphi(S)$ by restricting π to $\text{Mor}(n_1 \oplus \cdots \oplus n_r, m_1 \oplus \cdots \oplus m_k)$. It is now obvious that φ is a unitary fiber functor on \mathbb{G} .

Suppose conversely that φ is a unitary fiber functor on $A_o(F)$. We continue to use the concrete identification $\widehat{\mathbb{G}} = \mathbb{N}$ introduced above. Up to isomorphism, we may assume that $H_{\varphi(1)} = \mathbb{C}^{n_1}$ and we denote $K = \mathbb{C}^{n_1}$. We define the *-homomorphism $\pi : (\text{Mor}(U^n, U^m))_{n, m} \rightarrow (\text{B}(K^n, K^m))_{n, m}$ by restricting φ . Define the matrix F_1 such that $\varphi(t_F) = t_{F_1}$. Then, $F_1 \overline{F_1} = c_1 1$ with $c_1 = c$ and $\text{Tr}(F_1^*F_1) = \text{Tr}(F^*F)$. Since t generates the graded C*-algebra $(\text{Mor}(U^n, U^m))_{n, m}$, π coincides with the *-homomorphism constructed in the first part of the proof starting with F_1 . Denoting by $T_n \in \text{Mor}(n, 1 \oplus \cdots \oplus 1)$ the embedding, we get unitary operators $\varphi(T_n) : H_{\varphi(n)} \rightarrow \pi(P_n)K^n$ who implement the isomorphism between φ and φ_{F_1} . \square

Corollary 5.4. *Let $F \in \text{GL}(n, \mathbb{C})$ with $F \overline{F} = c 1$ and $c \in \mathbb{R}$. Consider $\mathbb{G} = A_o(F)$. A compact quantum group \mathbb{G}_1 is monoidally equivalent with \mathbb{G} if and only if there exists $F_1 \in \text{GL}(n_1, \mathbb{C})$ satisfying $F_1 \overline{F_1} = c_1 1$ and $\frac{c}{\text{Tr}(F^*F)} = \frac{c_1}{\text{Tr}(F_1^*F_1)}$ such that $\mathbb{G}_1 \cong A_o(F_1)$.*

Proof. The unitary fiber functor φ_{F_1} constructed in Theorem 5.3 yields a monoidal equivalence $A_o(F) \underset{\text{mon}}{\sim} A_o(F_1)$. Since these fiber functors φ_{F_1} are, up to isomorphism, the only unitary fiber functors on $A_o(F)$, we are done. \square

So, we exactly know when the compact quantum groups $A_o(F_1)$ and $A_o(F_2)$ are monoidally equivalent. If this is the case, Proposition 3.13 provides us with a universal C*-algebra B_u and a pair of ergodic coactions of full quantum multiplicity. It is possible to give an intrinsic description of this C*-algebra B_u .

Theorem 5.5. *Let $F_i \in \text{GL}(n_i, \mathbb{C})$ by such that $F_i \overline{F_i} = c_i 1$ for $c_i \in \mathbb{R}$ ($i = 1, 2$). Assume that $c_1 = c_2$ and $\text{Tr}(F_1^*F_1) = \text{Tr}(F_2^*F_2)$.*

- Denote by $A_o(F_1, F_2)$ the universal unital C*-algebra generated by the coefficients of

$$Y \in M_{n_2, n_1}(\mathbb{C}) \otimes A_o(F_1, F_2) \quad \text{with relations } Y \text{ unitary and } Y = (F_2 \otimes 1) \overline{Y} (F_1^{-1} \otimes 1).$$

Then, $A_o(F_1, F_2) \neq 0$ and there exists a unique pair of commuting universal ergodic coactions of full quantum multiplicity, δ_1 of $A_o(F_1)$ and δ_2 of $A_o(F_2)$, such that

$$(\iota \otimes \delta_1)(Y) = Y_{12}(U_1)_{13} \quad \text{and} \quad (\iota \otimes \delta_2)(Y) = (U_2)_{12} Y_{13}.$$

Here, U_i denotes the fundamental corepresentation of $A_o(F_i)$.

- $(A_o(F_1, F_2), \delta_1, \delta_2)$ is isomorphic with the C^* -algebra B_u and the coactions thereon given by Proposition 3.13 and the monoidal equivalence $A_o(F_1) \underset{\text{mon}}{\sim} A_o(F_2)$ of Corollary 5.4.
- The multiplicity of the fundamental corepresentation U_1 in the coaction δ_1 equals n_2 .

Remark that the condition on the matrices F_1 and F_2 is not really less general than the condition in Theorem 5.3, but just a normalization: if $\frac{c_1}{\text{Tr}(F_1^* F_1)} = \frac{c_2}{\text{Tr}(F_2^* F_2)}$, we multiply F_2 by a scalar and obtain $c_1 = c_2$ and $\text{Tr}(F_1^* F_1) = \text{Tr}(F_2^* F_2)$.

Proof. Take F_i as in the statement of the theorem and denote by φ the unitary fiber functor on $\mathbb{G}_1 := A_o(F_1)$ given by Theorem 5.3 and (5.1). We continue to use the identification of $\widehat{\mathbb{G}}_1$ with \mathbb{N} . Theorem 3.9 provides us with a $*$ -algebra \mathcal{B} generated by the coefficients of unitary operators $X^n \in \text{B}(H_n, H_{\varphi(n)}) \otimes \mathcal{B}$, $n \in \mathbb{N}$. Define $X := X^1$. Since every element of $\widehat{\mathbb{G}}_1$ appears in a tensor power of the fundamental corepresentation and since $\bar{1} = 1$, it follows that the coefficients of X generate \mathcal{B} as an algebra. Moreover (3.7) precisely says that $X = (F_2 \otimes 1) \bar{X} (F_1^{-1} \otimes 1)$. It follows that $A_o(F_1, F_2) \neq 0$. Denoting by \mathcal{C} the unital $*$ -subalgebra of $A_o(F_1, F_2)$ generated by the coefficients of Y , we get a surjective $*$ -homomorphism $\rho : \mathcal{C} \rightarrow \mathcal{B}$ satisfying $(\iota \otimes \rho)(Y) = X$. It remains to show that ρ is a $*$ -isomorphism.

Denote by U the fundamental corepresentation of $A_o(F_1)$ on $H = \mathbb{C}^{n_1}$ and by U^n its n -th tensor power, on H^n . As in the proof of Theorem 5.3, we denote by $P_n \in \text{Mor}(U^n, U^n)$ the projection onto the irreducible corepresentation U_n . Denote $K = \mathbb{C}^{n_2}$ and denote by K^n the n -th tensor power of K . Recall that we constructed a faithful $*$ -homomorphism $\pi : (\text{Mor}(U^n, U^m))_{n,m} \rightarrow (\text{B}(K^n, K^m))_{n,m}$ and that φ is defined by restricting π to the relevant subspaces. The graded C^* -algebra $(\text{Mor}(U^n, U^m))_{n,m}$ is generated by the elements $t \in \text{Mor}(\epsilon, U^2)$ and $1_r \otimes t \otimes 1_s$.

Put $Y^n = Y_{1,n+1} \cdots Y_{n,n+1} \in \text{B}(H^n, K^n) \otimes A_o(F_1, F_2)$. Since $Y^2(t \otimes 1) = \pi(t) \otimes 1$, it follows that $Y^m(S \otimes 1) = (\pi(S) \otimes 1) Y^m$ for all $S \in \text{Mor}(U^n, U^m)$. Define a linear map $\gamma : \mathcal{B} \rightarrow \mathcal{C}$ by the formula $(\iota \otimes \gamma)(X^n) = Y^n(P_n \otimes 1) = (\pi(P_n) \otimes 1) Y^n$. We claim that γ is multiplicative. Take $a, n, m \in \mathbb{N}$ and $S \in \text{Mor}(a, n \oplus m)$. Then,

$$\begin{aligned}
(\iota \otimes \iota \otimes \gamma)(X_{13}^n X_{23}^m)(S \otimes 1) &= (\varphi(S) \otimes 1)(\iota \otimes \gamma)(X^a) \\
&= (\varphi(S) \otimes 1)(P_a \otimes 1) Y^a \\
&= (\pi(S) \otimes 1) Y^a = Y^{n+m}(S \otimes 1) \\
&= (Y^n(P_n \otimes 1))_{13} (Y^m(P_m \otimes 1))_{23} (S \otimes 1) \\
&= (\iota \otimes \gamma)(X^n)_{13} (\iota \otimes \gamma)(X^m)_{23} (S \otimes 1).
\end{aligned}$$

Because $(\iota \otimes \gamma \rho)(Y) = Y$, because γ is multiplicative and because the coefficients of Y generate \mathcal{C} as an algebra, it follows that γ is the inverse of ρ . So, ρ is indeed a $*$ -isomorphism. \square

Remark 5.6. A combination of Proposition 6.2.6 in [6] and the results in [7] yields an alternative proof for the fact that $A_o(F_1, F_2) \neq 0$.

Remark 5.7. Combining the co-amenability of $\text{SU}_q(2)$ with Remark 2.6, we obtain the following. If either F_1 or F_2 is a two by two matrix (but not necessarily both), the (unique) invariant state on the universal C^* -algebra $A_o(F_1, F_2)$ is faithful. So, the ergodic coactions of $A_o(F_1)$ and $A_o(F_2)$ are both universal and reduced. This is somehow remarkable, because $A_o(F)$ is not co-amenable when F has dimension strictly bigger than 2.

Moreover, still supposing that either F_1 or F_2 is a two by two matrix, we also get that $A_o(F_1, F_2)$ is a nuclear C^* -algebra. Indeed, supposing that $F_2 \in \text{GL}(2, \mathbb{C})$, $A_o(F_1, F_2)$ is Morita equivalent with its double crossed product, i.e. the crossed product of the compact operators with the co-amenable quantum group $A_o(F_2)$, yielding a nuclear C^* -algebra. See [13] for details.

A precise parameterisation of the unitary fiber functors (and hence, the ergodic coactions of full quantum multiplicity) on the quantum groups $A_o(F)$, amounts to the study of matrices $F \in \text{GL}(n, \mathbb{C})$ satisfying $F \bar{F} = \pm 1$, up to the equivalence relation

$$F_1 \sim F \quad \text{if and only if there exists a unitary } v \in \text{U}(n) \text{ such that } F_1 = v F v^t. \quad (5.2)$$

Let $F \in \text{GL}(n, \mathbb{C})$ with $F\bar{F} = \pm 1$. Denote $H = \mathbb{C}^n$ and $J : H \rightarrow H$ the complex conjugation. We rather look at the anti-linear operator $\mathcal{F} = JF$, satisfying $\mathcal{F}^2 = \pm 1$. In the case where $\mathcal{F}^2 = 1$, our data come down to giving a real vector space together with a Hilbert space structure on the complexification. In the case where $\mathcal{F}^2 = -1$, H becomes a right module on the quaternions such that, restricting the quaternions to \mathbb{C} , we get a Hilbert space. In particular H is even-dimensional.

It is then straightforward to provide a fundamental domain for the equivalence relation (5.2) (see e.g. [33]). Take $F \in \text{GL}(n, \mathbb{C})$ with $F\bar{F} = \pm 1$. Let $\mathcal{F} = \mathcal{J}|F|$ be the polar decomposition of \mathcal{F} . Then, \mathcal{J} is an anti-unitary, $\mathcal{J}^2 = \pm 1$ and $\mathcal{J}|F|\mathcal{J}^* = |F|^{-1}$. Define $H_<$ as the subspace of H spanned by the eigenvectors of $|F|$ with eigenvalue $\lambda < 1$. Define $H_> = \mathcal{J}H_<$, which is as well the subspace of H spanned by the eigenvectors of $|F|$ with eigenvalue $\lambda > 1$. Finally, let H_1 be the subset of eigenvectors of $|F|$ with eigenvalue 1. Take an orthonormal basis ξ_1, \dots, ξ_k of $H_<$ of eigenvectors of $|F|$ with eigenvalues $0 < \lambda_1 \leq \dots \leq \lambda_k < 1$.

If $F\bar{F} = 1$, we have $\mathcal{J}^2 = 1$ and we take an orthonormal basis μ_1, \dots, μ_{n-2k} for H_1 of real vectors: $\mathcal{J}\mu_i = \mu_i$. If (e_i) denotes the standard basis for \mathbb{C}^n and $w : (e_i) \rightarrow (\xi_i, \mathcal{J}\xi_i, \mu_i)$ denotes the transition unitary, we find that

$$w^t F w = \begin{pmatrix} 0 & D(\lambda_1, \dots, \lambda_k) & 0 \\ D(\lambda_1, \dots, \lambda_k)^{-1} & 0 & 0 \\ 0 & 0 & 1_{n-2k} \end{pmatrix}. \quad (5.3)$$

Here, $D(\lambda_1, \dots, \lambda_k)$ denotes the diagonal matrix with the λ_i along the diagonal.

If $F\bar{F} = -1$, we have $\mathcal{J}^2 = -1$, H_1 has even dimension and we take an orthonormal basis μ_1, \dots, μ_r , $\mathcal{J}\mu_1, \dots, \mathcal{J}\mu_r$ for H_1 . If $w : (e_i) \rightarrow (\xi_i, \mu_i, \mathcal{J}\xi_i, \mathcal{J}\mu_i)$ denotes the transition unitary, we find that

$$w^t F w = \begin{pmatrix} 0 & D(\lambda_1, \dots, \lambda_{n/2}) \\ -D(\lambda_1, \dots, \lambda_{n/2})^{-1} & 0 \end{pmatrix}, \quad (5.4)$$

where $0 < \lambda_1 \leq \dots \leq \lambda_{n/2} \leq 1$.

Since the spectrum of F^*F is invariant under the equivalence relation (5.2), a fundamental domain is given by the matrices in (5.3) with $2k \leq n$ and $0 < \lambda_1 \leq \dots \leq \lambda_k < 1$ and the matrices in (5.4) with $0 < \lambda_1 \leq \dots \leq \lambda_{n/2} \leq 1$.

Corollary 5.8. *Let $0 < q \leq 1$. For every even natural number n with $2 \leq n \leq q + \frac{1}{q}$, the quantum group $\text{SU}_q(2)$ admits an ergodic coaction of full quantum multiplicity such that the multiplicity of the fundamental corepresentation is n . If $-1 \leq q < 0$, the same statement holds for every natural number n with $2 \leq n \leq |q + \frac{1}{q}|$.*

Corollary 5.9. *Let $F \in \text{GL}(n, \mathbb{C})$ with $F\bar{F} = c1$ and $c \in \mathbb{R}$. Denote $\mathbb{G} = A_o(F)$. For all $F_1 \in \text{GL}(n_1, \mathbb{C})$ satisfying $F_1\bar{F}_1 = c1$ and $\text{Tr}(F_1^*F_1) = \text{Tr}(F^*F)$, we denote by δ_{F_1} the coaction of \mathbb{G} on $A_o(F, F_1)$ defined in Theorem 5.5.*

- *Up to isomorphism, the δ_{F_1} yield all universal ergodic coactions of full quantum multiplicity of \mathbb{G} . Moreover, δ_{F_1} is isomorphic with δ_{F_2} if and only if $n_1 = n_2$ and there exists a unitary $v \in \text{U}(n_1)$ such that $F_2 = vF_1v^t$.*
- *For all F_1 as above with $n_1 = n$, we denote by $\Omega(F_1)$ the unitary 2-cocycle on the dual of \mathbb{G} associated with the unitary fiber functor φ_{F_1} . The $\Omega(F_1)$ describe, up to coboundary, all unitary 2-cocycles on the dual of \mathbb{G} . Moreover $\Omega(F_1)$ and $\Omega(F_2)$ differ by a coboundary if and only if there exists a unitary $v \in \text{U}(n)$ such that $F_2 = vF_1v^t$.*

Corollary 5.10. *Every unitary 2-cocycle on the dual of $\text{SU}_q(2)$ is a coboundary.*

6 Monoidal equivalence for $A_u(F)$

In this section, we prove, for the unitary quantum groups $A_u(F)$ studied by Banica [3], analogous results as for $A_o(F)$.

Again, we give a complete classification of unitary fiber functors, monoidally equivalent quantum groups, ergodic coactions of full quantum multiplicity and 2-cohomology.

Recall the following definition.

Definition 6.1. For all $n \in \mathbb{N}$ and $F \in \text{GL}(n, \mathbb{C})$, we define $A_u(F)$ as the universal compact quantum group generated by the coefficients of the corepresentation $U \in M_n(\mathbb{C}) \otimes A_u(F)$ with relations

$$U \text{ and } (F \otimes 1)\overline{U}(F^{-1} \otimes 1) \text{ are unitary .}$$

Then, $(A_u(F), U)$ is a compact (matrix) quantum group.

Banica [3] has shown that the irreducible corepresentations of $A_u(F)$ can be labelled by the free monoid $\mathbb{N} \star \mathbb{N}$ generated by α and β . He also computed the corresponding fusion rules.

Defining $U^\alpha := U$ and $U^\beta := (F \otimes 1)\overline{U}(F^{-1} \otimes 1)$, the spaces $\text{Mor}(\epsilon, U^\alpha \oplus U^\beta)$ and $\text{Mor}(\epsilon, U^\beta \oplus U^\alpha)$ are one-dimensional and generated by

$$t_F := \sum_i e_i \otimes F e_i \quad \text{resp.} \quad s_F := \sum_i e_i \otimes \overline{F}^{-1} e_i .$$

Theorem 6.2. Let $F \in \text{GL}(n, \mathbb{C})$ be normalized such that $\text{Tr}(F^*F) = \text{Tr}((F^*F)^{-1})$. Let $\mathbb{G} = A_u(F)$.

- If $F_1 \in \text{GL}(n_1, \mathbb{C})$ satisfies $\text{Tr}(F_1^*F_1) = \text{Tr}((F_1^*F_1)^{-1}) = \text{Tr}(F^*F)$, there exists a unitary fiber functor φ_{F_1} on \mathbb{G} , uniquely determined up to isomorphism, such that

$$\varphi(t_F) = t_{F_1} \quad \text{and} \quad \varphi(s_F) = s_{F_1} .$$

- Every unitary fiber functor φ on \mathbb{G} is isomorphic with one of the form φ_{F_1} . Moreover, φ_{F_1} is isomorphic with φ_{F_2} if and only if $n_1 = n_2$ and there exist unitary elements $v, w \in \text{U}(n_1)$ such that $F_2 = vF_1w$.

Proof. Let $\mathbb{N} \star \mathbb{N}$ be the free monoid generated by α and β . Denote by e the empty word. Elements of $\mathbb{N} \star \mathbb{N}$ are words in α and β .

Take a parameter $c > 0$. Let $(A(p, q))_{p, q \in \mathbb{N} \star \mathbb{N}}$ be the universal graded C^* -algebra generated by elements

$$V_x(p, q) \in A(pq, pxq) \quad \text{for} \quad p, q \in \mathbb{N} \star \mathbb{N}, \quad x \in \{\alpha\beta, \beta\alpha\}$$

with relations (denoting by 1_p the unit of the C^* -algebra $A(p) := A(p, p)$)

$$\begin{aligned} V_x(p, q)^* V_x(p, q) &= 1_{pq} \\ V_{\alpha\beta}(p, \alpha q)^* V_{\beta\alpha}(p\alpha, q) &= c 1_{p\alpha q} \\ V_{\beta\alpha}(p, \beta q)^* V_{\alpha\beta}(p\beta, q) &= c 1_{p\beta q} \\ V_y(p, qxr) V_x(pq, r) &= V_x(pyq, r) V_y(p, qr) \\ V_x(p, qyr)^* V_y(pxq, r) &= V_y(pq, r) V_x(p, qr)^* \end{aligned}$$

Take $F \in \text{GL}(n, \mathbb{C})$ normalized in such a way that $\text{Tr}(F^*F) = \text{Tr}((F^*F)^{-1})$. Put $c = \text{Tr}(F^*F)$. Consider the quantum group $\mathbb{G} = (A_u(F), \Delta)$. Define for every $p \in \mathbb{N} \star \mathbb{N}$ the unitary corepresentation U^p of \mathbb{G} inductively by $U^{pq} := U^p \oplus U^q$.

Defining $t \in \text{Mor}(\epsilon, U^{\alpha\beta})$ and $s \in \text{Mor}(\epsilon, U^{\beta\alpha})$ by the formulas $t = \frac{1}{\sqrt{\text{Tr}(F^*F)}} t_F$ and $s = \frac{1}{\sqrt{\text{Tr}(F^*F)}} s_F$, we get a natural $*$ -homomorphism

$$\pi : (A(p, q))_{p, q \in \mathbb{N} \star \mathbb{N}} \rightarrow (\text{Mor}(U^p, U^q))_{p, q \in \mathbb{N} \star \mathbb{N}}$$

given by $\pi(V_{\alpha\beta}(p, q)) = 1_p \otimes t \otimes 1_q$ and $\pi(V_{\beta\alpha}(p, q)) = 1_p \otimes s \otimes 1_q$. It follows from Proposition 4, the proof of Théorème 1 and Proposition 3 in [3] that π is an isomorphism of C^* -algebras.

Take $F_1 \in \mathrm{GL}(n_1, \mathbb{C})$ satisfying $\mathrm{Tr}(F_1^* F_1) = \mathrm{Tr}((F_1^* F_1)^{-1}) = \mathrm{Tr}(F^* F)$. Write $K^\alpha = K^\beta = \mathbb{C}^{n_1}$ and define inductively K^p , for all $p \in \mathbb{N} \star \mathbb{N}$ such that $K^{pq} = K^p \otimes K^q$. We take $K^e = \mathbb{C}$. From the preceding discussion, we obtain a faithful $*$ -homomorphism

$$\pi : (\mathrm{Mor}(U^p, U^q))_{p,q \in \mathbb{N} \star \mathbb{N}} \rightarrow (\mathrm{B}(K^p, K^q))_{p,q \in \mathbb{N} \star \mathbb{N}}$$

satisfying $\pi(t) = \frac{1}{\sqrt{\mathrm{Tr}(F_1^* F_1)}} t_{F_1}$, $\pi(s) = \frac{1}{\sqrt{\mathrm{Tr}(F_1^* F_1)}} s_{F_1}$ and $\pi(1 \otimes S \otimes 1) = 1 \otimes \pi(S) \otimes 1$ for all S .

We choose a concrete identification of $\widehat{\mathbb{G}}$ with $\mathbb{N} \star \mathbb{N}$ as follows. We define, for $p \in \mathbb{N} \star \mathbb{N}$, $P_p \in \mathrm{Mor}(U^p, U^p)$ as the unique projection satisfying $P_p T = 0$ for all $r \in \mathbb{N} \star \mathbb{N}$ with $\mathrm{length} r < \mathrm{length} p$ and all $T \in \mathrm{Mor}(U^r, U^p)$. We define U_p as the restriction of U^p to the image of P_p . We then identify

$$\mathrm{Mor}(p_1 \oplus \cdots \oplus p_r, q_1 \oplus \cdots \oplus q_k) = (P_{q_1} \otimes \cdots \otimes P_{q_k}) \mathrm{Mor}(U^{p_1 \cdots p_r}, U^{q_1 \cdots q_k}) (P_{p_1} \otimes \cdots \otimes P_{p_r}).$$

Define $H_{\varphi(p)} := \pi(P_p) K^p$ and define $\varphi(S)$ by restricting π to $\mathrm{Mor}(p_1 \oplus \cdots \oplus p_r, q_1 \oplus \cdots \oplus q_k)$. It is obvious that φ is a unitary fiber functor.

The converse statement is proven in exactly the same way as in the proof of Theorem 5.3. \square

The next two results are proven in exactly the same way as the corresponding results for $A_o(F)$.

Corollary 6.3. *Let $F \in \mathrm{GL}(n, \mathbb{C})$ with $\mathrm{Tr}(F^* F) = \mathrm{Tr}((F^* F)^{-1})$ and consider $\mathbb{G} = A_o(F)$. A compact quantum group \mathbb{G}_1 is monoidally equivalent with \mathbb{G} if and only if there exists $F_1 \in \mathrm{GL}(n_1, \mathbb{C})$ satisfying $\mathrm{Tr}(F_1^* F_1) = \mathrm{Tr}((F_1^* F_1)^{-1}) = \mathrm{Tr}(F^* F)$ such that $\mathbb{G}_1 \cong A_o(F_1)$.*

So, we exactly know when the compact quantum groups $A_u(F_1)$ and $A_u(F_2)$ are monoidally equivalent. If this is the case, Proposition 3.13 provides us with a universal C^* -algebra B_u and a pair of ergodic coactions of full quantum multiplicity. It is again possible to give an intrinsic description of this C^* -algebra B_u . The proof is analogous to the proof of Theorem 5.5. Again, the fact that $A_u(F_1, F_2) \neq 0$ can be deduced from Proposition 6.2.6 in [6] and the results in [8].

Theorem 6.4. *Let $F_i \in \mathrm{GL}(n_i, \mathbb{C})$ by such that $\mathrm{Tr}(F_1^* F_1) = \mathrm{Tr}((F_1^* F_1)^{-1}) = \mathrm{Tr}(F_2^* F_2) = \mathrm{Tr}((F_2^* F_2)^{-1})$.*

- Denote by $A_u(F_1, F_2)$ the universal unital C^* -algebra generated by the coefficients of

$$X \in M_{n_2, n_1}(\mathbb{C}) \otimes A_u(F_1, F_2) \quad \text{with relations} \quad X \text{ and } (F_2 \otimes 1) \overline{X} (F_1^{-1} \otimes 1) \text{ are unitary.}$$

Then, $A_u(F_1, F_2) \neq 0$ and there exists a unique pair of commuting universal ergodic coactions of full quantum multiplicity, δ_1 of $A_u(F_1)$ and δ_2 of $A_u(F_2)$, such that

$$(\iota \otimes \delta_1)(X) = X_{12} (U_1)_{13} \quad \text{and} \quad (\iota \otimes \delta_2)(X) = (U_2)_{12} X_{13}.$$

Here, U_i denotes the fundamental corepresentation of $A_u(F_i)$.

- $(A_u(F_1, F_2), \delta_1, \delta_2)$ is isomorphic with the C^* -algebra B_u and the coactions thereon given by Proposition 3.13 and the monoidal equivalence $A_u(F_1) \underset{\mathrm{mon}}{\sim} A_u(F_2)$ of Corollary 6.3.

Remark 6.5. Exactly as in Corollary 5.9, a combination of Theorems 6.2 and 6.4 gives a complete classification of the ergodic coactions of full quantum multiplicity of $A_u(F)$ and of the 2-cohomology of the dual of $A_u(F)$.

A precise parameterisation of the unitary fiber functors on the quantum groups $A_u(F)$ is easy. If $F_1, F \in \mathrm{GL}(n, \mathbb{C})$, we write

$$F_1 \sim F \quad \text{if and only if there exist unitary } v, w \in \mathrm{U}(n) \text{ such that } F_1 = v F w.$$

We study matrices $F \in \mathrm{GL}(n, \mathbb{C})$ satisfying $\mathrm{Tr}(F^* F) = \mathrm{Tr}((F^* F)^{-1})$ up to the equivalence relation \sim . It is obvious that for any such F , there exist unique $0 < \lambda_1 \leq \cdots \leq \lambda_n$ satisfying $\sum_i \lambda_i^2 = \sum_i \lambda_i^{-2}$ such that $F \sim D(\lambda_1, \dots, \lambda_n)$. Here $D(\lambda_1, \dots, \lambda_n)$ denotes again the diagonal matrix with the λ_i along the diagonal.

References

- [1] S. BAAJ & G. SKANDALIS, Unitaires multiplicatifs et dualité pour les produits croisés de C^* -algèbres. *Ann. Scient. Ec. Norm. Sup.* **26** (1993), 425–488.
- [2] T. BANICA, Théorie des représentations du groupe quantique compact libre $O(n)$. *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996), 241–244.
- [3] T. BANICA, Le groupe quantique compact libre $U(n)$. *Commun. Math. Phys.* **190** (1997), 143–172.
- [4] T. BANICA, Quantum automorphism groups of homogeneous graphs. *J. Funct. Anal.*, to appear.
- [5] T. BANICA, Quantum automorphism groups of small metric spaces. *Pac. J. Math.*, to appear.
- [6] J. BICHON, Galois extension for a compact quantum group. *Preprint*.
- [7] J. BICHON, The representation category of the quantum group of a non-degenerate bilinear form. *Comm. Algebra* **31** (2003), 4831–4851.
- [8] J. BICHON, Corepresentation theory of universal cosovereign Hopf algebras. *Preprint*.
- [9] J. BICHON, Quantum automorphism groups of finite graphs. *Proc. Amer. Math Soc.* **131** (2003), 665–673.
- [10] F. BOCA, Ergodic actions of compact matrix pseudogroups on C^* -algebras. In *Recent advances in operator algebras (Orléans, 1992)*. *Astérisque* **232** (1995), 93–109.
- [11] A. BRUGUIÈRES, Dualité tannakienne pour les quasi-groupoïdes quantiques. *Comm. Algebra* **25** (1997), 737–767.
- [12] J.S. CARTER, D.E. FLATH & M. SAITO, The classical and quantum $6j$ -symbols. *Mathematical Notes* **43**, Princeton University Press, Princeton, NJ, 1995.
- [13] S. DOPLICHER, R. LONGO, J.E. ROBERTS & L. ZSIDÓ, A remark on quantum group actions and nuclearity. *Rev. Math. Phys.* **14** (2002), 787–796.
- [14] V.G. DRINFEL'D, Quantum groups. *Proceedings of the International Congress of Mathematicians*, AMS, Providence, 1987, pp. 798–820.
- [15] P. ETINGOF & S. GELAKI, On cotriangular Hopf algebras. *Amer. J. Math.* **123** (2001), 699–713.
- [16] R. HØEGH-KROHN, M.B. LANDSTAD & E. STØRMER, Compact ergodic groups of automorphisms. *Ann. of Math. (2)* **114** (1981), 75–86.
- [17] M. JIMBO, A q -difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation. *Lett. Math. Phys.* **10** (1985), 63–69.
- [18] M. LANDSTAD, Simplicity of crossed products from ergodic actions of compact matrix pseudogroups. In *Recent advances in operator algebras (Orléans, 1992)*. *Astérisque* **232** (1995), 111–114.
- [19] M. LANDSTAD, Ergodic actions of nonabelian compact groups. In *Ideas and methods in mathematical analysis, stochastic, and applications (Oslo, 1988)*, Cambridge Univ. Press, Cambridge, 1992, pp. 365–388.
- [20] S. MACLANE, Categories for the working mathematician. *Graduate Texts in Mathematics* **5**, Springer-Verlag, New York–Berlin, 1971.
- [21] M. ROSSO, Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif. *Duke Math. J.* **61** (1990), 11–40.
- [22] P. SCHAUENBURG, Hopf bi-Galois extensions. *Comm. Algebra* **24** (1996), 3797–3825.

- [23] R. TOMATSU, Compact quantum ergodic systems. *Preprint* (2004).
- [24] K.-H. ULBRICH, Galois extensions as functors of comodules. *Manuscripta Math.* **59** (1987), 391–397.
- [25] A. VAN DAELE & S. WANG, Universal quantum groups. *Internat. J. Math.* **7** (1996), 255–263.
- [26] S. WANG, Ergodic actions of universal quantum groups on operator algebras. *Comm. Math. Phys.* **203** (1999), 481–498.
- [27] A. WASSERMANN, Ergodic actions of compact groups on operator algebras. I. General theory. *Ann. of Math. (2)* **130** (1989), 273–319.
- [28] A. WASSERMANN, Ergodic actions of compact groups on operator algebras. II. Classification of full multiplicity ergodic actions. *Canad. J. Math.* **40** (1988), 1482–1527.
- [29] A. WASSERMANN, Ergodic actions of compact groups on operator algebras. III. Classification for $SU(2)$. *Invent. Math.* **93** (1988), 309–354.
- [30] S.L. WORONOWICZ, Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, North-Holland, Amsterdam, 1998, pp. 845–884.
- [31] S.L. WORONOWICZ, Twisted $SU(2)$ group. An example of a non-commutative differential calculus. *Publ. Res. Inst. Math. Sci.* **23** (1987), 117–181.
- [32] S.L. WORONOWICZ, Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups. *Invent. Math.* **93** (1988), 35–76.
- [33] S. YAMAGAMI, Fiber functors on Temperley-Lieb categories. *Preprint* (2004).